Preprint April 2001
SfB 288, No. 512

LIOUVILLE THEORY REVISITED

by **J. Teschner**

Abstract:

We try to develop a coherent picture of Liouville theory as a two-dimensional conformal field theory that takes into account the perspectives of path-integral approach, bootstrap, canonical quantization and operator approach. To do this, we need to develop further some of these approaches. This includes in particular a construction of general exponential field operators from a set of covariant chiral operators. The latter are shown to satisfy braid relations that allow one to prove the locality of the former.

1. INTRODUCTION

1.1. Motivation

Liouville theory seems to be a kind of universal building block for a variety of models for two-dimensional gravity and non-trivial backgrounds in string theory. Some aspects of it were important for understanding what the matrix models of 2D gravity actually describe (see e.g. [GM]), and it keeps popping up in sometimes unexpected circumstances such as the the physics of membranes in string theory (e.g. [SW]). In the context of string theory, Liouville theory and close relatives such as the SL(2) or SL(2)/U(1) WZNW models seem to be the simplest examples where the new qualitative features of nontrivial (possibly curved) non-compact backgrounds can be studied.

For all this it is crucial that Liouville theory, as indeed supported by many investigations of this issue, can be quantized as a *conformal field theory* (CFT), implying in particular that the space of states forms a representation of the Virasoro algebra. What makes the analysis of the quantized theory much more difficult as compared to other conformal field theories is the fact that the set of Virasoro representations that make up the space of states is *continuous*. This can be viewed as a reflection of the noncompactness of the space in which the Liouville zero mode $q \equiv \int_0^{2\pi} d\sigma \Phi(\sigma)$ takes values.

Liouville theory may furthermore be seen as probably the simplest prototypical example for a class of conformal field theories called non-compact CFT which have continuous spectrum of representations of the Virasoro algebra. It may well be expected to play a role in the development of a general theory of such CFT's that is analogous to the role of the minimal models as prototype for rational CFT. This is in fact one of our main motivations: We believe that other non-compact CFT will share many features with Liouville theory that distinguish non-compact from rational CFT. Moreover, once the technical tools for the proper investigation of Liouville theory are established, it should not be too difficult to generalize them to other non-compact CFT. For example, many results from Liouville theory can be carried over fairly directly to the H_3^+-WZNW model [Te3].

1.2. **Aims and scope**

This paper focuses on the understanding of Liouville theory on a (space-time) cylinder with circumference 2π, time-coordinate t and (periodic) space-coordinate σ as a two dimensional quantum field theory in its own right. (Semi-)classically the theory is defined by the action

$$S_c \;=\; \int\limits_{-\infty}^{\infty} dt \int\limits_{0}^{2\pi} d\sigma \left(\frac{1}{16\pi}\big((\partial_t\varphi)^2 - (\partial_\sigma\varphi)^2\big) - \mu_c e^{\varphi} \right). \tag{1}$$

We will attempt to describe the corresponding quantum theory that will depend on the parameter $\hbar$ which we will write as $\hbar = b^2$. It turns out that there are two interesting regimes for the parameter b that one may consider: $b \in (0,1] \subset \mathbb{R}$ ("weak coupling") and $b = e^{i\varphi}$, $\varphi \in [0, \frac{\pi}{2})$ ("strong coupling"). One aim is to construct a quantum Liouville field ϕ that in the semi-classical limit $b \to 0$ corresponds to φ via the rescaling $\phi \underset{b\to 0}{\sim} \frac{1}{2b}\varphi$.

This paper grew out of an attempt to write a review of Liouville theory. We tried to develop a coherent picture that takes into account the different perspectives of path-integral approach, bootstrap, canonical quantization and operator approach. In doing so we felt forced to go beyond the existing literature as the listed approaches had not been sufficiently developed to allow for their mutual comparison. The character of this paper will therefore be somewhat intermediate between review, preview and original contribution: It has the character of a review in the sense that we try to exhibit the main ideas and results rather than all technical details. However, since many of the results and arguments presented here have not appeared in the literature yet, we have included short presentations of at least some key technical points.

We will not try to cover Liouville theory in all of its aspects and ramifications. We will *not* discuss important and interesting topics such as

- OTHER BOUNDARY CONDITIONS AND SUPERSYMMETRIC VERSIONS OF LIOUVILLE THEORY — Instead of periodic boundary conditions in the spatial direction one may also consider Liouville theory on the real line [dHJ], or the strip with boundary conditions that preserve conformal invariance at its end-points, see e.g. [GN2][CG] and references therein for early work, and [FZZ][TB][ZZ2] for more recent progress. Some exact results on $N = 1$ Liouville theory have been obtained in [RS][Po]. It would be quite important to have similar results for $N = 2$ Liouville (for some discussion see e.g. [KS]).
- LIOUVILLE THEORY ON THE LATTICE — Early works include [FT][B1][B2]. Substantial progress in this direction was made quite recently, see [FKV] and references therein, but we felt unable to explain the precise connections to the material present here at the present stage of development. Nevertheless, we would like to at least mention the very recent result of L. Faddeev and R. Kashaev [FK], where the spectrum of the Hamiltonian in the lattice Liouville

theory [FKV] is determined on the basis of [Ka2]. This result nicely confirms our claim that the spectrum is purely continuous both in the weak and strong coupling regimes.

- CONNECTIONS WITH TEICHMÜLLER THEORY AND QUANTIZATION OF TEICHMÜLLER SPACE — (For the former see e.g. the review [Ta], on the latter topic see e.g. [V][Ka1][CF][Ka2] and references therein). Quantization of Teichmüller space is expected [V] to encode topological information on the space of conformal blocks of Liouville theory, which should be equivalent (see [Ka2] for strong evidence) to the description of the duality transformations on spaces of conformal blocks that was proposed in [PT1] and which will be discussed in Part IV.
- NON-UNITARY SPECTRA, "ELLIPTIC SECTOR" There exist proposals for versions of quantum Liouville theory that have spectra involving non-unitary representations of the Virasoro algebra, see e.g. [GS3] and references therein. Related to the lack of unitarity one finds unusual hermiticity properties of the exponential field operators. In the present paper we will be exclusively concerned with the possibility of having a quantization of Liouville theory that preserves unitarity.
- CLOSED DISCRETE SUB-ALGEBRAS OF THE ALGEBRA OF VERTEX OPERATORS THAT CREATE UNITARY REPRESENTATIONS — It was proposed in [G2] (see also [GR1][GR2]) that for certain special values of the parameter b spectrum and vertex operator algebra of Liouville theory admit a "unitary truncation" in the following sense: There exists a Hilbert space $\mathcal{H}_d$ spanned by a discrete set of unitary representations of the Virasoro algebra, equipped with a realization of a discrete family of local vertex operators with real positive conformal dimensions. This would allow to construct non-critical bosonic string models. Let us remark, however, that our discussion *will* apply to the values of b, for which the proposal of [G2] was made. Nevertheless, it could be that the operator algebra that we discuss here shows some kind of "reducibility" for certain values of b, which would lead to the proposal of [G2] — see also our remark in Part IV, Subsection 19.7.
- APPLICATIONS TO MODELS OF 2D GRAVITY — (see e.g. [GM])

The author will be grateful to anybody pointing out further omissions or missing references concerning the material that is covered in the present paper.

1.3. **Overview**

If a conformal field theory fits into the framework introduced in [BPZ], it is essentially fully characterized by its spectrum of primary fields and the full set of three point functions. What we will call the "DOZZ-proposal" (where DOZZ stands for Dorn, Otto, Al. and A. Zamolodchikov) amounts to the proposal that Liouville theory fits into the framework of [BPZ], together with an explicit formula for the three point functions of the primary fields [DO][ZZ].

And indeed, as we will discuss in more detail, one can show that essentially all of Liouville theory is encoded in these pieces of information: Thanks to conformal symmetry it is possible to reconstruct all correlation functions of primary fields from the three-point function by summing over intermediate states. Primary fields and the energy momentum tensor generate the operator algebra of the theory. Mixed correlation functions involving the energy momentum tensor together with primary fields are reduced to correlation functions of only primary fields by the conformal Ward identities.

From that point of view the main problem for establishing the validity of the DOZZ-proposal is to show crossing symmetry of the amplitudes that can be reconstructed by means of conformal symmetry in terms of the particular formula for the three point functions that was proposed in [DO][ZZ]. Once this is established, one has ample reason to view the theory that is characterized by the DOZZ-proposal as a quantization of classical Liouville theory: Let us only mention that the semi-classical limit of the formula proposed in [DO][ZZ] indeed matches the result of direct semi-classical calculations [ZZ].

In the present paper we will try to explain why the DOZZ-proposal works and what the resulting picture of the physics of quantum Liouville theory looks like.

In Part I we begin by formulating more precisely what we refer to as the "DOZZ-proposal". Afterwards some of the original motivation for that proposal will be discussed which came from the path-integral approach to Liouville theory as initiated in [GL]. Our main objective in that Part I is to explain how Liouville theory can be reconstructed on the basis of the DOZZ-proposal: How to reconstruct the Hilbert space, operators corresponding to the fields, their correlation functions etc.. Some features arise that are unfamiliar from rational conformal field theories: For example, one does not find the $SL(2,\mathbb{C})$-invariant state $|0\rangle$ in the spectrum. However, it will be shown that a distributional interpretation of the "state" $|0\rangle$ is not only natural in a theory with continuous spectrum, but also makes operator-state correspondence and the interpretation of correlation functions as "vacuum expectation values" work in much the same way as in rational conformal field theories.

The following Part II discusses some aspects of the quantum Liouville physics as encoded in the DOZZ-proposal. We begin by discussing to what extend the DOZZ-proposal can be shown to represent a canonical quantization of Liouville theory: It is possible to reconstruct a field ϕ that weakly (within matrix elements) satisfies the canonical commutation relations and a natural quantum version of the Liouville equation of motion. One might then hope to have a representation where the Liouville zero mode q is diagonal, so that one could describe the physics of Liouville theory in terms of wave-functions $\psi(q)$ on "target-space". And indeed, such a representation would allow one to get natural interpretations for many features of the DOZZ-proposal in terms of scattering off the Liouville potential. However, at present we only have good control over such a representation in the region corresponding to zero mode $q \to -\infty$, the "asymptotic boundary of target space" where the Liouville interaction vanishes. This is good enough for setting up the scattering picture that we had mentioned above. But it is not clear so far how to describe Liouville physics for finite values of q, in the "bulk of target space". It is not even clear whether such a representation with diagonal zero mode q exists at all: The zero mode operator q has a rather complicated description in a representation where the Hamiltonian is diagonal, making it difficult to control properties such as self-adjointness.

Our third part is devoted to the operator approach to Liouville theory. Classically, one has a canonical transformation from free field theory to the interacting Liouville theory. In the operator approach one tries to quantize this transformation, i.e. to construct Liouville field operators in terms of the quantized free field. We outline recent results that lead to a complete construction of general exponential Liouville fields within such a framework (more details will be given in a forthcoming publication). Their matrix elements are given by the three point function proposed in [DO][ZZ].

This construction is based on the construction of a class of covariant chiral operators that form the building blocks of the exponential fields. Locality of the exponential fields can be controlled thanks to the existence of exchange- (braid-) relations that are satisfied by the covariant chiral operators. These results form the technical core for the construction of Liouville theory that we propose.

Conceptually it is interesting to compare this approach with the discussion in Part II: This furnishes another interpretation for the reflection operator R that describes the scattering of wave-packets off the Liouville-potential "wall": Classically one finds that the map from the free field to the Liouville field is two-to-one. This ambiguity is expressed in the quantum theory by the existence of an operator S that commutes with the Liouville field operators. The operator S coincides precisely with the reflection operator R. This observation may lead one to identify the free field theory on which the operator approach is based with the free field theory that describes the asymptotic in- and out-states of the scattering off the Liouville potential. Alternatively, one may view the operator approach as providing a reconstruction of Liouville theory from the free field theory "living on the asymptotic boundary of target space".

We finally discuss in Part IV how Liouville theory fits into a "chiral bootstrap" framework such as the one introduced for rational conformal field theories in [MS][FFK]. This not only yields insight into the mathematics behind the consistency of Liouville theory (fusion of unitary Virasoro representations, relation to quantum group representation theory), but also provides a useful framework to complete the verification of locality and crossing symmetry for the exponential Liouville fields constructed in Part III.

1.4. **Notational conventions**

Throughout we will use the convention that for a local field F, the notation $F(z,\bar{z})$ denotes the euclidean field on the Riemann sphere, $F(\tau,\sigma)$ or $F(w,\bar{w})$, $w = \tau + i\sigma$ its counterpart on the euclidean cylinder, $F(t,\sigma)$ the minkowskian version, and $F(\sigma) \equiv F(0,\sigma)$.

Part I. THE DOZZ PROPOSAL

We will start by discussing what we call the DOZZ-proposal. In a nutshell it consists of two ingredients: (i) Liouville theory fits into a rather mild generalization of the BPZ-formalism for two-dimensional conformal field theories, and (ii) a proposal for an explicit representation of the three point function ("DOZZ-formula"). The DOZZ-proposal will be formulated more more explicitly in the next section. Afterwards we will present some motivation for the DOZZ-formula from the path-integral point of view. In the remainder of this part we will try to explain how really all of Liouville theory is encoded in these two pieces of information.

2. CONFORMAL SYMMETRY AND THREE POINT FUNCTION

Within a formalism for conformal field theories such as that introduced in [BPZ] one assumes Liouville theory to be fully characterized by the set of all vacuum expectation values of the form

$$\Omega\left(\prod_{r=1}^{R} T(w_r)\prod_{s=1}^{S}\bar{T}(\bar{w}_r)\prod_{i=1}^{N} V_{\alpha_i}(z_i,\bar{z}_i)\right). \tag{2}$$

$T(z)$ and $\bar{T}(\bar{z})$ are the holomorphic and antiholomorphic components of the energy-momentum tensor respectively, and the $V_\alpha(z)$, $\alpha\in\mathbb{C}$ are the primary fields. The vacuum expectation values (2) are assumed to satisfy the following conditions:

(1) CONFORMAL WARD IDENTITIES —

$$\begin{aligned}
&\Omega\left(T(w)\prod_{r=1}^{R} T(w_r)\prod_{s=1}^{S}\bar{T}(\bar{w}_s)\prod_{i=1}^{N} V_{\alpha_i}(z_i,\bar{z}_i)\right)= \\
&\quad=\sum_{r=1}^{R}\Omega\left(T(w_R)\dots\big\{T(w)T(w_r)\big\}\dots T(w_1)\prod_{s=1}^{S}\bar{T}(\bar{w}_s)\prod_{i=1}^{N} V_{\alpha_i}(z_i,\bar{z}_i)\right)\\
&\quad+\sum_{i=1}^{N}\Omega\left(\prod_{r=1}^{R} T(w_r)\prod_{s=1}^{S}\bar{T}(\bar{w}_s)\; V_{\alpha_N}(z_N,\bar{z}_N)\dots\big\{T(w)V_{\alpha_i}(z_i,\bar{z}_i)\big\}\dots V_{\alpha_1}(z_1,\bar{z}_1)\right),
\end{aligned}$$

where $\big\{T(w)T(w)\big\}$ and $\big\{T(w)V_\alpha(z,\bar{z})\big\}$ are defined as

$$\begin{aligned}
\big\{T(w)T(w)\big\} &= \frac{c}{2(z-w)^4}+\frac{2}{(z-w)^2}T(w)+\frac{1}{z-w}\partial_w T(w),\\
\big\{T(w)V_\alpha(z,\bar{z})\big\} &= \frac{\Delta_\alpha}{(z-w)^2}V_\alpha(w)+\frac{1}{z-w}\partial_w V_\alpha(w),
\end{aligned}$$

together with a similar equation for $\Omega(\bar{T}(\bar{w})\dots)$, and furthermore:

(2) GLOBAL $SL(2,\mathbb{C})$-INVARIANCE —

$$\Omega\left(\prod_{i=1}^{N} V_{\alpha_i}(z_i)\right) = \Omega\left(\prod_{i=1}^{N}\big|\beta z_i+\delta\big|^{-4\Delta_{\alpha_i}} V_\alpha\Big(\frac{\alpha z_i+\gamma}{\beta z_i+\delta}\Big)\right).$$

The verification of the conformal Ward identities is possible, but nontrivial in low orders of the semiclassical expansion for the path-integral [Ta], where it is possible to identify $T(z)$, $\bar{T}(\bar{z})$ with the following functions of the Liouville field $\phi(z,\bar{z})$:

$$\begin{aligned} T(z) &\underset{b\to 0}{\simeq} -(\partial_z\phi)^2 + b^{-1}\partial_z^2\phi, \\ \bar{T}(\bar{z}) &\underset{b\to 0}{\simeq} -(\partial_{\bar{z}}\phi)^2 + b^{-1}\partial_{\bar{z}}^2\phi, \end{aligned} \qquad \text{and} \quad V_\alpha(z,\bar{z}) \underset{b\to 0}{\simeq} e^{2\alpha\phi}.$$

From that point of view we assume here that $T(z)$, $\bar{T}(\bar{z})$ and the V_α, $\alpha\in\mathbb{C}$ have quantum counterparts for which the above formulation of conformal invariance still holds up to quantum corrections of the parameters c and Δ_α. Let us anticipate the following relations between the parameters b, α and c, Δ_α:

$$c \;=\; 1+6Q^2 \qquad \Delta_\alpha \;=\; \alpha(Q-\alpha), \qquad Q=b+b^{-1} \tag{3}$$

REMARK 1. — Let us remember two simple consequences of these assumptions: First, the conformal Ward identities can be read as a rule that allows to recursively express general vacuum expectation values (2) in terms of those which contain only the field V_α. Second, the property of global $SL(2,\mathbb{C})$-invariance allows one to determine part of the dependence of the vacuum expectation values on the variables z_i, in particular for $n=1,2,3,4$:

$$\begin{aligned} \Omega\big(V_{\alpha_1}(z_1)\big) \;&=\; 0 \\ \Omega\big(V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)\big) \;&=\; |z_{21}|^{-4\Delta_1}\big(N(\alpha_1)\delta_{\alpha_2,Q-\alpha_1}+\delta_{\alpha_2,\alpha_1}B(\alpha_1)\big) \\ \Omega\big(V_{\alpha_3}(z_3)\ldots V_{\alpha_1}(z_1)\big) \;&=\; |z_{32}|^{2\Delta_{32}}|z_{31}|^{2\Delta_{31}}|z_{21}|^{2\Delta_{21}}C(\alpha_3,\alpha_2,\alpha_1) \\ \Omega\big(V_{\alpha_4}(z_4)\ldots V_{\alpha_1}(z_1)\big) \;&=\; |z_{43}|^{2(\Delta_2+\Delta_1-\Delta_4-\Delta_3)}|z_{42}|^{-4\Delta_2}|z_{41}|^{2(\Delta_3+\Delta_2-\Delta_4-\Delta_1)} \\ &\qquad |z_{32}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}G_{\alpha_4,\alpha_3,\alpha_2,\alpha_1}(z,\bar{z}), \end{aligned} \tag{4}$$

where $z_{ij}=z_i-z_j$, $\Delta_i=\Delta_{\alpha_i}$, $\Delta_{ij}=\Delta_k-\Delta_i-\Delta_j$ if $i\neq j$, $j\neq k$, $k\neq i$ and $z=\frac{z_{43}z_{21}}{z_{42}z_{32}}$.

