FREE FIELD CONSTRUCTION FOR THE ABF MODELS IN REGIME II

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Dedicated to Professor Rodney Baxter on the occasion of his sixtieth birthday

Abstract. The Wakimoto construction for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ admits a reduction to the q-deformed parafermion algebras. We interpret the latter theory as a free field realization of the Andrews-Baxter-Forrester models in regime II. We give multi-particle form factors of some local operators on the lattice and compute their scaling limit, where the models are described by a massive field theory with $\mathbb{Z}_k$ symmetric minimal scattering matrices.

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1. Introduction

In conformal field theory (CFT), free field construction (or the 'Coulomb gas' representation) [1] is the most effective calculational tool for physical quantities. The same can be said about the vertex operator approach to solvable lattice models (see e.g. [2]). The latter can be viewed as a q-deformation of some conformal field models, the most typical example being the Andrews-Baxter-Forrester (ABF) models in regime III [3] which corresponds to the minimal unitary series. Despite its technical importance, however, no uniform recipe is known at present for finding a free field representation of a given CFT. Even when it is known, an equally non-trivial task is to do the same for the corresponding off-critical solvable lattice models. In this article we address this issue in the case of the ABF models in regime II [4].

By the level-rank duality for the Boltzmann weights [5], the ABF models in regime II at level k are equivalent to the $A^{(1)}_{k-1}$ face models in regime III at level 2. For the latter, a free field realization is already available [6]. However this picture is rather complicated, since one has to deal with $k-1$ kinds of oscillators. The Felder complex in higher rank is also quite cumbersome [7]. On the other hand, the ABF models in regime II are described in the conformal limit by the $\mathbb{Z}_k$ parafermionic CFT [8], and in the massive scaling limit by its perturbation by the first energy operator [9, 10, 11, 12, 13]. Therefore one naturally expects that an alternative construction on the lattice is to invoke the q-deformation of the parafermion theory [14, 15], where it is sufficient to deal with only two kinds of oscillators and the resolution of Fock modules has a simpler structure. In this paper we work out this point in detail. Our purpose is to show that the known results in representation theory fits nicely the interpretation as a bosonization of lattice models.

The text is organized as follows. In section 2 we recall the ABF models and set up the notation. In section 3 we review the free field realization of the deformed parafermion theory. We then present various commutation relations of operators. Comparing them with those in the lattice theory, we identify the deformed counterpart of the primary fields and the parafermionic currents with the vertex operators (VO's) of type I and type II, respectively. A simplifying feature is that these deformed parafermionic currents mutually commute by scalars. To our knowledge, this is the only known example in the vertex operator approach[1)] where the type II VO's do not involve integrals of screening operators. We note also the existence of a deformation of the

[1)] as well as the lattice counterpart of the non-unitary minimal models $M_{2,2n+1}$ which are expected to have the same property.

W_k currents, which we believe to be equivalent to the one in [16, 17] with specialization $r = k+2$. This conforms with the known equivalence between the $\mathbb{Z}_k$ parafermionic CFT and the first member in the W_k minimal series. In section 4 we calculate form factors of height variables on one and neighboring two lattice sites. In section 5 we take their continuous limit. Section 6 is devoted to the summary and discussions about open problems. Some technical formulas are gathered in the Appendices.

2. ABF model in regime II

2.1. **Boltzmann weights.** In this section we recall the ABF model in regime II and set up the notation. Throughout this article, we fix a positive integer $k \geq 2$.

The Boltzmann weights of the ABF model [4] have the form

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) = \rho(u)\overline{W}\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right). \tag{2.1}$$

Here a, b, c, d denote positive integers which we refer to as height variables. Besides, the weights depend on two real parameters u and x. If $1 \leq a, b, c, d \leq k+1$ and $|a-b| = |b-d| = |d-c| = |c-a| = 1$, then we set

$$\overline{W}\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a\pm 2 \end{matrix}\middle| u\right) = 1, \tag{2.2}$$

$$\overline{W}\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a \end{matrix}\middle| u\right) = \frac{[a\pm u][1]}{[a][1-u]}, \tag{2.3}$$

$$\overline{W}\left(\begin{matrix} a & a\pm 1 \\ a\mp 1 & a \end{matrix}\middle| u\right) = \frac{[a\mp 1][-u]}{[a][1-u]}. \tag{2.4}$$

Here $[u]$ stands for the function

$$[u] = x^{\frac{u^2}{k+2}-u}\Theta_{x^{2(k+2)}}(x^{2u}),$$

and we use the standard symbols

$$\Theta_p(z) = (z;p)_\infty (pz^{-1};p)_\infty (p;p)_\infty,$$

$$(z_1, \cdots, z_n; p_1, \cdots, p_m)_\infty = \prod_{j=1}^{n} \prod_{l_1,\cdots,l_m \geq 0} (1 - p_1^{l_1} \cdots p_m^{l_m} z_j).$$

For all other values of a, b, c, d, we set the weight (2.1) to 0.

In this note we restrict to regime II defined by

$$0 < x < 1, \qquad -\frac{k}{2} < u < 0.$$

We choose the scalar factor $\rho(u)$ in (2.1) to ensure that the partition function per site equals to 1. Explicitly

$$\rho(u) = x^{\frac{2u}{k(k+2)}} \frac{\rho_+(u)}{\rho_+(-u)},$$
$$\rho_+(u) = \frac{(x^{2k+2+2u}, x^{2k+2+2u}; x^{2k}, x^{2(k+2)})_\infty}{(x^{2k+2u}, x^{2k+4+2u}; x^{2k}, x^{2(k+2)})_\infty}.$$

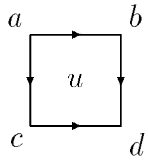
We have

$$\rho(u)\rho(-u) = 1, \qquad \rho(u)\rho(-k-u) = \frac{[1-u]^2}{[-u][2-u]}.$$

In [3], a free field construction of the ABF model was found in regime III defined by $0 < x < 1, 0 < u < 1$. The Boltzmann weights in regime II differs from that case only in the choice of $\rho(u)$ made above.

We represent the Boltzmann weights graphically as follows.

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) =$$



2.2. **Vertex operators in the naïve picture.** In this subsection we summarize the vertex operator approach in the naïve picture [18].

In regime II, there are $2k$ different ground states labeled by $m \in \mathbb{Z}/2k\mathbb{Z}$. Each of them is invarinat under the shift in the NE-SW direction, and hence is specified by a sequence of heights on a column. Fix a reference column and a site (say, site 1) on it. Then the m-th ground state is given by the following sequence $\{\bar{l}^{(m)}_j\}_{j\in\mathbb{Z}}$ on that column

$$\bar{l}^{(m)}_{j-m} = \begin{cases} j & (1 \le j \le k) \\ 2k+2-j & (k+1 \le j \le 2k) \end{cases},$$
$$\bar{l}^{(m)}_{j+2k} = \bar{l}^{(m)}_j \ (j \in \mathbb{Z}).$$

Here the sites on the reference column are numbered by j from south to north in the increasing order.

Let $\mathcal{H}_{m,a}$ denote the space of states of the half-infinite lattice, in the sector where the central height (i.e. the one on the reference site 1) is fixed to a and the boundary heights are in the ground state m. Namely we set formally

$$\mathcal{H}_{m,a} = \text{ span } \left\{ (l_j)_{j=1}^{\infty} \mid 1 \le l_j \le k+1, \ l_1 = a, \ l_j = \bar{l}^{(m)}_j \ (j \gg 1) \right\}.$$

Notice that $\mathcal{H}_{m,a} = 0$ if $a \equiv m \bmod 2$. The corner transfer matrices (CTM's) associated with the north-west, south-west, south-east and north-east quadrants operate respectively as

$$\begin{aligned} A^{(a)}_{NW}(u) &: \mathcal{H}_{m,a} \longrightarrow \mathcal{H}_{m,a}, \\ A^{(a)}_{SW}(u) &: \mathcal{H}_{m,a} \longrightarrow \mathcal{H}_{-m,a}, \\ A^{(a)}_{SE}(u) &: \mathcal{H}_{-m,a} \longrightarrow \mathcal{H}_{-m,a}, \\ A^{(a)}_{NE}(u) &: \mathcal{H}_{-m,a} \longrightarrow \mathcal{H}_{m,a}. \end{aligned}$$

In the limit of an infinitely large lattice, they have the simple form [4]

$$\begin{aligned} &A^{(a)}_{NW}(u) = x^{-2uH^{(a)}_C}, \\ &A^{(a)}_{SW}(u) = M^{(a)} x^{2uH^{(a)}_C}, \\ &A^{(a)}_{SE}(u) = M^{(a)} x^{-2uH^{(a)}_C} {M^{(a)}}^{-1}, \\ &A^{(a)}_{NE}(u) = [a] x^{2(u+k)H^{(a)}_C} {M^{(a)}}^{-1}, \end{aligned}$$

where the 'corner Hamiltonian' $H^{(a)}_C : \mathcal{H}_{m,a} \to \mathcal{H}_{m,a}$ has a discrete and equidistant spectrum, and $M^{(a)} : \mathcal{H}_{m,a} \to \mathcal{H}_{-m,a}$ is an operator which does not play a role in the following. Both of them are independent of u.

Likewise let

$$\begin{aligned} \Phi^{(a,b)}_N(u) &: \mathcal{H}_{m,b} \longrightarrow \mathcal{H}_{m+1,a}, \\ \Phi^{(b,a)}_W(-u) &: \mathcal{H}_{m+1,a} \longrightarrow \mathcal{H}_{m,b}, \\ \Phi^{(a,b)}_S(u) &: \mathcal{H}_{-m,b} \longrightarrow \mathcal{H}_{-m+1,a}, \\ \Phi^{(b,a)}_E(-u) &: \mathcal{H}_{-m+1,a} \longrightarrow \mathcal{H}_{-m,b} \end{aligned}$$

be the half-infinite transfer matrices extending to north, west, south and east directions, respectively. (Note that the $(m+1)$-th ground state is obtained from the m-th one by shifting it one step to the left.) Normally we suppress the dependence on the boundary condition m in the notation. Unlike the case of regime III, our Boltzmann weights do not have the crossing symmetry. Accordingly there is no simple relation between the operators $\Phi^{(a,b)}_N(u)$ and $\Phi^{(a,b)}_W(u)$.

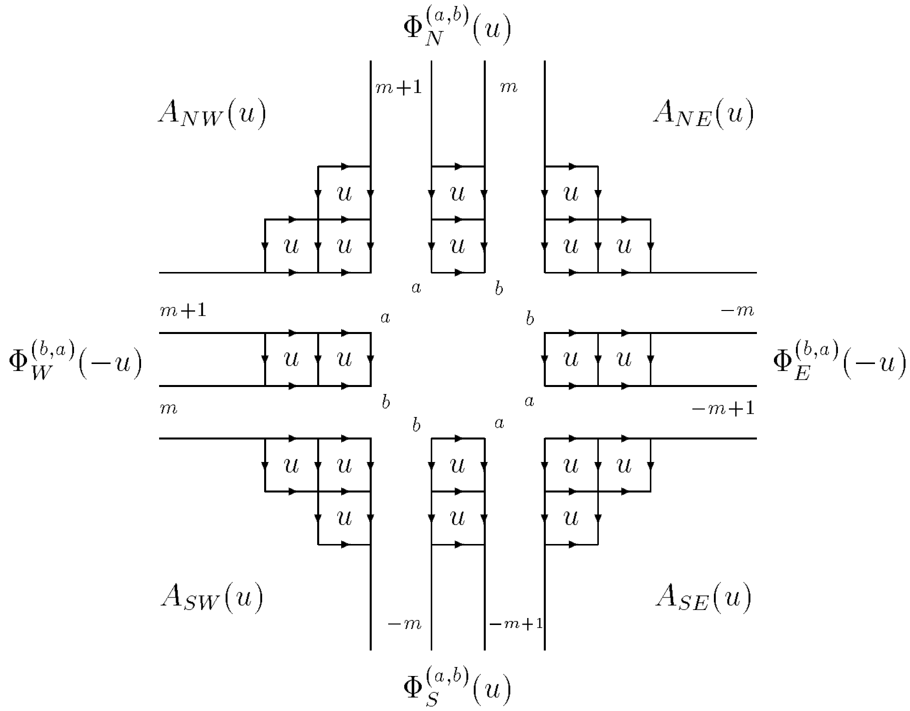


Figure : Corner transfer matrices and VO's of type I.

