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Identifying the Operator Content, the Homogeneous Sine-Gordon models

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Abstract

We address the general question of how to reconstruct the field content of a quantum field theory from a given scattering theory in the context of the form factor program. For the $SU(3)_2$-homogeneous Sine-Gordon model we construct systematically all n-particle form factors for a huge class of operators in terms of general determinant formulae. We investigate how different operators are interrelated by the momentum space cluster property. Finally we compute several two-point correlation functions and carry out the ultraviolet limit in order to identify each operator with its corresponding partner in the underlying conformal field theory.
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1 Introduction

The central concepts of relativistic quantum field theory, like Einstein causality and Poincaré covariance, are captured in local field equations and commutation relations. As a matter of fact local quantum physics (algebraic quantum field theory) [1] takes the collection of all operators localized in a particular region, which generate a von Neumann algebra, as its very starting point (for recent reviews see e.g. [2]).

On the other hand, in the formulation of a quantum field theory one may alternatively start from a particle picture and investigate the corresponding scattering theories. In particular for 1+1 dimensional integrable quantum field theories this latter approach has proved to be impressively successful. As its most powerful tool one exploits here the bootstrap principle [3], which allows to write down exact scattering matrices. Ignoring subtleties of non-asymptotic states, it is essentially possible to obtain the latter picture from the former by means of the LSZ-reduction formalism [4]. However, the question of how to reconstruct at least part of the field content from the scattering theory is in general still an outstanding issue. Recently a link between scattering theory and local interacting fields in terms of polarization-free generators has been developed [5]. Unfortunately, they involve subtle domain properties and are therefore objects which concretely can only be handled with great difficulties.

In the context of 1+1 dimensional integrable quantum field theories the identification of the operators is based on the assumption, dating back to the initial papers [6], that each solution to the form factor consistency equations [6, 7, 8, 9] corresponds to a particular local operator. Based on this numerous authors [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] have used various ways to identify and constrain the specific nature of the operator, e.g. by looking at asymptotic behaviours, performing perturbation theory, taking symmetries into account, formulating quantum equations of motion, etc. Our analysis will make especially exploit the conjecture that each local operator has a counterpart in the ultraviolet conformal field theory.

In the present manuscript we show for a concrete model, the $SU(3)_2$-homogeneous Sine-Gordon model (HSG), that, by means of the form factor program, it is possible to reconstruct the field content starting from its scattering matrix. Our analysis is based on the assumption [12, 13] that each solution to the form factor consistency equations [6, 7, 8, 9] corresponds to a particular local operator. We take furthermore into account that the $SU(3)_2$-HSG model, like numerous other 1+1 dimensional integrable models, may be viewed as a perturbed conformal field theory whose entire field content is well classified. Assuming now a one-to-one correspondence between operators in the conformal and in the perturbed theory, we can carry out an identification on this level, that is we associate each solution of the form factor consistency equations a local operator which is labeled according to the the ultra-violet conformal field theory. We therefore construct systematically all possible solutions for the

n-particle form factors related to a huge class of operators in terms of some general building blocks which consist out of determinants of matrices whose entries are elementary symmetric polynomials depending on the rapidities. We demonstrate how these general solutions are interrelated by the momentum space cluster property. In particular we show that the cluster property serves also as a construction principle, in the sense that from one solution to the consistency equations we may obtain a huge class, almost all, of new solutions. Finally we compute the corresponding two-point correlation functions and carry out the ultraviolet limit in order to identify the corresponding conformal dimensions.

Our manuscript is organized as follows: In section 2 we recall [16] the solutions for the minimal form factors and the recursive equation which is central for the determination of the form factors. We describe the general structure of the n-particle form factors for a huge class of operators. In section 3 we provide a rigorous proof for all solutions. In section 4 we investigate the cluster property. In section 5 we compute several two-point correlation functions and carry out the ultraviolet limit on the base of a sum rule and the explicit two-point correlation function in order to identify the conformal dimensions of each operator. We state our conclusions in section 6. The appendix contains a collection of useful properties of elementary symmetric polynomials and some explicit formulae for the first non-vanishing form factors.

2 The $SU(3)_2$-HSG model form factors

The $SU(3)_2$-HSG model contains only two self-conjugate solitons which we denote, following the conventions of [16], by "+" and "−". The two particle scattering matrix as a function of the rapidity θ related to this model was found [17] to be

$$S_{\pm\pm} = -1 \qquad \text{and} \qquad S_{\pm\mp}(\theta) = \pm \tanh \frac{1}{2}\left(\theta \pm \sigma - i\frac{\pi}{2}\right) \,. \tag{1}$$

Here σ is a real constant and corresponds to a resonance parameter. The system (1) constitutes probably the simplest example of a massive quantum field theory involving two particles of distinct type. Nonetheless, despite the simplicity of the scattering matrix we expect to find a relatively involved operator content, since for finite resonance parameter the $SU(3)_2$-HSG model describes a WZNW-coset model with central charge $c = 6/5$ perturbed by an operator with conformal dimension $\Delta = 3/5$. Since this is true as long as σ is finite, we shall be content in the following mostly by setting $\sigma = 0$. It is expected from the classical analysis that for finite value of σ we always have the same ultraviolet central charge and therefore the same operator content. The TBA-analysis carried out in [18] supports this analysis. Thus, a finite variation of σ at the ultraviolet fixed point and away from it is not very illuminating and we therefore only distinguish the behaviour $\sigma \to \infty$ and σ

finite. In the former case one trivially observes that the S-matrices $S_{\pm\mp}$ tend to one and the theory decouples into two Ising models. The related form factors have to respect this behaviour and all combinations involving different types of particles vanish, see [16]. One may see also directly that the form factor solutions behave this way by employing the Riemann-Lebesgue theorem.

The underlying conformal field theory has recently [19] found an interesting application in the context of the construction of quantum Hall states which carry a spin and fractional charges.

Taking the scattering matrix as an input, it is in principle possible to compute form factors, by solving certain consistency equations [6, 7, 8, 9], and thereafter to evaluate correlation functions. Form factors are tensor valued functions, representing matrix elements of some local operator $\mathcal{O}(\vec{x})$ located at the origin between a multiparticle in-state and the vacuum, which we denote by

$$F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n) := \left\langle 0|\mathcal{O}(0)|V_{\mu_1}(\theta_1)V_{\mu_2}(\theta_2)\ldots V_{\mu_n}(\theta_n)\right\rangle_{\text{in}} . \tag{2}$$

Here the $V_\mu(\theta)$ are some vertex operators representing a particle of species μ. We commence now by recalling the basic ansatz for solutions of the form factors for the $SU(3)_2$-HSG model from [16]. We used the parameterization

$$F_n^{\mathcal{O}|\overbrace{\mu_1\cdots\mu_l}^{l\times\pm}\overbrace{\mu_{l+1}\cdots\mu_n}^{m\times\mp}}(\theta_1\ldots\theta_n) = H_n^{\mathcal{O}|\mu_1\ldots\mu_n} Q_n^{\mathcal{O}|\mu_1\ldots\mu_n}(x_1\ldots x_n)\prod_{i<j}\hat{F}^{\mu_i\mu_j}(\theta_{ij}) \tag{3}$$

where $\hat{F}^{\mu_i\mu_j}(\theta_{ij}) := F_{\text{min}}^{\mu_i\mu_j}(\theta_{ij})/\left(x_i^{\mu_i}+x_j^{\mu_j}\right)^{\delta_{\mu_i\mu_j}}$. The rapidities enter through the variable $x_i=\exp(\theta_i)$ and the functions $F_{\text{min}}^{\mu_i\mu_j}(\theta_{ij})$ denote the so-called minimal form factors. They were found to be

$$F_{\text{min}}^{\pm\pm}(\theta) = -i\sinh\frac{\theta}{2} \tag{4}$$

$$F_{\text{min}}^{\pm\mp}(\theta) = 2^{\frac{1}{4}}e^{\frac{i\pi(1\mp1)\pm\theta}{4}-\frac{G}{\pi}}\exp\left(-\int_0^\infty\frac{dt}{t}\frac{\sin^2\left((i\pi-\theta\mp\sigma)\frac{t}{2\pi}\right)}{\sinh t\cosh t/2}\right) = e^{\pm\frac{\theta}{4}}\tilde{F}_{\text{min}}^{\pm\mp}(\theta) . \tag{5}$$

Here G is the Catalan constant. For the overall constants we obtained

$$H^{\mathcal{O}|2s+\tau,m} = i^{sm}2^{s(2s-m-1+2\tau)}e^{sm\sigma/2}H^{\mathcal{O}|\tau,m}, \qquad \tau=0,1 . \tag{6}$$

Note that at this point an unknown constant, that is $H^{\mathcal{O}|\tau,m}$, enters into the procedure. This quantity is not constrained by the form factor consistency equations and has to be obtained from elsewhere. The polynomials Q have to satisfy the recursive equations

$$Q^{\mathcal{O}|l+2,m}(-x,x,\ldots,x_n) = D_\vartheta^{l,m}(x,x_1,\ldots,x_n)Q^{\mathcal{O}|l,m}(x_1,\ldots,x_n) \tag{7}$$

$$D_\vartheta^{l,m}(x,x_1,\ldots,x_n) = \frac{1}{2}(-ix)^{l+1}\sigma_l^+\sum_{k=0}^m x^{-k}(1-(-1)^{l+k+\vartheta})\hat{\sigma}_k^- . \tag{8}$$

In particular

$$D_\zeta^{2s+\tau,2t+\tau'}(x,x_1,\ldots,x_n) = (-i)^{2s+\tau+1}\sigma^+_{2s+\tau}\sum_{p=0}^{t} x^{2s-2p+\tau+1-\zeta}\hat{\sigma}^-_{2p+\zeta} \ . \tag{9}$$

Here ϑ is related to the factor of local commutativity $\omega = (-1)^\vartheta = \pm 1$. We introduced also the function ζ which is 0 or 1 for the sum $\vartheta+\tau$ being odd or even, respectively. We shall use various notations for elementary symmetric polynomials (see appendix for some essential properties). We employ the symbol σ_k when the polynomials depend on the variables x_i, the symbol $\bar{\sigma}_k$ when they depend on the inverse variables x_i^{-1}, the symbol $\hat{\sigma}_k$ when they depend on the variables $x_i e^{-\sigma+i\pi/2}$ and $\tilde{\sigma}_k$ when we set the first two variables to $x_1=-x$, $x_2=x$. The number of variables the polynomials depend upon is defined always in an unambiguous way through the l.h.s. of our equations, where we assume the first l variables to be associated with $\mu=+$ and the last m variables with $\mu=-$. In case no superscript is attached to the symbol the polynomials depend on all $m+l$ variables, in case of a "+" they depend on the first l variables and in case of a "−" on the last m variables.