2.1. Descendants

It will also be useful to keep in mind that further fields can be generated from the field $V_\alpha(z,\bar{z})$ by "acting on it with $T(z)$, $\bar{T}(\bar{z})$". More precisely, let $\mathcal{W}_\alpha$ be the representation of the Virasoro algebra that is generated by acting with the generators L_n, $\bar{\mathsf{L}}_n$ $n<0$ on the vector v_α that satisfies $\mathsf{L}_0v_\alpha=\bar{\mathsf{L}}_0v_\alpha=\Delta_\alpha v_\alpha$, $\mathsf{L}_nv_\alpha=0=\bar{\mathsf{L}}_nv_\alpha$ for $n>0$. Clearly $\mathcal{W}_\alpha\equiv\mathcal{V}_\alpha\otimes\mathcal{V}_\alpha$ where $\mathcal{V}_\alpha$ is a Verma-module over the Virasoro algebra (see Section 6 for a summary of some relevant results on the representation theory of the Virasoro algebra).

One may then define fields $V_\alpha(\zeta|z,\bar{z})$ parameterized by vectors $\zeta\in\mathcal{W}_\alpha$ by means of the following recursive definition: Let $V_\alpha(v_\alpha|z,\bar{z})\equiv V_\alpha(z,\bar{z})$, and extend the definition to a basis for $\mathcal{W}_\alpha$ by means of the recursion relation

$$T(w)V_\alpha(\zeta|z,\bar{z}) \;=\; \sum_{n=-N(\zeta_\alpha)}^{\infty}(w-z)^{n-2}V_\alpha(\mathsf{L}_{-n}\zeta|z,\bar{z}), \tag{5}$$

and its obvious counterpart with $\bar{T}(\bar{z})$. The number $N(\zeta)$ appearing in (5) is the smallest positive integer such that $\mathsf{L}_n\zeta=0$ for any $n>N(\zeta)$. Equation (5) is to be understood as definition of

$V_\alpha(\mathsf{L}_{-n}\zeta|z,\bar{z})$ in terms of $V_\alpha(\mathsf{L}_{-m}\zeta|z,\bar{z})$, $m < n$ by shuffling the terms with $m < n$ to the left hand side of the equation, taking $n-2$ derivatives w.r.t. w and finally the limit $w \to z$, explicitly:

$$
\begin{aligned}
&(n-2)!\, V_\alpha(\mathsf{L}_{-n}\zeta|z,\bar{z}) \;= \\
&\qquad = \lim_{w\to z} \partial_w^{n-2} \left(T(w) V_\alpha(\zeta|z,\bar{z}) - \sum_{m=-N(\zeta_\alpha)}^{n-1} (w-z)^{m-2} V_\alpha(\mathsf{L}_{-m}\zeta|z,\bar{z}) \right).
\end{aligned} \tag{6}
$$

It is clear from the definition (5) of the descendants that the conformal Ward identities allow one to express the vacuum expectation values of fields $V_\alpha(\zeta|z,\bar{z})$ in terms of those for $V_\alpha(z,\bar{z})$. It will also be useful to note that this fact together with global $SL(2,\mathbb{C})$-invariance imply the existence of

$$
\Omega\big(V_\alpha(\zeta|\infty)\ldots\big) \;\equiv\; \lim_{z\to\infty} z^{2\Delta(\zeta)} \bar{z}^{2\bar{\Delta}(\zeta)} \Omega\big(V_\alpha(\zeta|z,\bar{z})\ldots\big) \tag{7}
$$

whenever $\zeta \in \mathcal{W}_\alpha$ is such that both L_0 and $\bar{\mathsf{L}}_0$ act diagonally according to $\mathsf{L}_0\zeta = \Delta(\zeta)\zeta$ and $\bar{\mathsf{L}}_0\zeta = \bar{\Delta}(\zeta)\zeta$.

2.2. **The DOZZ-formula**

The second main ingredient of the DOZZ-proposal is an explicit formula for the function $C(\alpha_3,\alpha_2,\alpha_1)$ that represents the part of the three point function which is not determined by $SL(2,\mathbb{C})$-invariance:

Let us define the special function $\Upsilon(x)$ by the integral representation

$$
\log \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q}{2}-x\right)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}} \right] \tag{8}
$$

The formula as proposed in [ZZ] is the following:

$$
\begin{aligned}
C(\alpha_1,\alpha_2,\alpha_3) = &\left[\pi\mu\gamma(b^2) b^{2-2b^2}\right]^{(Q-\sum_{i=1}^3 \alpha_i)/b} \times \\
&\times \frac{\Upsilon_0 \Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)}.
\end{aligned} \tag{9}
$$

For later use let us summarize a couple of relevant properties of the Υ-function. First, it is useful to note that it can also be constructed out of the Barnes Double Gamma function $\Gamma_2(x|\omega_1,\omega_2)$ [Ba] as

$$
\begin{aligned}
&\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}, \qquad \Gamma_b(x) = \Gamma_2(x|b,b^{-1}), \quad \text{where} \\
&\log\Gamma_2(s|\omega_1,\omega_2) = \left(\frac{\partial}{\partial t} \sum_{n_1,n_2=0}^{\infty} (s + n_1\omega_1 + n_2\omega_2)^{-t} \right)_{t=0}.
\end{aligned} \tag{10}
$$

One may thereby benefit from the existence of some literature on the Barnes Double Gamma function, cf. e.g. [Ba, Sh].

The following basic properties follow then directly from the integral representation (8) or results on Γ_b found in [Sh]:

FUNCTIONAL EQUATIONS —

$$
\Upsilon(x+b) = \gamma(bx) b^{1-2bx} \Upsilon(x). \tag{11}
$$

SELF-DUALITY —

(12) $$\Upsilon_b(x) = \Upsilon_{b^{-1}}(x).$$

REFLECTION PROPERTY —

(13) $$\Upsilon_b(x) = \Upsilon_b(Q - x).$$

ANALYTICITY — $\Upsilon_b(x)$ is entire analytic with zeros at $x = -nb - mb^{-1}$ and $x = Q + nb + mb^{-1}$, $n, m \in \mathbb{Z}^{\geq 0}$.

ASYMPTOTIC BEHAVIOR —

(14) $$\Upsilon_b(x) \sim x^2 \log x + \tfrac{3}{2}x^2 \mp \pi i x^2 + Qx \log x + \mathcal{O}(x) \quad \text{for } \Im(x) \to \pm\infty.$$

3. PATH INTEGRAL APPROACH

We will now discuss some of the motivation that has led Dorn, Otto and Al.B., A.B. Zamolodchikov to propose the formula (9): Let us consider fields $V_\alpha(w, \bar{w})$ on the euclidean cylinder, which correspond to the classical functions $e^{2\alpha\phi}$ of the Liouville field. One looks for a representation for the correlation functions of the fields $V_\alpha(w, \bar{w})$ as an integral over all possible field configurations,

(15) $$\Omega\big(V_{\alpha_N}(w_N, \bar{w}_N) \dots V_{\alpha_1}(w_1, \bar{w}_1)\big) \;=\; \lim_{T \to \infty} \int [\mathcal{D}\phi] \; e^{-S_T[\phi]} \prod_{i=1}^{N} e^{2\alpha_i \phi(w_i, \bar{w}_i)},$$

where the configuration $\phi(\sigma, \tau)$ is weighted with a measure $[\mathcal{D}\phi] e^{-S_T[\phi]}$ which is written in terms of the euclidean action

(16) $$S_T[\phi] \;=\; \int_{-T}^{T} d\tau \int_{0}^{2\pi} d\sigma \; \left(\frac{1}{\pi} |\partial_w \phi|^2 + \mu e^{2b\phi} \right).$$

We refer to [Ta] for a discussion of conformal symmetry in the perturbative path integral framework, with interesting links to Teichmüller theory. Here we restrict ourselves to a discussion of the information that can be obtained when the measure $[\mathcal{D}\phi]$ factorizes as $[\mathcal{D}\phi] = d\phi_0 [\mathcal{D}\bar{\phi}]$, where the Liouville field has been split into zero mode and oscillator parts, $\phi(w, \bar{w}) = \phi_0 + \bar{\phi}(w, \bar{w})$. This approach goes back to [GL] and was further developed by [D2][DK].

3.1. Path integral on the Riemann sphere

For many purposes it is convenient not to perform the path integral over fields ϕ defined on the cylinder, but instead to integrate over configurations that are defined on the Riemann-sphere, which is related to the cylinder by the conformal mapping $z = e^w$. A bit of care is needed when transforming the action (16).

Corresponding to a conformal transformation $z = z(w)$ one may consider the following transformation law of the Liouville field ϕ:

(17) $$\phi(z) = \phi'(w(z)) - \frac{Q}{2} \log \left| \frac{dz}{dw} \right|^2.$$

If the parameter Q is chosen equal to $Q_c = b^{-1}$, one has *invariance* of the classical action (16) *up to boundary terms*. In the case of the transformation $z = e^w$ one finds that (16) is transformed into the following expression:

$$S_T[\phi] \;=\; \frac{1}{4\pi}\int_{A_R} d^2z\; \left[(\partial_a\phi)^2 + 4\pi\mu e^{2b\phi}\right] \;+\; \frac{Q}{\pi R}\int_{\partial A_R} dl\;\phi \;+\; 2Q^2\log R, \tag{18}$$

where A_R is an annulus around $z = 0$ with outer and inner radii given by $R = e^T$ and $1/R$ respectively. The boundary term is often interpreted as describing the effect of a “background charge $-Q$ at infinity”. In the case of the quantum theory one will have to consider values of Q that differ from the classical value $Q_c = b^{-1}$.

3.2. **Scaling behavior**

We will now assume that the measure $[\mathcal{D}\phi]e^{-S[\phi]}$ factorizes as

$$[\mathcal{D}\phi]e^{-S[\phi]} = d\phi_0[\mathcal{D}\bar{\phi}]_{\mu,\phi_0}, \tag{19}$$

and that the measure for the integration over the zero mode ϕ_0 is translationally invariant. The latter assumption does not only require translational invariance of $d\phi_0$, but also that

$$[\mathcal{D}\bar{\phi}]_{\mu,\phi_0} \;=\; [\mathcal{D}\bar{\phi}]_{e^{2ba}\mu,\phi_0-a}, \tag{20}$$

as is satisfied by the weight $e^{\mu\int d^2z e^{2b\phi}}$. By introducing a new integration variable $\phi_0' = \phi_0 + \frac{1}{2b}\ln\mu$ it is then possible to extract the μ-dependence of vacuum expectation values (15):

$$\begin{aligned}\Omega\big(V_{\alpha_N}(z_N,\bar{z}_N)\dots V_{\alpha_1}(z_1,\bar{z}_1)\big)_\mu \;=&\\ =\; \mu^s\;\Omega\big(V_{\alpha_N}(z_N,\bar{z}_N)\dots V_{\alpha_1}(z_1,\bar{z}_1)\big)_{\mu=1},&\end{aligned} \qquad s \;\equiv\; \frac{1}{b}\Big(Q - \sum_{i=1}^{N}\alpha_i\Big). \tag{21}$$

The fact that the dependence of vacuum expectation values on μ is generically not analytic in μ indicates that perturbation theory w.r.t. the variable μ can not lead to convergent series expansions for the expectation values.

3.3. **Relation to free field?**

Let us consider the question of convergence of the integration over ϕ_0. We will assume that the upper limit in the ϕ_0 does yield any problems due to the factor $e^{-\mu\int d^2z e^{2b\phi_0}}$ coming from the Liouville-interaction. The leading asymptotic behavior for $\phi_0 \to -\infty$ is proportional to $e^{-2sb\phi_0}$, so we expect convergence of the integration over ϕ_0 as long as $\Re(s) < 0$.

The fact that the dependence of the fields $e^{2\alpha\phi}$ on the variable α is analytic suggests that the dependence of the vacuum expectation values on the variable s might be analytic as long as the ϕ_0-integration converges. In the same spirit one might hope to get a *meromorphic* continuation to $\Re(s) > 0$ by standard regularization of the ϕ_0-integration [GS]: This would be possible if the asymptotic behavior of $[\mathcal{D}\bar{\phi}]_{\mu,\phi_0}$ for $\phi_0 \to -\infty$ was known. The vanishing of the Liouville interaction for $\phi_0 \to -\infty$ leads one to suspect that $[\mathcal{D}\bar{\phi}]_{\mu,\phi_0}$ is asymptotic to the Gaussian measure of the path integral for a free field theory, with corrections given by the interaction term:

$$[\mathcal{D}\bar{\phi}]_{\mu,\phi_0} \underset{q\to-\infty}{\sim} [\mathcal{D}\bar{\phi}]_{\mathrm{F},Q}\sum_{n=0}^{\infty}\frac{(-\mu)^n e^{2bn\phi_0}}{n!}\left(\int d^2z\, e^{2b\bar{\phi}}\right)^n. \tag{22}$$

We use the notation $[\mathcal{D}\bar{\phi}]_{\mathrm{F},Q}$ to denote the standard Gaussian measure in free field theory with background charge $-Q$, see [DF1] or equations (25), (26) below for its definition. We have included in (22) corrections which are subleading for $\phi_0 \to -\infty$. These corrections come from the expansion of the weight $e^{\mu\int d^2z e^{2b\phi}}$ as power series in μ. One should note that the previous argument indicating the failure of perturbation theory w.r.t. μ does not apply here: It amounts to the statement that it does not make sense to consider μ as small as long as it may be changed by shifting ϕ_0. Here one may take $\mu e^{2b\phi_0}$ as small variable that is invariant under $(\mu,\phi_0) \to (e^{2ba}\mu, \phi_0 - a)$.

The integration over ϕ_0 can then be regularized in a standard way by subtracting the leading divergencies:

$$
\begin{aligned}
&\Omega\big(V_{\alpha_N}(z_N,\bar{z}_N)\dots V_{\alpha_1}(z_1,\bar{z}_1)\big) \;= \\
&\quad = \lim_{q_0\to-\infty}\Bigg(\int_{q_0}^{\infty} d\phi_0\, e^{-2sb\phi_0}\int[\mathcal{D}\bar{\phi}]_{\mu,\phi_0}\prod_{i=1}^{N} e^{2\alpha_i\bar{\phi}(z_i,\bar{z}_i)} \\
&\qquad -\sum_{n=0}^{\infty}\frac{(-\mu)^n}{n!}\frac{e^{-2(s-n)bq_0}}{2b(s-n)}\int[\mathcal{D}\bar{\phi}]_{\mathrm{F},Q}\prod_{i=1}^{N}e^{2\alpha_i\bar{\phi}(z_i,\bar{z}_i)}\left(\int d^2z\, e^{2b\bar{\phi}}\right)^n\Bigg).
\end{aligned}
\tag{23}
$$

Poles w.r.t. the variable s are explicitly exhibited. Their residues are given by path integrals in free field theory with a background charge $-Q$.

3.4. **Evaluation of the residues**

Let us study the path-integrals that describe the residues of Liouville expectation values at $s = n$:

$$
\mathcal{G}^n_{\alpha_N,\dots,\alpha_1}(z_N,\dots,z_1) \;=\; \frac{(-\mu)^n}{2bn!}\int[\mathcal{D}\bar{\phi}]_{\mathrm{F},Q}\prod_{i=1}^{N}e^{2\alpha_i\bar{\phi}(z_i,\bar{z}_i)}\left(\int d^2z\, e^{2b\bar{\phi}}\right)^n,
\tag{24}
$$

where $\alpha_1,\dots,\alpha_N$ are subject to the relation $s = n$. It is well-known how to define the path-integrals on the right hand side [DF1]: The result can be represented in the form

$$
\begin{aligned}
\mathcal{G}^n_{\alpha_N,\dots,\alpha_1}(z_N,\dots,z_1) \;&=\frac{(-\mu)^n}{n!}\int d^2t_n\dots d^2t_1\,\langle 0|:e^{2\alpha_N\phi(z_N)}:\dots:e^{2\alpha_1\phi(z_1)}: \\
&\qquad\qquad :e^{2b\phi(t_n)}:\dots:e^{2b\phi(t_1)}:|0\rangle_{\mathrm{F},Q},
\end{aligned}
\tag{25}
$$

where correlators of the form $\langle 0|:e^{2\alpha_N\phi(z_N)}:\dots e^{2\alpha_1\phi(z_1)}:|0\rangle_{\mathrm{F},Q}$ are nonvanishing only if $Q-\sum_{i=1}^N\alpha_i = 0$, and in that case given by

$$
\langle 0|:e^{2\alpha_N\phi(z_N)}:\dots:e^{2\alpha_1\phi(z_1)}:|0\rangle_{\mathrm{F},Q} \;=\; \prod_{i>j}|z_i-z_j|^{-4\alpha_i\alpha_j}.
\tag{26}
$$

If the integrals (25) are to represent the residues of correlation functions that satisfy the conformal Ward identities irrespective of the choice of the α_i, $i = 1,\dots,N$, one evidently needs that the operator $:e^{2b\phi(t)}:$ that appears in (25) transforms as a tensor of weight $(1,1)$ under conformal transformations, so that $\int d^2t\, :e^{2b\phi(t)}:$ is conformally invariant. This is the case iff the parameters b and Q are related by $Q = b + b^{-1}$ [DF1], as will be assumed from now on.