Formal manipulations [18] show that these operators should satisfy various commutation relations. We have

$$\Phi_N^{(a,b)}(u) = x^{2uH_C^{(a)}}\Phi_N^{(a,b)}(0)x^{-2uH_C^{(b)}},$$
$$\Phi_W^{(a,b)}(u) = x^{2uH_C^{(a)}}\Phi_W^{(a,b)}(0)x^{-2uH_C^{(b)}}.$$

The VO's satisfy the commutation relations

$$\Phi_N^{(a,b)}(u_2)\Phi_N^{(b,c)}(u_1) = \sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| u_1-u_2\right)\Phi_N^{(a,g)}(u_1)\Phi_N^{(g,c)}(u_2), \tag{2.5}$$

$$\Phi_W^{(a,b)}(u_2)\Phi_W^{(b,c)}(u_1) = \sum_g W\left(\begin{matrix} c & b \\ g & a \end{matrix}\middle| u_1-u_2\right)\Phi_W^{(a,g)}(u_1)\Phi_W^{(g,c)}(u_2), \tag{2.6}$$

$$\Phi_N^{(a,b)}(u_1)\Phi_W^{(b,c)}(u_2) = \sum_g W\left(\begin{matrix} g & c \\ a & b \end{matrix}\middle| u_1-u_2\right)\Phi_W^{(a,g)}(u_2)\Phi_N^{(g,c)}(u_1). \tag{2.7}$$

We have in addition

$$\sum_g \Phi_W^{(a,g)}(u)\Phi_N^{(g,a)}(u) = \mathrm{id}, \tag{2.8}$$

$$\sum_g \frac{[g]}{[a]}\Phi_N^{(a,g)}(u)\Phi_W^{(g,a)}(k+u) = \mathrm{id}. \tag{2.9}$$

The last two equations are consequences of the unitarity of the Boltzmann weights

$$\sum_g W\Big(\begin{matrix} a & b \\ g & c\end{matrix}\Big|u\Big) W\Big(\begin{matrix} a & g \\ d & c\end{matrix}\Big|-u\Big) = \delta_{bd},$$

and the second inversion relation

$$\sum_g \frac{[g]}{[c]} W\Big(\begin{matrix} d & c \\ b & g\end{matrix}\Big|-k-u\Big) W\Big(\begin{matrix} a & b \\ c & g\end{matrix}\Big|u\Big) = \delta_{ad}\frac{[b]}{[d]}.$$

In Section 3 we shall present a bosonic realization of the relations (2.5)–(2.9).

2.3. **Local height probabilities.** Consider successive n sites $i_1, \cdots, i_n$ on a row of the lattice, numbered from right to left. We regard i_1 as the reference site used to label the ground states. Fixing the boundary heights to the ground state m, we let $P_{a_n,\cdots,a_1}(m)$ denote the joint probability that the height variable l_{i_j} takes the value a_j, $j = 1, \cdots, n$. These are the n point local height probabilities (LHP's). In terms of the CTM and VO's, they can be expressed as

$$\begin{aligned} P_{a_n,\cdots,a_1}(m) &= \frac{1}{Z_m}[a_1] \\ &\times \mathrm{tr}_{\mathcal{H}_{m,a_1}}\Big(x^{2kH_C^{(a_1)}}\Phi_W^{(a_1,a_2)}(0)\cdots\Phi_W^{(a_{n-1},a_n)}(0)\Phi_N^{(a_n,a_{n-1})}(0)\cdots\Phi_N^{(a_2,a_1)}(0)\Big), \\ Z_m &= \sum_{a=1}^{k+1}[a]\,\mathrm{tr}_{\mathcal{H}_{m,a}}\big(x^{2kH_C^{(a)}}\big). \end{aligned}$$

It turns out that (with an appropriate normalization of $H_C^{(a)}$) $Z = Z_m$ is independent of m. Obviously $P_{a_n,\cdots,a_1}(m) = 0$ for $a_1 \equiv m \bmod 2$.

The following result for $n = 1$ is due to Andrews, Baxter and Forrester [4].

$$P_a(m) = \frac{[a]\,\mathrm{tr}_{\mathcal{H}_{m,a}}\big(x^{2kH_C^{(a)}}\big)}{Z} = x^{\frac{k+2}{4}}[a]\,c^{\Lambda_k(a-1)}_{\Lambda_k(m)}(\tau), \tag{2.10}$$

where $c^{\Lambda_k(l)}_{\Lambda_k(m)}(\tau)$ stands for the string function [19] for integrable $\widehat{\mathfrak{sl}}_2$-modules. Explicitly it is

$$\begin{aligned} c^{\Lambda_k(l)}_{\Lambda_k(m)}(\tau) &= \eta(\tau)^{-3} \\ &\times\Big(\sum_{n_1\geq|n_2|}-\sum_{-n_1>|n_2|}\Big)(-1)^{2n_1}\left(x^{2k}\right)^{\frac{(l+1+2n_1(k+2))^2}{4(k+2)}-\frac{(m+2n_2k)^2}{4k}} \end{aligned}$$

for $l\equiv m \bmod 2$ and $c^{\Lambda_k(l)}_{\Lambda_k(m)}(\tau)=0$ for $l\not\equiv m \bmod 2$. In the above, we set $x^{2k}=e^{2\pi i\tau}$, $\eta(\tau)=(x^{2k})^{1/24}(x^{2k};x^{2k})_\infty$, $\Lambda_k(j)=(k-j)\Lambda_0+j\Lambda_1$, and the sum is taken over $n_1,n_2\in\frac{1}{2}\mathbb{Z}$ with $n_1-n_2\in\mathbb{Z}$. We have

$$\begin{aligned} &P_a(m)=0 \quad (a\equiv m \bmod 2),\\ &\sum_{a=1}^{k+1}P_a(m)=1,\\ &P_a(-m)=P_a(m). \end{aligned}$$

As observed in [3] for regime III, the nearest neighbor LHP's ($n=2$) can be written in terms of the one point LHP. To see this it suffices to note the following relations which are obvious consequences of the definition.

$$\begin{aligned} &P_{1,2}(m)=P_1(m+1), \qquad P_{2,1}(m)=P_1(m),\\ &P_{a-1,a}(m)+P_{a+1,a}(m)=P_a(m),\\ &P_{a,a-1}(m)+P_{a,a+1}(m)=P_a(m+1). \end{aligned}$$

These properties fix $P_{a\pm1,a}(m)$ uniquely in terms of the one point LHP (2.10) as follows.

$$P_{a+1,a}(m)=\sum_{\substack{1\leq s\leq a\\ s\equiv a \bmod 2}}P_s(m)-\sum_{\substack{1\leq s\leq a\\ s\not\equiv a \bmod 2}}P_s(m+1), \tag{2.11}$$

$$P_{a-1,a}(m)=-\sum_{\substack{1\leq s< a\\ s\equiv a \bmod 2}}P_s(m)+\sum_{\substack{1\leq s< a\\ s\not\equiv a \bmod 2}}P_s(m+1). \tag{2.12}$$

The right hand sides of (2.11)–(2.12) automatically satisfy $P_{k+2,k+1}(m)=P_{k+1,k+2}(m)=0$, as it should be.

3. Free Field Realization

The q-deformation of the Wakimoto module over the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ was found by several authors [14] (See refs. in [14] as to the other variants of free field realizations.), using three kinds of bosonic fields. The deformed parafermion theory is obtained by dropping one of these fields. In this section we give some details of the latter theory, and interpret it as a free field realization of the ABF model in regime II. A part of the results are given in [15]. The various relations given in Propositions 3.1, 3.3–3.6 are new.

3.1. **Basic operators.** The main objects in the deformed parafermion theory are the vertex operators (VO's) of type I

$$\Phi^*_\pm(u), \quad \Phi_\pm(u), \tag{3.1}$$

and of type II

$$\Psi_+(u) = \Psi^\dagger(u), \quad \Psi_-(u) = \Psi(u). \tag{3.2}$$

The VO's of type I (3.1) are a lattice counterpart of the simplest primary fields in conformal theory, while those of type II (3.2) corresponds to the parafermions. In addition, there are also some auxiliary operators: the so-called screening current and the ξ-η system which we denote by

$$S(u), \quad \xi(u), \quad \eta(u),$$

respectively. All these operators act on the direct sum of Fock modules $\mathcal{F} = \oplus_{m\equiv l \bmod 2}\mathcal{F}_{m,l}$ labeled by $m, l \in \mathbb{Z}$. Their explicit formulas are given in Appendix A. Here we only mention their basic features.

The operators

$$\Psi_\pm(u),\ \Phi^*_-(u),\ \Phi_-(u),\ S(u),\ \xi(u),\ \eta(u)$$

are either an exponential of a bosonic field or a sum of two such terms. In contrast, the formulas for $\Phi^*_+(u)$, $\Phi_+(u)$ comprise integrals. To elaborate on the last point, let us introduce the screening charge

$$X(u) = \oint_{C_X(z)} \underline{dz'} S(u') \frac{[u-u'-\frac{1}{2}-P_1]}{[u-u'-\frac{1}{2}]}.$$

Here we set $z = x^{2u}$, $z' = x^{2u'}$, $\underline{dz'} = dz'/(2\pi i z')$, and P_1 denotes a 'zero-mode' operator. For more details we refer to Appendix A. The contour $C_X(z)$ is a simple closed curve encircling the origin counterclockwise in the z'-plane, such that $z' = zx^{-1+2(k+2)n}$ $(n \geq 0)$ are inside $C_X(z)$ and $z' = zx^{-1+2(k+2)n}$ $(n < 0)$ are outside $C_X(z)$. Using the screening charge, the 'plus' components of the type I VO's are given by

$$\Phi_+(u) = \Phi_-(u)X(u+k+1),$$
$$\Phi^*_+(u) = -\Phi^*_-(u)X(u),$$

where the product $A(u)B(v)$ is defined to be the analytic continuation from the domain $|x^{2u}| \gg |x^{2v}|$.

Each of the operators $\Phi_\pm(u)$, $\Phi^*_\pm(u)$, $\Psi_\pm(u)$, $S(u)$, $\xi(u)$, $\eta(u)$ sends one Fock space $\mathcal{F}_{m,l}$ to another $\mathcal{F}_{m',l'}$ with a different label (m',l'). The change of label is listed in the following table.

	$\Phi_\pm(u)$	$\Phi^*_\pm(u)$	$\Psi_\pm(u)$	$S(u)$	$\xi(u)$	$\eta(u)$
$\mathcal{F}_{m',l'}$	$\mathcal{F}_{m+1,l\mp1}$	$\mathcal{F}_{m-1,l\mp1}$	$\mathcal{F}_{m\mp2,l}$	$\mathcal{F}_{m,l-2}$	$\mathcal{F}_{m-k,l-(k+2)}$	$\mathcal{F}_{m+k,l+k+2}$

3.2. **Commutation relations.** In the next three propositions we state the commutation relations among the operators $\Phi_\pm(u)$, $\Phi^*_\pm(u)$ and $\Psi_\pm(u)$. Set

$$P = P_1 + 1.$$

Proposition 3.1. *The operators $\Phi_\pm(u)$, $\Phi^*_\pm(u)$ satisfy the commutation relations*

$$\Phi_{\varepsilon_2}(u_2)\Phi_{\varepsilon_1}(u_1) = \sum W\left(\begin{matrix} P & P+\varepsilon'_1 \\ P+\varepsilon_2 & P+\varepsilon_1+\varepsilon_2 \end{matrix}\middle| u_1-u_2\right)\Phi_{\varepsilon'_1}(u_1)\Phi_{\varepsilon'_2}(u_2),$$

$$\Phi^*_{\varepsilon_2}(u_2)\Phi^*_{\varepsilon_1}(u_1) = \sum W\left(\begin{matrix} P+\varepsilon_1+\varepsilon_2 & P+\varepsilon_2 \\ P+\varepsilon'_1 & P \end{matrix}\middle| u_1-u_2\right)\Phi^*_{\varepsilon'_1}(u_1)\Phi^*_{\varepsilon'_2}(u_2),$$

$$\Phi_{\varepsilon_1}(u_1)\Phi^*_{\varepsilon_2}(u_2) = \sum W\left(\begin{matrix} P+\varepsilon'_2 & P+\varepsilon_1+\varepsilon_2 \\ P & P+\varepsilon_1 \end{matrix}\middle| u_1-u_2\right)\Phi^*_{\varepsilon'_2}(u_2)\Phi_{\varepsilon'_1}(u_1),$$

where the Boltzmann weights are given in (2.2)–(2.4). *The sums are taken over $\varepsilon'_1, \varepsilon'_2 = \pm1$ such that $\varepsilon'_1+\varepsilon'_2 = \varepsilon_1+\varepsilon_2$. Moreover we have the inversion relations*

$$\sum_\varepsilon \Phi^*_\varepsilon(u)\Phi_{-\varepsilon}(u) = \mathrm{id}, \tag{3.3}$$

$$\sum_\varepsilon \Phi_\varepsilon(u)[P]\Phi^*_{-\varepsilon}(k+u) = [P]\,\mathrm{id}. \tag{3.4}$$

In the next propositions we set

$$[u]^* = x^{\frac{u^2}{k}-u}\Theta_{x^{2k}}(x^{2u}).$$

Proposition 3.2. *The following commutation relations hold.*

$$\Psi_\pm(u_2)\Psi_\pm(u_1) = \frac{[u_1-u_2+1]^*}{[u_1-u_2-1]^*}\Psi_\pm(u_1)\Psi_\pm(u_2),$$

$$\Psi_\pm(u_2)\Psi_\mp(u_1) = \frac{[u_1-u_2-1+\frac{k}{2}]^*}{[u_1-u_2+1+\frac{k}{2}]^*}\Psi_\mp(u_1)\Psi_\pm(u_2).$$

Moreover, as $u_2 \to u_1 - \frac{k}{2}$ *we have*

$$\Psi_\pm(u_2)\Psi_\mp(u_1) = \frac{k}{2\pi}\frac{g^*}{u_2 - u_1 + \frac{k}{2}} \times \mathrm{id} + O(1),$$

where $g^* = \dfrac{\pi}{k \log x}\dfrac{(x^{2k-2}; x^{2k})_\infty}{(x^2; x^{2k})_\infty}$.