Solving recursive equations of the type (7) in complete generality is still an entirely open problem. Ideally one would like to reach a situation similar to the one in the boostrap construction procedure of the scattering matrices, where one can state general building blocks, e.g. particular combinations of hyperbolic functions whenever backscattering is absent [20], infinite products of gamma functions when backscattering occurs or elliptic functions when infinite resonances are present. At least for all operators in the model we consider here this goal has been achieved. It will turn out that all solutions to the recursive equations (7) may be constructed from some general building blocks consisting out of determinants of matrices whose entries are elementary symmetric polynomials in some particular set of variables. Let us therefore define the $(t+s)\times(t+s)$-matrix

$$\left(\mathcal{A}^{\mu,\nu}_{l,m}(s,t)\right)_{ij} := \begin{cases} \sigma^+_{2(j-i)+\mu} & \text{for} \quad 1\le i\le t \\ \hat{\sigma}^-_{2(j-i)+2t+\nu} & \text{for} \quad t<i\le s+t \end{cases} . \tag{10}$$

More explicitly the matrix $\mathcal{A}$ reads

$$\mathcal{A}^{\mu,\nu}_{l,m} = \begin{pmatrix} \sigma^+_\mu & \sigma^+_{\mu+2} & \sigma^+_{\mu+4} & \sigma^+_{\mu+6} & \cdots & 0 \\ 0 & \sigma^+_\mu & \sigma^+_{\mu+2} & \sigma^+_{\mu+4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sigma^+_{2s+\mu} \\ \hat{\sigma}^-_\nu & \hat{\sigma}^-_{\nu+2} & \hat{\sigma}^-_{\nu+4} & \hat{\sigma}^-_{\nu+6} & \cdots & 0 \\ 0 & \hat{\sigma}^-_\nu & \hat{\sigma}^-_{\nu+2} & \hat{\sigma}^-_{\nu+4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \hat{\sigma}^-_{2t+\nu} \end{pmatrix} . \tag{11}$$

The superscripts μ, ν may take the values 0 and 1 and the subscripts l, m characterize the number of different variables related to the particle species "+", "−", respectively. The different combinations of the integers μ, ν, l, m will correspond to different kind of local operators $\mathcal{O}$.
In addition, the form factors will involve a function depending on two further indices $\bar{\mu}$ and $\bar{\nu}$

$$g_{l,m}^{\bar{\mu},\bar{\nu}} := (\sigma_l)^{\frac{l-m+\bar{\mu}}{2}} (\sigma_m)^{\frac{\bar{\nu}-m}{2}} \,. \tag{12}$$

Here the $\bar{\mu}, \bar{\nu}$ are integers whose range, unlike the one for μ, ν, is in principle not restricted. However, it will turn out that due to the existence of certain constraining relations, to be specified in detail below, it is sufficient to characterize a particular operator by the four integers μ, ν, l, m only. Then, as we shall demonstrate, all Q-polynomials acquire the general form

$$Q^{\mathcal{O}|l,m} = Q_{l,m}^{\mu,\nu} = Q_{2s+\tau,2t+\tau'}^{\mu,\nu} = i^{s\nu}(-1)^{s(\tau+t+1)} g_{2s+\tau,2t+\tau'}^{\bar{\mu},\bar{\nu}} \ \det \mathcal{A}_{2s+\tau,2t+\tau'}^{\mu,\nu} \,. \tag{13}$$

We used here already a parameterization for l, m which will turn out to be most convenient. The subscripts in g and $\mathcal{A}$ are only needed in formal considerations, but in most cases the number of particles of species "+" and "−" are unambiguously defined through the l.h.s. of our equations. This is in the same spirit in which we refer to the number of variables in the elementary symmetric polynomials. We will therefore drop them in these cases, which leads to simpler, but still precise, notations. To illustrate this with examples, we consider for instance the solutions to the recursive equations (7) related to the trace of the energy momentum tensor Θ, the order operator Σ and the disorder operator μ which were already stated in [16]

$$\begin{aligned}
Q^{\Theta|2s+2,2t+2} &= i^{s(2t+3)} e^{-(t+1)\sigma} \sigma_1 \bar{\sigma}_1 \, g^{0,2} \ \det \mathcal{A}^{1,1} && (14)\\
Q^{\Sigma|2s,2t+1} &= i^{s(2t+3)} \, g^{-1,1} \ \det \mathcal{A}^{0,1} && (15)\\
Q^{\mu|2s,2t} &= i^{2s(t+1)} \, g^{-1,1} \ \det \mathcal{A}^{0,0} \,. && (16)
\end{aligned}$$

Here $\mathcal{A}$ is always taken to be a $(t+s)\times(t+s)$-matrix. Notice that in comparison with (13) the factor of proportionality in (15) and (16) is only a constant, whereas in (14) also the term $\sigma_1\bar{\sigma}_1$ appears. Terms of this type may always be added since they satisfy the consistency equations trivially. This is also the reason why in comparison with [16] we can safely drop in $Q^{\Sigma|2s,2t+1}$ the factor $(\sigma_1)^{\frac{1}{2}}(\sigma_1^-)^{-\frac{1}{2}}$. Additional reasons for this modification will be provided below. For Θ we were forced [16] to introduce the factor $\sigma_1\bar{\sigma}_1$ in order to recover the solution of the thermally perturbed Ising model for $2s+2=0$. Note that for Θ the value $s=-1$ formally makes sense.

3 Solution procedure

We shall now recall the principle steps of the general solution procedure for the form factor consistency equations [6, 7, 8, 9]. For any local operator $\mathcal{O}$ one may anticipate the pole structure of the form factors and extract it explicitly in form of an ansatz of the type (3). This might turn out to be a relatively involved matter due to the occurrence of higher order poles in some integrable theories, e.g. [14], but nonetheless it is possible. Thereafter the task of finding solutions may be reduced to the evaluation of the minimal form factors and to solving a (or two if bound states may be formed in the model) recursive equation of the type (7). The first task can be carried out relatively easily, especially if the related scattering matrix is given as a particular integral representation [6]. Then an integral representation of the type (5) can be deduced immediately. The second task is rather more complicated and the heart of the whole problem. Having a seed for the recursive equation, that is the lowest non-vanishing form factor*, one can in general compute from them several form factors which involve more particles. However, the equations become relatively involved after several steps. Aiming at the solution for all n-particle form factors, it is therefore highly desirable to unravel a more generic structure which enables one to formulate rigorous proofs. Several examples [21, 8, 11] have shown that often the general solution may be cast into the form of determinants whose entries are elementary symmetric polynomials. Presuming such a structure which, at present, may be obtained by extrapolating from lower particle solutions to higher ones or by some inspired guess, one can rigorously formulate proofs as we now demonstrate for the $SU(3)_2$-HSG-model, for which some solutions were merely stated in [16].

We have two universal structures† at our disposal. We could either exploit the integral representation for the determinant $\mathcal{A}$, as presented in [16], or exploit simple properties of determinants. Here we shall pursue the latter possibility. For this purpose it is convenient to define the operator $C^x_{i,j}$ ($R^x_{i,j}$) which acts on the j^{th} column (row) of an $(n \times n)$-matrix $\mathcal{A}$ by adding x times the i^{th} column (row) to it

$$C^x_{i,j}\mathcal{A} \quad : \qquad \mathcal{A}_{kj} \mapsto \mathcal{A}_{kj} + x\mathcal{A}_{ki} \qquad 1 \leq i,j,k \leq n \tag{17}$$
$$R^x_{i,j}\mathcal{A} \quad : \qquad \mathcal{A}_{jk} \mapsto \mathcal{A}_{jk} + x\mathcal{A}_{ik} \qquad 1 \leq i,j,k \leq n \ . \tag{18}$$

Naturally the determinant of $\mathcal{A}$ is left invariant under the actions of $C^x_{i,j}$ and $R^x_{i,j}$ on $\mathcal{A}$, such that we can use them to bring $\mathcal{A}$ into a suitable form for our purposes. Furthermore, it is convenient to define the ordered products, i.e. operators related to the lowest entry act first,

$$\mathcal{C}^x_{a,b} := \prod_{p=a}^{b} C^x_{p,p+1} \qquad \text{and} \qquad \mathcal{R}^x_{a,b} := \prod_{p=a}^{b} R^x_{p,p-1} \ . \tag{19}$$

*For the case at hand this is provided for some operators by the well known solutions of the Ising model. In general this is also a difficult hurdle to take as, for instance, one might need to know vacuum expectation values.

†There exist also different types of universal expressions like for instance the integral representations presented in [9]. However, these type of expressions are sometimes only of a very formal nature since to evaluate them concretely for higher n-particle form factors requires still a considerable amount of computational effort.