The integrals (25) are too complicated to carry out in general, but in a case of fundamental importance, namely $N = 3$, Dotsenko and Fateev have been able to compute the integrals (25) explicitly and found the following result [DF2] (see [D1] for an explanation of the techniques to calculate such integrals)

$$\mathcal{G}^n_{\alpha_3,\alpha_2,\alpha_1}(z_3,z_2,z_1) \;=\; |z_{32}|^{2\Delta_{32}}|z_{31}|^{2\Delta_{31}}|z_{21}|^{2\Delta_{21}} I_n(\alpha_3,\alpha_2,\alpha_1)_{\sum_{i=1}^3 \alpha_i = Q-nb}, \tag{27}$$

where $z_{ij} = z_i - z_j$, $\Delta_{ij} = \Delta(\alpha_k) - \Delta(\alpha_i) - \Delta(\alpha_j)$ if $i \neq j$, $j \neq k$, $k \neq i$, and furthermore

$$I_n(\alpha_1,\alpha_2,\alpha_3) = \left(\frac{-\pi\mu}{\gamma(-b^2)}\right)^n \frac{\prod_{j=1}^n \gamma(-jb^2)}{\prod_{k=0}^{n-1}\left[\gamma(2\alpha_1 b + kb^2)\gamma(2\alpha_2 b + kb^2)\gamma(2\alpha_3 b + kb^2)\right]}. \tag{28}$$

In (28) we have used the notation

$$\gamma(x) = \Gamma(x)/\Gamma(1-x). \tag{29}$$

3.5. **Continuation in** s

It is then natural to look for a "continuation" of the results for $s \in \mathbb{Z}^{\geq 0}$ to all of $\mathbb{C}$ as a candidate for the three point function at generic values of s. More precisely, the task is to find an expression that depends meromorphically on s and has poles for $s \in \mathbb{Z}^{\geq 0}$ with residues given by (27)(28). It is natural and important to demand that this continuation preserves the z_i-dependence of the three point function as given in (27), i.e. that it takes the general $SL(2,\mathbb{C})$-invariant form

$$\Omega\big(V_{\alpha_3}(z_3,\bar{z}_3)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_1}(z_1,\bar{z}_1)\big) \;=\; |z_{32}|^{2\Delta_{32}}|z_{31}|^{2\Delta_{31}}|z_{21}|^{2\Delta_{21}} C(\alpha_3,\alpha_2,\alpha_1). \tag{30}$$

The task is then to find an expression for $C(\alpha_3,\alpha_2,\alpha_1)$ that has poles for $s \in \mathbb{Z}^{\geq 0}$ with residues given by (28). A very promising candidate for such a continuation has been proposed by Dorn and Otto [DO] and independently by AL.B. and A.B. Zamolodchikov [ZZ]. A possible starting point for finding their proposal may be the observation that the known result for $s \in \mathbb{Z}^{\geq 0}$ has an interesting recursive structure w.r.t. shifts of one of the arguments by an amount of b:

$$\begin{aligned} &\frac{C(\alpha_3,\alpha_2,\alpha_1+b)}{C(\alpha_3,\alpha_2,\alpha_1)} = \\ &= -\frac{\gamma(-b^2)}{\pi\mu}\frac{\gamma(b(2\alpha_1+b))\gamma(2b\alpha_1)\gamma(b(\alpha_2+\alpha_3-\alpha_1-b))}{\gamma(b(\alpha_1+\alpha_2+\alpha_3-Q))\gamma(b(\alpha_1+\alpha_2-\alpha_3))\gamma(b(\alpha_1+\alpha_3-\alpha_2))}. \end{aligned} \tag{31}$$

The functional relation (31) is a priori only known for $\alpha_3,\alpha_2,\alpha_1$ subject to the constraint $\sum_{i=1}^3 \alpha_i = Q - sb$ for some $s \in \mathbb{Z}^{\geq 0}$. However, it seems to be a natural guess that this recursive relation may even be valid for general values of $\alpha_3,\alpha_2,\alpha_1$. One may then easily convince oneself that a solution to (31) can be assembled if one was given as building block a function called $\Upsilon(x)$ that satisfies the functional equation (11) of the Υ-function introduced in (10).

These facts alone can hardly be considered as strong motivation for considering the DOZZ-proposal as a promising candidate for the three point function of exponential fields in Liouville theory. One should therefore emphasize that this formula has passed a couple of rather nontrivial checks: For example, it was shown to imply a quantum version of the Liouville equation of motion in [DO] (see also our discussion in Part II). The checks performed in [ZZ] include comparison with semiclassical calculations in two different limiting regimes, comparison with results from the thermodynamic Bethe ansatz for the sinh-Gordon model (which is related to Liouville theory in the ultraviolet limit), as well as a numerical check of crossing symmetry for the four-point functions constructed from the three point functions via factorization.

3.6. **Duality** $b \to b^{-1}$

The expression (9) has (at least) one amazing feature: It is left invariant if one replaces $b \to \tilde{b} \equiv b^{-1}$ and furthermore

$$\mu \to \tilde{\mu} \quad \text{where } \tilde{\mu} \text{ is defined by} \quad \pi\tilde{\mu}\gamma(\tilde{b}^2) = \big(\pi\mu\gamma(b^2)\big)^{b^{-2}}. \tag{32}$$

Remembering that b^2 was playing the role of a coupling constant or alternatively the role of $\hbar$, this indicates a rather remarkable and profound self-duality of Liouville theory.

But it also raises a puzzle: The expression (9) has more poles than expected on the basis of our previous discussion in Subsection 3.3. Poles occur if

$$Q - \alpha_1 - \alpha_2 - \alpha_3 = nb + mb^{-1}, \qquad n, m \in \mathbb{Z}^{\geq 0} \tag{33}$$

and all cases obtained by the reflections $\alpha_i \to Q - \alpha_i$, $i = 1, 2, 3$ from (33). Is there a way to explain these poles from the path-integral point of view? According to our arguments in Subsection 3.3 one has a relation between the asymptotic behavior of the path integral measure $[\mathcal{D}\bar{\phi}]_{\mu,\phi_0}$ for $\phi_0 \to -\infty$ and the poles of the three point function. The additional poles in (33) should therefore be attributed to a modification of the asymptotic expansion (22). In fact, if one replaces the right hand side of (22) by

$$[\mathcal{D}\bar{\phi}]_{\mathrm{F},Q} \sum_{m,n=0}^{\infty} \frac{(-\mu)^n(-\tilde{\mu})^m e^{2(bn+b^{-1}m)\phi_0}}{m!n!} \left(\int d^2z\, e^{2b\bar{\phi}}\right)^n \left(\int d^2z'\, e^{2\tilde{b}\bar{\phi}}\right)^m, \tag{34}$$

and continues as in subsections 3.3 and 3.4, one would find additional poles at (33) with residues represented by the Dotsenko-Fateev integrals [DF2]

$$\begin{aligned} \mathcal{G}^{n,m}_{\alpha_n,\dots,\alpha_1}(z_n,\dots,z_1) \;=\; & \frac{(-\mu)^n(-\tilde{\mu})^m}{m!n!} \int_{\mathbb{C}} d^2t_n \dots d^2t_1 \int_{\mathbb{C}} d^2s_m \dots d^2s_1 \\ & \langle 0| \prod_{i=1}^{3} :e^{2\alpha_i\phi(z_i)}: \prod_{j=1}^{n} :e^{2b\phi(t_j)}: \prod_{k=1}^{m} :e^{2\tilde{b}\phi(s_k)}: |0\rangle_{\mathrm{F},Q}. \end{aligned} \tag{35}$$

And indeed, the residues of the proposal (33) coincide precisely with the result of the explicit evaluation of (35) performed in [DF2]. This was observed in [DO][ZZ], and elaborated upon in [OPS1].

It seems natural to interpret the modification (34) as describing a quantum correction to the path integral measure that could be described by adding a second interaction term $\tilde{\mu}\int d^2z e^{2b^{-1}\phi}$ to the action. Such a modification is compatible with conformal invariance due to the fact [DF1] that the normal ordered exponential $:e^{2b^{-1}\phi}:$ transforms under conformal transformations the same way as $:e^{2b\phi}:$, namely as $(1,1)$-tensor field. The modified action is clearly self-dual under $b \to b^{-1}$, $\mu \to \tilde{\mu}$.

3.7. **Measure in the bulk of ϕ_0-space?**

It has become clear that the appearance of poles in the dependence of vacuum expectation values in their dependence w.r.t. the parameters $\alpha_N, \dots, \alpha_1$ and the explicit form of the corresponding residues are entirely explained in terms of the asymptotic behavior of the path-integral measure for zero mode $\phi_0 \to -\infty$. The free-field vacuum expectation values that represent the residues may be called "resonant amplitudes" following [DK]. Are there any hints concerning the definition of the path-integral for the "non-resonant" amplitudes?

The following observation from [OPS2] is intriguing: Let us tentatively assume that the measure $d\phi_0[\mathcal{D}\bar{\phi}]_{\mu,\phi_0}$ is of the form

$$d\phi_0[\mathcal{D}\bar{\phi}]_{\mu,\phi_0} = d\phi_0[\mathcal{D}\bar{\phi}]' e^{-\int d^2z(\mu e^{2b\phi(z)}+\tilde{\mu}e^{2\tilde{b}\phi(z)})}, \tag{36}$$

where $[\mathcal{D}\bar{\phi}]'$ is independent of ϕ_0. In order to reproduce the scaling behavior (21), we will require it to be μ-independent as well [1]. $[\mathcal{D}\bar{\phi}]'$ must then coincide with $[\mathcal{D}\bar{\phi}]_{\mathrm{F},Q}$ for $d\phi_0[\mathcal{D}\bar{\phi}]_{\mu,\phi_0}$ to have asymptotics given by (34). One may then rewrite the condition of translation invariance of the ϕ_0-measure, $0=\int_{-\infty}^{\infty} d\phi_0\partial_{\phi_0}$, as a relation between vacuum expectation values with different numbers of operator insertions:

$$\begin{aligned} 2b\mu \int_{\mathbb{C}} d^2z\; \Omega\Big(V_b(z,\bar{z})\prod_{i=1}^{N} V_{\alpha_i}(z_i,\bar{z}_i)\Big) + 2b^{-1}\tilde{\mu}\int_{\mathbb{C}} d^2z\; \Omega\Big(V_{b^{-1}}(z,\bar{z})\prod_{i=1}^{N} V_{\alpha_i}(z_i,\bar{z}_i)\Big) = \\ = -2\Big(\sum_{i=1}^{N}\alpha_i - Q\Big)\Omega\Big(\prod_{i=1}^{N} V_{\alpha_i}(z_i,\bar{z}_i)\Big). \end{aligned} \tag{37}$$

This condition can be evaluated more explicitly in the case N=2 [OPS2]. One gets a relation between $C(b,\alpha,\alpha)$, $C(b^{-1},\alpha,\alpha)$ and the two-point function

$$\Omega\big(V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_1}(z_1,\bar{z}_1)\big) \;=\; \big(2\pi\delta(\alpha_2+\alpha_1-Q)+S(\alpha)\delta(\alpha_2-\alpha_1)\big)|z_2-z_1|^{-4\Delta_{\alpha_1}}, \tag{38}$$

which was observed to be fulfilled by the DOZZ-proposal in [OPS2].[2] The objects that appear in (38) are not expected to be determined by the ($\phi_0\to-\infty$)-asymptotics of the measure alone. The fact that the DOZZ-proposal satisfies the condition in the case $N=2$ may therefore be taken as a hint that (36) is correct, i.e. that there are no further corrections to the measure besides adding the dual interaction $\tilde{\mu}\int d^2ze^{2b^{-1}\phi}$.

Nevertheless it is not clear to us what conclusion to draw from these observations. First, it is not clear whether one should expect the "sum-rule" (37) to be valid in general. The case $N=2$ for which consistency with the DOZZ-proposal was verified is still somewhat special. Moreover, one should observe that the compatibility of the DOZZ-proposal with a literal interpretation of (36) is not obvious: The former seems to imply that $\tilde{\mu}$ as given in (32) can become negative for certain values of b. But this would imply trouble with the *upper* limit of the integration over ϕ_0!

We will come back to this problem from a different point of view in Part II.

4. **RECONSTRUCTION**

For rational conformal field theories it is well-known that the two- and three point functions of the set $\{V_\imath;\imath\in\mathcal{I}\}$ of all primary fields suffices to reconstruct the Hilbert-space $\mathcal{H}$ of the theory and

[1] Note that the inclusion of the "dual interaction" $\tilde{\mu}\int d^2ze^{2b^{-1}\phi}$ preserves the scaling (21) if $\tilde{\mu}$ and μ are related by (32)

[2] In fact, it is argued in [OPS2] that the DOZZ-proposal is the unique expression that satisfies (37) and has residues given by the Dotsenko-Fateev integrals (35). We did not understand the assumptions underlying the uniqueness argument of [OPS2] well enough to include a discussion here.

to fully characterize the operators $\mathsf{V}_\imath$ that correspond to the fields $V_\imath$. One may therefore suspect that really all that one might want to know about Liouville theory can (at least in principle) be extracted from the DOZZ formula and conformal invariance. In the present section we will discuss an adaption of the reconstruction procedure from rational conformal field theories to Liouville theory. The DOZZ-proposal will be found to encode the spectrum of Liouville theory in a natural way. Moreover, the identification between three point functions of fields V_α and matrix elements of the corresponding operators V_α will be found to work straightforwardly for α with $Q > \Re(\alpha) > 0$. However, subtleties that are unfamiliar from rational conformal field theories arise due to the fact that the $SL(2,\mathbb{C})$-invariant state $|0\rangle$ is not found in the spectrum. Furthermore, the description of the operators V_α in terms of matrix elements turns out to be more subtle in the cases where $\Re(\alpha)$ is not in $(0,Q)$.

We believe the following considerations to be important for understanding how particular features of the DOZZ-formula (like analyticity, pole structure, values of residues) are crucial for the success of such a reconstruction procedure.

4.1. **Reconstruction of rational conformal field theories**

Consider a rational conformal field theory C with set $\mathcal{P}_C \equiv \{V_\imath ; \imath \in \mathcal{I}\}$ of all primary fields. Let us assume for simplicity that left- and right conformal dimensions of the primary field $V_\imath$ coincide: $\Delta_\imath = \bar{\Delta}_\imath$. It is often convenient to assume that $\mathcal{P}_C$ includes the identity corresponding to the label $\imath = 0$. Assume furthermore to be given the set of all three-point functions of the form

$$\Omega\big(V_{\imath_3}(z_3)\dots V_{\imath_1}(z_1)\big) \;=\; |z_{32}|^{2\Delta_{32}}|z_{31}|^{2\Delta_{31}}|z_{21}|^{2\Delta_{21}}C(\imath_3,\imath_2,\imath_1), \tag{39}$$

where $C(\imath_3,\imath_2,\imath_1)$ are real numbers. The two-point functions are obtained by setting any of $\imath_3,\imath_2,\imath_1$ to 0. The conformal Ward identities allow one to recover two- and three point functions of all descendants $V_\imath(\zeta|z)$, $\zeta \in \mathcal{R}_{\Delta_\imath}\otimes\mathcal{R}_{\Delta_\imath}$ from the data (39), where $\mathcal{R}_\Delta$ denotes the irreducible representation of the Virasoro algebra with highest weight Δ. One wants to identify

$$\Omega\big(V_{\imath_n}(\zeta_n|z_n)\dots V_{\imath_1}(\zeta_1|z_1)\big) \;\equiv\; \langle 0|\mathsf{V}_{\imath_n}(\zeta_n|z_n)\dots\mathsf{V}_{\imath_1}(\zeta_1|z_1)|0\rangle. \tag{40}$$

The vacuum state $|0\rangle$ must have the property $\mathsf{L}_n|0\rangle = \bar{\mathsf{L}}_n|0\rangle = 0$ $n=-1,0,1$ for the right hand side of (40) to share the property of $SL(2,\mathbb{C})$-invariance. This property of $|0\rangle$ assures existence of the following limits:

$$|\imath,\zeta\rangle_{\rm in} \;\equiv\; \lim_{z\to 0}\mathsf{V}_\imath(\zeta|z)|0\rangle, \qquad {}_{\rm out}\langle\imath,\zeta| \;\equiv\; \lim_{z\to\infty} z^{2\Delta(\zeta)}\bar{z}^{2\bar{\Delta}(\zeta)}\langle 0|\mathsf{V}_\imath(\zeta|z), \tag{41}$$

where it was assumed that $\zeta \in \mathcal{R}_{\Delta_\imath}\otimes\mathcal{R}_{\Delta_\imath}$ is an eigenvector of both L_0 and $\bar{\mathsf{L}}_0$ with eigenvalues $\Delta(\zeta)$ and $\bar{\Delta}(\zeta)$ respectively. The identification (40) together with operator-state correspondence therefore allow one to recover matrix elements of $\mathsf{V}_\imath$ as

$$\begin{aligned} {}_{\rm out}\langle\imath_3,\zeta_3|\mathsf{V}_{\imath_2}(\zeta_2|z_2)|\imath_1,\zeta_1\rangle_{\rm in} \;\equiv\; & \\ \equiv\; \lim_{z_1\to 0}\lim_{z_3\to\infty} & z_3^{2\Delta(\zeta_3)}\bar{z}_3^{2\bar{\Delta}(\zeta_3)}\Omega\big(V_{\imath_3}(\zeta_3|z_3)V_{\imath_2}(\zeta_2|z_2)V_{\imath_1}(\zeta_1|z_1)\big). \end{aligned} \tag{42}$$

Let us next observe that compatibility of

$$\big({}_{\rm out}\langle\imath_3,\zeta_3|\mathsf{V}_{\imath_2}(\zeta_2|z_2)|\imath_1,\zeta_1\rangle_{\rm in}\big)^* = {}_{\rm in}\langle\imath_1,\zeta_1|\big(\mathsf{V}_{\imath_2}(\zeta_2|z_2)\big)^\dagger|\imath_3,\zeta_3\rangle_{\rm out}$$

with (39) requires that

$$\big(V_\imath(\zeta|z)\big)^\dagger \;=\; \bar{z}^{-2\Delta(\zeta)}z^{-2\bar{\Delta}(\zeta)}V_\imath\big(\bar{\zeta}|\bar{z}^{-1}\big), \tag{43}$$

where $\bar{\zeta}$ is the complex conjugate of the vector ζ. This leads to the following relation between in- and out states:

$$\begin{aligned} {}_{\rm in}\langle\imath,\zeta| \;\equiv\; \big(|\imath,\zeta\rangle_{\rm in}\big)^\dagger \;&=\big(\lim_{z\to 0}\, V_\imath(\zeta|z)|0\rangle\big)^\dagger \;=\; \lim_{z\to 0}\,\langle 0|\big(V_\imath(\zeta|z)\big)^\dagger \\ &= \lim_{z\to 0}\,\langle 0|\,\bar{z}^{-2\Delta(\zeta)}z^{-2\bar{\Delta}(\zeta)}V_\imath\big(\bar{\zeta}|\tfrac{1}{\bar{z}}\big) \\ &= {}_{\rm out}\langle\imath,\bar{\zeta}|. \end{aligned} \tag{44}$$

Taken together this means that the Hilbert space $\mathcal{H}_C$ is given as

$$\mathcal{H}_C \;=\; \bigoplus_{\imath\in\mathcal{I}} \mathcal{V}_{\Delta_\imath}\otimes\mathcal{V}_{\Delta_\imath}, \tag{45}$$

with scalar product given by

$${}_{\rm in}\langle\imath_2,\zeta_2|\imath_1,\zeta_1\rangle_{\rm in} \;=\; \lim_{z_1\to 0}\lim_{z_2\to\infty} z_2^{2\Delta(\zeta_2)}\bar{z}_2^{2\bar{\Delta}(\zeta_2)}\Omega\big(V_{\imath_2}(\bar{\zeta}_2|z_2)V_{\imath_1}(\zeta_1|z_1)\big). \tag{46}$$

The matrix elements of the operators $V_\imath(\zeta|z)$ are finally recovered as in (42), taking into account (44).