Proposition 3.3.

$$\Psi_+(u_2)\Phi_\pm(u_1) = \frac{[u_1 - u_2 - \frac{1}{2} + \frac{k}{2}]^*}{[u_1 - u_2 + \frac{1}{2} + \frac{k}{2}]^*}\Phi_\pm(u_1)\Psi_+(u_2),$$

$$\Psi_+(u_2)\Phi^*_\pm(u_1) = \frac{[u_1 - u_2 + \frac{1}{2} + \frac{k}{2}]^*}{[u_1 - u_2 - \frac{1}{2} + \frac{k}{2}]^*}\Phi^*_\pm(u_1)\Psi_+(u_2),$$

$$\Psi_-(u_2)\Phi_\pm(u_1) = \frac{[u_1 - u_2 + \frac{1}{2}]^*}{[u_1 - u_2 - \frac{1}{2}]^*}\Phi_\pm(u_1)\Psi_-(u_2),$$

$$\Psi_-(u_2)\Phi^*_\pm(u_1) = \frac{[u_1 - u_2 - \frac{1}{2}]^*}{[u_1 - u_2 + \frac{1}{2}]^*}\Phi^*_\pm(u_1)\Psi_-(u_2).$$

3.3. **Free fields resolution.** The space of states for the deformed parafermion theory is constructed following a procedure well known in conformal field theory [20]. This is done in two steps. The first step is to introduce a certain subspace $\widetilde{\mathcal{F}}_{m,l}$ of the Fock space $\mathcal{F}_{m,l}$ using $\eta(u)$. The second step is to consider a complex consisting of these $\widetilde{\mathcal{F}}_{m,l}$, in which the coboundary maps are given by powers of the screening charge $X(u)$. The space of states is then defined as the 0-th cohomology of this complex.

Let

$$\eta_0 : \mathcal{F}_{m,l} \longrightarrow \mathcal{F}_{m+k,l+k+2}$$

denote the zeroth Fourier coefficient of $\eta(u)$. It is well defined provided $m \equiv l \bmod 2$. We set

$$\widetilde{\mathcal{F}}_{m,l} = \mathrm{Ker}\ \eta_0\big|_{\mathcal{F}_{m,l}}.$$

Then we have the following resolution of $\widetilde{\mathcal{F}}_{m,l}$ by Fock spaces (see Appendix B)

(3.5)
$$0 \longrightarrow \widetilde{\mathcal{F}}_{m,l} \longrightarrow \mathcal{F}_{m,l} \xrightarrow{\eta_0} \mathcal{F}_{m+k,l+k+2} \xrightarrow{\eta_0} \mathcal{F}_{m+2k,l+2(k+2)} \xrightarrow{\eta_0} \cdots,$$

(3.6)
$$\cdots \xrightarrow{\eta_0} \mathcal{F}_{m-2k,l-2(k+2)} \xrightarrow{\eta_0} \mathcal{F}_{m-k,l-(k+2)} \xrightarrow{\eta_0} \widetilde{\mathcal{F}}_{m,l} \longrightarrow 0.$$

The following proposition shows that the operators $S(u)$, $\Psi_\pm(u)$, $\Phi_\pm(u)$ and $\Phi^*_\pm(u)$ have a well-defined action on the subspace $\widetilde{\mathcal{F}}_{m,l}$.

Proposition 3.4. *We have*

$$\begin{aligned}\eta_0 S(u) &= -S(u)\eta_0,\\ \eta_0\Psi_\pm(u) &= -\Psi_\pm(u)\eta_0,\\ \eta_0\Phi_\pm(u) &= \Phi_\pm(u)\eta_0,\\ \eta_0\Phi^*_\pm(u) &= \Phi^*_\pm(u)\eta_0.\end{aligned}$$

Next we fix $l, m \in \mathbb{Z}$ with $0 \le l \le k$, $l \equiv m \bmod 2$. Consider a sequence $\mathcal{C}_{m,l}$

(3.7)

$$\cdots \xrightarrow{X_{-2}} \widetilde{\mathcal{F}}_{m,-l-2+2(k+2)} \xrightarrow{X_{-1}} \widetilde{\mathcal{F}}_{m,l} \xrightarrow{X_0} \widetilde{\mathcal{F}}_{m,-l-2} \xrightarrow{X_1} \widetilde{\mathcal{F}}_{m,l-2(k+2)} \xrightarrow{X_2} \cdots,$$

defined by appropriate powers of the screening charge $X(u)$, i.e.,

$$\begin{aligned}X_{2j} &= X(u)^{l+1} \ : \ \widetilde{\mathcal{F}}_{m,l-2j(k+2)} \longrightarrow \widetilde{\mathcal{F}}_{m,-l-2-2j(k+2)},\\ X_{2j+1} &= X(u)^{k-l+1} \ : \ \widetilde{\mathcal{F}}_{m,-l-2-2j(k+2)} \longrightarrow \widetilde{\mathcal{F}}_{m,l-2(j+1)(k+2)}.\end{aligned}$$

The following can be shown in exactly the same way as in [21].

Proposition 3.5. *The maps X_j are independent of u, and $\mathcal{C}_{m,l}$ is a cochain complex:*

$$X_j X_{j-1} = 0 \qquad (j \in \mathbb{Z}).$$

The following statement concerning the cohomology group of this complex seems quite plausible (cf. [20]).

$$H^j(\mathcal{C}_{m,l}) = 0 \qquad (j \neq 0). \tag{3.8}$$

As we do not have a rigorous mathematical proof, we assume henceforth the validity of (3.8). By the Euler-Poincaré principle, the character of the remaining cohomology $H^0(\mathcal{C}_{m,l})$ then becomes

$$\begin{aligned}\mathrm{tr}_{H^0(\mathcal{C}_{m,l})}\big(x^{2kD}\big) &= x^{\frac{k^2}{4(k+2)}}(x^{2k};x^{2k})_\infty\, c^{\Lambda_k(l)}_{\Lambda_k(m)}(\tau),\\ Z = \sum_{\substack{0\le l\le k\\ l\equiv m \bmod 2}} [l+1]\,\mathrm{tr}_{H^0(\mathcal{C}_{m,l})}\big(x^{2kD}\big) &= x^{-\frac{k+1}{k+2}}(x^{2k};x^{2k})_\infty,\end{aligned}$$

giving the same formula as that of the one point LHP (2.10).

Proposition 3.6.

$$\begin{aligned}X_j\Psi_\pm(u) &= \Psi_\pm(u)X_j,\\ X_j\Phi_\pm(u) &= \Phi_\mp(u)X_j,\\ X_j\Phi^*_\pm(u) &= \Phi^*_\mp(u)X_j.\end{aligned}$$

This proposition ensures that $\Phi_\pm(u)$, $\Phi^*_\pm(u)$ and $\Psi_\pm(u)$ give rise to well-defined operators on the cohomology $H^0(\mathcal{C}_{m,l})$. We abuse the notation and denote them by the same letters.

3.4. **Identification with lattice theory.** The construction of this Section is related to the lattice theory in Section 2 as follows. We make the following identification:

(i) The space of states of the lattice model (with central height a and boundary condition m) with the 0-th cohomology of $\mathcal{C}_{m,a-1}$,

$$\mathcal{H}_{m,a} = H^0(\mathcal{C}_{m,a-1}).$$

(ii) The corner Hamiltonian $H_C^{(a)}$ with the grading operator D.

(iii) The half-infinite transfer matrices with the type I VO's

$$\Phi_N^{(a-\varepsilon,a)}(u) = \Phi_\varepsilon(u)\big|_{H^0(\mathcal{C}_{m,a-1})},$$
$$\Phi_W^{(a-\varepsilon,a)}(u) = \Phi^*_\varepsilon(u)\big|_{H^0(\mathcal{C}_{m,a-1})}.$$

(iv) The creation/annihilation operators of particles and anti-particles with the type II VO's $\Psi_\pm(u)$.

As we already mentioned, with this identification the characters of the two spaces match. The commutation relations for VO's of type I expected from the lattice theory (2.5)–(2.9) are recovered in Proposition 3.1. In Section 5, we will comment about the agreement with the known results on the excitation spectrum and the S-matrix [11] in the scaling limit.

Correlation functions and form factors of local operators are given as traces of operators acting on the physical space $H^0(\mathcal{C}_{m,l})$. Let $\mathcal{O}$ stand for x^{2kD} times a product of operators of the form $\Psi_\varepsilon(u)$, $\Phi_\varepsilon(u)$ and $\Phi^*_\varepsilon(u)$. For simplicity we consider the case where there are an equal number of $\Psi_+(u)$'s and $\Psi_-(u)$'s. Proposition 3.6 shows that

$$\mathcal{O}_i\eta_0 = \eta_0\mathcal{O}_i,$$
$$X_j\mathcal{O}_i = \mathcal{O}_{1-i}X_j \qquad (j \equiv i \bmod 2),$$

where $\mathcal{O}_0 = \mathcal{O}$, and $\mathcal{O}_1$ signifies the operator obtained from $\mathcal{O}$ by negating the indices ε of $\Phi_\varepsilon(u)$ and $\Phi^*_\varepsilon(u)$. The resolutions (3.5),(3.6),(3.7) afford a procedure for computing the trace as follows.

$$\text{(3.9)}\quad \mathrm{tr}_{H^0(\mathcal{C}_{m,l})}(\mathcal{O}) = \sum_{s\in\mathbb{Z}} \mathrm{tr}_{\widetilde{\mathcal{F}}_{m,l-2s(k+2)}}(\mathcal{O}_0) - \sum_{s\in\mathbb{Z}} \mathrm{tr}_{\widetilde{\mathcal{F}}_{m,-l-2-2s(k+2)}}(\mathcal{O}_1),$$

$$
\begin{aligned}
\text{(3.10)} \qquad \mathrm{tr}_{\tilde{\mathcal{F}}_{m,l}}(\mathcal{O}_i) &= \sum_{n\geq 0}(-1)^n\,\mathrm{tr}_{\mathcal{F}_{m+kn,l+(k+2)n}}(\mathcal{O}_i)\\
&= -\sum_{n<0}(-1)^n\,\mathrm{tr}_{\mathcal{F}_{m+kn,l+(k+2)n}}(\mathcal{O}_i).
\end{aligned}
$$

Taking $\mathcal{O} = x^{2kD}\Phi^*_\varepsilon(u)\Phi_{-\varepsilon}(u)$, we have verified directly that the formula thus obtained reproduces the LHP (2.11)–(2.12) for neighboring lattice sites.

3.5. **Fusion of parafermions and W currents.** The type II VO's $\Psi_-(u)$, $\Psi_+(u)$ play the role of creation operators of excitations over the ground state. Besides them, there are altogether $k-1$ kinds of 'elementary' particles in regime II, as expected from the level-rank duality [5]. The standard fusion procedure provides us with a free field realization $\Psi_a(u)$ defined recursively as

$$
\begin{aligned}
&\Psi_1(u'-\frac{a}{2})\Psi_a(u+\frac{1}{2})\\
&= \frac{1}{1-\frac{z}{z'}}x^{\frac{a(a+1)}{k}}\frac{(x^{2a},x^{2a+2k+2};x^{2k})_\infty}{(x^{2k},x^{2k+2};x^{2k})_\infty}\Psi_{a+1}(u)+O(1) \quad (u'\to u),
\end{aligned}
$$

where $\Psi_1(u) = -(x-x^{-1})\Psi_-(u)$. A more explicit expression is given in (A.1). We have the following commutation relations

$$
\Psi_a(u_2)\Phi_\pm(u_1) = \frac{[u_1-u_2+\frac{a}{2}]^*}{[u_1-u_2-\frac{a}{2}]^*}\Phi_\pm(u_1)\Psi_a(u_2),
$$

$$
\begin{aligned}
\text{(3.11)}\ &\Psi_a(u_2)\Psi_b(u_1)\\
&= \prod_{s=1}^{a}\prod_{s'=1}^{b}\frac{[u_1-u_2+1+\frac{a-b}{2}-(s-s')]^*}{[u_1-u_2-1+\frac{a-b}{2}-(s-s')]^*}\cdot\Psi_b(u_1)\Psi_a(u_2)\\
&= \frac{[u+\frac{a+b}{2}]^*}{[u-\frac{a+b}{2}]^*}\frac{[u+\frac{|a-b|}{2}]^*}{[u-\frac{|a-b|}{2}]^*}\prod_{s=1}^{\min(a,b)-1}\frac{{[u+\frac{|a-b|}{2}+s]^*}^2}{{[u-\frac{|a-b|}{2}-s]^*}^2}\cdot\Psi_b(u_1)\Psi_a(u_2),
\end{aligned}
$$

where $u = u_1 - u_2$. These relations agree with the known results about the excitation spectra (5.1) and the scattering matrices (5.6) in the scaling field theory discussed in Section 5.