It will be our strategy to use these operators in such a way that we produce as many zeros as possible in one column or row of a matrix of interest to us. In order to satisfy (7) we have to set now the first variables in $\mathcal{A}$ to $x_1 = -x$, $x_2 = x$, which we denote as $\tilde{\mathcal{A}}$ thereafter and relate the matrices $\tilde{\mathcal{A}}^{\mu,\nu}_{l+2,m}$ and $\mathcal{A}^{\mu,\nu}_{l,m}$. Taking relation (70) for the elementary symmetric polynomials into account, we can bring $\tilde{\mathcal{A}}^{\mu,\nu}_{l+2,m}$ into the form

$$\left(\mathcal{R}^{-x^2}_{t+2,s+t+1}\mathcal{C}^{x^2}_{1,s+t-1}\tilde{\mathcal{A}}^{\mu,\nu}_{l+2,m}\right)_{ij} = \begin{cases} \sigma^{+}_{2(j-i)+\mu} & 1 \le i \le t \\ \hat{\sigma}^{-}_{2(j-i)+2t+\nu} & t < i \le s+t \\ \sum\limits_{p=1}^{j} x^{2(j-p)}\hat{\sigma}^{-}_{2(p-s-1)+\nu} & i = s+t+1 \end{cases} . \qquad (20)$$

It is now crucial to note that since the number of variables has been reduced by two, several elementary polynomials may vanish. As a consequence, for $2s+2+\mu > l$ and $2t+2+\nu > m$, the last column takes on the simple form

$$\left(\mathcal{R}^{-x^2}_{t+2,s+t+1}\mathcal{C}^{x^2}_{1,s+t-1}\tilde{\mathcal{A}}^{\mu,\nu}_{l+2,m}\right)_{i(s+t+1)} = \begin{cases} 0 & 1 \le i \le s+t \\ \sum\limits_{p=0}^{t} x^{2(t-p)}\hat{\sigma}^{-}_{2p+\nu} & i = s+t+1 \end{cases} . \qquad (21)$$

Therefore, developing the determinant of $\tilde{\mathcal{A}}^{\mu,\nu}_{l+2,m}$ with respect to the last column, we are able to relate the determinants of $\tilde{\mathcal{A}}^{\mu,\nu}_{l+2,m}$ and $\mathcal{A}^{\mu,\nu}_{l,m}$ as

$$\det \tilde{\mathcal{A}}^{\mu,\nu}_{l+2,m} = \left(\sum_{p=0}^{t} x^{2(t-p)}\hat{\sigma}^{-}_{2p+\nu}\right) \det \mathcal{A}^{\mu,\nu}_{l,m} \quad . \qquad (22)$$

We are left with the task to specify the behaviour of the function g with respect to the "reduction" of the first two variables

$$\tilde{g}^{\bar{\mu},\bar{\nu}}_{l+2,m} = i^{l-m+\bar{\mu}+2}\, x^{l-m+\bar{\mu}+2}\, \sigma^{+}_{l}\, g^{\bar{\mu},\bar{\nu}}_{l,m} \ . \qquad (23)$$

Assembling the two factors (22) and (23), we obtain, in terms of the parameterization (13)

$$\tilde{Q}^{\bar{\mu},\bar{\nu},\mu,\nu}_{2s+2+\tau,2t+\tau'} = (-i)^{2s+\tau+1}\sigma^{+}_{2s+\tau}\left(\sum_{p=0}^{t} x^{2(s-p+1)+\tau-\tau'+\bar{\mu}}\hat{\sigma}^{-}_{2p+\nu}\right) Q^{\bar{\mu},\bar{\nu},\mu,\nu}_{2s+\tau,2t+\tau'} \ . \qquad (24)$$

We are now in the position to compare our general construction (24) with the recursive equation for the Q-polynomials of the $SU(3)_2$-HSG model (9). We read off directly the following restrictions

$$\nu = \zeta \qquad \text{and} \qquad \zeta = \tau' - \bar{\mu} - 1 \ . \qquad (25)$$

A further constraint results from relativistic invariance, which implies that the overall power in all variables x_i of the form factors has to be zero for a spinless operator. Introducing the short hand notation $[F_n^{\mathcal{O}}]$ for the total power, we have to evaluate

$$\left[Q^{\bar{\mu},\bar{\nu},\mu,\nu}_{2s+\tau,2t+\tau'}\right] = \left[g^{\bar{\mu},\bar{\nu}}_{2s+\tau,2t+\tau'}\right] + \left[\det \mathcal{A}^{\mu,\nu}_{2s+\tau,2t+\tau'}\right] \ . \tag{26}$$

Combining (25) and (26) with the explicit expressions $[\det \mathcal{A}^{\mu,\nu}_{2s+\tau,2t+\tau'}] = s(2t+\nu)+\mu t$, $[g^{\bar{\mu},\bar{\nu}}_{l,m}] = l(l-m+\bar{\mu})/2+m(\bar{\nu}-m)/2$ and $[Q^{\bar{\mu},\bar{\nu},\mu,\nu}_{2s+\tau,2t+\tau'}] = l(l-1)/2-m(m-1)/2$, we find the additional constraints

$$\mu = 1+\tau-\bar{\nu} \qquad \text{and} \qquad \tau\nu = \tau'(\bar{\nu}-1) \ . \tag{27}$$

Collecting now everything we conclude that different solutions to the form factor consistency equations can be characterized by a set of four distinct integers. Assuming that each solution corresponds to a local operator, there might be degeneracies of course, we can label the operators by μ,ν,τ,τ', i.e. $\mathcal{O}\to\mathcal{O}^{\mu,\nu}_{\tau,\tau'}$, such that we can also write $Q^{\mu,\nu}_{m,l}$ instead of $Q^{\bar{\mu},\bar{\nu},\mu,\nu}_{m,l}$. Then each Q-polynomial takes on the general form

$$Q^{\mathcal{O}|2s+\tau,2t+\tau'} = Q^{\mathcal{O}^{\mu,\nu}_{\tau,\tau'}|2s+\tau,2t+\tau'} = Q^{\mu,\nu}_{2s+\tau,2t+\tau'} \sim g^{\tau'-1-\nu,\tau+1-\mu}_{2s+\tau,2t+\tau'} \ \det \mathcal{A}^{\mu,\nu}_{2s+\tau,2t+\tau'} \tag{28}$$

and the integers μ,ν,τ,τ' are restricted by

$$\tau\nu+\tau'\mu = \tau\tau', \qquad 2+\mu>\tau, \qquad 2+\nu>\tau' \ . \tag{29}$$

We combined here (25) and (27) to get the first relation in (29). The inequalities result from the requirement in the proof which we needed to have the form (21). We find 12 admissible solutions to (29), i.e. potentially 12 different local operators, whose quantum numbers are presented in table 1.

μ	ν	τ	τ'	$[F^{\mu\nu}_{\tau\tau'}]_+$	$[F^{\mu\nu}_{\tau\tau'}]_-$	Δ
0	0	0	0	0	0	1/10
0	0	1	0	0	0	1/10
0	0	0	1	0	0	1/10
0	1	0	1	-1/2	0	1/10
0	1	1	1	-1/2	0	1/10
0	1	0	2	-1/2	0	1/10
1	0	1	1	0	-1/2	1/10
1	0	2	0	0	-1/2	1/10
1	0	1	0	0	-1/2	1/10
1	1	2	2	-1/2	-1/2	1/10
1	1	0	0	-1/2	-1	*
1	0	0	0	0	-1/2	*

Table 1: Operator content of the $SU(3)_2$-HSG model.

Comparing with our previous results, we have according to this notation $F^{\mathcal{O}^{0,0}_{0,0}|2s,2t} = F^{\mu|2s,2t}$, $F^{\mathcal{O}^{0,1}_{0,1}|2s,2t+1} = F^{\Sigma|2s,2t+1}$ and $F^{\mathcal{O}^{1,1}_{2,2}|2s,2t+1} \sim F^{\Theta|2s+2,2t+2}$. The last two solutions are only formal in the sense that they solve the constraining equations (29), but the corresponding explicit expressions turn out to be zero.

In summary, by taking the determinant of the matrix (11) as the ansatz for the general building block of the form factors, we constructed systematically generic formulae for the n-particle form factors possibly related to 12 different operators.

4 Momentum space cluster properties

Cluster properties in space, i.e. the observation that far separated operators do not interact, are quite familiar in quantum field theories [23] for a long time. In 1+1 dimensions a similar property has also been noted in momentum space. For the purely bosonic case this behaviour can be explained perturbatively by means of Weinberg's power counting theorem, see e.g. [6, 22][‡]. This property has been analysed explicitly for several specific models [21, 8, 12, 24]. It states that whenever the first, say κ, rapidities of an n-particle form factor are shifted to infinity, the n-particle form factor factorizes into a κ and an $(n-\kappa)$-particle form factor which are possibly related to different types of operators

$$\mathcal{T}^{\lambda}_{1,\kappa} F^{\mathcal{O}}_{n}(\theta_1,\ldots,\theta_n) \sim F^{\mathcal{O}'}_{\kappa}(\theta_1,\ldots,\theta_\kappa) F^{\mathcal{O}''}_{n-\kappa}(\theta_{\kappa+1},\ldots,\theta_n) \; . \tag{30}$$

For convenience we have introduced here the operator

$$\mathcal{T}^{\lambda}_{a,b} := \lim_{\lambda\to\infty} \prod_{p=a}^{b} T^{\lambda}_{p} \tag{31}$$

which will allow for concise notations. It is composed of the translation operator T^{λ}_{a} which acts on a function of n variables as

$$T^{\lambda}_{a} \, f(\theta_1,\ldots,\theta_a,\ldots,\theta_n) \mapsto f(\theta_1,\ldots,\theta_a+\lambda,\ldots,\theta_n) \; . \tag{32}$$

Whilst Watson's equations and the residue equations, see e.g. [6, 7, 8, 16], are operator independent features of form factors, the cluster property captures part of the operator nature of the theory. The cluster property (30) does not only constrain the solution, but eventually also serves as a construction principle in the sense that

‡There exists also a heuristic argument which provides some form of intuitive picture of this behaviour[15] by appealing to the ultraviolet conformal field theory. However, the argument is based on various assumptions, which need further clarification. For instance it remains to be proven rigorously that the particle creation operator $V_\mu(\theta)$ tends to a conformal Zamolodchikov operator for $\theta\to\infty$ and that the local field factorizes equally into two chiral fields in that situation. The restriction in there that $\lim_{\theta\to\infty} S_{ij}(\theta)=1$, for i being a particle which has been shifted and j one which has not, excludes a huge class of interesting models, in particular the one at hand.

when given $F_n^{\mathcal{O}}$ we may employ (30) and construct form factors related to $\mathcal{O}'$ and $\mathcal{O}''$. Hence, (30) constitutes a closed mathematical structure, which relates various solutions and whose abstract nature still needs to be unraveled.

We shall now systematically investigate the cluster property (30) for the $SU(3)_2$-HSG model. Choosing w.l.g. the upper signs for the particle types in equation (3), we have four different options to shift the rapidities

$$\mathcal{T}^{\pm\lambda}_{1,\kappa\leq l} F_n^{\mathcal{O}|l\times+,m\times-} = \mathcal{T}^{\mp\lambda}_{\kappa+1<l,n} F_n^{\mathcal{O}|l\times+,m\times-} \tag{33}$$
$$\mathcal{T}^{\pm\lambda}_{1,\kappa>l} F_n^{\mathcal{O}|l\times+,m\times-} = \mathcal{T}^{\mp\lambda}_{\kappa+1\geq l,n} F_n^{\mathcal{O}|l\times+,m\times-} \tag{34}$$

which a priori might all lead to different factorizations on the r.h.s. of equation (30). The equality signs in the equations (33) and (34) are a simple consequence of the relativistic invariance of form factors, i.e. we may shift all rapidities by the same amount, for $\mathcal{O}$ being a scalar operator.