4.2. **Preliminaries**

To prepare for our discussion of the reconstruction procedure in the case of Liouville theory let us make two observations concerning the DOZZ formula:

First, it satisfies reflection relations such as

$$C(\alpha_3,\alpha_2,\alpha_1) \;=\; S(\alpha_3)C(Q-\alpha_3,\alpha_2,\alpha_1), \tag{47}$$

where the the *reflection amplitude* $S(\alpha)$ is given as

$$S(\alpha) \;=\; \frac{\big(\pi\mu\gamma(b^2)\big)^{b^{-1}(Q-2\alpha)}}{b^2}\,\frac{\gamma(2b\alpha-b^2)}{\gamma(2-2b^{-1}\alpha+b^{-2})}. \tag{48}$$

By symmetry of $C(\alpha_3,\alpha_2,\alpha_1)$ one finds the corresponding relations for the other arguments. This allows one to restrict attention to values of the variables $\alpha_i,\ i = 1,2,3$ that satisfy the so-called Seiberg-bound [Se]:

$$\Re(\alpha_i) \le \tfrac{Q}{2}, \quad i = 1,2,3. \tag{49}$$

Second, let us observe that the DOZZ-formula is indeed *analytic* as long as the condition for convergence of the zero mode integration in the path-integral is satisfied, namely

$$-b\Re(s) \;=\; \Re(\alpha_1+\alpha_2+\alpha_3) - Q \;>\; 0, \tag{50}$$

but has singularities otherwise. This suggests that an interpretation of the tree point functions as a matrix element,

$$\Omega\big(V_{\alpha_3}(z_3)V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)\big) \;\equiv\; \langle 0|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_1}(z_1)|0\rangle \tag{51}$$

will be most straightforward if one starts with the range given by (50). Taking into account (49) one finds that (50) can only be satisfied if $0 < \Re(\alpha_i) \le \frac{Q}{2}$ $i = 1,2,3$.

4.3. **Two-point function?**

So can we define states $|\alpha\rangle_{\rm in}$, ${}_{\rm out}\langle\alpha|$ that are created via

$$(52) \qquad |\alpha\rangle_{\rm in} \;\equiv\; \lim_{z\to 0} \mathsf{V}_\alpha(z)|0\rangle, \qquad {}_{\rm out}\langle\alpha| \;\equiv\; \lim_{z\to\infty} |z|^{4\Delta_\alpha}\langle 0|\mathsf{V}_\alpha(z)\ ?$$

When trying to adapt the reconstruction procedure from rational conformal field theories to the present case, the first question to address is: What is the unit field, or equivalently: How to recover the two-point functions from $C(\alpha_3,\alpha_2,\alpha_1)$? The unit field should clearly have vanishing conformal dimension $\Delta_\alpha = \alpha(Q-\alpha)$. Under the restriction (49) this is only found for $\alpha = 0$ which lies on the boundary of our "allowed" region $0 < \Re(\alpha_i) \leq \frac{Q}{2}$. So let us consider the behavior of $C(\alpha_2,\epsilon,\alpha_1)$ for small $|\epsilon|$ with $\Re(\epsilon) > 0$: It is given as

$$(53) \quad C(\alpha_2,\epsilon,\alpha_1) \;\simeq\; \frac{2\epsilon S(\alpha_1)}{(\alpha_2-\alpha_1+\epsilon)(\alpha_1-\alpha_2+\epsilon)} + \frac{2\epsilon}{(Q-\alpha_2+\alpha_1+\epsilon)(\alpha_1+\alpha_2-Q+\epsilon)},$$

where $S(\alpha)$ is the reflection amplitude introduced above. $C(\alpha_2,\epsilon,\alpha_1)$ vanishes for $\epsilon \to 0$ unless $\alpha_2 = \alpha_1$ or $\alpha_2 = Q - \alpha_1$ and becomes infinite otherwise. The two-point function can therefore only be defined in the distributional sense. To actually define it as a distribution from (53) one needs to specify a contour over which α_2, α_1 are supposed to be integrated. One clearly can only get distributions proportional to delta-distributions, but the precise pre-factors depend on the direction from which ϵ approaches zero.

The distributional character of the two-point function is of course just what was to be expected when having continuous sets of primary fields, since the vanishing of the two-point function for $\alpha_2 \neq \alpha_1$ and $\alpha_2 \neq Q - \alpha_1$ is required by conformal invariance. This also implies that we can not hope to find any state that is normalizable in the strict sense, but only normalizability in the distributional sense.

4.4. **Scalar product**

Let us therefore ask for which values of α it is possible to define a reasonable scalar product from the two-point function. To this aim we need to know the hermiticity properties of V_α: It follows from $(C(\alpha_3,\alpha_2,\alpha_1))^* = C(\alpha_3^*,\alpha_2^*,\alpha_1^*)$ that V_α behaves as follows under hermitian conjugation:

$$(54) \qquad \big(\mathsf{V}_\alpha(z)\big)^\dagger \;=\; |z|^{-4\Delta(\alpha)} V_{\alpha^*}(\bar{z}^{-1}).$$

We should therefore obtain the scalar product from the three-point function as

$$(55) \qquad {}_{\rm in}\langle\alpha_2|\alpha_1\rangle_{\rm in} \;=\; \lim_{\alpha\to 0} C(\alpha_2^*,\alpha,\alpha_1).$$

As the right hand side vanishes unless $\alpha_2^* = \alpha_1$ or $\alpha_2^* = Q - \alpha_1$, it is easy to see that no reasonable scalar product can be obtained unless either $\alpha_i \in \mathbb{R}$ or $\alpha_i \in \frac{Q}{2} + i\mathbb{R}$. This would also follow when considering the extension of the scalar product (55) to states generated from descendants: The representations $\mathcal{V}_{\alpha_i}$ are unitary only if $\alpha_i \in \mathbb{R}$ or $\alpha_i \in \frac{Q}{2} + i\mathbb{R}$.

In the first case one would need to choose in (53) an ϵ with $\Im(\epsilon) \neq 0$, which would lead to

$$(56) \qquad {}_{\rm in}\langle\alpha_2|\alpha_1\rangle_{\rm in} \;=\; \pm 2\pi i S(\alpha_1)\delta(\alpha_2-\alpha_1).$$

Due to the unavoidable factor of i in (56) one does not get a reasonable scalar product for $|\alpha\rangle$ with $\alpha \in \mathbb{R}$.

The other case $\alpha_i = \frac{Q}{2} + iP_i$, $P_i \in \mathbb{R}$ is better: The reflection property (47) allows one to restrict to $P_i > 0$. One then needs to choose ϵ with $\Re(\epsilon) > 0$, in which case (53) gives

$$
{}_{\rm in}\langle \tfrac{Q}{2} + iP_2 | \tfrac{Q}{2} + iP_1 \rangle_{\rm in} \; = \; 2\pi\delta(P_2 - P_1). \tag{57}
$$

Let us identify $|P\rangle \equiv |\frac{Q}{2} + iP_1\rangle_{\rm in}$. It is straightforward to generalize the discussion to states $|P, \zeta\rangle$, $\zeta \in \mathcal{W}_P \equiv \mathcal{W}_{\frac{Q}{2}+iP}$ that are created by the vertex operators $\mathsf{V}_{\frac{Q}{2}+iP}$: The scalar product is then given as

$$
\langle P', \zeta_2 | P, \zeta_1 \rangle \; = \; 2\pi\delta(P' - P)(\zeta_2, \zeta_1)_{\mathcal{W}_P}, \tag{58}
$$

where $(\zeta_2, \zeta_1)_{\mathcal{W}_P}$ denotes the scalar product in $\mathcal{W}_P$ that is normalized such that $(v_P, v_P)_{\mathcal{W}_P} = 1$ if v_P is the highest weight vector in $\mathcal{W}_P$.

We conclude that conformal symmetry and DOZZ-formula imply that the Liouville Hilbert space takes the form

$$
\mathcal{H} \; \simeq \; \int_{\mathbb{R}^+}^{\oplus} \frac{dP}{2\pi} \, \mathcal{W}_P. \tag{59}
$$

REMARK 2. — Let us note that a calculation like (44) now implies that

$$
|P, \bar{\zeta}\rangle_{\rm out} \; = \; |-P, \zeta\rangle_{\rm in} \; = \; R(-P)|P, \zeta\rangle_{\rm in}, \tag{60}
$$

where $R(-P) \equiv S(\frac{Q}{2} - iP)$ is the reflection amplitude encountered earlier. But this means that the *scattering operator* S that relates in- and out states is diagonal in the basis $\{|P, \zeta\rangle; P \in \mathbb{R}^+, \zeta \in \mathcal{W}_P\}$ and given by multiplication with $R(-P)$. The unitarity of S follows from $|R(p)|^2 = 1$. We will see later that S indeed has an interpretation as a scattering operator that describes the scattering of wave-packets off the Liouville potential.

4.5. **Matrix elements**

Having identified the set of normalizable states, we may then recover the matrix elements of operators V_α, with α in the range $0 < \Re(\alpha) \leq \frac{Q}{2}$ identified in subsection (4.2), as follows:

$$
\begin{aligned}
&\langle P_3, \zeta_3 | \mathsf{V}_{\alpha_2}(z_2) | P_1, \zeta_1 \rangle \; \equiv \\
&\qquad \equiv \lim_{z_1 \to 0} \lim_{z_3 \to \infty} z_3^{2\Delta(\zeta_3)} \bar{z}_3^{2\bar{\Delta}(\zeta_3)} \Omega\big(V_{\bar{\alpha}_3}(\zeta_3|z_3) V_{\alpha_2}(z_2) V_{\alpha_1}(\zeta_1|z_1)\big).
\end{aligned} \tag{61}
$$

We use the notation $\alpha_i = \frac{Q}{2} + iP_i$, $\bar{\alpha}_i \equiv Q - \alpha_i = \frac{Q}{2} - iP_i$, $i = 1, 2, \ldots$. Knowing the matrix elements of the operators V_α should of course allow one to represent the matrix elements of products of these operators by summing over intermediate states: Let $\mathbb{B}_P$ be a basis for $\mathcal{W}_P$, which may be chosen to consist of vectors $\zeta \in \mathcal{W}_P$ that diagonalize L_0, $\bar{\mathsf{L}}_0$. Each element $\zeta \in \mathbb{B}_P$ has a unique "dual" ζ^t which is the vector defined by the property $(\zeta^t, \zeta')_{\mathcal{W}_P} = \delta_{\zeta,\zeta'}$ for all $\zeta' \in \mathbb{B}_P$. An example for the representation of matrix elements by summing over intermediate states can then be written as follows:

$$
\begin{aligned}
&\langle P_4 | \mathsf{V}_{\alpha_3}(z_3) \mathsf{V}_{\alpha_2}(z_2) | P_1 \rangle \; \equiv \\
&\qquad \equiv \int_{\mathbb{R}^+} \frac{dP}{2\pi} \sum_{\zeta \in \mathbb{B}_P} \langle P_4 | \mathsf{V}_{\alpha_3}(z_3) | P, \zeta \rangle \langle P, \zeta^t | \mathsf{V}_{\alpha_2}(z_2) | P_1 \rangle,
\end{aligned} \tag{62}
$$

We may furthermore assume that the elements of $\mathbb{B}_P$ factorize as $\zeta = \xi_2 \otimes \xi_1$. This leads to a factorized representation of the matrix element (62) of the following form:

$$
\begin{aligned}
\langle P_4|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle \;\equiv\; \int_{\mathbb{S}} \frac{d\alpha}{2\pi}\, C\big(\bar{\alpha}_4,\alpha_3,\alpha\big)C\big(\bar{\alpha},\alpha_2,\alpha_1\big) \times \\
\times \mathcal{F}^s_{\alpha}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](z_3,z_2)\mathcal{F}^s_{\alpha}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](\bar{z}_3,\bar{z}_2),
\end{aligned}
\tag{63}
$$

where the variable α was introduced by $\alpha = \frac{Q}{2} + iP$, such that the integral over $P \in \mathbb{R}^+$ becomes an integral over $\alpha \in \mathbb{S} \equiv \frac{Q}{2} + i\mathbb{R}^+$. The conformal blocks $\mathcal{F}^s_{\alpha}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](z_3,z_2)$ are given by the following power series:

$$
\mathcal{F}^s_{\alpha}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](z_3,z_2) \;=\; z_3^{\Delta_{\alpha_4}-\Delta_{\alpha}-\Delta_{\alpha_3}}\, z_2^{\Delta_{\alpha}-\Delta_{\alpha_2}-\Delta_{\alpha_1}} \sum_{n=0}^{\infty}\Big(\frac{z_2}{z_3}\Big)^n \mathcal{F}^s_{\alpha}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](n).
\tag{64}
$$

The coefficients $\mathcal{F}^s_{\alpha}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](n)$ are given by sums over vectors $\xi \in \mathcal{V}_\alpha$ with fixed eigenvalues $\Delta_\alpha + n$. Some more information concerning the definition of the conformal blocks can be found in Subsection 7.1. Here let us only note that the series that represent $\mathcal{F}_\alpha$ actually converge for $|z_2| < |z_3|$.

4.6. **Meromorphic continuation of matrix elements**

It will be important in the following to note that the matrix elements such as (62) admit a meromorphic continuation to arbitrary complex values of P_4, P_1, α_3, α_2. For notational convenience let us identify $|\alpha\rangle \equiv |P\rangle$ if α and P are related by $\alpha = \frac{Q}{2} + iP$. We will consider the example of $\langle\alpha_4|\mathsf{V}_{\alpha_3}(w)\mathsf{V}_{\alpha_2}(z)|\alpha_1\rangle$. It is shown in Section 7 below that this matrix element has a meromorphic continuation w.r.t. all four variables $\alpha_4,\dots,\alpha_1$ with poles if and only if

$$
Q + \sum_{i=1}^{4} s_i\big(\alpha_i - \tfrac{Q}{2}\big) \;=\; -nb - mb^{-1}, \quad s_i \in \{1,-1\}.
\tag{65}
$$

The poles with $s_i = 1$ $i = 1,\dots,4$ are in precise correspondence with the singularities that one would expect on the basis of the path-integral arguments discussed in section 3. All others are generated by reflection relations like (47).

This meromorphic continuation can be represented as in (62) as long as the conditions

$$
\begin{aligned}
&|\Re(\alpha_1-\alpha_2)| < Q/2; \qquad |\Re(Q-\alpha_1-\alpha_2)| < Q/2;\\
&|\Re(\alpha_3-\alpha_4)| < Q/2; \qquad |\Re(Q-\alpha_3-\alpha_4)| < Q/2,
\end{aligned}
\tag{66}
$$

are satisfied. Otherwise one has a representation of the form

$$
\begin{aligned}
&\langle P_4|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle \;\equiv \\
&\qquad \equiv \frac{1}{2}\int_{\mathcal{C}} \frac{dP}{2\pi} \sum_{\zeta\in\mathbb{B}_P} \langle P_4|\mathsf{V}_{\alpha_3}(z_3)|P,\zeta\rangle\langle P,\zeta^t|\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle,
\end{aligned}
\tag{67}
$$

where the contour $\mathcal{C}$ can generically be taken as $\mathbb{R}$ plus a finite sum of small circles around certain poles of the three-point functions that appear in (62).

4.7. **Normalizable vs. non-normalizable states**

Our identification of the set of normalizable states seems to create puzzles: For example, the states $|P\rangle$ have conformal dimensions larger than $Q^2/4$, so one does not find the state $|0\rangle$ among them. But if $|0\rangle$ is not in the spectrum, what meaning does the state-operator correspondence have?

To start with, let us recall that $|P\rangle \notin \mathcal{H}$. This means in particular that scalar products such as $\langle P|\psi\rangle$ will not be defined for all $|\psi\rangle \in \mathcal{H}$ (square-integrability only requires $\langle P|\psi\rangle$ to be defined up to P from a set of measure zero). One needs to consider subspaces $\mathcal{T} \subset \mathcal{H}$ which are such that $\langle P|\psi\rangle$ is defined for any $|\psi\rangle \in \mathcal{T}$. $|P\rangle$ is then interpreted as an element of the hermitian dual $\mathcal{T}^\dagger$ of $\mathcal{T}$, the space of all anti-linear forms on $\mathcal{T}$. The triple of spaces $\mathcal{T} \subset \mathcal{H} \subset \mathcal{T}^\dagger$ is often called a "Gelfand triple". Let us remark that for elements of the spectrum, $|P\rangle$ with $P \in \mathbb{R}^+$, it does not matter which subspace $\mathcal{T} \subset \mathcal{H}$ one chooses.

But one may also consider particularly "nice" subspaces $\mathcal{T}$ which are such that the wave-function $\psi(P) \equiv \langle P|\psi\rangle$ admits an analytic continuation into some region $R \subset \mathbb{C}$ around $\mathbb{R}^+$. On such a subspace one may of course consider the forms $|P\rangle$ to be defined by $\langle\psi|P\rangle \equiv (\psi(P))^*$ for any complex $P \in R$. So let us try to see whether such a distributional interpretation of the states $|P\rangle$ for $P \in \mathbb{C}$ is useful in the present context.

The matrix element $\langle P_2|\mathsf{V}_\alpha(z)|P_1\rangle$, $\Re(\alpha) > 0$, $|z| < 1$, can be interpreted as the wave-function of the state $\mathsf{P}_0\mathsf{V}_\alpha(z)|P_1\rangle$, where P_0 denotes the projection onto the subspace $\mathcal{H}_0 \subset \mathcal{H}$ of vectors that satisfy $\mathsf{L}_n|\psi\rangle = 0 = \bar{\mathsf{L}}_n|\psi\rangle$. The very existence of that wave-function means that the operator $\mathsf{V}_\alpha(z)$ must act *smoothing* on states $|\psi\rangle$: It creates a state with smooth wave-function when acting on the distribution $|P_1\rangle$, which would have a delta-distribution as "wave-function". For $\Re(\alpha) > 0$, the wave-function $\langle P_2|\mathsf{V}_\alpha(z)|P_1\rangle$ is not only smooth w.r.t. P_2, but even analytic within the strip $\{P_2 \in \mathbb{C}; |\Im(P_2)| < \Re(\alpha)\}$, as follows from the DOZZ formula. The distributional interpretation of the states $|P\rangle$ with $P \notin \mathbb{R}$ is therefore very natural in this context.