The operator $\Psi_k(u)$ commutes with $\Psi_\pm(v)$ and X_j, commutes or anticommutes with η_0, and anticommutes with $\Phi_\pm(v)$. It can be shown that it is also invertible. We expect that on the cohomology it is independent of u and defines an isomorphism $\iota : H^0(\mathcal{C}_{m,l}) \simeq H^0(\mathcal{C}_{m+2k,l})$. We also expect that $\Psi_{k-1}(u)$ defines the same operator as $\iota\circ\Psi_+(u)$.

The level-rank duality also suggests the existence of the deformed W_k currents [16, 17] in the parafermionic description of the ABF models.

Indeed, the first deformed W current $W^1(u)$ can be obtained either as fusion of type I VO's ($z_2 \to x^{2(k+2)} z_1$)

$$\begin{aligned} &\Phi_-(u_2)\Phi_+^*(u_1) \\ &= \Big(1 - x^{2(k+2)}\frac{z_1}{z_2}\Big)\frac{x^{\frac{1}{k}}}{[k+1]_x}\frac{(x^{2k}, x^{-4}; x^{2k})_\infty}{(x^{-2}, x^{-2}; x^{2k})_\infty} W^1(u_1) + \cdots , \end{aligned}$$

or as that of type II VO's ($z_2 \to x^{k+2} z_1$)

$$\begin{aligned} &\Psi_-(u_2)\Psi_+(u_1) \\ &= \frac{1}{1 - x^{k+2}\frac{z_1}{z_2}}\frac{x^{\frac{2}{k}+1}}{(x - x^{-1})^2 [k+1]_x}\frac{(x^{-4}; x^{2k})_\infty}{(x^{2k}; x^{2k})_\infty} W^1(u_1 - \tfrac{k+1}{2}) + \cdots . \end{aligned}$$

Explicit formulas of $W^1(u)$, as well as those of the higher currents $W^j(u)$ $(j = 2, 3, \cdots)$, are given in Appendix A. We expect that these $W^j(u)$ $(j \geq 1)$ generate the same deformed W algebra for $\mathfrak{sl}_N$ in [16], under the following identification

$$N = k, \quad q = x^{2(k+1)}, \quad t = x^{2(k+2)} \quad (p = qt^{-1} = x^{-2}).$$

We have checked a part of the relations for $W^j(u)$ (eq.(8) in [16]), but have not verified such relations as $W^k(u) = 1$ which are expected to hold only at the level of the cohomology.

4. Form Factors

4.1. **Traces of type II operators.** As a simplest example of form factors of local operators, let us consider the quantity

$$\begin{aligned} Q_a^{(n,n)}(m) &= \big(Zg^{*n}\big)^{-1}[a] \\ &\times \mathrm{tr}_{\mathcal{H}_{m,a}}\Big(x^{2kD}\Psi_+(v_1)\cdots\Psi_+(v_n)\Psi_-(v'_1)\cdots\Psi_-(v'_n)\Big). \end{aligned} \tag{4.1}$$

To simplify the notation, we have not exhibited explicitly the dependence of $Q_a^{(n,n)}(m)$ on the parameters v_j, v'_j. For the same reason we will often suppress the superscript (n,n). Note that $Q_a(m) = 0$ for $a \equiv m \bmod 2$.

Hereafter we set

$$\tau = -\frac{ik}{\pi}\log x\,.$$

It is a standard task to compute traces of bosonic operators over the Fock space. After the working outlined in Appendix C, we find the following expression.

$$
\begin{aligned}
(4.2) \qquad Q_a(m) = & \sum_{\substack{\mu_1,\cdots,\mu_n=\pm 1\\ \nu_1,\cdots,\nu_n=\pm 1}} \kappa(v,\mu,\nu) \prod_{i=1}^{n} \mu_i\nu_i \\
& \times \prod_{1\le i<j\le n} \frac{[v_j - v_i + \frac{\mu_i-\mu_j}{2}]^*}{[v_j - v_i - 1]^*} \frac{F(v_j - v_i)}{F(0)} \\
& \times \prod_{1\le i<j\le n} \frac{[v'_j - v'_i - \frac{\nu_i-\nu_j}{2}]^*}{[v'_j - v'_i - 1]^*} \frac{F(v'_j - v'_i)}{F(0)} \\
& \times \prod_{i,j=1}^{n} \frac{[v'_j - v_i - \frac{k}{2} - \frac{\mu_i+\nu_j}{2}]^*}{[v'_j - v_i - \frac{k}{2}]^*} \frac{F(0)}{F(v'_j - v_i - \frac{k}{2})} \\
& \times \Gamma_{m,a-1}\Big(\frac{\tau}{2k}(\mu-\nu), \frac{\tau}{k}(\frac{2v}{k} - \frac{\mu+\nu}{2})\Big|\tau\Big).
\end{aligned}
$$

In the above, we have set $v = \sum_{i=1}^{n}(v'_i - v_i)$, $\mu = \sum_{i=1}^{n}\mu_i$, $\nu = \sum_{i=1}^{n}\nu_i$. The function $\Gamma_{m,l}(y_1,y_2|\tau) = \Gamma^{(0,0)}_{m,l}(y_1,y_2|\tau)$ is defined in (C.6). The functions $F(v)$ and $\kappa(v,\mu,\nu)$ are defined as follows.

$$
\begin{aligned}
(4.3) \quad F(v) &= \frac{(x^{2(k+1+v)}, x^{2(k+1-v)}; x^{2k}, x^{2k})_\infty}{(x^{2(k-1+v)}, x^{2(k-1-v)}; x^{2k}, x^{2k})_\infty}, \\
\kappa(v,\mu,\nu) &= \frac{[a]}{Z}\Big(-i\tau\frac{\eta(\tau)^3}{[1]^*}\Big)^n x^{(\frac{\mu+\nu}{k} - \frac{2}{k}n)v + \frac{n^2}{2} + \frac{\mu^2+\nu^2}{4k} - \frac{kn}{4} - \frac{k+1}{2k}\mu\nu + \frac{kc}{12}}
\end{aligned}
$$

with $c = 2(k-1)/(k+2)$.

Using (C.7),(C.8) we see that $Q_a(m+2k) = Q_{k+2-a}(m+k) = Q_a(m)$.

Remark. We see from (C.3), (C.4) that the contribution of the oscillator part to the trace (4.1) is convergent if $x^{\varepsilon_2-\varepsilon_1} < |z_1/z_2| < x^{\varepsilon_2-\varepsilon_1-2k}$. For $k > 2$ there is a non-empty domain of convergence common to all $\varepsilon_1, \varepsilon_2$. The case $k = 2$ of the Ising model is exceptional and needs a separate treatment by analytic continuation. *For the rest of the paper we assume $k \geq 3$.*

4.2. **Neighboring heights.** As the next example, let us consider the trace involving two type I VO's,

$$
\begin{aligned}
Q_{b,a}(m) \;=\; & \big(Zg^{*n}\big)^{-1}[a]\,\mathrm{tr}_{\mathcal{H}_{m,a}}\Big(x^{2kD}\Phi^{*(a,b)}(u)\Phi^{(b,a)}(u) \\
& \times \Psi_+(v_1)\cdots\Psi_+(v_n)\Psi_-(v'_n)\cdots\Psi_-(v'_n)\Big).
\end{aligned}
$$

Here $\Phi^{(a-\varepsilon,a)}(u)$ means $\Phi_\varepsilon(u)|_{\mathcal{H}_{m,a}}$, and similarly for $\Phi^{*(a,b)}(u)$.

We have already mentioned that the two point LHP for the neighboring height variables (2.11),(2.12) can be expressed simply in terms of one point LHP's. Let us apply the same argument to compute $Q_{b,a}(m)$. The first inversion identity (3.3) entails that

$$Q_{a+1,a}(m) + Q_{a-1,a}(m) = Q_a(m), \tag{4.4}$$

with $Q_a(m)$ given by (4.1). On the other hand, the second inversion relation (3.4) together with the cyclicity of the trace implies

$$Q_{b,b-1}(m) + Q_{b,b+1}(m) = Q_b(m+1)\, G. \tag{4.5}$$

The factor

$$G = \prod_{j=1}^{n} \frac{[u - v_j + \frac{1}{2} + \frac{k}{2}]^*}{[u - v_j - \frac{1}{2} + \frac{k}{2}]^*} \frac{[u - v'_j - \frac{1}{2}]^*}{[u - v'_j + \frac{1}{2}]^*}$$

arises from the commutation relations between type I and type II VO's. Solving the relations (4.4),(4.5) under the condition $Q_{2,1}(m) = Q_1(m)$, $Q_{1,2}(m) = Q_1(m+1)$ (which can be verified using the integral representations), we obtain

$$Q_{a+1,a}(m) = \sum_{\substack{1 \le s \le a \\ s \equiv a \bmod 2}} Q_s(m) - \sum_{\substack{1 \le s \le a \\ s \not\equiv a \bmod 2}} Q_s(m+1)\, G,$$

$$Q_{a-1,a}(m) = -\sum_{\substack{1 \le s < a \\ s \equiv a \bmod 2}} Q_s(m) + \sum_{\substack{1 \le s < a \\ s \not\equiv a \bmod 2}} Q_s(m+1)\, G.$$

The above reasoning does not seem to generalize easily to the case where more than two VO's of type I are present.

5. Scaling limit

Basic facts about the general RSOS models and their scaling limit have been worked out by Bazhanov and Reshetikhin [11], by diagonalizing the row-to-row transfer matrix. First we briefly review their results specializing to the present case of the ABF models in regime II. Then we work out new expressions for form factors by taking the scaling limit of the formulas on the lattice.

5.1. Review of known results.

Denote by H the Hamiltonian of the lattice model related with the row-to-row transfer matrix T as

$$H = \frac{1}{4\pi\delta} \frac{d}{du} \log T(u) \Big|_{u=0},$$

where δ is the lattice spacing. The excitation spectrum over the ground state found from the Bethe ansatz has the form

$$\epsilon_a(v) = \frac{1}{4\pi\delta}\frac{d}{dv}\log\frac{[v+\frac{a}{2}]^*}{[v-\frac{a}{2}]^*}, \qquad (1 \le a \le k-1). \tag{5.1}$$

The energies $|\epsilon_a(v)|$ are periodic functions with the period $-\frac{i\pi}{\log x}$, and have a minimum at $v = -\frac{i\pi}{2\log x}$. The analysis of this function shows that there is a non-zero gap in the spectrum, and the corresponding continuous model is a massive field theory.

Introduce the rapidity variable β by

$$v = \frac{ik}{2\pi}\beta - \frac{i\pi}{2\log x}, \tag{5.2}$$

and let

$$p = e^{\frac{\pi^2}{(k+2)\log x}} \tag{5.3}$$

be the temperature parameter of the lattice model. In the scaling limit

$$\log x \to 0, \quad \delta \to 0, \quad p^{\frac{k+2}{k}}\delta^{-1} \to const = \frac{kM}{2\sin\frac{\pi}{k}}, \tag{5.4}$$

while keeping β fixed, (5.1) gives a massive relativistic spectrum

$$\lim_{\log x\to 0}\epsilon_a\Bigl(\frac{ik}{2\pi}\beta - \frac{i\pi}{\log x}\Bigr) = M_a\cosh\beta.$$

Here the mass M_a of the particle a is defined by

$$M_a = \frac{\sin\frac{\pi a}{k}}{\sin\frac{\pi}{k}}M.$$

The scattering matrices $S_{ab}(\beta)$ of these particles are diagonal and have a very simple form. For the fundamental particle/antiparticle, it is given by 'minimal' S matrices of the $\mathbb{Z}_k$ symmetric model proposed for the first time in [22],

$$\begin{aligned} S_{11}(\beta) = S_{\bar{1}\bar{1}}(\beta) &= \frac{\sinh(\frac{\beta}{2}+\frac{i\pi}{k})}{\sinh(\frac{\beta}{2}-\frac{i\pi}{k})}, \\ S_{1\bar{1}}(\beta) = S_{\bar{1}1}(\beta) &= S_{11}(i\pi-\beta). \end{aligned} \tag{5.5}$$

In general, we have

$$S_{ab}(\beta) = f_{a+b}(\beta)f_{|a-b|}(\beta)\prod_{s=1}^{\min(a,b)-1} f_{|a-b|+2s}(\beta)^2, \tag{5.6}$$

where

$$f_A(\beta) = \frac{\sinh(\frac{\beta}{2} + \frac{i\pi A}{2k})}{\sinh(\frac{\beta}{2} - \frac{i\pi A}{2k})}.$$

The particles in the scaling theory are not self-conjugate except for the case $k = 2$ of the Ising model. The charge conjugation identifies the antiparticle $\bar{a}$ of a with the particle $k - a$, reflecting the additional $\mathbb{Z}_2$ symmetry of the model.