Considering now the ansatz (3) we may first carry out part of the analysis for the terms which are operator independent. Noting that

$$\mathcal{T}^{\pm\lambda}_{1,1} F^{++}_{\min}(\theta) = \mathcal{T}^{\pm\lambda}_{1,1} F^{--}_{\min}(\theta) \sim e^{\frac{(\lambda\pm\theta)}{2}} \qquad \text{and} \qquad \mathcal{T}^{\pm\lambda}_{1,1} F^{+-}_{\min}(\theta) \sim \begin{cases} \mathcal{O}(1) \\ e^{\frac{(\theta-\lambda)}{2}} \end{cases}, \tag{35}$$

we obtain for the choice of the upper signs for the particle types in the ansatz (3)

$$\mathcal{T}^{\pm\lambda}_{1,\kappa\leq l} \prod_{i<j} \hat{F}^{\mu_i\mu_j}(\theta_{ij}) \sim \prod_{1\leq i<j\leq\kappa} \hat{F}^{++}(\theta_{ij}) \prod_{\kappa<i<j\leq l+m} \hat{F}^{\mu_i\mu_j}(\theta_{ij}) \begin{cases} \dfrac{\sigma_\kappa(x_1,\ldots,x_\kappa)^{\frac{\kappa-l}{2}} e^{\frac{\lambda\kappa(1-l)}{2}}}{\sigma_{l-\kappa}(x_{\kappa+1},\ldots,x_l)^{\frac{\kappa}{2}}} \\ \dfrac{\sigma_\kappa(x_1,\ldots,x_\kappa)^{\frac{m-l+\kappa}{2}} e^{\frac{\lambda\kappa(l-m-1)}{2}}}{\sigma_{n-\kappa}(x_{\kappa+1},\ldots,x_n)^{\frac{\kappa}{2}}} \end{cases}$$

$$\mathcal{T}^{\pm\lambda}_{n+1-\kappa<m,n} \prod_{i<j} \hat{F}^{\mu_i\mu_j}(\theta_{ij}) \sim \prod_{1\leq i<j\leq n-\kappa} \hat{F}^{\mu_i\mu_j}(\theta_{ij}) \prod_{n-\kappa<i<j\leq n} \hat{F}^{--}(\theta_{ij}) \begin{cases} \dfrac{\sigma_{n-\kappa}(x_1,\ldots,x_{n-\kappa})^{\frac{\kappa}{2}} e^{\frac{\lambda\kappa(m-l-1)}{2}}}{\sigma_\kappa(x_{n+1-\kappa},\ldots,x_n)^{\frac{l-m+\kappa}{2}}} \\ \dfrac{\sigma_{m-\kappa}(x_{l+1},\ldots,x_{n-\kappa})^{\frac{\kappa}{2}} e^{\frac{\lambda\kappa(1-m)}{2}}}{\sigma_\kappa(x_{n+1-\kappa},\ldots,x_n)^{\frac{\kappa-m}{2}}} \end{cases}.$$

The remaining cases can be obtained from the equalities (33) and (34). Turning now to the behaviour of the function g as defined in (12) under these operations, we observe with help of the asymptotic behaviour of the elementary symmetric polynomials (71) and (72)

$$\mathcal{T}^{\pm\lambda}_{1,\kappa\leq l}\, g^{\bar{\mu},\bar{\nu}}_{l,m} = [e^{\pm\lambda\kappa}\sigma_\kappa(x_1,\ldots,x_\kappa)\sigma_{l-\kappa}(x_{\kappa+1},\ldots,x_l)]^{\frac{l-m+\bar{\mu}}{2}} (\sigma_m)^{\frac{\bar{\nu}-m}{2}} \tag{36}$$
$$\mathcal{T}^{\pm\lambda}_{n+1-\kappa<m,n} g^{\bar{\mu},\bar{\nu}}_{l,m} = (\sigma_l)^{\frac{l-m+\bar{\mu}}{2}} [e^{\pm\lambda\kappa}\sigma_\kappa(x_{n+1-\kappa},\ldots,x_n)\sigma_{m-\kappa}(x_{l+1},\ldots,x_{n-\kappa})]^{\frac{\bar{\nu}-m}{2}}. \tag{37}$$

In a similar fashion we compute the behaviour of the determinants

$$\mathcal{T}^{\lambda}_{1,2\kappa+\xi\leq l} \det \mathcal{A}^{\mu,\nu}_{l,m} = e^{\lambda t(2\kappa+\xi)}(\sigma_{2\kappa+\xi})^t((-1)^t\hat{\sigma}^-_\nu)^{\kappa+\xi(1-\mu)} \det \mathcal{A}^{1-\mu,\nu}_{l-2\kappa-\xi,m} \tag{38}$$
$$\mathcal{T}^{-\lambda}_{1,2\kappa+\xi\leq l} \det \mathcal{A}^{\mu,\nu}_{l,m} = (\hat{\sigma}^-_{2t+\nu})^{\kappa+\xi} \det \mathcal{A}^{\mu,\nu}_{l-2\kappa-\xi,m} \tag{39}$$
$$\mathcal{T}^{\lambda}_{n+1-2\kappa-\xi<m,n} \det \mathcal{A}^{\mu,\nu}_{l,m} = e^{\lambda s(2\kappa+\xi)}(\sigma^+_\mu)^{\kappa+\xi(1-\nu)}(\hat{\sigma}_{2\kappa+\xi})^s \det \mathcal{A}^{\mu,1-\nu}_{l,m-2\kappa-\xi} \tag{40}$$
$$\mathcal{T}^{-\lambda}_{n+1-2\kappa-\xi<m,n} \det \mathcal{A}^{\mu,\nu}_{l,m} = ((-1)^s\sigma^+_{2s+\mu})^{\kappa+\xi} \det \mathcal{A}^{\mu,\nu}_{l,m-2\kappa-\xi}\ . \tag{41}$$

We have to distinguish here between the odd and even case, which is the reason for the introduction of the integer ξ taking on the values 0 or 1.

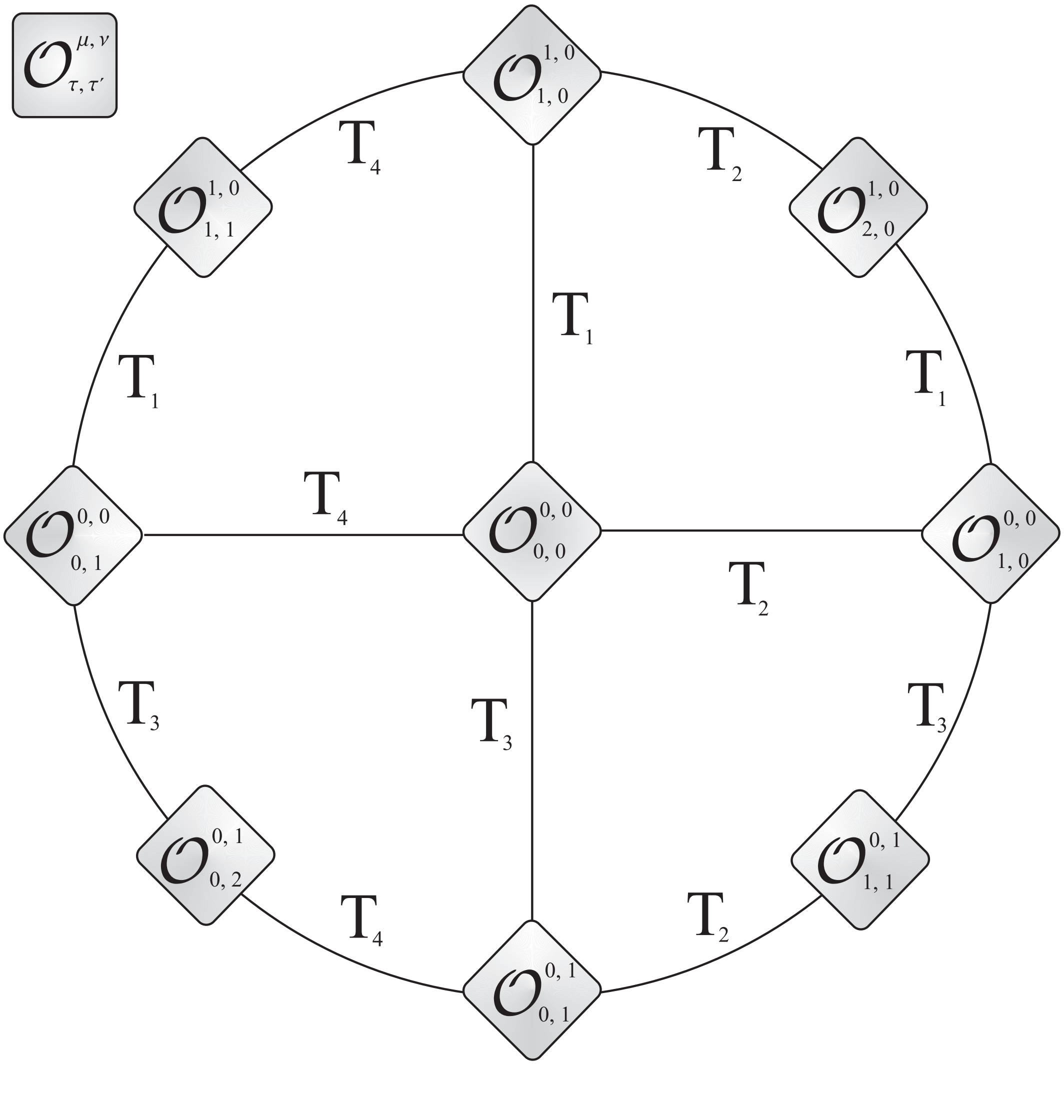

Figure 1: Interrelation of various operators via clustering. In this figure we use the abbreviations $T_1 \equiv \mathcal{T}^{\lambda}_{1,2\kappa+1\leq l}$, $T_2 \equiv \mathcal{T}^{-\lambda}_{1,2\kappa+1\leq l}$, $T_3 \equiv \mathcal{T}^{\lambda}_{n-2\kappa<m,n}$, $T_4 \equiv \mathcal{T}^{-\lambda}_{n-2\kappa<m,n}$. We also drop the 2s and 2t in the subscripts of the $\mathcal{O}$'s. The T_i on the links operate in both directions.