However, if the $|P\rangle$ are to be interpreted in a distributional sense both for $P \in \mathbb{R}$ and $P \notin \mathbb{R}$, why can't the $|P\rangle$ with $P \notin \mathbb{R}$ appear in the spectrum? It can be shown on rather general functional analytic grounds that a state such as $|P\rangle$ can appear in a spectral decomposition only if the distribution $\langle P|$ is defined on *any* space $\mathcal{T} \subset \mathcal{H}$ of test functions that allows the wave-functions $\langle P|\psi\rangle$ to be *pointwise* defined for any $\psi \in \mathcal{T}$, not only up to a set of measure zero [3]. This is clearly not the case for the $|P\rangle$ with $P \notin \mathbb{R}$: The condition of analyticity in some strip that determines the domain of $\langle P|$ is much too strong. A related fact is that "scalar products" like $\langle P'|P\rangle$ can be defined in the distributional sense as long as $P, P' \in \mathbb{R}$, but do not have a canonical definition as soon as P or P' have non-vanishing imaginary part.

4.8. **The vacuum $|0\rangle$**

Let us now check that the distributional interpretation of $|P\rangle$ for complex P yields a self-consistent understanding of the $SL(2,\mathbb{C})$ -invariant "state" $|0\rangle$. To this aim we need to check if (i) the state $\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$ is in the domain of $\langle 0|$, and (ii) The value of $\langle 0|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$ is given by

$$\langle 0|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle \;=\; \lim_{z_1\to 0}\Omega\big(\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_1}(z_1)\big), \tag{68}$$

[3] For a more precise mathematical discussion and proofs see e.g. the introduction and first section of [Be].

where $\alpha_1 = \frac{Q}{2} + iP_1$.

So let us consider $\langle P_4|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$, the wave-function of the state $\mathsf{P}_0\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$. It follows from our observation in Subsection 4.6 that $\langle P_4|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$ is analytic w.r.t. P_4 in a strip of width $\Re(\alpha_1+\alpha_2)$. So if $\Re(\alpha_1+\alpha_2) > \frac{Q}{2}$ one indeed finds that the state $\mathsf{P}_0\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$ is in the domain of $\langle 0| \equiv \langle P|_{P=-i\frac{Q}{2}}$.

In order to determine its value, one needs to return to the representation (62) for $\langle P_4|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$. When continuing P_4 to $-i\frac{Q}{2}$ one necessarily leaves the range (66), so that the more general representation of the form (67) has to be used. The residual terms include the contribution from the pole of $\langle P_4|\mathsf{V}_{\alpha_3}(z_3)|P,\zeta\rangle$ at $\frac{Q}{2} - iP_4 = \alpha_3 + \frac{Q}{2} + iP$, which yields a contribution that can be identified with $\lim_{z_1\to 0}\Omega\big(\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_1}(z_1)\big)$. All other contributions are found to vanish due to the zero of $C(\alpha_3,\alpha_2,\alpha_1)$ at $\alpha_3 = 0$.

So for $\Re(\alpha_1+\alpha_2) > \frac{Q}{2}$, the smoothing effect of $\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)$ is strong enough to map $|P_1\rangle$ into the domain of $\langle 0|$. This discussion is easily generalized to the case where $\Re(\alpha_1) \neq \frac{Q}{2}$, $\alpha_1 = \frac{Q}{2}+iP_1$: $\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|\alpha_1\rangle$ will be in the domain of $\langle 0|$ if $\Re(\alpha_1+\alpha_2+\alpha_3) > Q$. Moreover, by a very similar reasoning one would find that the smoothing effect of $\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_1}(z_1)$ is strong enough to map $|0\rangle$ into the domain of $\langle 0|$. We have thereby obtained the precise meaning of the relation

$$\Omega\big(\mathsf{V}_{\alpha_3}(\zeta_3|z_3)\mathsf{V}_{\alpha_2}(\zeta_2|z_2)\mathsf{V}_{\alpha_1}(\zeta_2|z_1)\big) \;\equiv\; \langle 0|\mathsf{V}_{\alpha_3}(\zeta_3|z_3)\mathsf{V}_{\alpha_2}(\zeta_2|z_2)\mathsf{V}_{\alpha_1}(\zeta_1|z_1)|0\rangle. \tag{69}$$

It is straightforward to interpret vacuum expectation values of more than three operators along these lines. Let us note that the condition for $\langle 0|$ to be in the domain of the state created by acting with $\prod_{i=1}^n \mathsf{V}_{\alpha_i}(z_i)$ on $|0\rangle$ is precisely the condition for convergence of zero mode integration found in Subsection 3.3.

4.9. Operators V_α with $\Re(\alpha) \leq 0$

So far we had only discussed the connection between fields V_α and operators V_α for $0 \leq \Re(\alpha) \leq \frac{Q}{2}$. Let us now generalize to $\Re(\alpha) \leq 0$.

The main point is best seen when considering the wave-function $\langle P_2|\mathsf{V}_\alpha(z)|\psi\rangle$ of the state created by acting with $\mathsf{P}_0\mathsf{V}_\alpha(z)$ on a state $|\psi\rangle$ of the form $|\psi\rangle = \frac{1}{2\pi}\int_0^\infty dP_1\,\psi(P_1)|P_1\rangle$. It is clearly well-defined as long as $0 < \Re(\alpha) \leq \frac{Q}{2}$ and given by

$$\langle P_2|\mathsf{V}_\alpha(z)|\psi\rangle \;=\; \int_{\mathbb{R}^+}\frac{dP_1}{2\pi}\;\psi(P_1)\;\langle P_2|\mathsf{V}_\alpha(z)|P_1\rangle. \tag{70}$$

When $\Re(\alpha) \to 0$, it will generically cease to be defined due to the poles of $\langle P_2|\mathsf{V}_\alpha(z)|P_1\rangle$ at $\frac{Q}{2} \pm iP_2 = \alpha + \frac{Q}{2} \pm iP_1$. One may, however, consider states $|\psi\rangle$ whose wave-functions $\psi(P)$ admit an analytic continuation into a strip of width w. For such $|\psi\rangle$ it is possible to analytically continue the expression (70) to all α with $-w < \Re(\alpha) \leq 0$: This is done by first using the reflection property $|P\rangle = R(P)|-P\rangle$ to extend the integration over $\mathbb{R}^+$ to an integral over $\mathbb{R}$, and then deforming the contour of integration over P_1 suitably around the poles of $\langle P_2|\mathsf{V}_\alpha(z)|P_1\rangle$ that cross the axis $\mathbb{R}$ when $\Re(\alpha)$ becomes negative. We conclude that $\mathsf{V}_\alpha(z)$ for $\Re(\alpha) \leq 0$ makes sense as an unbounded operator with domain restricted to states with wave-functions analytic in strips of width larger or equal to $|\Re(\alpha)|$.

The following alternative point of view is useful: Consider $\mathsf{P}_0\mathsf{V}_{\alpha_2}(z_1)|P_1\rangle$, which for $0 < \Re(\alpha) \leq \frac{Q}{2}$ can be represented as

$$\mathsf{P}_0\mathsf{V}_{\alpha_2}(z_1)|P_1\rangle \;=\; \frac{1}{2}\int_{\mathbb{R}} \frac{dP}{2\pi}\;|P\rangle\langle P|\mathsf{V}_{\alpha_2}(z_1)|P_1\rangle. \tag{71}$$

The matrix element $\langle P|\mathsf{V}_{\alpha_2}(z_1)|P_1\rangle$ has poles at $\frac{Q}{2} \pm iP = \alpha_2 + \frac{Q}{2} \pm iP_1$, which approach the contour $\mathbb{R}$ of integration in (71) when $\Re(\alpha) \to 0$.

On a formal level it is straightforward to perform the continuation to $\Re(\alpha) \leq 0$: We had previously discussed the continuation of the $|P\rangle$ to complex values of P in the sense of distributions in $\mathcal{T}^\dagger$. In this spirit one would perform the continuation of (71) simply by again deforming the contour of integration over P. This generically yields a representation of the form (71) but with contour $\mathbb{R}$ replaced by $\mathbb{R}$ plus a finite sum of circles around the poles of $\langle P|\mathsf{V}_{\alpha_2}(z_1)|P_1\rangle$ that have crossed the contour $\mathbb{R}$. For example, if $-b < \Re(\alpha) \leq 0$ one would get instead of (71) the expression

$$\mathsf{P}_0\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle \;= |P_1 - i\alpha_2\rangle \;+\; \frac{1}{2}\int_{\mathbb{R}} \frac{dP}{2\pi}\;|P\rangle\langle P|\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle, \tag{72}$$

where the value of the residues has been worked out from the DOZZ-formula taking into account the reflection property (47). Formula (72) clearly makes sense as an equation in $\mathcal{T}^\dagger$ if $\mathcal{T}$ is such that the wave-functions $\langle P|\psi\rangle$ of $|\psi\rangle \in \mathcal{T}$ are analytic in a strip of width larger than $|\Re(\alpha_2)|$. The unboundedness of $\mathsf{V}_{\alpha_2}(z_2)$ for $\Re(\alpha_2) \leq 0$ is now reflected in the appearance of non-normalizable states such as e.g. the state $|P_1 - i\alpha_2\rangle$ in (72).

4.10. **Null vector decoupling**

Something interesting happens if one considers $\mathsf{V}_\alpha(z)$ for $2\alpha = 2\alpha_{m,n} \equiv -mb - nb^{-1}$, $n, m \in \mathbb{Z}^{\geq 0}$, where the Verma module $\mathcal{V}_\alpha$ contains a singular vector $s_{n,m}$. The matrix element $\langle P_3|\mathsf{V}_{\alpha_2}(z)|P_1\rangle$ is proportional to $\Upsilon(2\alpha_2)$, which vanishes for $\alpha_2 = \alpha_{m,n}$. This means that the expansion (72) of $\mathsf{P}_0\mathsf{V}_{\alpha_2}(z)|P_1\rangle$ over states $|P\rangle$ does not contain the part represented as an integral over $\mathbb{R}$. What is non-vanishing, however, are the residue terms that were picked up in the process of defining the analytic continuation. These are found to be given by an expression of the form

$$\mathsf{P}_0\mathsf{V}_\alpha(z)|P\rangle \;=\; \sum_{r=0}^{m}\sum_{s=0}^{n} C_{r,s}^{m,n}(P)\;|P - i(\alpha + rb + sb^{-1})\rangle. \tag{73}$$

The values of $P - i(\alpha + rb + sb^{-1})$ that appear in (73) are in precise correspondence to the so-called fusion rules for decoupling of the null vector in the representation $\mathcal{V}_{\alpha_2}$: It follows from a theorem of Feigin and Fuchs [FF], which is quoted below in Section 6 as Theorem 1, that

$$\mathsf{V}_\alpha(s_{n,m} \otimes v_{n,m}|z) \;=\; 0, \qquad \mathsf{V}_\alpha(v_{n,m} \otimes s_{n,m}|z) \;=\; 0, \tag{74}$$

if $v_{n,m}$ and $s_{n,m}$ are the highest weight and the singular vectors of the Verma module $\mathcal{V}_{\alpha_{m,n}}$ respectively. Vacuum expectation values that contain these fields will satisfy differential equations of the type discussed in [BPZ].

We find it remarkable to see how deep information on the representation theory of the Virasoro algebra is encoded in the analytic structure of the DOZZ three point function.

REMARK 3. — A special case of this result reproduces the identification of the identity operator as the primary field with vanishing conformal dimension:

$$\lim_{\alpha\to 0} \mathsf{V}_\alpha(z) \;=\; \text{id}, \tag{75}$$

which may be seen as an example of null vector decoupling in the case $n = 0 = m$.

5. LOCALITY AND CROSSING SYMMETRY?

A major issue of consistency arises: If follows from the preceding discussions that vacuum expectation values such as $\langle 0|\prod_i^n \mathsf{V}_{\alpha_i}(z_i)|0\rangle$ are uniquely given by conformal symmetry and the DOZZ formula: Summing over intermediate states as in (62) produces power series with coefficients that can all be expressed in terms of the matrix elements $\langle P_3,\zeta_3|\mathsf{V}_{\alpha_2}(z_2)|P_1,\zeta_1\rangle$, which characterize the operators V_α uniquely. But are the operators that are characterized in such a way also local, i.e. do they satisfy $\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_1}(z_1) = \mathsf{V}_{\alpha_1}(z_1)\mathsf{V}_{\alpha_2}(z_2)$?

It clearly suffices to consider the case of $n = 4$ which corresponds to the matrix element $\langle P_4|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$. In view of the power series expansions for this matrix element discussed in Subsection 4.5 one sees that this question involves the following issues: First, it was stated there that the power series that represent $\langle P_4|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$ and $\langle P_4|\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_3}(z_3)|P_1\rangle$ are convergent for $|z_2| < |z_3|$ and $|z_3| < |z_2|$ respectively. In order for the question of locality to have any sense, one evidently needs that the conformal blocks can be analytically continued into $|z_2| > |z_3|$ and $|z_3| > |z_2|$ respectively. Given that such an analytic continuation exists, locality amounts to a highly nontrivial identity between the conformal blocks, which are fully given by conformal symmetry, and the coefficients $C(\alpha_3,\alpha_2,\alpha_1)$ that represent the measure with which the conformal blocks are weighted in (63).

By now we believe to have a proof for this crucial property, which will be sketched in Parts III and IV, with details to be presented elsewhere. Until then we will simply assume that the DOZZ-proposal indeed describes local operators V_α.

5.1. Crossing symmetry

Locality is closely related (almost equivalent) to another property of vacuum expectation values $\langle 0|\prod_i^n \mathsf{V}_{\alpha_i}(z_i)|0\rangle$ that is usually called crossing symmetry. Let us again restrict to the case $n = 4$, which is good enough. Inserting a complete set of intermediate states yields an expansion

$$\begin{aligned}
&\langle 0|\mathsf{V}_{\alpha_4}(z_4)\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_1}(z_1)|0\rangle \;\equiv \\
&\qquad\equiv \int_{\mathbb{S}} \frac{d\alpha}{2\pi} \sum_{\zeta\in\mathbb{B}_\alpha} \langle 0|\mathsf{V}_{\alpha_4}(z_4)\mathsf{V}_{\alpha_3}(z_3)|\alpha,\zeta\rangle\langle\alpha,\zeta^t|\mathsf{V}_{\alpha_2}(z_2)\mathsf{V}_{\alpha_1}(z_1)|0\rangle.
\end{aligned} \tag{76}$$

Alternatively one may use locality to move V_{α_1} to the right of V_{α_4}, and then insert a complete set of states between $\mathsf{V}_{\alpha_4}\mathsf{V}_{\alpha_1}$ and $\mathsf{V}_{\alpha_3}\mathsf{V}_{\alpha_2}$. Noting that $\mathsf{V}_\alpha(z)|0\rangle = e^{zL_{-1}+\bar{z}\bar{L}_{-1}}|\alpha\rangle$ one may rewrite the resulting expansion as follows:

$$\langle 0|\mathsf{V}_{\alpha_4}(z_4)\dots\mathsf{V}_{\alpha_1}(z_1)|0\rangle \;\equiv\; \int_{\mathbb{S}}\frac{d\alpha}{2\pi}\sum_{\zeta\in\mathbb{B}_\alpha}\Omega\big(V_{\alpha_4}(z_4)V_{\alpha_1}(z_1)V_\alpha(\zeta|z_2)\big)\,\times \\ \times\,\Omega\big(V_{\bar\alpha}(\zeta^t|\infty)V_{\alpha_3}(z_3-z_2)V_{\alpha_2}(0)\big), \tag{77}$$

where we used the identification (69) and the notation (7). If one finally uses locality of $\mathsf{V}_{\alpha_1}(z_1)$ and $\mathsf{V}_\alpha(\zeta|z_2)$ one gets an expansion that can be read as the result of expanding the product of operators $\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)$ according to the OPE:

$$\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2) \;=\; \int_{\mathbb{S}}\frac{d\alpha}{2\pi}\sum_{\zeta\in\mathbb{B}_\alpha}\mathsf{V}_\alpha(\zeta|z_2)\;\Omega\big(V_{\bar\alpha}(\zeta^t|\infty)V_{\alpha_3}(z_3-z_2)V_{\alpha_2}(0)\big). \tag{78}$$

The fact that the power series expansion obtained by inserting (78) into $\Omega\big(\mathsf{V}_{\alpha_4}(z_4)\dots\mathsf{V}_{\alpha_1}(z_1)\big)$ also yields a valid representation for that vacuum expectation value is usually referred to as the property of *crossing symmetry*. It can be written as

$$\Omega\big(V_{\alpha_4}(\infty)V_{\alpha_3}(1)V_{\alpha_2}(z,\bar z)V_{\alpha_1}(0)\big) \;= \\ =\int_{\mathbb{S}}\frac{d\alpha}{2\pi}\,C(\alpha_4,\alpha,\alpha_1)C(\bar\alpha,\alpha_3,\alpha_2)\;\mathcal{F}^t_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](z)\mathcal{F}^t_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](\bar z), \tag{79}$$

where the *t-channel conformal blocks* $\mathcal{F}^t_\alpha$ (as opposed to the *s-channel conformal blocks* $\mathcal{F}^s_\alpha$ that appear in (63)) are given as power series expansions of the form

$$\mathcal{F}^s_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](z) \;=\; (1-z)^{\Delta_\alpha-\Delta_{\alpha_3}-\Delta_{\alpha_2}}\sum_{n=0}^{\infty}(1-z)^n\;\mathcal{F}^t_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](n). \tag{80}$$

This relationship can be generalized to the expressions obtained by using the operator product expansion iteratively. One gets expansions of the form

$$\Omega\big(\dots V_{\alpha_k}(\zeta_k|z_k)\dots V_{\alpha_l}(\zeta_l|z_l)\dots\big) = \int_{\mathbb{S}}\frac{d\alpha}{2\pi}\sum_{\zeta\in\mathbb{B}_\alpha}\Omega\big(\dots V_\alpha(\zeta|z_l)\dots\big) \\ \times\;\Omega\big(V_{Q-\alpha}(\zeta^t|\infty)V_{\alpha_k}(\zeta_k|z_{kl})\dots V_{\alpha_l}(\zeta_l|0)\big),$$

where $z_{kl}\equiv z_k-z_l$.

6. APPENDIX A: VERMA MODULES OF THE VIRASORO ALGEBRA

6.1. Verma modules

Let $\mathcal{V}$ be the infinite dimensional vector space with basis $\mathcal{B}=\{v_\nu;\nu\in\mathcal{T}\}$, where $\mathcal{T}$ is the set of all tuples $\nu=(r_1,\dots,r_i,\dots)$ with all but finitely many r_i being zero. The element of $\mathcal{B}$ that corresponds to the tuple with $r_i\equiv 0$ for all i will be denoted v. $\mathcal{V}$ is the direct sum of finite dimensional vector spaces $\mathcal{V}[m]$ with fixed "level" m, which are spanned by the vectors v_ν with $n(\nu)=m$, where $n(\nu)=\sum_{i=0}^{\infty} ir_i$.