It has been found in [9, 10, 11, 12, 13] that the ultraviolet properties of the scaling theory are described by the parafermionic CFT [8] with the central charge

$$c = 2\frac{k-1}{k+2}\,.$$

According to these works, one can treat the model in the scaling region as the CFT perturbed by the first energy operator with the left and right conformal dimensions

$$\Delta_{0,2} = \frac{2}{k+2}\,.$$

All these results agree well with the algebraic picture we have followed. The vertex operators of type II are identified with operators that create eigenstates of the row-to-row transfer matrix [2]. Writing the formal limit (5.4) of $\Psi_a(v)$ as $\mathcal{Z}_a(\beta)$, where v and β are related as in (5.2), we find that the commutation relations of Proposition 3.2 and (3.11) become

$$\mathcal{Z}_a(\beta_1)\mathcal{Z}_b(\beta_2) = S_{ab}(\beta_1 - \beta_2)\,\mathcal{Z}_b(\beta_2)\mathcal{Z}_a(\beta_1)\,. \tag{5.7}$$

Thus the operators $\mathcal{Z}_a$ can be interpreted as generators of the Zamolodchikov-Faddeev algebra in the angular quantization approach [23, 24]. The commutation relations between $\Phi_\pm(0)$ and $\Psi_a(v)$ (Proposition 3.3) ensure that the states created by acting with the type II operators on the vacuum have the eigenvalues (5.1) of the corresponding Hamiltonian.

Our present aim is to take the continuous limit of the formulas in the previous section, and to interpret them as $\mathbb{Z}_k$-neutral form factors of some operators in the $\mathbb{Z}_k$-symmetric model. We will however discuss neither the Lagrangian description of the present model nor the problems of identification and normalization of local operators.

General aspects of the form factors in diagonal scattering theories with $\mathbb{Z}_k$ symmetry were discussed in [25], where a recursive system of functional equations have been written. Another well-known example of theories with such a symmetry is the affine A_{k-1} Toda models. Its S

matrix for fundamental particles differs from S_{11} (5.5) by a coupling-dependent factor which has no poles and zeros in the physical region. For the affine Toda models, the two- and four-particle form factors have been determined in [25, 26]. Our results below are very similar to those for the affine Toda case.

5.2. **Projection operators.** Let us denote by $|0\rangle_m$ the $2k$ degenerate ground states in regime II. They are related to each other by a spatial translation. For the discussion of the continuous limit it is more convenient to deal with eigenstates with respect to the translation operator on the lattice,

$$\frac{1}{\sqrt{2k}} \sum_{m'=-k}^{k-1} e^{\frac{i\pi m m'}{k}} |0\rangle_{m'} . \tag{5.8}$$

Let Pr_a stand for the projection operator onto the sector where the central height takes the value a. We shall focus attention to form factors of the linear transform

$$\widehat{\mathrm{Pr}}_a = \frac{-1}{\sqrt{2k}} \sum_{a'=1}^{k+1} \frac{\sin \frac{\pi a a'}{k+2}}{\sin \frac{\pi a'}{k+2}} \mathrm{Pr}_{a'}$$

with respect to the translationally invariant vacuum $m = 0$ in (5.8). Thus the quantities we study in the continuous limit are

$$\widehat{\mathcal{Q}}_a^{(n,n)} = \frac{-1}{2k} \sum_{a'=1}^{k+1} \sum_{m=-k}^{k-1} \frac{\sin \frac{\pi a a'}{k+2}}{\sin \frac{\pi a'}{k+2}} Q_{a'}^{(n,n)}(m), \tag{5.9}$$

where $Q_{a'}^{(n,n)}(m)$ is given by (4.1). In the angular quantization approach to affine Toda field theory, similar operators are attributed to the exponential operators [26]. Note that $\widehat{\mathcal{Q}}_a^{(n,n)} = 0$ if a is even.

5.3. **Two-particle form factors.** To illustrate the procedure of taking the scaling limit, let us consider in some details the simplest case $n = 1$, corresponding to the particle-antiparticle form factors.

To find the limit (5.4) of $\widehat{\mathcal{Q}}_a^{(1,1)}$ it is convenient to introduce the conjugate modulus transformation. The standard formulas for the theta functions give

$$[u]=\sqrt{\frac{ik}{\tau(k+2)}}e^{-i\frac{k+2}{4k}\pi\tau}\theta_1\Big(\frac{\pi u}{k+2};-\frac{k}{\tau(k+2)}\Big),$$
$$[u]^*=\sqrt{\frac{i}{\tau}}e^{-\frac{i\pi}{4}\tau}\theta_1\Big(\frac{\pi u}{k};-\frac{1}{\tau}\Big),$$
$$\theta_1(u;\tau)=2\sum_{n=1}^{\infty}(-1)^n e^{i\pi\tau(n-\frac{1}{2})^2}\sin(2n-1)u.$$

To deal with the function $\Gamma_{m,l}$ we borrow the technique of [27], as mentioned in Appendix C. With the notation $\beta=\beta_1-\beta_1'$, the expression (4.2) can be rewritten in the following form.

$$\begin{aligned}Q_a^{(1,1)}(m)=&\frac{e^{\frac{i\pi}{2k}\tau(\frac{i\beta}{\pi}+1)^2}}{k+2}\eta(-\tfrac{1}{\tau})^2\frac{\theta_1(\frac{\pi a}{k+2};-\frac{k}{\tau(k+2)})}{\theta_1(\frac{\pi}{k};-\frac{1}{\tau})}\frac{F(0)}{F\big(\frac{k}{2\pi i}(\beta-i\pi)\big)}\\ &\times\sum_{\mu,\nu=\pm1}\mu\nu\frac{\theta_1(\frac{i\beta}{2}+\frac{\pi}{2k}(\mu+\nu)+\frac{\pi}{2};-\frac{1}{\tau})}{\theta_1(\frac{i\beta}{2}+\frac{\pi}{2};-\frac{1}{\tau})}2\sum_{l'=0}^{k}\sum_{m'=0}^{k-1}\sin\frac{\pi a(l'+1)}{k+2}\\ &\times e^{-\frac{i\pi mm'}{k}}\Gamma_{m',l'}\Big(\frac{\mu-\nu}{2k},-\frac{i\beta}{\pi k}-\frac{\mu+\nu}{2k}\Big|-\frac{1}{\tau}\Big)\ .\end{aligned}$$

From the sum over m and a', the linear transform (5.9) selects only one term ($m'=0, l'=a-1$). Thus we have to analyze the limiting behavior of $\Gamma_{0,l}$,

$$\begin{aligned}&\Gamma_{0,l}\Big(\frac{\mu-\nu}{2k},-\frac{i\beta}{\pi k}-\frac{\mu+\nu}{2k}\Big|-\frac{1}{\tau}\Big)\\ =&\ \frac{1}{\eta(-\frac{1}{\tau})^2}\Big(\sum_{n_1\geq|n_2|}-\sum_{-n_1>|n_2|}\Big)(-1)^{2n_1}p^{2\frac{k+2}{k}\left(\frac{(l+1+2n_1(k+2))^2}{4(k+2)}-n_2^2k\right)}\\ &\times e^{\frac{\pi i}{2k}(\mu-\nu)(l+1+2(k+2)n_1)+2\pi i(\frac{i\beta}{\pi}+\frac{\mu+\nu}{2})n_2}\ ,\end{aligned}$$

where p is defined in (5.3) and the sum is taken over $n_1,n_2\in\frac{1}{2}\mathbb{Z}$ with $n_1-n_2\in\mathbb{Z}$. The leading contribution to the sums comes from the term with $n_1=n_2=0$. In the limit (5.4), we thus find

$$\Gamma_{0,l}\Big(\frac{\mu-\nu}{2k},-\frac{i\beta}{\pi k}-\frac{\mu+\nu}{2k}\Big|-\frac{1}{\tau}\Big)=p^{2\frac{k+2}{k}(\Delta_{0,l}-\frac{c}{24})}\Big(e^{\frac{\pi i}{2k}(\mu-\nu)(l+1)}+O(p^{2\frac{k+2}{k}})\Big).$$

Here $\Delta_{0,l}$ stands for the conformal dimension of $\mathbb{Z}_k$-neutral primary fields in the parafermionic CFT

$$\Delta_{0,l}=\frac{(l+1)^2-1}{4(k+2)}\qquad(l\equiv 0 \bmod 2).$$

The limit of the remaining terms can be computed directly. For example,

$$\lim_{\log x\to 0}\frac{F(0)}{F\left(\frac{k}{2\pi i}(\beta-\pi i)\right)}=F^{min}_{1\bar{1}}(\beta)\,,$$

where we introduced the minimal form factor for the particle-antiparticle scattering [28]

$$F^{min}_{1\bar{1}}(\beta)=\prod_{n=1}^{\infty}\frac{\Gamma(\frac{i\beta}{2\pi}+\frac{1}{k}+n+\frac{1}{2})\Gamma(-\frac{i\beta}{2\pi}+\frac{1}{k}+n-\frac{1}{2})\Gamma(n-\frac{1}{k})^2}{\Gamma(\frac{i\beta}{2\pi}-\frac{1}{k}+n+\frac{1}{2})\Gamma(-\frac{i\beta}{2\pi}-\frac{1}{k}+n-\frac{1}{2})\Gamma(n+\frac{1}{k})^2}\,,$$

normalized as $F^{min}_{1\bar{1}}(i\pi)=1$. As for the sum over different components of vertex operators we use

$$\begin{aligned}&\sum_{\mu,\nu=\pm 1}\mu\nu\, e^{\frac{\pi i}{2k}(\mu-\nu)a}\cosh\Big(\frac{\beta}{2}+\frac{i\pi}{2k}(\mu+\nu)\Big)\\ &=4\cosh\frac{\beta}{2}\,\sin\frac{\pi(a-1)}{2k}\,\sin\frac{\pi(a+1)}{2k}.\end{aligned}$$

This leads us to the simple expression for the two-point form factor of the projection operators

$$\widehat{\mathcal{Q}}^{(1,1)}_a=-2\,\frac{\sin\frac{\pi(a-1)}{2k}\sin\frac{\pi(a+1)}{2k}}{\sin\frac{\pi}{k}}\,p^{2\frac{k+2}{k}\Delta_{0,a-1}}F^{min}_{1\bar{1}}(\beta)\times\big(1+o(1)\big).$$

As we expect from the results of [25], there is no singularity at $\beta=i\pi$.

In the next subsection, we compute the scaling limit for the case of multi-particles in a similar manner. We note in passing that

$$\lim_{\log x\to 0}\frac{F\left(\frac{k}{2\pi i}\beta\right)}{F(0)}=\frac{F^{min}_{11}(\beta)}{\sinh\frac{\beta}{2}\sinh(\frac{\beta}{2}+\frac{i\pi}{k})}\,,$$

where the minimal two particle form factor $F^{min}_{11}(\beta)$ reads [28]

$$\begin{aligned}F^{min}_{11}(\beta)&=\sinh\frac{\beta}{2}\sinh\Big(\frac{\beta}{2}+\frac{i\pi}{k}\Big)\\ &\times\prod_{n=1}^{\infty}\frac{\Gamma(\frac{i\beta}{2\pi}-\frac{1}{k}+n)\Gamma(-\frac{i\beta}{2\pi}-\frac{1}{k}+n)\Gamma(n+\frac{1}{k})^2}{\Gamma(\frac{i\beta}{2\pi}+\frac{1}{k}+n)\Gamma(-\frac{i\beta}{2\pi}+\frac{1}{k}+n)\Gamma(n-\frac{1}{k})^2}.\end{aligned}$$

5.4. **Many particles.** To present the result in the general case, it is convenient to use the formal bosonization rule as explained in [26, 29]. Let us introduce

$$\mathcal{B}^a_1(\beta)=-\sum_{\mu=\pm 1}\mu\, e^{-\frac{\mu\pi i a}{2k}}\mathcal{Z}_{1,\mu}(\beta),\tag{5.10}$$

$$\mathcal{B}^a_{\bar{1}}(\beta)=\sum_{\mu=\pm 1}\mu\, e^{\frac{\mu\pi i a}{2k}}\mathcal{Z}_{\bar{1},\mu}(\beta).\tag{5.11}$$