Collecting now all the factors, we extract first the leading order behaviour in λ

$$\mathcal{T}^{\pm\lambda}_{1,\kappa\leq l} F^{\mu,\nu}_{2s+\tau,2t+\tau'} \sim e^{-\lambda\kappa\left(\pm\nu+\tau'\frac{(1\mp1)}{2}\right)} \qquad \mathcal{T}^{\pm\lambda}_{n+1-\kappa<m,n} F^{\mu,\nu}_{2s+\tau,2t+\tau'} \sim e^{-\lambda\kappa\left(\pm\mu+\tau\frac{(1\mp1)}{2}\right)} . \tag{42}$$

Notice that, if we require that all possible actions of $\mathcal{T}^{\pm\lambda}_{a,b}$ should lead to finite expressions on the r.h.s. of (30), we have to impose two further restrictions, namely

$\tau' \geq \nu$ and $\tau \geq \mu$. These restrictions would also exclude the last two solutions from table 1. We observe further that $F_{2,2}^{1,1}$ tends to zero under all possible shifts. Seeking now solutions for the set μ, ν, τ, τ' of (42) which at least under some operations leads to finite results and in all remaining cases tends to zero, we end up precisely with the first 9 solutions in table 1.

Concentrating now in more detail on these latter cases which behave like $\mathcal{O}(1)$, we find from the previous equations the following cluster properties

$$\begin{aligned}
\mathcal{T}^{\lambda}_{1,2\kappa+\xi\leq l} F^{\mu,0}_{2s+\tau,2t+\tau'} &\sim F^{0,0}_{2\kappa+\xi,0} F^{\mu+\xi(1-2\mu),0}_{2s+\tau-2\kappa-\xi,2t+\tau'} && (43)\\
\mathcal{T}^{-\lambda}_{1,2\kappa+\xi\leq l} F^{\mu,\nu}_{2s+\tau,2t+\nu} &\sim F^{0,0}_{2\kappa+\xi,0} F^{\mu,\nu}_{2s+\tau-2\kappa-\xi,2t+\nu} && (44)\\
\mathcal{T}^{\lambda}_{n+1-2\kappa-\xi<m,n} F^{0,\nu}_{2s+\tau,2t+\tau'} &\sim F^{0,\nu+\xi(1-2\nu)}_{2s+\tau,2t+\tau'-2\kappa-\xi} F^{0,0}_{0,2\kappa+\xi} && (45)\\
\mathcal{T}^{-\lambda}_{n+1-2\kappa-\xi<m,n} F^{\mu,\nu}_{2s+\mu,2t+\tau'} &\sim F^{\mu,\nu}_{2s+\mu,2t+\tau'-2\kappa-\xi} F^{0,0}_{0,2\kappa+\xi} \,. && (46)
\end{aligned}$$

We may now use (43)-(46) as a means of constructing new solutions, i.e. we can start with one solution and use (43)-(46) in order to obtain new ones. Figure 1 demonstrates that when knowing just one of the first nine operators in table 1 it is possible to (re)-construct all the others in this fashion.

4.1 The energy momentum tensor

As we observed from our previous discussion the solution $F_{2,2}^{1,1}$ is rather special. In fact this solution is part of the expression which in [16] was identified as the trace of the energy momentum tensor

$$Q^{\Theta|2s+2,2t+2} = i^{s(2t+3)} e^{-(t+1)\sigma} \sigma_1 \bar{\sigma}_1 F^{1,1}_{2s+2,2t+2} \,. \tag{47}$$

The pre-factor $\sigma_1 \bar{\sigma}_1$ will, however, alter the cluster property. The leading order behaviour reads now

$$\mathcal{T}^{\pm\lambda}_{1,\kappa\leq 2s} F^{\Theta|2s,2t} \sim \mathcal{T}^{\pm\lambda}_{n+1-\kappa<2t,n} F^{\Theta|2s,2t} \sim e^{\lambda(1-\kappa/2)} \,. \tag{48}$$

We observe that still in most cases the shifted expressions tend to zero, unless $\kappa = 1$ for which it tends to infinity as a consequence of the introduction of the $\sigma_1 \bar{\sigma}_1$. There is now also the interesting case $\kappa = 2$, for which the λ-dependence drops out completely. Considering this case in more detail we find

$$\begin{aligned}
\mathcal{T}^{\pm\lambda}_{1,2} F^{\Theta|2s,2t} &\sim F^{\Theta|2,0} F^{\Theta|2s-2,2t} \times \begin{cases} \frac{\sigma_1(x_{2s+1},\ldots,x_{2s+2t})}{\sigma_1(x_3,\ldots,x_{2s+2t})} \\ \frac{\bar{\sigma}_1(x_{2s+1},\ldots,x_{2s+2t})}{\bar{\sigma}_1(x_3,\ldots,x_{2s+2t})} \end{cases} && (49)\\
\mathcal{T}^{\pm\lambda}_{n+1-\kappa,n} F^{\Theta|2s,2t} &\sim F^{\Theta|2s,2t-2} F^{\Theta|0,2} \times \begin{cases} \frac{\sigma_1(x_1,\ldots,x_{2s})}{\sigma_1(x_1,\ldots,x_{2s+2t-2})} \\ \frac{\bar{\sigma}_1(x_1,\ldots,x_{2s})}{\bar{\sigma}_1(x_1,\ldots,x_{2s+2t-2})} \end{cases} . && (50)
\end{aligned}$$

Note that unless $s = 1$ in (49) or $t = 1$ in (50) the form factors do not "purely" factorize into known form factors, but in all cases a parity breaking factor emerges. We now turn to the cases $\kappa = 2s$ or $\kappa = 2t$ for which we derive

$$\mathcal{T}_{1,2s}^{\pm\lambda} F^{\Theta|2s,2t} \sim \mathcal{T}_{n+1-2t,n}^{\pm\lambda} F^{\Theta|2s,2t} \sim e^{\lambda(2-t-s)} \ . \tag{51}$$

We observe that once again in most cases these expressions tend to zero. However, we also encounter several situations in which the λ-dependence drops out altogether. It may happen whenever $t = 2$, $s = 0$ or $s = 2$, $t = 0$, which simply expresses the relativistic invariance of the form factor. The other interesting situation occurs for $t = 1$, $s = 1$. Choosing temporarily (in general we assume $m_- = m_+$) $H_2^{\Theta|0,2} = 2\pi m_-^2$, $m = m_- = m_+ e^{2G/\pi}$, we derive in this case

$$\mathcal{T}_{1,2}^{\lambda} F_4^{\Theta|2,2} = \frac{F_2^{\Theta|2,0} F_2^{\Theta|0,2}}{2\pi m^2} \ . \tag{52}$$

In general when shifting the first $2s$ or last $2t$ rapidities we find the following factorization

$$\mathcal{T}_{1,2s}^{\pm\lambda} F^{\Theta|2s,2t} \sim \mathcal{T}_{n+1-2t,n}^{\pm\lambda} F^{\Theta|2s,2t} \sim \ F^{\Theta|2s,0} F^{\Theta|0,2t} \ . \tag{53}$$

This equation holds true when keeping in mind that the r.h.s. of this equation vanishes once it involves a form factor with more than two particles. Note that only in these two cases the form factors factorize "purely" into two form factors without the additional parity breaking factors as in (49) and (50).

5 Identifying the operator content

Having solved Watson's and the residue equations one has still little information about the precise nature of the operator corresponding to a particular solution. There exist, however, various non-perturbative (in the standard coupling constant sense) arguments which provide this additional information and which we now wish to exploit for the model at hand. Basically all these arguments rely on the assumption that the superselection sectors of the underlying conformal field theory remain separated after a mass scale has been introduced. We will therefore first have a brief look at the operator content of the $G_k/U(1)^r$-WZNW coset models and attempt thereafter to match them with the solutions of the form factor consistency equations. For these theories the different conformal dimensions in one model can be parameterized by two quantities [26]: a highest dominant weight Λ of level smaller or equal to k and their corresponding lower weights λ obtained in the usual way by subtracting multiples of simple roots α_i from Λ until the lowest weight is reached

$$\Delta(\Lambda, \lambda) = \frac{(\Lambda \cdot (\Lambda + 2\rho))}{2(k+h)} - \frac{(\lambda \cdot \lambda)}{2k} \ . \tag{54}$$

Here h is the Coxeter number of G and ρ the Weyl vector, i.e. the sum over all fundamental weights. Denoting the highest root of G by ψ, the conformal dimension related to the adjoint representation $\Delta(\psi, 0)$ is of special interest since it corresponds to the one of the perturbing operator which leads to the massive HSG-models. Taking the length of ψ to be 2 and recalling the well known fact that the height of ψ, that is $\mathrm{ht}(\psi)$, is the Coxeter number minus one, such that $(\psi \cdot \rho) = ht(\psi) = h - 1$, it follows that $O^{\Delta(\psi,0)}$ is a unique operator with conformal dimension $\Delta(\psi, 0) = h/(k + h)$. Note that uniqueness demands in addition that we do not take the multiplicities of the λ-states into account. For $SU(3)_2$ the expression (54) is easily computed and since we could not find the explicit values in the literature we report them for reference in table 2.

$\lambda \backslash \Lambda$	λ_1	λ_2	$\lambda_1 + \lambda_2$	$2\lambda_1$	$2\lambda_2$
Λ	1/10	1/10	1/10	0	0
$\Lambda - \alpha_1$	1/10	$*$	1/10	1/2	$*$
$\Lambda - \alpha_2$	$*$	1/10	1/10	$*$	1/2
$\Lambda - \alpha_1 - \alpha_2$	1/10	1/10	3/5	1/2	1/2
$\Lambda - 2\alpha_1$	$*$	$*$	$*$	0	$*$
$\Lambda - 2\alpha_2$	$*$	$*$	$*$	$*$	0
$\Lambda - 2\alpha_1 - \alpha_2$	$*$	$*$	1/10	1/2	$*$
$\Lambda - \alpha_1 - 2\alpha_2$	$*$	$*$	1/10	$*$	1/2
$\Lambda - 2\alpha_1 - 2\alpha_2$	$*$	$*$	1/10	0	0

Table 2: Conformal dimensions for $\mathcal{O}^{\Delta(\Lambda,\lambda)}$ in the $SU(3)_2/U(1)^2$ -coset model.