There is a standard family $\mathcal{V}_\alpha$, $\alpha\in\mathbb{C}$ of representations of the Virasoro algebra that can be defined on $\mathcal{V}$: It is uniquely defined by the requirements that

(i) $L_n v=0$ for $n>0$ and $L_0 v_0=\Delta_\alpha v$, where $\Delta_\alpha=\alpha(Q-\alpha)$, and

(ii) $v_\nu \;=\; \prod_{i=1}^{\infty}(L_{-i})^{r_i} v$ if $\nu = (r_1, \dots, r_i, \dots)$.

The representation $\mathcal{V}_\alpha$ depends of course on the *conformal dimension* Δ_α only[4], so $\mathcal{V}_\alpha \equiv \mathcal{V}_{Q-\alpha}$. There is a standard bilinear form $\langle .,. \rangle_\alpha$ on $\mathcal{V}$ which is defined by $\langle v, v\rangle_\alpha = 1$ and $\langle L_{-n}\xi, \zeta\rangle_\alpha = \langle \xi, L_n \zeta\rangle_\alpha$. The representation $\mathcal{V}_\alpha$ is *irreducible* if and only if $\langle .,. \rangle_\alpha$ is non-degenerate. A criterion for the latter is vanishing of the determinant of the matrix with elements $B_{\nu\mu}(\alpha) = \langle v_\nu, v_\mu\rangle_\alpha$. This matrix is block-diagonal with blocks $B_{\nu\mu}(\alpha, n)$ for each subspace $\mathcal{V}_\alpha[n]$ ($\equiv \mathcal{V}[n]$ as a vector space). The formula for the determinant $D_n(\alpha)$ of the matrix $B_{\nu\mu}(\alpha, n)$ was conjectured in [K1] and proven in [FF]. It may be written as

$$
\begin{aligned}
&D_n(\alpha) \;=\; C \prod_{r,s=0}^{\infty} (\Delta_\alpha - \Delta_{r,s})^{p(n-rs)}, \text{ where} \\
&\Delta_{r,s} = \alpha_{r,s}(Q - \alpha_{r,s}), \qquad \alpha_{r,s} = -\tfrac{b}{2} r - \tfrac{1}{2b} s,
\end{aligned}
\tag{81}
$$

where C is a constant independent of α, c, and $p(n)$ denotes the dimension of $\mathcal{V}[n]$.

With the help of the determinant formula (81) it is possible to determine the cases where one has a scalar product $(.,.)_\alpha$ on $\mathcal{V}_\alpha$ such that $(L_{-n}\xi, \zeta)_\alpha = (\xi, L_n\zeta)_\alpha$ (unitarity). We are interested in the case $c > 1$, in which the necessary and sufficient condition for unitarity of the representation $\mathcal{V}_\alpha$ was found to be $\Delta_\alpha > 0$ [K2].

If $\alpha \neq \alpha_{r,s}$ one has a unique basis $\mathcal{B}^t_\alpha$ that is dual to $\mathcal{B}$ w.r.t. $\langle .,. \rangle_\alpha$: It has elements $v^t_{\alpha,\nu}$ that are defined by $\langle v^t_{\alpha,\mu}, v_\nu\rangle_\alpha = \delta_{\mu,\nu}$. The expansion of the vectors $v^t_{\alpha,\nu}$ w.r.t. the canonical basis for $\mathcal{V}$ can be written in terms of the inverse $B^{\nu\mu}(\alpha)$ of the matrix $B_{\nu\mu}(\alpha)$:

$$
v^t_{\alpha,\nu} \;=\; \sum_{\mu\in\mathcal{T}(\nu)} B^{\nu\mu}(\alpha)\, v_\mu,
\tag{82}
$$

where $\mathcal{T}(\nu)$ is the set of all tuples $\mu = (r_1, \dots, r_i, \dots)$ with $n(\mu) \equiv n(\nu)$. It is clear that the dependence of the $v^t_{\alpha,\nu}$ on α is rational with poles at $\alpha = \alpha_{r,s}$. We will need to know the singular behavior at $\alpha = \alpha_{r,s}$ more precisely.

To this aim let us first consider $\mathcal{V}_{r,s} \equiv \mathcal{V}_{\alpha_{r,s}}$. Vanishing of the Kac-determinant $D_n(\alpha)$ for $\alpha = \alpha_{r,s}$. is equivalent to the existence of a subspace $\mathcal{S}_{r,s} \subset \mathcal{V}_{r,s}$ that consists of vectors ξ with the property $\langle \xi, \zeta\rangle_{\alpha_{r,s}} = 0$ for all $\zeta \in \mathcal{V}_{r,s}$. The singular subspace $\mathcal{S}_{r,s} \subset \mathcal{V}_{r,s}$ is generated from a so-called null-vector $s_{r,s} \in \mathcal{V}_{r,s}$ that satisfies the highest weight property $L_n s_{r,s} = 0$ for $n > 0$ and $L_0 s_{r,s} = (\Delta_{r,s} + rs)s_{r,s}$.

We would now like to show that $v^t_{\alpha,\nu}$ has a pole of first order at $\alpha = \alpha_{r,s}$ and $\lim_{\alpha\to\alpha_{r,s}}(\alpha - \alpha_{r,s})\; v^t_{\alpha,\nu} \in \mathcal{S}_{r,s}$. To verify the claim, one may argue as follows: Since $v^t_{\alpha,\nu}$ is rational there exists an integer $p_{r,s}(\nu) > 0$ such that the limit $v^t_{r,s;\nu} \equiv \lim_{\alpha\to\alpha_{r,s}}(\alpha - \alpha_{r,s})^{p_{r,s}(\nu)}\; v^t_{\alpha,\nu}$ exists. One finds $\langle v^t_{r,s;\nu}, v_\mu\rangle_{\alpha_{r,s}} = 0$ for all $\mu \in \mathcal{T}$ by combining the definition of $v^t_{r,s;\nu}$ with $\langle v^t_{\alpha,\nu}, v_\mu\rangle_\alpha = \delta_{\mu,\nu}$. Therefore $v^t_{r,s;\nu} \in \mathcal{S}_{r,s}$. It remains to show that $p_{r,s}(\nu) = 1$ for all $\nu \in \mathcal{T}$. But this follows from the fact that the order of the zero of $D_n(\alpha)$ at $\alpha = \alpha_{r,s}$ coincides with the dimension of $\mathcal{S}^n_{r,s} \equiv \mathcal{S}_{r,s} \cap \mathcal{V}[n]$.

---

[4] We mostly consider the central charge c as fixed parameter in what follows

6.2. The trilinear form ρ

Let us define a family of trilinear forms $\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1} : \mathcal{V}_{\alpha_3}\otimes\mathcal{V}_{\alpha_2}\otimes\mathcal{V}_{\alpha_1}\to\mathbb{C}$ such that

$$\begin{aligned}\Omega\big(V_{\alpha_3}(\xi_3,\bar{\xi}_3|z_3)V_{\alpha_2}(\xi_2,\bar{\xi}_2|z_2)V_{\alpha_1}(\xi_1,\bar{\xi}_1|z_1)\big) \;=\; \\ =\; C(\alpha_3,\alpha_2,\alpha_1)\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(\xi_3,\xi_2,\xi_1)\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(\bar{\xi}_3,\bar{\xi}_2,\bar{\xi}_1).\end{aligned}$$

Recall that the left hand side is defined by the conformal Ward identities and (4). Let us spell out the corresponding definition of $\rho(\xi_3,\xi_2,\xi_1)$ a bit more explicitly: One first of all needs to have

$$\begin{aligned}\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(v,v,v) \;=\; \\ =\; (z_3-z_2)^{\Delta_{\alpha_1}-\Delta_{\alpha_2}-\Delta_{\alpha_3}}(z_3-z_1)^{\Delta_{\alpha_2}-\Delta_{\alpha_1}-\Delta_{\alpha_3}}(z_2-z_1)^{\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1}}.\end{aligned} \tag{83}$$

The conformal Ward identities imply rules of the form

$$\begin{aligned}(n-2)!\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(L_{-n}\xi_3,\xi_2,\xi_1) \;=\; \\ =\; \rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}\big(\xi_3,\partial^{n-2}_{z_3}T_>(z_3-z_2)\xi_2,\xi_1\big)+\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}\big(\xi_3,\xi_2,\partial^{n-2}_{z_3}T_>(z_3-z_1)\xi_1\big),\end{aligned}$$

where $n > 1$, $T_>(z) = \sum_{n=-1}^{\infty}L_n z^{-n-2}$, together with analogous equations for $\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(\xi_3,L_{-n}\xi_2,\xi_1)$ and $\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(\xi_3,\xi_2,L_{-n}\xi_1)$. These rules allow one to express $\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(\xi_3,\xi_2,\xi_1)$ in terms of $\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(L^{n_3}_{-1}v,L^{n_2}_{-1}v,L^{n_1}_{-1}v)$. The evaluation of $\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}$ is therefore completed by noting that

$$\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(L_{-1}\xi_3,\xi_2,\xi_1) \;=\partial_{z_3}\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(\xi_3,\xi_2,\xi_1), \tag{84}$$

and analogously for $\partial_{z_2}\rho$ and $\partial_{z_1}\rho$. It is a consequence of these definitions that

$$\begin{aligned}\rho^{\alpha_3,\alpha_2,\alpha_1}_{\infty,z_2,0}(v_{\nu_3},v_{\nu_2},v_{\nu_1}) \;&\equiv\; \lim_{z_3\to\infty}\lim_{z_1\to 0} z_3^{2\Delta^{\nu_3}_{\alpha_3}}\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(v_{\nu_3},v_{\nu_2},v_{\nu_1}) \\ &=\; z_2^{\Delta^{\nu_3}_{\alpha_3}-\Delta^{\nu_2}_{\alpha_2}-\Delta^{\nu_1}_{\alpha_1}}\rho^{\alpha_3,\alpha_2,\alpha_1}(v_{\nu_3},v_{\nu_2},v_{\nu_1}),\end{aligned} \tag{85}$$

where $\Delta^{\nu}_{\alpha} \equiv \Delta_\alpha + n(\nu)$ and $\rho^{\alpha_3,\alpha_2,\alpha_1}(v_{\nu_3},v_{\nu_2},v_{\nu_1})$ is a polynomial in $\alpha_3,\alpha_2,\alpha_1$ and c. The following important result is proven in [FF]:

THEOREM 1. — NULL VECTOR DECOUPLING:
Let $i,j,k\in\{1,2,3\}$ be chosen such that $j\neq i$, $k\neq i$, $j\neq k$. Assume that (i) $\alpha_i=\alpha_{r,s}$, and (ii) ξ_i lies in the singular subspace $\mathcal{S}_{r,s}$. One then finds that $\rho^{\alpha_3,\alpha_2,\alpha_1}_{z_3,z_2,z_1}(\xi_3,\xi_2,\xi_1)=0$ if and only if α_j and α_k satisfy the fusion-rules $\Delta_{\alpha_k}=\Delta_{\alpha_j+mb+nb^{-1}}$, where $m\in\{-\frac{r}{2},-\frac{r}{2}+1,\dots,\frac{r}{2}\}$, $n\in\{-\frac{s}{2},-\frac{s}{2}+1,\dots,\frac{s}{2}\}$.

7. APPENDIX B: MEROMORPHIC CONTINUATION OF MATRIX ELEMENTS

7.1. Analytic properties of conformal blocks

Our definition of ρ allows us to write the coefficients of the power series (64) that define the conformal blocks as follows:

$$\mathcal{F}^s_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](n) \;=\; \sum_{\mu,\nu\in\mathcal{T}_n}\rho^{\alpha_4,\alpha_3,\alpha}(v,v,v_\mu)\,B^{\mu\nu}(\alpha)\,\rho^{\alpha,\alpha_2,\alpha_1}(v_\nu,v,v). \tag{86}$$

The convergence of the power series (64) for $|z_2| < |z_3|$ can be shown [TO] by means of the free field representation for the chiral vertex operators (cf. Part III, Subsection 15.2 and Part IV, Subsection 18.1). This fact yields important information on the dependence of the conformal blocks $\mathcal{F}^s_\alpha$ w.r.t. the variables $\alpha_1, \dots, \alpha_4$ and α:

The conformal blocks $\mathcal{F}^s_\alpha$ are entire analytic as functions of $\alpha_1, \dots, \alpha_4$ and meromorphic as function of α, with poles for $\Delta_\alpha = \Delta_{r,s}$. The residues of these poles vanish iff either $(\alpha_4, \alpha_3, \alpha)$ or $(\alpha, \alpha_2, \alpha_1)$ satisfy the fusion rules from Theorem 1.

By convergence of the power series (64) it suffices to verify the corresponding claims for the coefficients $\mathcal{F}^s_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](n)$. The product $\rho^{\alpha_4,\alpha_3,\alpha}(v, v, v_\mu)\rho^{\alpha,\alpha_2,\alpha_1}(v_\nu, v, v)$ is a polynomial in α and $\alpha_1, \dots, \alpha_4$. $B^{\mu\nu}(\alpha)$ depends rationally on α with poles iff $\Delta_\alpha = \Delta_{r,s}$. By our observation at the end of Subsection 6.1 one may express the resulting residues of $\mathcal{F}^s_\alpha\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](n)$ as sum of terms containing either $\rho^{\alpha_4,\alpha_3,\alpha}_{\infty,1,0}(v, v, s)$ or $\rho^{\alpha,\alpha_2,\alpha_1}_{\infty,1,0}(s, v, v)$, where s is an element of the singular subspace $\mathcal{S}_{r,s}$. The claim concerning vanishing of the residues is therefore a direct consequence of Theorem 1.

7.2. **Proof of meromorphic continuation of correlation functions**

To establish the meromorphic continuation of the correlation function $\langle P_4|\mathsf{V}_{\alpha_3}(z_3)\mathsf{V}_{\alpha_2}(z_2)|P_1\rangle$ let us first consider the integrand in (62): By the remark at the end of the previous subsection and the analytic properties of the Υ-function it takes the form of a product of functions that are meromorphic w.r.t. $\alpha_4, \dots, \alpha_1$ and α. The double poles from the conformal blocks $\mathcal{F}_\alpha$ are cancelled against zeros of the factors $\Upsilon(Q + 2iP)$ and $\Upsilon(Q - 2iP)$. One is therefore left with poles coming from the DOZZ three point functions. The resulting pattern of poles is as follows: Let us use the variable $\alpha \equiv \frac{Q}{2} + iP$ instead of P. One has strings of poles at

$$
\begin{aligned}
\alpha &= \sigma_1(\alpha_1 - Q/2) + \sigma_2(\alpha_2 - Q/2) - nb - mb^{-1},\\
\alpha &= \sigma_3(\alpha_3 - Q/2) + \sigma_4(\alpha_4 - Q/2) - nb - mb^{-1},\\
\alpha &= Q - \sigma_1(\alpha_1 - Q/2) - \sigma_2(\alpha_2 - Q/2) + nb + mb^{-1},\\
\alpha &= Q - \sigma_3(\alpha_3 - Q/2) - \sigma_4(\alpha_4 - Q/2) + nb + mb^{-1},
\end{aligned}
\tag{87}
$$

where $n, m \in \mathbb{Z}^{\geq 0}$, $\sigma_i \in \{+, -\}$, $i = 1, \dots, 4$. As long as

$$
\begin{aligned}
&|\Re(\alpha_1 - \alpha_2)| < Q/2; \qquad &|\Re(Q - \alpha_1 - \alpha_2)| < Q/2;\\
&|\Re(\alpha_3 - \alpha_4)| < Q/2; \qquad &|\Re(Q - \alpha_3 - \alpha_4)| < Q/2,
\end{aligned}
\tag{88}
$$

one finds that all the poles in (87) are strictly to the left or right of the contour $\frac{Q}{2} + i\mathbb{R}^+$ of integration over α. The integral over α is analytic w.r.t. $\alpha_4, \dots, \alpha_1$ in this case. When continuing outside (88) one finds that poles from (87) would cross the axis $\frac{Q}{2} + i\mathbb{R}$. To define the meromorphic continuation one may use the reflection property (47) together with the fact that $\mathcal{F}_\alpha = \mathcal{F}_{Q-\alpha}$, to "unfold" the α-integration to an integral over the full axis $\frac{Q}{2} + i\mathbb{R}$. Let us furthermore introduce

$$
\alpha_{21}^{\pm} = \tfrac{Q}{2} - \alpha_1 \pm \big(\tfrac{Q}{2} - \alpha_2\big), \qquad \alpha_{43}^{\pm} = \tfrac{Q}{2} - \alpha_3 \pm \big(\tfrac{Q}{2} - \alpha_4\big).
\tag{89}
$$

Thanks to the reflection symmetry $\alpha_i \to Q - \alpha_i$ it suffices to consider the case that $\arg(\alpha_{21}^{\pm}) \in [0, \frac{\pi}{2}]$ and $\arg(\alpha_{43}^{\pm}) \in [0, \frac{\pi}{2}]$. The definition of the meromorphic continuation is straightforward in the case that the imaginary parts of $\alpha_{21}^{\pm}$ and $\alpha_{43}^{\pm}$ are all different from zero and from each other. In this case one simply has to deform the original contour $\frac{Q}{2}+i\mathbb{R}$ to a contour that is indented around the strings of poles that have crossed $\frac{Q}{2}+i\mathbb{R}$. Equivalently one may use a contour that is the sum of $\frac{Q}{2}+i\mathbb{R}$ and a finite sum of small circles around the poles just mentioned. For concreteness let us consider the case that $\Re(\alpha_{21}^{\pm})>0$. One then has a contour consisting of $\frac{Q}{2}+i\mathbb{R}$ and small circles around the poles at

$$\begin{aligned} \alpha &= \alpha_{21}^{\pm}-nb-mb^{-1}; \quad n,m\in\mathbb{Z}^{\geq 0}, \quad \Re(\alpha)>\tfrac{Q}{2}, \\ \alpha &= \alpha_{43}^{\pm}-nb-mb^{-1}; \quad n,m\in\mathbb{Z}^{\geq 0}, \quad \Re(\alpha)>\tfrac{Q}{2}, \end{aligned} \tag{90}$$

together with their “reflected partners” obtained by $\alpha\to Q-\alpha$.