Here $\mathcal{Z}_{1,\mu}(\beta)$ and $\mathcal{Z}_{\bar{1},\mu}(\beta)$ are some operators, for which we assume the contraction rules

$$\begin{aligned}
&\langle\!\langle \mathcal{Z}_{1,\mu}(\beta_1)\mathcal{Z}_{1,\nu}(\beta_2)\rangle\!\rangle = \langle\!\langle \mathcal{Z}_{\bar{1},-\mu}(\beta_1)\mathcal{Z}_{\bar{1},-\nu}(\beta_2)\rangle\!\rangle \\
&\quad = \frac{F_{11}^{min}(\beta)}{\sinh(\frac{\beta}{2}+\frac{i\pi}{k})\sinh(\frac{\beta}{2}-\frac{i\pi}{k})}\frac{\sinh\left(\frac{\beta}{2}-\frac{i\pi}{2k}(\mu-\nu)\right)}{2\sin\frac{\pi}{k}\ \sinh\frac{\beta}{2}}, \\
&\langle\!\langle \mathcal{Z}_{1,\mu}(\beta_1)\mathcal{Z}_{\bar{1},\nu}(\beta_2)\rangle\!\rangle = \langle\!\langle \mathcal{Z}_{\bar{1},-\mu}(\beta_1)\mathcal{Z}_{1,-\nu}(\beta_2)\rangle\!\rangle \\
&\quad = F_{1\bar{1}}^{min}(\beta)\frac{\cosh\left(\frac{\beta}{2}+\frac{i\pi}{2k}(\mu+\nu)\right)}{2\sin\frac{\pi}{k}\ \cosh\frac{\beta}{2}},
\end{aligned}$$

where $\beta = \beta_1 - \beta_2$, and the Wick theorem as in (C.2) where C is replaced by

$$C_1 = \frac{1}{\sqrt{2\sin\frac{\pi}{k}}}.$$

The prescription (5.10), (5.11) is analogous to the one for the free field representation of form factors of the 'exponential fields' [29, 26] in the sine-Gordon and affine Toda theories. The exponential factors $e^{\pm\frac{\pi i a}{2k}}$ are the only remnant of the 'zero mode' part.

The limit of the formula (5.9) can be written compactly as

$$\begin{aligned}
(5.12)\quad &\lim_{\log x\to 0}\left(p^{-2\frac{k+2}{k}\Delta_{0,a-1}}\,\widehat{\mathcal{Q}}_a^{(n,n)}\right) \\
&= \langle\!\langle \mathcal{B}_{\bar{1}}^{a}(\beta_1)\cdots\mathcal{B}_{\bar{1}}^{a}(\beta_n)\mathcal{B}_{1}^{a}(\beta'_1)\cdots\mathcal{B}_{1}^{a}(\beta'_n)\rangle\!\rangle \\
&= \prod_{1\le i<j\le n}\frac{F_{11}^{min}(\beta_i-\beta_j)}{(x_i-\omega^2x_j)(x_i-\omega^{-2}x_j)}\cdot\prod_{1\le i<j\le n}\frac{F_{11}^{min}(\beta'_i-\beta'_j)}{(y_i-\omega^2y_j)(y_i-\omega^{-2}y_j)} \\
&\quad\times\prod_{1\le i,j\le n}\frac{F_{1\bar{1}}^{min}(\beta_i-\beta'_j)}{x_i+y_j}\times C_1^{2n}\,2^{2n(n-1)}\sigma_n^{n-1}\tau_n^{n-1}R^{(n,n)}_{\frac{a+1}{2}}(x\,;y).
\end{aligned}$$

The notation is as follows. We set $x_i = e^{\beta_i}$, $y_i = e^{\beta'_i}$, $\sigma_r = \sigma_r(x)$, $\tau_r = \sigma_r(y)$ with

$$\prod_{j=1}^{n}(t+x_j) = \sum_{r=0}^{n} t^{n-r}\sigma_r(x),$$

$\omega = e^{\pi i/k}$, and $\{l\} = \omega^l - \omega^{-l}$. The polynomials $R_\alpha^{(m,n)}(x\,;y)$ are defined by

$$R_\alpha^{(m,n)}(x\,;y) = \sum_{\substack{\mu_1,\cdots,\mu_m=\pm 1\\ \nu_1,\cdots,\nu_n=\pm 1}} \prod_{i=1}^m \mu_i \omega^{(\alpha-m+n)\mu_i} \cdot \prod_{j=1}^n \nu_j \omega^{(\alpha+m-n)\nu_j}$$
$$\times \prod_{1\le i<j\le m} \frac{x_i\omega^{\mu_i} - x_j\omega^{\mu_j}}{x_i - x_j} \cdot \prod_{1\le i<j\le n} \frac{y_i\omega^{\nu_i} - y_j\omega^{\nu_j}}{y_i - y_j} \cdot \prod_{\substack{1\le i\le m\\ 1\le j\le n}} (x_i\omega^{-\mu_i} + y_j\omega^{-\nu_j}).$$

For example,

$$\begin{aligned}
R_\alpha^{(1,1)} &= \{\alpha\}\{\alpha-1\}(\sigma_1+\tau_1),\\
R_\alpha^{(2,2)} &= \{\alpha\}\{\alpha-1\}\big(\{\alpha\}\{\alpha-1\}(\sigma_1+\tau_1)(\sigma_2\tau_1+\sigma_1\tau_2)\\
&\qquad +\{\alpha+1\}\{\alpha-2\}(\sigma_2-\tau_2)^2\big),\\
R_\alpha^{(3,3)} &= \{\alpha\}^3\{\alpha-1\}^3(\sigma_1+\tau_1)(\sigma_3+\sigma_2\tau_1+\sigma_1\tau_2+\tau_3)(\sigma_3\tau_2+\sigma_2\tau_3)\\
&\quad -\{\alpha+2\}\{\alpha\}^2\{\alpha-1\}^2\{\alpha-3\}(\sigma_1+\tau_1)(\sigma_3+\tau_3)(\sigma_3\tau_2+\sigma_2\tau_3)\\
&\quad +\{\alpha+1\}^2\{\alpha\}\{\alpha-1\}\{\alpha-2\}^2(\sigma_2\tau_1+\sigma_1\tau_2)(\sigma_3+\tau_3)^2\\
&\quad +\{\alpha+1\}\{\alpha\}^2\{\alpha-1\}^2\{\alpha-2\}\big((\sigma_1+\tau_1)(\sigma_3\tau_1-\sigma_1\tau_3)^2\\
&\qquad +(\sigma_2-\tau_2)^2(\sigma_3\tau_2+\sigma_2\tau_3)-3(\sigma_1+\tau_1)(\sigma_3+\tau_3)(\sigma_3\tau_2+\sigma_2\tau_3)\big)\\
&\quad +\{\alpha+2\}\{\alpha+1\}\{\alpha\}\{\alpha-1\}\{\alpha-2\}\{\alpha-3\}(\sigma_3+\tau_3)^3.
\end{aligned}$$

We note that $R_1^{(n,n)} = 0$ holds for all n.

It can be shown that $R_\alpha^{(m,n)}(x\,;y)$ is a sum of products of determinants. Namely let $\Lambda(m,n)$ denote the set of partitions $\lambda = (\lambda_1,\cdots,\lambda_m)$ satisfying $n \ge \lambda_1 \ge \cdots \ge \lambda_m \ge 0$. For $\lambda \in \Lambda(m,n)$ we write $\widetilde{\lambda'} = (m-\lambda'_n,\cdots,m-\lambda'_1)$, where $\lambda' = (\lambda'_1,\cdots,\lambda'_n)$ denotes the conjugate partition. Then

$$\begin{aligned}
R_\alpha^{(m,n)}(x\,;y) \;=\; &\prod_{i=1}^m \{\alpha-i+n\} \cdot \prod_{j=1}^n \{\alpha-j+m\}\\
&\times \sum_{\lambda\in\Lambda(m,n)} S_\lambda(x\,;\alpha+n,\omega) S_{\widetilde{\lambda'}}(y\,;\alpha+m,\omega),
\end{aligned}$$

where

$$S_\lambda(x\,;\alpha,\omega) = \det\left(\frac{\{\alpha-\lambda'_i+i-2j\}}{\{\alpha-i\}}\sigma_{\lambda'_i-i+j}(x)\right)_{1\le i,j\le N},$$

with $N \ge n$.

6. Discussions

In this paper we have developed an algebraic approach to the ABF models in regime II, using free fields. We have found the following.

First, our results immediately give an integral representation for correlation functions of these non-critical models. We computed exactly the simplest integrals for the nearest neighbor correlation functions. The result supports the validity of our construction. Beyond this case, it remains a technical open problem to perform multiple contour integrals explicitly.

Second, following the prescription of [2] we found states which diagonalize the row-to-row transfer matrix in the thermodynamic limit. These are obtained by acting with type II vertex operators on the vacuum. Compared with other lattice models studied so far, a nice feature is that these operators do not contain contour integrals. This gives us a hope for handling them more effectively in further analysis.

As an application we studied the continuous limit of the traces of vertex operators of type II. We obtained a family of functions satisfying Watson's equations as well as kinematical pole conditions. The former is a consequence of the commutation relation (5.7) and the property of the corner Hamiltonian as the grading operator, while the latter follows from

$$i\mathcal{B}^{a}_{\bar{1}}(\beta_1)\mathcal{B}^{a}_{1}(\beta_2) = \frac{1}{\beta_1-\beta_2-i\pi}\times \mathrm{id} + O(1) \quad (\beta_1\to\beta_2+i\pi).$$

On these grounds, we interpreted the resulting formulas as form-factors of some local operators in the $\mathbb{Z}_k$ invariant massive scattering theory with minimal S-matrices. Our analysis is not complete, since we have considered only the $\mathbb{Z}_k$ neutral sector, which corresponds to having the same number of $\Psi_+(v)$'s and $\Psi_-(v)$'s in the trace. The bound state conditions also remain to be worked out.

There are other unsolved problems which deserve attention. Of particular interest is the problem of identifying the local operators corresponding to the projection operators. The form-factors $\widehat{\mathcal{Q}}_a^{(n,n)}$ are proportional to $M^{2\Delta_{0,a-1}}$. This mass dependence indicates that the projection operators correspond to perturbations of the neutral primary fields in the parafermionic CFT with the respective conformal dimensions.[2)] An argument in favor of this proposal is that the cluster property characteristic to the 'exponential operators' seems to hold.

It is natural to identify the case $a=1$, $\Delta_{0,0}=0$, with the form-factor of the identity operator. The fact that $\widehat{\mathcal{Q}}_1^{(n,n)}=0$ for all n agrees

[2)] A similar study has been made in [30] for the scaling limit of the XXZ model.

with this identification. In the case $a = 3$, $\Delta_{0,2} = \frac{2}{k+2}$, we infer that the $\widehat{\mathcal{Q}}_3^{(n,n)}$ give the form factors of the first energy operator. Since this operator perturbs the parafermionic CFT into the massive region, the corresponding form factors are proportional to those of the trace of stress-energy tensor. One can show that, for $a = 3$, the polynomial $R_2^{(n,n)}(x; y)$ in the formula (5.12) contains a factor

$$(\sigma_1 + \tau_1)\Big(\frac{\sigma_{n-1}}{\sigma_n} + \frac{\tau_{n-1}}{\tau_n}\Big),$$

as we expect for form factors of the stress-energy tensors [31, 25].

However, our understanding of this problem of identification of local operators, as well as that of deriving the vacuum expectation values of local fields, is still incomplete.

In the scaling limit we considered only the translationally invariant sector $m = 0$. From the lattice point of view, the other sectors have equal rights. It would be interesting to understand the field theoretical meaning of the traces corresponding to the $m \neq 0$ sectors.

Although practically all the technical elements of the construction have been known, we think that the interpretation as a free field realization of ABF models in regime II is rather interesting. It opens up a way for applying algebraic methods to the study of physical phenomena in these models.

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Appendix A. Formulas for operators

We give here explicit formulas for the operators in the deformed parafermion theory.