Turning now to the massive theory, a crude constraint which gives a first glimpse at possible solutions to the form factor consistency equations is provided by the bound [14]

$$\left[F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1, \ldots, \theta_n)\right]_i \leq \Delta^{\mathcal{O}} \quad . \tag{55}$$

We introduced here $\lim_{\theta_i\to\infty} f(\theta_1, \ldots, \theta_n) =: \text{const} \exp([f(\theta_1, \ldots, \theta_n)]_i \theta_i)$ as abbreviation and denote the conformal dimension of the operator $\mathcal{O}$ in the ultraviolet conformal limit by $\Delta^{\mathcal{O}}$. We use the notation $[\ ]_\pm$ when we take the limit in the variable x_i related to the particle species $\mu_i =$ " $\pm$ ", respectively. For the different solutions we constructed, we report the asymptotic behaviour in table 1. When we are in a position in which we already anticipate the conformal dimensions the bound (55) will severely restrict the possible inclusion of factors like $\sigma_1, \bar{\sigma}_1, \sigma_1^-, \sigma_1^+$, which as we mentioned above may always be added since they trivially satisfy the consistency equations.

More concrete and definite values for $\Delta^{\mathcal{O}}$ are obtainable when we exploit the knowledge about the underlying conformal field theory more deeply. Considering an operator which in the conformal limit corresponds to a primary field we can of course compute the conformal dimension by appealing to the ultraviolet limit of the two-point correlation function

$$\langle \mathcal{O}_i(r)\mathcal{O}_j(0)\rangle = \sum_k C_{ijk}\, r^{2\Delta_k-2\Delta_i-2\Delta_j}\ \langle \mathcal{O}_k(0)\rangle + \ldots \tag{56}$$

The three-point couplings C_{ijk} are independent of r. In particular when assuming that 0 is the smallest conformal dimension occurring in the model (which is the case for unitary models), we have

$$\lim_{r\to 0}\langle \mathcal{O}(r)\mathcal{O}(0)\rangle \sim r^{-4\Delta^{\mathcal{O}}} \qquad\qquad \text{for } r \ll \left(\frac{C_{\Delta^{\mathcal{O}}\Delta^{\mathcal{O}}0}}{C_{\Delta^{\mathcal{O}}\Delta^{\mathcal{O}}\Delta^{\mathcal{O}''}}\langle \mathcal{O}''\rangle}\right)^{1/2\Delta^{\mathcal{O}''}} . \tag{57}$$

Here $\mathcal{O}''$ is the operator with the second smallest dimension for which the vacuum expectation value is non-vanishing. Using a Lorentz transformation to shift the $\mathcal{O}(r)$ to the origin and expanding the correlation function in terms of form factors in the usual fashion

$$\begin{aligned}\langle \mathcal{O}(r)\mathcal{O}'(0)\rangle \;\; &= \;\; \sum_{n=1}^{\infty}\sum_{\mu_1\cdots\mu_n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_n}{n!(2\pi)^n}\exp\left(-r\sum_{i=1}^{n}m_{\mu_i}\cosh\theta_i\right)\\ &\quad\times F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n)\left(F_n^{\mathcal{O}'|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n)\right)^* . \end{aligned}\tag{58}$$

we can compute the l.h.s. of (57) and extract $\Delta^{\mathcal{O}}$ thereafter. The disadvantage to proceed in this way is many-fold. First we need to compute the multidimensional integrals in (58) for each value of r, which means to produce a proper curve requires a lot of computational (at present computer) time. Second we need already a relatively good guess for $\Delta^{\mathcal{O}}$. Third for very small r the n-th term within the sum is proportional to $(\log(r))^n$ such that we have to include more and more terms in that region and fourth we need the precise values of the lowest non-vanishing form factors, i.e. in general vacuum expectation values or one particle form factors to compute the r.h.s. of (58). However, the lowest non-vanishing form factor can be of an arbitrary particle number and one may still extract the value of $\Delta^{\mathcal{O}}$.

A short remark is also due concerning solutions related to different sets of μ's. The sum over the particle types simplifies considerably when taking into account that form factors corresponding to two sets, which differ only by a permutation, lead to the same contribution in the sum. This follows simply by using one of Watson's equations [6, 7, 8, 9], which states that when two particles are interchanged we will pick up the related two particle scattering matrix as a factor. Noting that the scattering matrix is a phase, the expression remains unchanged.

5.1 Δ-sum rules

Most of the disadvantages, which emerge when using (57) to compute the conformal dimensions, can be circumvented by formulating sum rules in which the

r-dependence has been eliminated. Such type of rule has for instance been formulated by Smirnov [21] already more than a decade ago. However, the rule stated there is slightly cumbersome in its evaluation and we will therefore resort to one found more recently by Delfino, Simonetti and Cardy [15]. In close analogy to the spirit and derivation of the c-theorem [25] these authors derived an expression for the difference between the ultraviolet and infrared conformal dimension of a primary field $\mathcal{O}$

$$\Delta_{uv}^{\mathcal{O}} - \Delta_{ir}^{\mathcal{O}} = -\frac{1}{2\langle \mathcal{O} \rangle} \int_0^\infty r \, \langle \Theta(r) \mathcal{O}(0) \rangle \; dr \; . \tag{59}$$

Using the expansion of the correlation function in terms of form factors (58) we may carry out the r-integration in (59) and obtain

$$\begin{aligned} \Delta_{uv}^{\mathcal{O}} - \Delta_{ir}^{\mathcal{O}} &= -\frac{1}{2\langle \mathcal{O} \rangle} \sum_{n=1}^{\infty} \sum_{\mu_1 \cdots \mu_n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n \left(\sum_{i=1}^n m_{\mu_i} \cosh \theta_i \right)^2} \\ &\quad \times F_n^{\Theta|\mu_1 \cdots \mu_n}(\theta_1, \ldots, \theta_n) \left(F_n^{\mathcal{O}|\mu_1 \cdots \mu_n}(\theta_1, \ldots, \theta_n) \right)^* \; . \end{aligned} \tag{60}$$

Notice also that unlike in the evaluation of the c-theorem, which deals with a monotonically increasing series, due to the fact that it only involves absolute values of form factors, the series (59) can in principle be alternating. Before the concrete evaluation of the expression (60) for the various solutions we constructed for the $SU(3)_2$-HSG model, we should pause for a while and appreciate the advantages of this formula in comparison with (57). First of all, since the r-dependence has been integrated out we only have to evaluate the multidimensional integrals once. Second the evaluation of (60) does not involve any anticipation of the value of $\Delta^{\mathcal{O}}$. Third one is very often in the comfortable position that despite the fact that the vacuum expectation value occurs explicitly in the sum rule, its explicit form is not needed. Once it is non-vanishing, the next lowest non-vanishing form factor may be normalized such that $\langle \mathcal{O} \rangle$ cancels from the whole expression. For a singular vacuum expectation value the value of the remaining integral guarantees that the sum rule maintains its form as for instance discussed in [27]. Finally and most important fourth, the difficulty to identify the suitable region in r which is governed by the $(\log r)^n$ behaviour of the n-th term in the sum in (58) and the upper bound in (56) has completely disappeared.

There are however little drawbacks for theories with internal symmetries and for the case when the lowest non-vanishing form factor of the operator we are interested in is not the vacuum expectation value. The first problem arises due to the fact that the sum rule is only applicable for primary fields $\mathcal{O}$ whose two-point correlation function with the energy momentum tensor is non-vanishing. Since in our model the n-particle form factors related to the energy momentum tensor are only non-vanishing for even particle numbers, we may only use it for the operators $\mathcal{O}_{0,0}^{0,0}$, $\mathcal{O}_{0,2}^{0,1}$, $\mathcal{O}_{2,0}^{1,0}$ and $\mathcal{O}_{2,2}^{1,1}$, where the latter operator is plagued be the second problem.

We will now compute the sum rule for the operators $\mathcal{O}^{0,0}_{0,0}$, $\mathcal{O}^{0,1}_{0,2}$, $\mathcal{O}^{1,0}_{2,0}$ up to the 6-particle contribution. We commence with the two particle contribution which is always evaluated effortlessly. Noting that

$$F_2^{\Theta}(\theta) = -2\pi i m^2 \sinh(\theta/2) \tag{61}$$

and the fact that $\Delta^{\mathcal{O}}_{ir}$ is zero in a purely massive model, the two particle contribution acquires a particular simple form

$$(\Delta^{\mathcal{O}})^{(2)} = \frac{i}{4\pi \langle \mathcal{O} \rangle} \int\limits_{-\infty}^{\infty} d\theta \frac{\tanh\theta}{\cosh\theta} \left(F_2^{\mathcal{O}|++}(2\theta) \right)^* \quad . \tag{62}$$

Using now the explicit expressions for the two-particle form factors (74), we immediately find

$$(\Delta^{\mathcal{O}^{0,0}_{0,0}})^{(2)} = (\Delta^{\mathcal{O}^{0,1}_{0,2}})^{(2)} = (\Delta^{\mathcal{O}^{1,0}_{2,0}})^{(2)} = 1/8 \ . \tag{63}$$

We recall, see also [16], that in the limit $\sigma \to \infty$ we obtain two copies of the thermally perturbed Ising model. This means that in the sum over the particle types in (60) there will be no contributions from terms involving different types of particles. We only obtain two equal contributions, namely 1/16 from $F_2^{\mathcal{O}|++}$ and $F_2^{\mathcal{O}|--}$, such that the operator $\mathcal{O}^{0,0}_{0,0}$ plays the role of the disorder operator, as we expect.