It remains to consider the following cases:

(1) *The imaginary part of one of $\alpha_{21}^{\pm}$, $\alpha_{43}^{\pm}$ becomes zero.* It is enough to consider the case $\Im(\alpha_{21}^{-})\to 0$. One finds a collision of the poles $\alpha=\alpha_{21}^{-}-nb-mb^{-1}$ that have crossed $\frac{Q}{2}+i\mathbb{R}$ from the left with the poles $\alpha=Q-\alpha_{21}^{-}+n'b+m'b^{-1}$ only if $2\alpha_{21}^{-}=Q+(n+n')b+(m+m')b^{-1}$, so that $2\alpha=(1+n'-n)b+(1+m'-m)b^{-1}$. The Verma module $\mathcal{V}_\alpha$ will therefore generically contain a singular subspace. However, in these cases it is easy to check that α_2 and α_1 satisfy the fusion rules of Theorem 1. By our discussion in the previous subsection one observes that the conformal blocks $\mathcal{F}_\alpha$ remain nonsingular at these values of $(\alpha,\alpha_2,\alpha_1)$. But this means that all of the poles that can potentially arise will be cancelled by the zeros of the factors $\Upsilon(2\alpha)$ and $\Upsilon(2Q-2\alpha)$ contained in $C(\alpha_4,\alpha_3,\alpha)C(\bar{\alpha},\alpha_2,\alpha_1)$.

(2) $\Im(\alpha_{21}^{+})=\Im(\alpha_{21}^{-})$ or $\Im(\alpha_{43}^{+})=\Im(\alpha_{43}^{-})$: The first case requires $\Delta_{\alpha_1}=\Delta_{r,s}$ for some $r,s\in\mathbb{Z}^{\geq 0}$. The zeros of the factor $\Upsilon(2\alpha_1)$ in $C(\bar{\alpha},\alpha_2,\alpha_1)$ prevent the occurrence of a singularity. The second case is treated analogously.

(3) $\Im(\alpha_{43}^{\pm})=\Im(\alpha_{21}^{\pm})$: Here it suffices to consider $\Im(\alpha_{43}^{+})=\Im(\alpha_{21}^{-})$, all other cases being related to this one by reflections $\alpha_i\to Q-\alpha_i$. Collision of poles $\alpha=\alpha_{21}^{+}-nb-mb^{-1}$ with $\alpha=Q-\alpha_{43}^{+}+n'b+m'b^{-1}$ occurs if $\sum_{i=1}^{4}\alpha_i=Q-rb-sb^{-1}$ for some $r,s\in\mathbb{Z}^{\geq 0}$. These cases indeed produce poles of the matrix element.

Part II. CANONICAL QUANTIZATION

One may view the DOZZ-proposal as providing a complete description of Liouville theory in the energy representation: It is trivial to rewrite $\mathcal{H} \simeq \int_{\mathbb{R}_+} \mathcal{W}_P$ as decomposition of $\mathcal{H}$ into eigenspaces of the Hamiltonian $\mathsf{H} = \mathsf{L}_0 + \bar{\mathsf{L}}_0$. However, it is difficult to get insight into the physics of Liouville theory on the basis of this representation only: One would like to have something like a Schrödinger- or coordinate representation, where states in Liouville theory are represented by wave-functions on "target-space". We are trying to address the following two questions in the present Part II of our paper:

(i) Does such a representation exist?
(ii) What can be learned from it?

The basis for our discussion will be the description of Liouville theory as provided by our previous discussion of the DOZZ-proposal. We will propose a certain picture, many consequences of which are found to be consistent with the DOZZ-proposal. Moreover, it provides a more intuitive interpretation for some of the otherwise mysterious consequences of the DOZZ-proposal, like the reflection property and the associated Seiberg-bound.

However, our discussion will remain in some respects inconclusive. In fact, we regard some of the questions that we are going to discuss as extremely interesting open problems for the future study of quantum Liouville theory.

8. THE PROBLEM OF CANONICAL QUANTIZATION

Let us start by formulating what we will mean when speaking of "canonical quantization" in the context of Liouville theory.

8.1. Classical theory

Classically one may introduce the canonical formalism by starting from the action (1) and defining the momentum conjugate to φ as $\Pi_\varphi = \frac{1}{8\pi}\partial_t\varphi$. By introducing the Poisson bracket

$$\{\Pi_\varphi(\sigma), \varphi(\sigma')\} \;=\; \delta(\sigma-\sigma') \tag{91}$$

and the canonical Hamiltonian H

$$H \;=\; \int_0^{2\pi} d\sigma\left(4\pi\Pi_\varphi^2 + \frac{1}{16\pi}(\partial_\sigma\varphi)^2 + \mu_c e^{\varphi}\right) \tag{92}$$

one may recast the Liouville equation of motion in the Hamiltonian form

$$\partial_t\varphi(\sigma,t) \;=\; \{H,\varphi(\sigma,t)\}, \qquad \partial_t\Pi_\varphi(\sigma,t) \;=\; \{H,\Pi_\varphi(\sigma,t)\}. \tag{93}$$

8.2. **Notion of canonical quantization**

Naively one would want to define the algebra of observables to be generated by operators $\varphi(\sigma)$ and $\Pi_\varphi(\sigma)$ that satisfy commutation relations obtained from (91) by replacing $\{\, ,\,\} \to \frac{i}{\hbar}[\, ,\,]$. Recall that we write $\hbar$ as $\hbar = b^2$ and use the rescaled fields $\phi = \frac{1}{2b}\varphi$ and $\Pi_\phi = \frac{2}{b}\Pi_\varphi$ when discussing the quantum theory. The canonical commutation relations would take the form $\big[\phi(\sigma), \phi(\sigma')\big] = 0$, $\big[\Pi_\phi(\sigma), \Pi_\phi(\sigma')\big] = 0$ and

$$\big[\phi(\sigma)\,,\, \Pi_\phi(\sigma')\big] = i\delta(\sigma - \sigma'). \tag{94}$$

The fields for nonzero time will then be given as solutions of the quantum equation of motion, which one would expect to be of the form

$$(\partial_t^2 - \partial_\sigma^2)\phi = -4\pi\mu b\; [e^{2b\phi}]_b. \tag{95}$$

The notation $[O]_b$ is supposed to indicate the quantum corrections (e.g. normal ordering, other renormalizations) that are necessary to properly define an operator $[O]_b$ that corresponds to the classical observable O upon taking the semi-classical limit $b \to 0$. We have introduced $\mu = b^{-2}\mu_c$.

There are of course many well-known subtleties associated with such a formulation: Due to short-distance singularities one can not expect $\phi(x), \Pi_\phi(x)$, $x = (t, \sigma)$ to represent well-defined operators.

One might, however, hope that the situation is better for the Fourier-modes of $\phi(\sigma)$, $\Pi_\phi(\sigma)$, which are good operators at least in the case of the free bosonic field field theory in two dimensions. The Fourier modes q, p, a_n, b_n of $\phi(\sigma)$, $\Pi_\phi(\sigma)$ will be introduced such that

$$\phi(\sigma) \;=\; \mathsf{q} + i\sum_{n\neq 0}\frac{1}{n}\big(\mathsf{a}_n e^{-in\sigma} + \mathsf{b}_n e^{in\sigma}\big) \qquad \Pi_\phi(\sigma) \;=\; 2\mathsf{p} + \sum_{n\neq 0}\big(\mathsf{a}_n e^{-in\sigma} + \mathsf{b}_n e^{in\sigma}\big). \tag{96}$$

Instead of the canonical commutation relations for $\phi(\sigma)$, $\Pi_\phi(\sigma)$ one would then consider the following commutation relations:

$$[\mathsf{p}, \mathsf{q}] = -\tfrac{i}{2} \qquad [\mathsf{a}_n\,, \mathsf{a}_m] = \tfrac{n}{2}\delta_{n,-m} \qquad [\mathsf{b}_n\,, \mathsf{b}_m] = \tfrac{n}{2}\delta_{n,-m}, \tag{97}$$

together with the hermiticity relations

$$\mathsf{q}^\dagger = \mathsf{q} \qquad \mathsf{p}^\dagger = \mathsf{p}, \qquad \mathsf{a}_n^\dagger = \mathsf{a}_{-n}, \qquad \mathsf{b}_n^\dagger = \mathsf{b}_{-n}. \tag{98}$$

The Liouville Hilbert space $\mathcal{H}$ would be required to form a representation of the commutation relations (97), with dynamics being generated by a Hamiltonian of the form

$$\mathsf{H} = 2\mathsf{p}^2 + 2\sum_{k>0}[\mathsf{a}_{-k}\mathsf{a}_k + \mathsf{b}_{-k}\mathsf{b}_k]_b \;+\; \mu\int_0^{2\pi} d\sigma\; [e^{2b\phi(\sigma)}]_b. \tag{99}$$

8.3. **Representation?**

But how to choose $\mathcal{H}$? The most natural choice might seem to be $\mathcal{H}^{\mathrm{F}} = L^2(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{F}$, where the zero modes q and p are realized on $L^2(\mathbb{R})$ as multiplication operator and $-\frac{i}{2}\partial_q$ respectively, and $\mathcal{F} \otimes \mathcal{F}$ is the Fock-space generated by acting with the oscillators a_{-n}, b_{-n}, $n > 0$ on the Fock-vacuum Ω. But it is well-know that there exist many inequivalent unitary representations of the canonical commutation relations. A large class of such representations may e.g. be obtained by improper Bogoliubov transformations of the Fock-space representation [Bz]. Moreover, for massive quantum field theories it is known to be impossible to define an interacting quantum dynamics when using a Fock-space representation for the canonical commutation relations (see e.g. [Ha], p.55).

This issue is of course closely related to the problem of defining the interaction term $\mu\int_0^{2\pi} d\sigma\ \big[e^{2b\phi(\sigma)}\big]_b$. How to choose the ordering of the a_{-n}, b_{-n}? Are there other quantum corrections (renormalizations of parameters, counterterms) necessary to define H? It could also be that there are many ways to define $(\mathcal{H},\mathsf{H})$ which would all represent canonical quantizations of Liouville theory in the sense of the previous subsection. Here, however, we are looking for a very particular one: $\mathcal{H}$ should form a representation of two commuting Virasoro algebras and $\mathsf{H} = \mathsf{L}_0 + \bar{\mathsf{L}}_0$. If conformal invariance is taken as the primary requirement, it is not clear whether it is possible to insist on the strict interpretation of "the rules of canonical quantization". It may become necessary to consider weak forms of the above requirement like only demanding that the canonical commutation relations hold between a dense set of states in $\mathcal{H}$.

To complete the confusion, let us note that it is not clear what the precise sense of the hermiticity relations (98) should be. For example, p might be symmetric, but not self-adjoint: Just think of quantum mechanics on the half-line, where p^2 can be made self-adjoint, but p can't (otherwise one could "leave" the half-line by means of some translation $e^{it\mathsf{p}}$). It could even be much worse: The operators q, p, a_n, b_n might be too singular to have a dense domain of definition.

REMARK 4. — For readers considering such discussions as pedantic let us make the following comment: A string background can abstractly be defined as BRST-cohomology of a collection of conformal field theories with total central charge 26 or 10, tensored with ghosts. Scattering amplitudes are constructed from correlation functions of the conformal field theories. One would of course like to have an interpretation of the amplitudes as coming from the perturbative expansion of a string field theory around some target space that is supposed to be described by the collection of conformal field theories. What is the target space to a given collection of abstractly defined conformal field theories? One might try to get coordinates for the target space from the operators that describe the motion of the center of mass of the string, the zero modes. But what if these operators cease to be well-defined in the case of interacting conformal field theories? This could indicate some obstruction to point-like localization in target space, some "fuzziness", "stringy uncertainty" or "non-commutativity of target space".

9. CANONICAL QUANTIZATION FROM DOZZ-PROPOSAL?

Let us now examine to what extend one may view the DOZZ-proposal as providing a canonical quantization of Liouville theory. The first task is of course to reconstruct the Liouville field itself. The identification $V_\alpha = e^{2\alpha\varphi}$ that one has semi-classically suggests that $\mathsf{V}_\alpha(x) = [e^{2\alpha\phi}]_b(x)$, $x=(t,\sigma)$, and furthermore

$$(100)\qquad \phi(\sigma) \,=\, \tfrac{1}{2}\partial_\alpha \mathsf{V}_\alpha(\sigma)|_{\alpha=0}, \qquad \Pi_\phi(\sigma) \,=\, \tfrac{1}{4\pi}\partial_t\big(\partial_\alpha\mathsf{V}_\alpha(t,\sigma)|_{\alpha=0}\big)_{t=0}.$$

We will see that this definition produces fields $\phi(\sigma)$, $\Pi_\phi(\sigma)$ that can be shown to represent the canonical commutation relations in a weak sense i.e. between states from a dense subset of $\mathcal{H}$. Moreover, the *euclidean* field $\phi(z,\bar{z})$ weakly solves a natural quantum version of the Liouville equation of motion.

9.1. **Canonical fields**

First of all, we need to recover fields $\mathsf{V}_\alpha(\tau,\sigma)$ on the euclidean cylinder from the fields $\mathsf{V}_\alpha(z,\bar{z})$ on the Riemann sphere that are furnished by the DOZZ-proposal. Fields $\mathsf{V}_\alpha(\tau,\sigma)$, on the euclidean cylinder are recovered from $\mathsf{V}_\alpha(z,\bar{z})$ by means of the conformal mapping $z=e^{\tau+i\sigma}$: $\mathsf{V}_\alpha(\tau,\sigma)=|z|^{2\Delta_\alpha}\mathsf{V}_\alpha(z,\bar{z})$. The fields $\mathsf{V}_\alpha(\tau,\sigma)$ are well-defined as operators for negative euclidean time $\tau<0$. We may then define euclidean fields $\phi(\tau,\sigma)$, $\Pi_\phi(\tau,\sigma)$ as follows

$$\phi(\tau,\sigma)\;=\;\tfrac{1}{2}\partial_\alpha\mathsf{V}_\alpha(\tau,\sigma)|_{\alpha=0},\qquad \Pi_\phi(\tau,\sigma)\;=\;\tfrac{i}{4\pi}\partial_\tau\big(\partial_\alpha\mathsf{V}_\alpha(\tau,\sigma)|_{\alpha=0}\big). \tag{101}$$

Matrix elements such as $\langle\psi_2|\phi(\tau_2,\sigma_2)\phi(\tau_1,\sigma_1)|\psi_1\rangle$, $\langle\psi_2|\Pi_\phi(\tau_2,\sigma_2)\phi(\tau_1,\sigma_1)|\psi_1\rangle$ etc. will be well-defined for $\tau_2>\tau_1$ and can be recovered from the matrix elements $\langle\psi_2|\mathsf{V}_{\alpha_2}(z_2,\bar{z}_2)\mathsf{V}_{\alpha_1}(z_1,\bar{z}_1)|\psi_1\rangle$.

Moreover, it is possible to recover the matrix elements of the Liouville field from the DOZZ three point function: Let us start by considering $\langle P|\phi(\tau,\sigma)|\psi\rangle$, where $|\psi\rangle$ is of the form $|\psi\rangle=\frac{1}{2\pi}\int_{\mathbb{R}^+}dP\psi(P)|P\rangle$. This matrix element is by our definition (101) represented as

$$\langle P|\phi(\tau,\sigma)|\psi\rangle\;=\;\frac{1}{4}\lim_{\alpha\to 0}\;\partial_\alpha\int\limits_{-\infty}^{\infty}\frac{dP'}{2\pi}\;C\big(\tfrac{Q}{2}-iP,\alpha,\tfrac{Q}{2}+iP'\big)e^{2\tau(\Delta(P)-\Delta(P'))}\;\psi(P'),$$

Taking the limit requires some care since the contour of integration will be pinched between poles approaching the real axis in the limit $\alpha\to 0$ (cf. our discussion in Section 7). The result may be written as

$$\begin{aligned}\langle P|\phi(\tau,\sigma)|\psi\rangle=\frac{1}{4}\int\limits_{-\infty}^{\infty}\frac{dP'}{2\pi}\;e^{2\tau(\Delta(P)-\Delta(P'))}\Bigg(&\frac{\lambda^{\frac{i}{b}(P-P')}\Upsilon(-2iP')\Upsilon(+2iP)}{|\Upsilon(i(P-P'))\Upsilon(i(P+P'))|^2}\;\psi(P')\\ &-\bigg(\frac{1}{(P-P')^2}+\frac{1}{(P+P')^2}\bigg)\psi(P)\Bigg),\end{aligned} \tag{102}$$

where $\lambda\equiv\pi\mu\gamma(b^2)b^{2-2b^2}$. Vacuum expectation values of arbitrary descendants of $\langle P_2|$ and $|\psi\rangle$ can then be obtained by using the commutation relations

$$\begin{aligned}[\mathsf{L}_n,\phi(w,\bar{w})]\;&=e^{nw}\big(\partial_w\phi(w,\bar{w})+Qn\big),\\ [\bar{\mathsf{L}}_n,\phi(w,\bar{w})]\;&=e^{n\bar{w}}\big(\partial_{\bar{w}}\phi(w,\bar{w})+Qn\big),\end{aligned} \tag{103}$$

where $w=\tau+i\sigma$, which follow from those for $V_\alpha(z,\bar{z})$.

One may then define operator-valued distributions $\mathsf{V}_\alpha(\sigma)$ at time $t=0$ by taking $\tau\uparrow 0$, in other words: Smeared operators $\mathsf{V}_\alpha(f)$ are obtained as

$$\mathsf{V}_\alpha(f)\;=\;\lim_{\tau\uparrow 0}\int_0^{2\pi}d\sigma\;f(\sigma)\mathsf{V}_\alpha(\tau,\sigma),\qquad f\in\mathcal{C}^\infty(S^1). \tag{104}$$

In a similar way one recovers operator-valued distributions $\phi(\sigma)$, $\Pi_\phi(\sigma)$ from $\phi(\tau,\sigma)$, $\Pi_\phi(\tau,\sigma)$.

REMARK 5. — It looks likely that the zero mode operator $\mathsf{q}\equiv\lim_{\tau\uparrow 0}\int_0^{2\pi}d\sigma\phi(\tau,\sigma)$ is at least densely defined, as the norm $||\mathsf{q}|\psi\rangle||^2$ would be given by the following expression

$$||\,\mathsf{q}|\psi\rangle\,||^2\;=\;\lim_{\tau_2\downarrow 0}\lim_{\tau_1\uparrow 0}\int_0^{2\pi}d\sigma_2 d\sigma_1\;\langle\psi|\phi(\tau_2,\sigma_2)\phi(\tau_1,\sigma_1)|\psi\rangle, \tag{105}$$

which should be finite for states $|\psi\rangle$ which are annihilated by some power of L_n for any n. However, the resulting description for q will be complicated, making it difficult to control properties such as self-adjointness, spectrum etc..