Oscillators We consider two kinds of oscillators $a_{j,n}$ ($j=1,2,\ n\in\mathbb{Z}\backslash\{0\}$) and 'zero-mode' operators P_j, Q_j ($j=1,2$) satisfying

$$\begin{aligned}
[a_{1,n},a_{1,n'}] &= \frac{[2n]_x[(k+2)n]_x}{n}\delta_{n+n',0},\\
[a_{2,n},a_{2,n'}] &= -\frac{[2n]_x[kn]_x}{n}\delta_{n+n',0},\\
[P_1,Q_1] &= 2(k+2), \quad [P_2,Q_2]=-2k,
\end{aligned}$$

where

$$[n]_x=\frac{x^n-x^{-n}}{x-x^{-1}}.$$

Fock space For $l,m\in\mathbb{Z}$ we set

$$\begin{aligned}
&\mathcal{F}_{m,l}=\mathbb{C}[a_{1,-n},a_{2,-n}\ (n>0)]\,|m,l\rangle,\\
&P_1|m,l\rangle=l|m,l\rangle, \quad P_2|m,l\rangle=-m|m,l\rangle,\\
&e^{\frac{Q_1}{2(k+2)}}|m,l\rangle=|m,l+1\rangle, \quad e^{\frac{Q_2}{2k}}|m,l\rangle=|m+1,l\rangle.
\end{aligned}$$

Grading operator We set $D=D^{osc}+D^{zero}$, with

$$\begin{aligned}
D^{osc} &= \sum_{n>0}\frac{n^2}{[2n]_x[(k+2)n]_x}a_{1,-n}a_{1,n}-\sum_{n>0}\frac{n^2}{[2n]_x[kn]_x}a_{2,-n}a_{2,n},\\
D^{zero} &= \frac{(P_1+1)^2-1}{4(k+2)}-\frac{P_2^2}{4k}.
\end{aligned}$$

We have

$$\begin{aligned}
&[D,a_{j,n}]=-na_{j,n},\\
&[D,Q_1]=P_1+1, \quad [D,Q_2]=P_2,\\
&D|m,l\rangle=\Big(\frac{(l+1)^2-1}{4(k+2)}-\frac{m^2}{4k}\Big)|m,l\rangle.
\end{aligned}$$

Notational convention We use the following notation.

$$\begin{aligned}
&\phi_j(A;B,C|z;C')=-\frac{A}{BC}(Q_j+P_j\log z)+\sum_{n\neq 0}\frac{[An]_x}{[Bn]_x[Cn]_x}a_{j,n}z^{-n}x^{C'|n|},\\
&\phi_j(B|z;C)=\phi_j(A;A,B|z;C),\\
&\phi_j^{(\pm)}(A;B|z)=\frac{A}{B}P_j\log x+(x-x^{-1})\sum_{n>0}\frac{[An]_x}{[Bn]_x}a_{j,\pm n}z^{\mp n}.
\end{aligned}$$

We use the 'additive' parameters $u,u',\cdots$ and the 'multiplicative' ones $z,z',\cdots$ on an equal footing. Unless otherwise stated explicitly, they are related by

$$z=x^{2u}, \quad z'=x^{2u'},\cdots.$$

The following are the list of operators used in the text.

Type II VO's

$$
\begin{aligned}
&\Psi_\pm(u) = \mp\frac{1}{x-x^{-1}}\left(\Psi_{\pm,+}(u)-\Psi_{\pm,-}(u)\right),\\
&\Psi_{\pm,\varepsilon}(u) = z^{-\frac{1}{k}}\times\\
&\quad :\exp\Big(\pm\phi_2(k|z;\pm\frac{k}{2})\pm\varepsilon\phi_1^{(\varepsilon)}(1;2|zx^{\pm\varepsilon\frac{k}{2}})-\varepsilon\phi_2^{(\varepsilon)}(1;2|zx^{\pm\varepsilon\frac{k+2}{2}})\Big):.
\end{aligned}
$$

More generally we set

$$
\begin{aligned}
\Psi_a(u) = {}& z^{-\frac{a(a-1)}{k}}\sum_{j=0}^{a}(-1)^j\, x^{(a-1)(j-\frac{a}{2})}\prod_{i=1}^{j}\frac{[a-i+1]_x}{[i]_x}\\
&\times:\prod_{i=1}^{a-j}\Psi_{-,-}(-\tfrac{a+1}{2}+i+u)\cdot\prod_{i=a-j+1}^{a}\Psi_{-,+}(-\tfrac{a+1}{2}+i+u):.
\end{aligned}
\tag{A.1}
$$

Screening current

$$
\begin{aligned}
&S(u) = -\frac{1}{x-x^{-1}}\left(S_+(u)-S_-(u)\right),\\
&S_\pm(u) =\\
&\quad :\exp\Big(\phi_1(k+2|z;-\frac{k+2}{2})\pm\phi_2^{(\pm)}(1;2|zx^{\mp\frac{k+2}{2}})\pm\phi_1^{(\pm)}(1;2|zx^{\mp\frac{k}{2}})\Big):.
\end{aligned}
$$

Type I VO's

$$
\begin{aligned}
&\Phi_-(u) = z^{\frac{k-1}{2k(k+2)}}\times\\
&\quad :\exp\Big(-\phi_1(1;2,k+2|zx^k;\frac{k+2}{2})-\phi_2(1;2,k|zx^k;\frac{k}{2})\Big):,\\
&\Phi_+(u) = \oint_{C_\Phi(z)}\underline{dz'}\,\Phi_-(u)S(u')\frac{[u-u'-\frac{3}{2}-P_1]}{[u-u'-\frac{3}{2}]},\\
&\Phi^*_-(u) = \frac{g}{[P_1+1]}\,z^{\frac{k-1}{2k(k+2)}}\times\\
&\quad :\exp\Big(-\phi_1(1;2,k+2|z;-\frac{k+2}{2})-\phi_2(-1;2,k|z;-\frac{k}{2})\Big):,\\
&\Phi^*_+(u) = -\oint_{C_{\Phi^*}(z)}\underline{dz'}\,\Phi^*_-(u)S(u')\frac{[u-u'-\frac{1}{2}-P_1]}{[u-u'-\frac{1}{2}]}.
\end{aligned}
$$

The contours $C_\Phi(z), C_{\Phi^*}(z)$ are specified by the rules

$$\begin{array}{llllll} C_\Phi(z): & z' & = & zx^{-3+2(k+2)n} & (n>0) & \text{inside,} \\ & & = & zx^{-1+2(k+2)n} & (n\le 0) & \text{outside,} \\ C_{\Phi^*}(z): & z' & = & zx^{-1+2(k+2)n} & (n\ge 0) & \text{inside,} \\ & & = & zx^{1+2(k+2)n} & (n\le 0) & \text{outside,} \end{array}$$

and

$$\begin{aligned} g &= (x-x^{-1})x^{-\frac{1}{k+2}} \\ &\times \frac{(x^{2k+2},x^{4k+2};x^{2k},x^{2k+4})_\infty}{(x^{2k},x^{4k+4};x^{2k},x^{2k+4})_\infty}(x^{2k+2};x^{2k+4})_\infty^2(x^{2k+4};x^{2k+4})_\infty. \end{aligned}$$

ξ-η system

$$\begin{aligned} \xi(u) &=: \exp\Big(\phi_1(2|z;\frac{k}{2})+\phi_2(2|z;\frac{k+2}{2})\Big):, \\ \eta(u) &=: \exp\Big(-\phi_1(2|z;\frac{k}{2})-\phi_2(2|z;\frac{k+2}{2})\Big):. \end{aligned}$$

Deformed W currents The first DWA current $W^1(u)$ is

$$W^1(u) = [k+1]_x\Lambda_-(u)+[k+2]_x\Lambda_0(u)+[k+1]_x\Lambda_+(u),$$

where $\Lambda_\pm(u)$ and $\Lambda_0(u)$ are

$$\begin{aligned} \Lambda_\pm(u) &= x^{\pm 1}g^{-1}x^{\frac{1-k}{k}}z^{\frac{1-k}{k(k+2)}} :\Phi_-(k+2+u)[P]\Phi_-^*(u)S_-(\tfrac{k+1}{2}\mp\tfrac{k+2}{2}+u):, \\ \Lambda_0(u) &= -g^{-1}x^{\frac{1-k}{k}}z^{\frac{1-k}{k(k+2)}} :\Phi_-(k+2+u)[P]\Phi_-^*(u)S_+(k+\tfrac{3}{2}+u):. \end{aligned}$$

In general the DWA currents $W^j(u)$ $(j=1,2,\cdots)$ are given by

$$\begin{aligned} W^j(u) &= \sum_{\substack{a,b,c\ge 0\\ a+b+c=j}} C^j_{a,b,c} :\prod_{i=1}^{a}\Lambda_-(\tfrac{j+1}{2}-i+u) \\ &\qquad\times\prod_{i=a+1}^{a+b}\Lambda_0(\tfrac{j+1}{2}-i+u)\cdot\prod_{i=a+b+1}^{j}\Lambda_+(\tfrac{j+1}{2}-i+u):, \end{aligned}$$

where $C^j_{a,b,c}$ is

$$C^j_{a,b,c} = \prod_{i=1}^{a}\frac{[k+2-i]_x}{[i]_x}\cdot\prod_{i=1}^{b}\frac{[k+1+i]_x}{[i]_x}\cdot\prod_{i=1}^{c}\frac{[k+2-i]_x}{[i]_x}.$$

Comparison of notation We have followed the notation in [15] with minor changes. Referring to the symbols in [15] by the subscript 'K', we have

$$\begin{aligned}
&x = q_K,\\
&\mathcal{F}_{m,l} = \mathcal{F}^{PF}_{l,m,K},\\
&S(u) = S_K(z),\\
&\eta(u) = \eta_K(z),\\
&\Phi_-(u) = \phi_{1,1,K}(q^k z) z^{\frac{k-1}{2k(k+2)}},\\
&\Phi^*_-(u) = \frac{g}{[P_1+1]}\phi_{1,-1,K}(z) z^{\frac{k-1}{2k(k+2)}},\\
&\Psi_+(u) = \Psi^\dagger_K(z) z^{-\frac{1}{k}},\\
&\Psi_-(u) = \Psi_K(z) z^{-\frac{1}{k}}.
\end{aligned}$$

We have introduced fractional powers of z in order that the homogeneity property

$$x^{2vD} Y(u) x^{-2vD} = Y(u+v)$$

holds for $Y = S, \Phi_\pm, \Phi^*_\pm, \Psi_\pm, \xi, \Psi_a, W^j$. As for η we have $x^{2vD}\eta(u)x^{-2vD} = \eta(u+v)x^{2v}$.

Appendix B. Resolution by ξ-η system

We summarize the main points concerning the resolution by the ξ-η system.

On the space $\mathcal{F}_{m,l}$ with $m \equiv l \bmod 2$, we have expansions of the form

$$\eta(u) = \sum_{n\in\mathbb{Z}} \eta_n z^{-n-1}, \qquad \xi(u) = \sum_{n\in\mathbb{Z}} \xi_n z^{-n}.$$

The Fourier components satisfy

$$[\eta_n, \eta_{n'}]_+ = 0, \quad [\xi_n, \xi_{n'}]_+ = 0, \quad [\eta_n, \xi_{n'}]_+ = \delta_{n+n',0}.$$

The components η_0, ξ_0 commute with D. Since $\xi_0^2 = \eta_0^2 = 0$ and $\xi_0\eta_0 + \eta_0\xi_0 = \mathrm{id}$, the complex

$$\cdots \xrightarrow{\eta_0} \mathcal{F}_{m-k,l-(k+2)} \xrightarrow{\eta_0} \mathcal{F}_{m,l} \xrightarrow{\eta_0} \mathcal{F}_{m+k,l+k+2} \xrightarrow{\eta_0} \mathcal{F}_{m+2k,l+2(k+2)} \xrightarrow{\eta_0} \cdots$$

is exact. Let

$$\begin{aligned}
\widetilde{\mathcal{F}}_{m,l} &= \mathrm{Ker}\,\eta_0\big|_{\mathcal{F}_{m,l}}\\
&= \mathrm{Coker}\,\eta_0\big|_{\mathcal{F}_{m-k,l-(k+2)}}.
\end{aligned}$$

Then, for an operator $\mathcal{O}$ on $\oplus_{m\equiv l \bmod 2}\mathcal{F}_{m,l}$ commuting with η_0, we have

$$\begin{aligned}\mathrm{tr}_{\widetilde{\mathcal{F}}_{m,l}}(\mathcal{O}) &= \sum_{n\geq 0}(-1)^n\,\mathrm{tr}_{\mathcal{F}_{m+kn,l+(k+2)n}}(\mathcal{O})\\ &= -\sum_{n<0}(-1)^n\,\mathrm{tr}_{\mathcal{F}_{m+kn,l+(k+2)n}}(\mathcal{O})\,.\end{aligned}$$

Appendix C. Calculation of the trace

We outline the computation of the trace of a product of type II operators focusing attention to the neutral case

$$\mathcal{O} = x^{2kD}\Psi_+(v_1)\cdots\Psi_+(v_n)\Psi_-(v_1')\cdots\Psi_-(v_n'). \tag{C.1}$$

It is convenient in what follows to consider the oscillator part and the zero mode part separately. Let us write

$$\begin{aligned}\Psi_{\pm,\varepsilon}(v) &= \Psi^{osc}_{\pm,\varepsilon}(v)\Psi^{zero}_{\pm,\varepsilon}(v),\\ \Psi^{zero}_{\pm,\varepsilon}(v) &= e^{\mp Q_2/k}z^{\mp P_2/k-1/k}x^{\pm\varepsilon P_1/2-\varepsilon P_2/2}.\end{aligned}$$