To distinguish the operators $\mathcal{O}^{0,0}_{0,0}$, $\mathcal{O}^{0,1}_{0,2}$, $\mathcal{O}^{1,0}_{2,0}$ from each other we have to proceed to higher particle contributions. At present there exist no analytical arguments for this and we therefore resort to a brute force numerical computation.
Denoting by $(\Delta^{\mathcal{O}})^{(n)}$ the contribution up to the n-th particle form factor, our numerical Monte Carlo integration[§] yields

$$\begin{aligned}
(\Delta^{\mathcal{O}^{0,0}_{0,0}})^{(4)} &= 0.0987 \qquad (\Delta^{\mathcal{O}^{0,0}_{0,0}})^{(6)} = 0.1004, && (64)\\
(\Delta^{\mathcal{O}^{0,1}_{0,2}})^{(4)} &= 0.0880 \qquad (\Delta^{\mathcal{O}^{0,1}_{0,2}})^{(6)} = 0.0895, && (65)\\
(\Delta^{\mathcal{O}^{1,0}_{2,0}})^{(4)} &= 0.0880 \qquad (\Delta^{\mathcal{O}^{1,0}_{2,0}})^{(6)} = 0.0895\ . && (66)
\end{aligned}$$

We shall be content with the precision reached at this point, but we will have a look at the overall sign of the next contribution. From the explicit expressions of the 8-particle form factors we see that for $\mathcal{O}^{0,0}_{0,0}$ the next contribution will reduce the value for Δ. For the other two operators we have several contributions with different signs, such that the overall value is not clear a priory. In this light, we conclude that the operators $\mathcal{O}^{0,0}_{0,0}$, $\mathcal{O}^{0,1}_{0,2}$, $\mathcal{O}^{1,0}_{2,0}$ all possess conformal dimension 1/10 in the ultraviolet limit. Unfortunately, the values for the latter two operators do not allow such a clear cut deduction as for the first one. Nonetheless, we base our statement on the knowledge of the operator content of the conformal field theory and confirm them also by elaborating directly on (57) and (58).

[§] We employed here the widely used numerical recipe routine VEGAS [28]. Typical standard deviations we achieve correspond to the order of the last digit we quote.

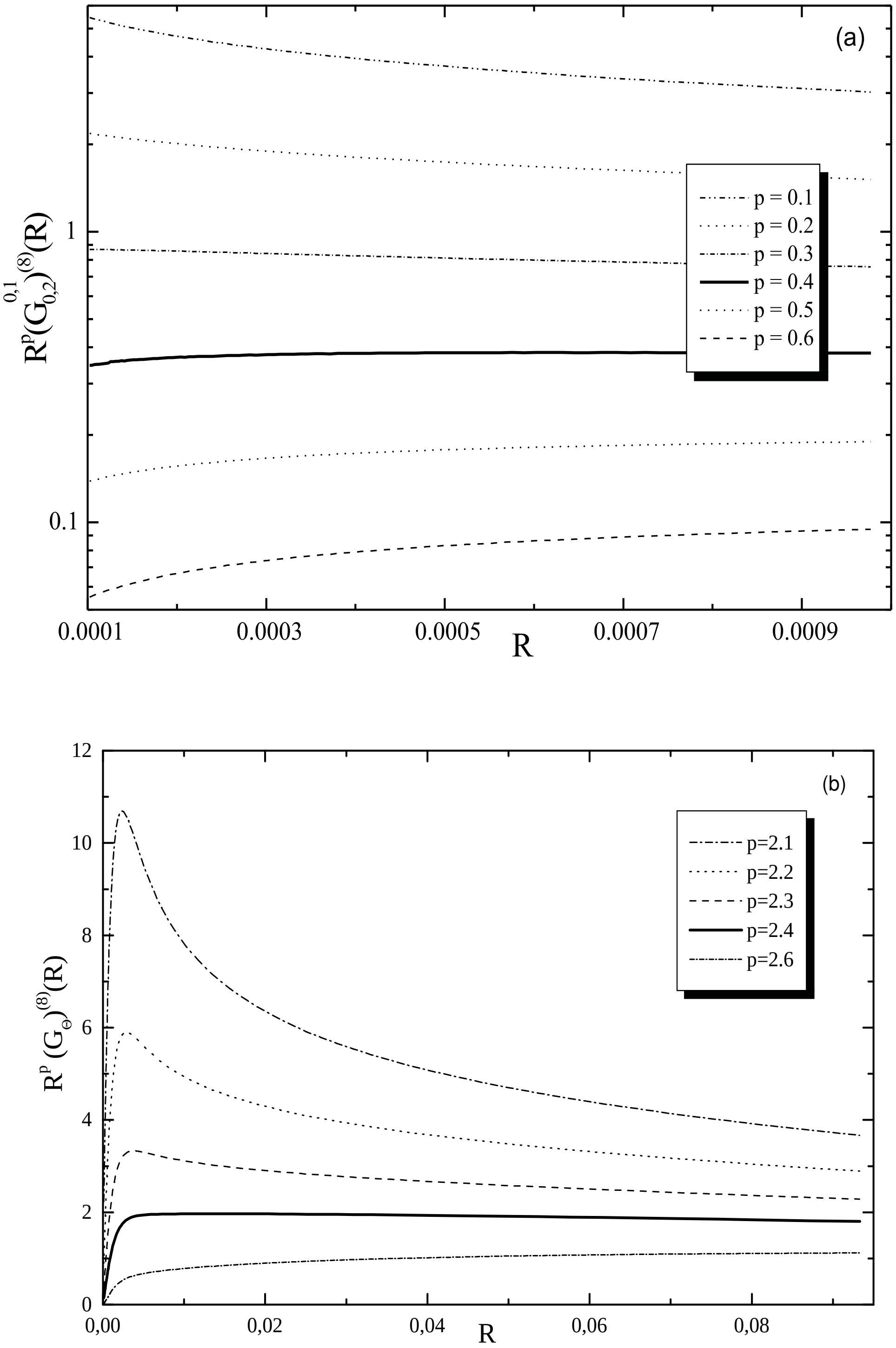

Figure 2: Rescaled correlation function $G_{0,2}^{0,1}(R) := \left\langle \mathcal{O}_{0,2}^{0,1}(R)\mathcal{O}_{0,2}^{0,1}(0)\right\rangle$ part (a) and $(G_\Theta)^{(8)}(R) := \langle\Theta(R)\Theta(0)\rangle$ part (b) summed up to the eight particle contribution as a function of $R = rm$.

5.2 Δ from correlation functions

First of all we do not presume anything about the conformal dimension of the operator $\mathcal{O}$ and multiply its two-point correlation function (58) by r^p with p being some arbitrary power. Once this combination behaves as a constant in the vicinity of $r = 0$ we take this value as the first non-vanishing three-point coupling divided by the vacuum expectation value of $\mathcal{O}$ and $p/4$ as its conformal dimension. This means even without knowing the vacuum expectation value we have a rational to fix p, but we can not determine the first term in (56). Figure 2a) exhibits this analysis for the operator $\mathcal{O}_{0,2}^{0,1}$ up to the 8-particle contribution and we conclude from there that its conformal dimension is $1/10$. For the other operators the figures look qualitatively the same.

The results of the same type of analysis for the energy momentum tensor is depicted in figure 2b), from which we deduce the conformal dimension $3/5$. Recalling that the energy-momentum tensor is proportional [29] to the dimension of the perturbing field this is precisely what we expected to find.

Furthermore, we observe that the relevant interval for r differs by two orders of magnitude, which by taking the upper bound for the validity of (57) into account should amount to $C_{\frac{1}{10}\frac{1}{10}0} C_{\frac{3}{5}\frac{3}{5}\frac{1}{10}} / (C_{\frac{3}{5}\frac{3}{5}0} C_{\frac{1}{10}\frac{1}{10}\frac{1}{10}}) \sim \mathcal{O}(10^{-2})$. Since to our knowledge these quantities have not been computed from the conformal side, this inequality can not be double checked at this stage.

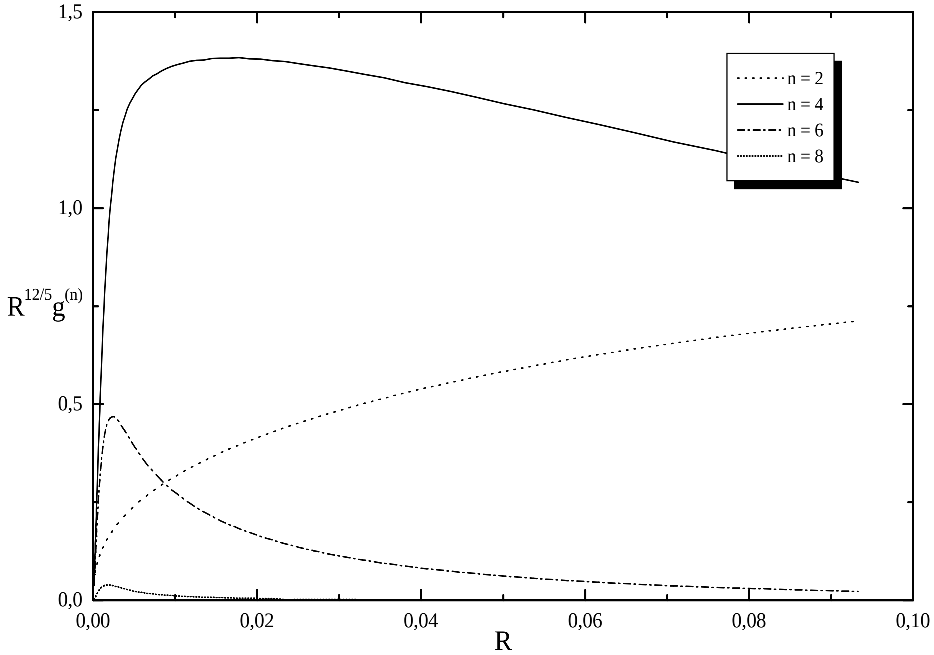

Figure 3: Rescaled individual n-particle contribution $g^{(n)}(R)$ to the correlation function.

In figure 3 we also exhibit the individual n-particle contributions. Excluding the two particle contribution, these data also confirm the proportionality of the n-th term to $(\log(r))^n$.

We have carried out similar analysis for the other solutions we have constructed and report our findings in table 1. We observe that the combination of the vacuum expectation value times the three-point coupling for these operators differ, which is the prerequisite for unraveling the degeneracy.

6 Conclusions

With regard to the main conceptual question addressed in this paper, we draw the overall conclusion that solutions of the form factor consistency equations can be identified with operators in the underlying ultraviolet conformal field theory. In this sense one can give meaning to the operator content of the integrable massive model. The quantity on which the identification is based is the conformal dimension of the operator. Naturally this implies that once the conformal field theory is degenerate in this quantity, as it is the case for the model we investigated, the identification can not be carried out in a one-to-one fashion and therefore the procedure has to be refined. In principle this would be possible by including the knowledge of the three-point coupling of the conformal field theory and the vacuum expectation value into the analysis. The former quantities are in principle accessible by working out explicitly the conformal fusion structure, whereas the computation of the latter still remains an open challenge. In fact what one would like to achieve ultimately is the identification of the conformal fusion structure within the massive models.

It would be desirable to put further constraints on the solutions by means of other arguments, that is exploiting the symmetries of the model, formulating quantum equations of motion, possibly performing perturbation theory etc.

Technically we have confirmed that the sum rule (59) is clearly superior to the direct analysis of the correlation function. It would therefore be highly desirable to develop arguments which also apply for theories with internal symmetries and possibly to resolve the mentioned degeneracies in the conformal dimensions.

It remains also an open question, whether the general solution procedure presented in this manuscript can be generalized to the degree that the type of determinants presented will serve as generic building blocks of form factors.

The specific conclusions for the $SU(3)_2$-homogeneous Sine-Gordon model are as follows: We have provided a rigorous proof for the solutions of the form factor consistency equations which were previously stated in [16]. In addition we found a huge number of new solutions. By means of the sum rule and a direct analysis of the correlation functions we identified the conformal dimension of these operators in the underlying conformal field theory. Considering the total number of operators present in the conformal field theory (see table 2) one still expects to find additional

solutions, in particular the identification of the fields possessing conformal dimension 1/2 is outstanding. Nonetheless, concerning the physical picture presented for this model one can surely claim that it rests now on quite firm ground. After the central charge of the conformal field theory had been reproduced by means of the thermodynamic Bethe ansatz [18] and the c-theorem in the context of the form factor program [16], we have now also identified the dimension of various operators. In particular the dimension of the perturbing operator was identified to be 3/5.
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7 Appendix

7.1 Elementary symmetric polynomials

In this appendix we assemble several properties of elementary symmetric polynomials to which we wish to appeal from time to time. Most of them may be found either in [30] or can be derived effortlessly. The elementary symmetric polynomials are defined as

$$\sigma_k(x_1,\ldots,x_n) = \sum_{l_1<\ldots<l_k} x_{l_1}\ldots x_{l_k} \ . \tag{67}$$

They are generated by

$$\prod_{k=1}^{n}(x+x_k) = \sum_{k=0}^{n} x^{n-k}\sigma_k(x_1,\ldots,x_n) \ . \tag{68}$$

and as a consequence may also be represented in terms of an integral representation

$$\sigma_k(x_1,\ldots,x_n) \ = \frac{1}{2\pi i}\oint_{|z|=\varrho}\frac{dz}{z^{n-k+1}}\prod_{k=1}^{n}(z+x_k)\,, \tag{69}$$

which is convenient for various applications. Here ϱ is an arbitrary positive real number.
With the help of (69) we easily derive the identity

$$\sigma_k(-x,x,x_1,\ldots,x_n) = \sigma_k(x_1,\ldots,x_n) - x^2\sigma_{k-2}(x_1,\ldots,x_n) \ , \tag{70}$$

which will be central for us. We will also require the asymptotic behaviours

$$\mathcal{T}_{1,\eta}^{\lambda}\,\sigma_k(x_1,\ldots,x_n) \sim \begin{cases} e^{\eta\lambda}\sigma_\eta(x_1,\ldots,x_\eta)\sigma_{k-\eta}(x_{\eta+1},\ldots,x_n) & \text{for } \eta<k \\ e^{k\lambda}\sigma_k(x_1,\ldots,x_\eta) & \text{for } \eta\geq k \end{cases} \tag{71}$$

and

$$\mathcal{T}_{1,\eta}^{-\lambda}\,\sigma_k(x_1,\ldots,x_n)\sim\begin{cases}\sigma_k(x_{\eta+1},\ldots,x_n) & \text{for } \eta\leq n-k\\ \frac{\sigma_{k+\eta-n}(x_1,\ldots,x_\eta)\,\sigma_{n-\eta}(x_{\eta+1},\ldots,x_n)}{e^{\lambda(k+\eta-n)}} & \text{for } \eta>n-k\end{cases} \tag{72}$$

which may be obtained from (69) as well.

7.2 Explicit form factor formulae

Having constructed the general solutions in terms of the parameterization (3), it is simply a matter of collecting all the factors to get explicit formulae. For the concrete computation of the correlation function, it is convenient to have some of the evaluated expressions at hand in form of hyperbolic functions.

7.2.1 One particle form factors

$$F_1^{\mathcal{O}_{1,0}^{0,0}|+}=F_1^{\mathcal{O}_{0,1}^{0,0}|-}=F_1^{\mathcal{O}_{0,1}^{0,1}|-}=F_1^{\mathcal{O}_{1,0}^{1,0}|+}=H^{1,0}=H^{0,1} \tag{73}$$

7.2.2 Two particle form factors

$$F_2^{\mathcal{O}|\pm\pm} = i\langle\mathcal{O}\rangle\tanh\tfrac{\theta}{2},\qquad \text{for}\quad \mathcal{O}=\ \mathcal{O}_{0,0}^{0,0},\,\mathcal{O}_{0,2}^{0,1},\mathcal{O}_{2,0}^{1,0}, \tag{74}$$

$$F_2^{\mathcal{O}_{1,1}^{0,1}|+-} = H^{1,1}e^{\theta_{21}/2}F_{\min}^{+-}(\theta),\qquad F_2^{\mathcal{O}_{1,1}^{1,0}|+-}=H^{1,1}F_{\min}^{+-}(\theta) \tag{75}$$

7.2.3 Three particle form factors

$$F_3^{\mathcal{O}|\pm\pm\pm}=\frac{H^{0,1}\prod_{i<j}F_{\min}^{\mu_i\mu_j}(\theta_{ij})}{\prod_{1\leq i<j\leq 3}\cosh(\theta_{ij}/2)}\quad\text{for}\quad \mathcal{O}_{1,0}^{0,0},\mathcal{O}_{0,1}^{0,0},\,\mathcal{O}_{0,1}^{0,1},\,\mathcal{O}_{1,0}^{1,0} \tag{76}$$

7.2.4 Four particle form factors

$$F_4^{\Theta|++--} = \frac{-\pi m^2(2+\sum_{i<j}\cosh(\theta_{ij}))}{2\cosh(\theta_{12}/2)\cosh(\theta_{34}/2)}\prod_{i<j}\tilde{F}_{\min}^{\mu_i\mu_j}(\theta_{ij}) \tag{77}$$

$$F_4^{\mathcal{O}_{0,0}^{0,0}|++--} = \frac{-\left\langle\mathcal{O}_{0,0}^{0,0}\right\rangle\cosh(\theta_{13}/2+\theta_{24}/2)}{2\cosh(\theta_{12}/2)\cosh(\theta_{34}/2)}\prod_{i<j}\tilde{F}_{\min}^{\mu_i\mu_j}(\theta_{ij}) \tag{78}$$

7.2.5 Five particle form factors

$$F_5^{\mathcal{O}|\pm\pm\pm\pm\pm}=\frac{H^{0,1}\prod_{i<j}F_{\min}^{\mu_i\mu_j}(\theta_{ij})}{\prod_{1\leq i<j\leq 5}\cosh(\theta_{ij}/2)}\quad\text{for } \mathcal{O}_{1,0}^{0,0},\,\mathcal{O}_{0,1}^{0,0},\,\mathcal{O}_{1,0}^{1,0},\,\mathcal{O}_{0,1}^{0,1} \tag{79}$$

7.2.6 Six particle form factors

$$F_6^{\Theta|++++--} = \frac{\pi m^2 (3 + \sum_{i<j} \cosh(\theta_{ij}))}{4 \prod_{1\le i<j\le 4} \cosh(\theta_{ij}/2)} \prod_{i<j} \tilde{F}_{\min}^{\mu_i \mu_j}(\theta_{ij}) \tag{80}$$

$$F_6^{\mathcal{O}_{0,0}^{0,0}|++++--} = \frac{\left\langle \mathcal{O}_{0,0}^{0,0} \right\rangle \left((\sigma_2^-)^2 + \sigma_4^+ + \sigma_2^+ \sigma_2^- / (\sigma_2^+ + \sigma_2^-) \right)}{16 \cosh(\theta_{56}/2) \prod_{1\le i<j\le 4} \cosh(\theta_{ij}/2)} \prod_{i<j} F_{\min}^{\mu_i \mu_j}(\theta_{ij}) \tag{81}$$

7.2.7 Seven particle form factors

$$F_7^{\mathcal{O}|\pm\pm\pm\pm\pm\pm\pm} = \frac{H^{0,1} \prod_{i<j} F_{\min}^{\mu_i \mu_j}(\theta_{ij})}{\prod_{1\le i<j\le 7} \cosh(\theta_{ij}/2)} \quad \text{for } \mathcal{O}_{1,0}^{0,0},\ \mathcal{O}_{0,1}^{0,0},\ \mathcal{O}_{1,0}^{1,0},\ \mathcal{O}_{0,1}^{0,1} \tag{82}$$

7.2.8 Eight particle form factors

$$F_8^{\Theta|++------} = \frac{-\pi m^2 (4 + \sum_{i<j} \cosh(\theta_{ij})) \cosh(\theta_{12}/2)}{8 \prod_{3\le i<j\le 8} \cosh(\theta_{ij}/2)} \prod_{i<j} \tilde{F}_{\min}^{\mu_i \mu_j}(\theta_{ij}) \tag{83}$$

$$F_8^{\Theta|++++----} = \frac{\pi m^2 (\sigma_4^-)^{1/2} (\sigma_1^- \sigma_3^+ + \sigma_1^+ \sigma_3^-)(4 + \sum_{i<j} \cosh(\theta_{ij}))}{2^7 (\sigma_4^+)^{3/2} \prod_{1\le i<j\le 4} \cosh(\theta_{ij}/2) \prod_{5\le i<j\le 8} \cosh(\theta_{ij}/2)} \prod_{i<j} F_{\min}^{\mu_i \mu_j}(\theta_{ij}) \tag{84}$$

$$F_8^{\Theta|++++++--} = \frac{-\pi m^2 (4 + \sum_{i<j} \cosh(\theta_{ij})) \cosh(\theta_{78}/2)}{8 \prod_{1\le i<j\le 6} \cosh(\theta_{ij}/2)} \prod_{i<j} \tilde{F}_{\min}^{\mu_i \mu_j}(\theta_{ij}) \tag{85}$$

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