Oscillators The contributions from the oscillator part

$$\langle\!\langle\mathcal{O}\rangle\!\rangle = \frac{\mathrm{tr}_{\mathcal{F}_{m,l}}\left(x^{2kD^{osc}}\mathcal{O}^{osc}\right)}{\mathrm{tr}_{\mathcal{F}_{m,l}}\left(x^{2kD^{osc}}\right)}$$

are given by the following rules:

$$\begin{aligned}&C^{-N}\langle\!\langle\Psi_{\varepsilon_1,\varepsilon_1'}(v_1)\cdots\Psi_{\varepsilon_N,\varepsilon_N'}(v_N)\rangle\!\rangle\\ &\qquad= \prod_{1\leq i<j\leq N} C^{-2}\langle\!\langle\Psi_{\varepsilon_i,\varepsilon_i'}(v_i)\Psi_{\varepsilon_j,\varepsilon_j'}(v_j)\rangle\!\rangle\end{aligned} \tag{C.2}$$

for $\sum_{i=1}^N\varepsilon_i = 0$, and

$$\begin{aligned}&\langle\!\langle\Psi_{+,\varepsilon_1}(v_1)\Psi_{+,\varepsilon_2}(v_2)\rangle\!\rangle = \langle\!\langle\Psi_{-,-\varepsilon_1}(v_1)\Psi_{-,-\varepsilon_2}(v_2)\rangle\!\rangle\\ &\qquad= C^2F(v)\frac{[v+\frac{\varepsilon_1-\varepsilon_2}{2}]^*}{[v-1]^*}\,x^{-\frac{2}{k}(1+\frac{\varepsilon_1-\varepsilon_2}{2})v+\frac{1+\varepsilon_1\varepsilon_2}{2k}+1+\frac{\varepsilon_1-\varepsilon_2}{2}},\end{aligned} \tag{C.3}$$

$$\begin{aligned}&\langle\!\langle\Psi_{+,\varepsilon_1}(v_1)\Psi_{-,\varepsilon_2}(v_2)\rangle\!\rangle = \langle\!\langle\Psi_{-,-\varepsilon_1}(v_1)\Psi_{+,-\varepsilon_2}(v_2)\rangle\!\rangle\\ &\qquad= C^2F(v-\tfrac{k}{2})^{-1}\frac{[v-\frac{k}{2}-\frac{\varepsilon_1+\varepsilon_2}{2}]^*}{[v-\frac{k}{2}]^*}\,x^{\frac{\varepsilon_1+\varepsilon_2}{k}v-\frac{1+k}{2k}(1+\varepsilon_1\varepsilon_2)-\frac{\varepsilon_1+\varepsilon_2}{2}},\end{aligned} \tag{C.4}$$

where $v = v_2 - v_1$, $F(v)$ is given in (4.3), and

$$C = (x^{2k};x^{2k})_\infty\frac{(x^{2+4k};x^{2k},x^{2k})_\infty}{(x^{-2+2k};x^{2k},x^{2k})_\infty}.$$

Zero mode The trace over the zero mode in the Fock space is taken in a standard fashion, following (3.9),(3.10). As we show at the end of this appendix, our operator (C.1) satisfies in addition

$$\mathrm{tr}_{\mathcal{F}_{m,l}}(\mathcal{O}_0)=\mathrm{tr}_{\mathcal{F}_{m,-l-2}}(\mathcal{O}_1). \tag{C.5}$$

Then, for any fixed real numbers $h,h'\in\mathbb{R}$, the trace can be rewritten as follows.

$$\begin{aligned}
&\mathrm{tr}_{H^0(\mathcal{C}_{m,l})}(\mathcal{O})\\
&=\Bigl(\sum_{s\le h}\sum_{n\ge 0}-\sum_{s>h}\sum_{n<0}\Bigr)(-1)^n\,\mathrm{tr}_{\mathcal{F}_{m+kn,l+(k+2)(n-2s)}}(\mathcal{O}_0)\\
&\quad-\Bigl(\sum_{s<-h'}\sum_{n\ge 0}-\sum_{s\ge -h'}\sum_{n<0}\Bigr)(-1)^n\,\mathrm{tr}_{\mathcal{F}_{m+kn,-l-2+(k+2)(n-2s)}}(\mathcal{O}_1)\\
&=\Bigl(\sum_{\substack{n\ge 0\\ s\le h}}-\sum_{\substack{n<0\\ s>h}}-\sum_{\substack{n\ge 0\\ s>n+h'}}+\sum_{\substack{n<0\\ s\le n+h'}}\Bigr)(-1)^n\,\mathrm{tr}_{\mathcal{F}_{m+kn,l+(k+2)(n-2s)}}(\mathcal{O}_0)\\
&=\Bigl(\sum_{-n_1-h'\le n_2\le n_1+h}-\sum_{n_1+h<n_2<-n_1-h'}\Bigr)(-1)^{2n_1}\,\mathrm{tr}_{\mathcal{F}_{m+2n_2k,l+2n_1(k+2)}}(\mathcal{O}_0),
\end{aligned}$$

where the last sum is taken over $n_1,n_2\in\frac{1}{2}\mathbb{Z}$ such that $n_1-n_2\in\mathbb{Z}$. Note that the result is independent of the choice of h,h'.

Accordingly, the sum of the zero-mode contributions coming from different Fock spaces can be expressed by the function

$$\begin{aligned}
&\Gamma^{(h,h')}_{m,l}(y_1,y_2|\tau)\quad (l\equiv m \bmod 2)\\
&=\frac{1}{\eta(\tau)^2}\Bigl(\sum_{-n_1-h'\le n_2\le n_1+h}-\sum_{n_1+h<n_2<-n_1-h'}\Bigr)(-1)^{2n_1}\\
&\qquad\times e^{2\pi i\tau\left(\frac{(l+1+2n_1(k+2))^2}{4(k+2)}-\frac{(m+2n_2k)^2}{4k}\right)}e^{\pi i(l+1+2n_1(k+2))y_1-\pi i(m+2n_2k)y_2},
\end{aligned}\tag{C.6}$$

where the sum is taken over $n_1,n_2\in\frac{1}{2}\mathbb{Z}$ with $n_1-n_2\in\mathbb{Z}$, and we set $\Gamma^{(h,h')}_{m,l}(y_1,y_2|\tau)=0$ for $l\not\equiv m \bmod 2$. Choosing $h=h'=0$, for example, we obtain the formula (4.2) for the $Q_a(m)$. We note the properties

$$\Gamma^{(h,h')}_{m+2k,l}(y_1,y_2|\tau)=\Gamma^{(h+1,h'-1)}_{m,l}(y_1,y_2|\tau), \tag{C.7}$$

$$\Gamma^{(h,h')}_{m+k,k-l}(-y_1,y_2|\tau)=\Gamma^{(-h'+\varepsilon',-h+\varepsilon)}_{m,l}(y_1,y_2|\tau), \tag{C.8}$$

where $\varepsilon=\begin{cases}0 & (h\notin\mathbb{Z})\\ -1 & (h\in\mathbb{Z})\end{cases}$ and $\varepsilon'=\begin{cases}1 & (h'\notin\mathbb{Z})\\ 0 & (h'\in\mathbb{Z})\end{cases}$.

Modular transform To study the continuous limit, it is useful to consider the modular transformation $\tau\to -1/\tau$. Unfortunately the functions $\Gamma^{(h,h')}_{m,l}(y_1,y_2|\tau)$ do not enjoy simple transformation properties individually. A way around this difficulty is proposed in [27]. Recall that up to an overall scalar factor the $Q^{(n,n)}_{l+1}(m)$ has the form

$$\sum_{\mu,\nu\in\mathbb{Z}} f^{(n)}_{\mu,\nu}\Gamma^{(h,h')}_{m,l}(y_1,y_2|\tau)\,, \tag{C.9}$$

where $f^{(n)}_{\mu,\nu}$ is a function of $\mu,\nu\in\mathbb{Z}$ given below (C.10), and $y_1 = \frac{\tau}{2k}(\mu-\nu)$, $y_2=\frac{\tau}{k}(\frac{2v}{k}-\frac{\mu+\nu}{2})$ with $v=\sum_{i=1}^n(\frac{k}{2}-v_i-u_i)$. Making use of the independence of (C.9) on h,h', it is shown in [27] that a certain generating function of (C.9) with respect to m,h,h' can be reexpressed in terms of theta functions and (the zero mode contribution of) the characters of the $N=2$ superconformal algebra. From the knowledge of the modular property for the latter, the following relation can be deduced for the sum (C.9).

$$\begin{aligned}&\sum_{\mu,\nu\in\mathbb{Z}} f^{(n)}_{\mu,\nu}\Gamma^{(h,h')}_{m,l}(y_1,y_2|\tau)\\ &=\frac{1}{\sqrt{k(k+2)}}\sum_{\mu,\nu\in\mathbb{Z}} f^{(n)}_{\mu,\nu}e^{-\frac{i\pi}{2\tau}\left((k+2)y_1^2-ky_2^2\right)}\\ &\quad\times\sum_{l'=0}^{k}\sum_{m'=-k}^{k-1}\sin\frac{\pi(l+1)(l'+1)}{k+2}\,e^{-\frac{i\pi mm'}{k}}\,\Gamma^{(h,h')}_{m',l'}\Bigl(\frac{y_1}{\tau},\frac{y_2}{\tau}\Big|-\frac{1}{\tau}\Bigr)\,.\end{aligned}$$

By noting (C.8) and the property $f^{(n)}_{\nu,\mu}=f^{(n)}_{\mu,\nu}$ (see below), the sums in the last line can also be written as

$$2\sum_{l'=0}^{k}\sum_{m'=0}^{k-1}\sin\frac{\pi(l+1)(l'+1)}{k+2}\,e^{-\frac{i\pi mm'}{k}}\,\Gamma^{(h,h')}_{m',l'}\Bigl(\frac{y_1}{\tau},\frac{y_2}{\tau}\Big|-\frac{1}{\tau}\Bigr).$$

An identity Let us verify the property (C.5). For $\mu,\nu\in\mathbb{Z}$, set

$$\begin{aligned}&f^{(n)}_{\mu,\nu}(u_1,\cdots,u_n;v_1,\cdots,v_n)\\ &=\sum\prod_{i=1}^{n}\mu_i\nu_i\cdot\prod_{1\le i,j\le n}[u_i+v_j+\tfrac{\mu_i+\nu_j}{2}]^*\\ &\quad\times\prod_{1\le i<j\le n}\frac{[u_i-u_j-\frac{\mu_i-\mu_j}{2}]^*}{[u_i-u_j]^*}\frac{[v_i-v_j-\frac{\nu_i-\nu_j}{2}]^*}{[v_i-v_j]^*}\,.\end{aligned} \tag{C.10}$$

Here the sum is taken over $\mu_i,\nu_i=\pm1$ $(i=1,\cdots,n)$ satisfying $\sum_{1=1}^n\mu_i=\mu$, $\sum_{i=1}^n\nu_i=\nu$. It is easy to see that this function is holomorphic and symmetric in $(u_1,\cdots,u_n)$ (resp. $(v_1,\cdots,v_n)$). The relation (C.5) reduces to the identity

$$f^{(n)}_{\mu,\nu}(u_1,\cdots,u_n;v_1,\cdots,v_n)=f^{(n)}_{\nu,\mu}(u_1,\cdots,u_n;v_1,\cdots,v_n). \tag{C.11}$$

Clearly (C.11) is true for $n=1$. To show it in general, let $g^{(n)}_{\mu,\nu}$ stand for the difference of the left hand side and the right hand side of (C.11). Then

$$
\begin{aligned}
&g^{(n)}_{\mu,\nu}(u_1,\cdots,u_n;v_1,\cdots,v_n)=-g^{(n)}_{\mu,\nu}(v_1,\cdots,v_n;u_1,\cdots,u_n),\\
&g^{(n)}_{\mu,\nu}(u_1+k,\cdots,u_n;v_1,\cdots,v_n)=(-1)^n g^{(n)}_{\mu,\nu}(u_1,\cdots,u_n;v_1,\cdots,v_n),\\
&g^{(n)}_{\mu,\nu}(u_1+\tfrac{k}{\tau},\cdots,u_n;v_1,\cdots,v_n)\\
&\quad=\left(-e^{\frac{i\pi}{\tau}}\right)^n e^{\frac{2\pi i}{k}(nu_1+v_1+\cdots+v_n+\frac{\mu+\nu}{2})} g^{(n)}_{\mu,\nu}(u_1,\cdots,u_n;v_1,\cdots,v_n),\\
&g^{(n)}_{\mu,\nu}(u_1,\cdots,u_{n-1},u;v_1,\cdots,v_{n-1},-u)\\
&\quad=\sum_{\mu_n=\pm1}[\mu_n]^*\prod_{i=1}^{n-1}[u_i-u+\mu_n]^*[v_i+u+\mu_n]^*\\
&\qquad\qquad\times g^{(n-1)}_{\mu-\mu_n,\nu-\mu_n}(u_1,\cdots,u_{n-1};v_1,\cdots,v_{n-1}).
\end{aligned}
$$

From these properties, we conclude that $g^{(n)}_{\mu,\nu}=0$ by induction on n.

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