9.2. **Equation of motion**

Again start by considering $|\psi\rangle$ of the form $|\psi\rangle = \int_{\mathbb{R}^+} dP\,\psi(P)|P\rangle$. The relation

$$\partial_w \partial_{\bar{w}} \langle P|\phi(w,\bar{w})|\psi\rangle \;=\; \pi\mu\gamma(b^2) b^{2-2b^2} \frac{\Upsilon_0}{\Upsilon(2b)} \langle P|\mathsf{V}_b(w,\bar{w})|\psi\rangle$$

follows from (102) and the DOZZ-formula for the matrix elements of V_b by a straightforward calculation using the functional equations for the Υ- and Γ-functions. Note furthermore that

$$\Upsilon(2b) = b^{1-2b^2}\gamma(b^2)\Upsilon(b) = b^{1-2b^2}\gamma(b^2) \lim_{\epsilon\to 0} b^{1-2b\epsilon} \frac{\Gamma(b\epsilon)}{\Gamma(1-b\epsilon)} \Upsilon(\epsilon) = b^{1-2b^2}\gamma(b^2)\Upsilon_0.$$

Since $\partial_w\partial_{\bar{w}}\phi(w,\bar{w})$ and $V_b(w,\bar{w})$ transform the same way under Virasoro transformations, one obtains the relation

$$\partial_w \partial_{\bar{w}} \langle P,\zeta|\phi(w,\bar{w})|\psi\rangle \;=\; \pi\mu b \langle P,\zeta|\mathsf{V}_b(w,\bar{w})|\psi\rangle$$

for all $|\psi\rangle$ of the form $|\psi\rangle = \sum_{\zeta\in\mathcal{B}^{\otimes 2}} \int_0^\infty dP\,\psi_\zeta(P)|P,\zeta\rangle$ with $\psi_\zeta(P)$ nonzero for finitely many ζ only.

9.3. **Canonical commutation relations**

In the present subsection it will be proved that

$$\langle P_2,\zeta_2|[\phi(\sigma),\partial_t\phi(\sigma')]|P_1,\zeta_1\rangle \;=\; i\,(2\pi)^2\,\delta(P_2-P_1)\delta(\sigma-\sigma')\,\big(\zeta_2,\zeta_1\big)_{\frac{Q}{2}+iP_1}. \tag{106}$$

Let us note that this result will essentially be a consequence of locality and crossing symmetry of the four point functions. We will begin by considering the distribution

$$D_{P_2,P_1}(\sigma,\sigma') \equiv \langle P_2|[\phi(\sigma),\partial_t\phi(\sigma')]|P_1\rangle.$$

The distribution $D_{P_2,P_1}(\sigma,\sigma')$ should be given in terms of euclidean correlation functions $E_{P_2P_1}(z,w) = \langle P_2|\phi(z,\bar{z})\phi(e^\tau w, e^\tau\bar{w})|P_1\rangle$ as

$$D_{P_2,P_1}(\sigma,\sigma') = \lim_{\tau\uparrow 0} i\partial_\tau E_{P_2P_1}(z,w_\tau) - \lim_{\tau\downarrow 0} i\partial_\tau E_{P_2P_1}(w_\tau,z),$$

where $z=e^{i\sigma}$, $w_\tau = e^{\tau+i\sigma'}$. The correlation functions $E_{P_2P_1}(z,w)$ may be represented as

$$E_{P_2P_1}(z,w) = \lim_{\alpha_1\to 0}\lim_{\alpha_2\to 0} \frac{1}{4}\partial_{\alpha_2}\partial_{\alpha_1} F^{\alpha_2\alpha_1}_{P_2P_1}(z,w),$$

where $F^{\alpha_2\alpha_1}_{P_2P_1}(z,w) \equiv \langle P_2|V_{\alpha_1}(z,\bar{z})V_{\alpha_2}(w,\bar{w})|P_1\rangle$. It turns out to be useful to employ the expression for the correlator F in terms of t-channel conformal blocks. For real α_1,α_2 near zero one may write the expansion of F into conformal blocks as (cf. Section 7)

$$\begin{aligned} F^{\alpha_2\alpha_1}_{P_2P_1}(z,w) = \int\limits_{\mathbb{S}} \frac{d\beta_t}{2\pi}\, C(\bar{\beta}_t,\alpha_2,\alpha_1)C(\beta_2,\beta_t,\beta_1)\,\Big|\mathcal{F}^t_{\beta_t}\big[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \beta_2 & \beta_1\end{smallmatrix}\big](z,w)\Big|^2 \\ +C(\beta_2,\alpha_{21},\beta_1)\Big|\mathcal{F}^t_{\alpha_{21}}\big[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \beta_2 & \beta_1\end{smallmatrix}\big](z,w)\Big|^2, \end{aligned} \tag{107}$$

where $\beta_2 = \frac{Q}{2} - iP_2$, $\beta_1 = \frac{Q}{2} + iP_1$ and $\alpha_{21} = \alpha_2 + \alpha_1$.

Let us note that $F^{\alpha_2\alpha_1}_{P_2P_1}(z,w) = F^{\alpha_1\alpha_2}_{P_2P_1}(w,z)$ (locality of the V_α) implies $E_{P_2P_1}(z,w) = E_{P_2P_1}(w,z)$. This implies that the distribution $D_{P_2P_1}(\sigma,\sigma')$ can have support for $\sigma = \sigma'$ only. Let us therefore focus on the singular behavior of $E_{P_2P_1}(z,w) = E_{P_2P_1}(w,z)$ for $z = w$. $E_{P_2P_1}(z,w)$ can be expanded as $E_{P_2P_1}(z,w) = E^c_{P_2P_1}(z,w) + E^d_{P_2P_1}(z,w)$, where

$$\begin{aligned} E^c_{P_2P_1}(z,w) \;&\equiv \frac{1}{4}\int_{\mathbb{S}} \frac{d\beta_t}{2\pi}\; C(\beta_2,\beta_t,\beta_1)\; \partial_{\alpha_2}\partial_{\alpha_1} C(\bar\beta_t,\alpha_2,\alpha_1)_{\substack{\alpha_1=0\\ \alpha_2=0}} \left|\mathcal{F}^t_{\beta_t}\left[\begin{smallmatrix} 0 & 0\\ \beta_2 & \beta_1\end{smallmatrix}\right](z,w)\right|^2,\\ E^d_{P_2P_1}(z,w) \;&\equiv \frac{1}{4}\partial_{\alpha_2}\partial_{\alpha_1}\big(C(\beta_2,\alpha_2+\alpha_1,\beta_1)|\mathcal{F}^t_{\alpha_{21}}\left[\begin{smallmatrix} \alpha_2 & \alpha_1\\ \beta_2 & \beta_1\end{smallmatrix}\right](z,w)|^2\big)_{\substack{\alpha_1=0\\ \alpha_2=0}}. \end{aligned}$$

We have simplified the expression for E^c slightly by noting that terms where not both derivatives w.r.t. α_2 and α_1 act on $C(\bar\beta_t,\alpha_2,\alpha_1)$ vanish when taking $\alpha_1 \to 0$ and $\alpha_2 \to 0$.

There are two types of singular behavior that one must consider: Power-like behavior of the form $|z-w|^{2\lambda}$ with *positive* λ, and logarithmic behavior of the form $\log|z-w|^2$. It is straightforward to verify that the former does not produce contributions to $D_{P_2P_1}(\sigma,\sigma')$. The logarithmic short-distance singularities come from $E^d_{P_2P_1}(z,w)$ only. They are produced when the derivatives $\partial_{\alpha_2}\partial_{\alpha_1}$ both act on the factor $|z-w|^{-4\alpha_1\alpha_2}$ in $|\mathcal{F}^t_{\alpha_{21}}|^2$. By observing that $\lim_{\alpha\to 0} C(\beta_2,\alpha,\beta_1) = 2\pi\delta(P_2 - P_1)$ and furthermore

$$\lim_{\tau\to 0}\partial_\tau\big(\ln|z-e^{-\tau}w|^2 + \ln|z-e^{\tau}w|^2\big) = \lim_{\tau\to 0}\frac{4\tau}{\tau^2+(\sigma-\sigma')^2} = 4\pi\delta(\sigma-\sigma'),$$

one finds that

$$\langle P_2|[\phi(\sigma),\partial_t\phi(\sigma')]|P_1\rangle \;=\; i\,(2\pi)^2\,\delta(P_2-P_1)\delta(\sigma-\sigma'). \tag{108}$$

This argument can easily be generalized to descendants of $\langle P_2|$, $|P_1\rangle$.

10. ZERO MODE SCHRÖDINGER REPRESENTATION

Meine Sätze erläutern dadurch, daß sie der, welcher mich versteht, am Ende als unsinnig erkennt, wenn er durch sie - auf ihnen - über sie hinausgestiegen ist. (Er muß sozusagen die Leiter wegwerfen, nachdem er auf ihr heraufgestiegen ist.) (L. Wittgenstein)

We are now going to discuss an assumption concerning the representation of the operators q, p, a_n, b_n that would lead to a representation of states by wave-functions on target-space: Assume that

$$\mathcal{H} \;\simeq\; \mathcal{H}^{\rm Schr} \;=\; L^2(\mathbb{R})\otimes\mathcal{F}\otimes\mathcal{F} \;\simeq\; \int_{\mathbb{R}}^{\oplus} dq\;\mathcal{F}_q\otimes\mathcal{F}_q, \tag{109}$$

where q and p are represented on functions $\psi(q)\in L^2(\mathbb{R})$ as the operator of multiplication with q and the operator $-\frac{i}{2}\partial_q$ respectively, and the nonzero modes a_n, b_n are represented in $\mathcal{F}\otimes\mathcal{F}$ by a standard Fock-representation generated from a vector $\Omega\in\mathcal{F}\otimes\mathcal{F}$ that satisfies $\mathsf{a}_n\Omega = 0 = \mathsf{b}_n\Omega$.

Within such a Schrödinger representation for the zero mode one could represent states by wave-functions $\psi(q)$ that take values in $\mathcal{F}\otimes\mathcal{F}$. The scalar product would be represented as

$$\langle\psi_2,\psi_1\rangle_{\mathcal{H}} \;=\; \int_{\mathbb{R}} dq\;\big(\psi_2(q),\psi_1(q)\big)_{\mathcal{F}\otimes\mathcal{F}}, \tag{110}$$

where $(.,.)_{\mathcal{F}\otimes\mathcal{F}}$ denotes the scalar product in $\mathcal{F}\otimes\mathcal{F}$.

Our assumption (109) has fair chances to be wrong (cf. our remarks in Subsections 8.3 and 9.1, as well as the discussion in Section 12 below). Nevertheless it will help us to develop a certain picture for the "target-space" physics of Liouville theory. Many features of that picture turn to be consistent with the DOZZ proposal in a rather nontrivial way. Let us therefore adopt (109) as a working hypothesis that is useful to discuss certain issues, but which will ultimately have to be replaced by some refined description.

10.1. **Asymptotic correspondence to the free field**

In the representation (109) one will find the Hamiltonian H to be represented as a second order differential operator of the form

$$\mathsf{H} = -\frac{1}{2}\partial_q^2 + N_b + 2\sum_{k>0}(\mathsf{a}_{-k}\mathsf{a}_k + \mathsf{b}_{-k}\mathsf{b}_k)\ +\ \mu\int_0^{2\pi} d\sigma\ [e^{2b\varphi(\sigma)}]_b, \tag{111}$$

where N_b is some normal-ordering constant. At least semi-classically one has $\mu\int_0^{2\pi} d\sigma\ [e^{2b\varphi(\sigma)}]_b \propto e^{2bq}$, which vanishes exponentially for $q\to-\infty$. It seems plausible to conjecture that quantum corrections in the interaction term $[e^{2b\varphi(\sigma)}]_b$ will preserve the zero mode dependence $\propto e^{2bq}$, at least in leading order for $q\to-\infty$. The role of the interaction will therefore become negligible if one considers wave-packets that have support in regions with large negative values of q. It should be possible to approximate the time-evolution of such wave-packets by the time-evolution generated by the free Hamiltonian

$$\mathsf{H}^{\mathrm{F}} = -\frac{1}{2}\partial_q^2 + N_b + 2\sum_{k>0}(\mathsf{a}_{-k}\mathsf{a}_k + \mathsf{b}_{-k}\mathsf{b}_k). \tag{112}$$

In this spirit one would also expect to have

$$[e^{2b\phi(\sigma)}]_b \underset{q\to-\infty}{\sim} :e^{2b\phi(\sigma)}:, \tag{113}$$

where $:\mathcal{O}:$ denotes the operator obtained by the usual free field normal ordering. Next-to-leading order corrections in the asymptotics for $q\to-\infty$ should then be represented by

$$\mathsf{H}^{\mathrm{F}}_{\mu} = -\frac{1}{2}\partial_q^2 + N_b + 2\sum_{k>0}(\mathsf{a}_{-k}\mathsf{a}_k + \mathsf{b}_{-k}\mathsf{b}_k)\ +\ \mu\int_0^{2\pi} d\sigma :e^{2b\varphi(\sigma)}: . \tag{114}$$

10.2. **Generalized eigenfunctions of $\mathcal{H}$**

Let us now examine how (generalized) eigenstates of H would be described in the representation (109). One would want to construct generalized eigenfunctions $\psi_{E,\nu}(q)$ to each eigenstate $|E,\nu\rangle$ of H, where ν is just some label for the degeneracy of the eigenvalue E for the moment. This is of course impossible unless one knows the precise definition for H. However, if the asymptotic correspondence with free field theory, as discussed in the previous subsection, really holds, one may at least discuss the asymptotic behavior of $\psi_{E,\nu}(q)$ for $q\to-\infty$, which is enough to get some important information on the spectrum.

The asymptotic behavior of $\psi_{E,\nu}(q)$ for $q \to -\infty$ should then of course be given by solutions $\psi_E^{\rm F}(q)$ of the free eigenvalue equation $\mathsf{H}^{\rm F}\psi_E^{\rm F}(q) = E\psi_E^{\rm F}(q)$, which take the form

$$\psi_E^{\rm F}(q) \;=\; e^{2iPq}{\rm f}_n^+ \;+\; e^{-2iPq}{\rm f}_n^-, \qquad E = P^2 + N_b + n, \tag{115}$$

where ${\rm f}_n^\pm \in \mathcal{F}\otimes\mathcal{F}$ are eigenstates of the number operator $\mathsf{N} = 2\sum_{k>0}(\mathsf{a}_{-k}\mathsf{a}_k + \mathsf{b}_{-k}\mathsf{b}_k)$ which have the eigenvalue n. Wave-functions $\psi_{E,\nu}(q)$ with asymptotic behavior (115) would correspond to generalized eigenstates in the continuous spectrum $\mathcal{H}^c$ of H. However, a priori it is not at all clear for which choices for the parameters P, ${\rm f}_n^+$, ${\rm f}_n^-$ it is possible to "integrate" the eigenvalue equation $\mathsf{H}\psi_E = E\psi_E$ to obtain a (plane-wave) normalizable wave-function $\psi[P,{\rm f}_n^+,{\rm f}_n^-](q)$ with asymptotics (115). But if $\mathcal{H}$ forms a representation of the canonical commutation relations for the nonzero modes one would need that existence of a wave-function $\psi[P,{\rm f}_n^+,{\rm f}_n^-](q)$ implies existence of $\psi[P,\mathsf{a}_n{\rm f}_n^+,\mathsf{a}_n{\rm f}_n^-](q)$ and $\psi[P,\mathsf{b}_n{\rm f}_n^+,\mathsf{b}_n{\rm f}_n^-](q)$. It follows that $\mathcal{H}^c$ must decompose into a collection of Fock-spaces parameterized by P:

$$\mathcal{H}^c \;\simeq\; \int_{\mathbb{R}}^{\oplus} d\mu(P)\, \mathcal{F}_P\otimes\mathcal{F}_P, \tag{116}$$

where the subscript P in the notation $\mathcal{F}_P\otimes\mathcal{F}_P$ indicates that the action of H on $\mathcal{F}_P\otimes\mathcal{F}_P$ is represented as $P^2 + N_b + \mathsf{N}$.

10.3. **Comparison with DOZZ-proposal**

This seems to be as far as one can get on the basis of the asymptotic correspondence to the free field. Let us now compare to the structure of the spectrum as given by the DOZZ-proposal. First of all, the latter is purely continuous. One would therefore need to identify $\mathcal{H}^c \equiv \mathcal{H}$. Second, instead of (116) we had found in Part I a similar expansion with Fock-spaces $\mathcal{F}_P$ replaced by Verma modules $\mathcal{V}_P$. But this is of course perfectly consistent since $\mathcal{F}_P \simeq \mathcal{V}_P$ as vector spaces (we will discuss the realization of conformal symmetry later).

The interesting point to observe is that comparison with (59) implies that $\mathbb{S} = \mathbb{R}^+$, only *half* of the spectrum of free field theory. This also implies that for each value of P it suffices to specify e.g. the vector ${\rm f}_n^+$ in (115). The vector ${\rm f}_n^-$ must be a function of ${\rm f}_n^+$ and P, ${\rm f}_n^- = \mathsf{R}(P){\rm f}_n^+$. This is very plausible from the point of view of the "quasi-quantum mechanical" picture that we are developing: Whatever possible quantum modifications of the potential may be, as long as they do not make it vanish in the opposite limit $q \to +\infty$, one will have to impose a boundary condition concerning the behavior of wave-functions for $q \to +\infty$ which selects a particular choice of ${\rm f}_n^-$ in dependence of ${\rm f}_n^+$ and P (or vice versa).

The resulting picture is that the wave-functions $\psi_{P,{\rm f}}(q)$ corresponding to generalized eigenstates $|P,{\rm f}\rangle$ are uniquely specified by the coefficient ${\rm f} \in \mathcal{F}\otimes\mathcal{F} \simeq \mathcal{V}_P\otimes\mathcal{V}_P$ of the "in-going" plane wave e^{2iPq}, with coefficient of the "reflected out-going" plane wave e^{-2iPq} being given by the *reflection operator* $\mathsf{R}(P)$:

$$\psi_{P,{\rm f}}(q) \underset{q\to-\infty}{\sim} \psi_{P,{\rm f}}^{\rm F}(q), \qquad \psi_{P,{\rm f}}^{\rm F}(q) \;=\; e^{2iPq}{\rm f} \;+\; e^{-2iPq}\mathsf{R}(P){\rm f}. \tag{117}$$

The wave-functions $\psi_{P,\mathrm{f}}(q)$ describe the generalized Fourier transformation from the zero mode Schrödinger representation (109) to the spectral representation

$$\mathcal{H} \simeq \mathcal{H}^{\mathrm{spec}} = \int_{\mathbb{R}+}^{\oplus} \frac{dP}{2\pi}\, \mathcal{F}_P \otimes \mathcal{F}_P. \tag{118}$$

Let us finally consider the wave-function $\psi_P(q) \equiv \psi_{P,\Omega}(q)$. For both terms in (117) to produce the same eigenvalue of $\mathsf{H} \sim \mathsf{H}^{\mathrm{F}}$ one needs to have $\mathsf{R}(P)\Omega = R(P)\Omega$, introducing the *reflection amplitude* $R(P)$. Therefore

$$\psi_P(q) \underset{q\to-\infty}{\sim} \psi_P^{\mathrm{F}}(q), \qquad \psi_P^{\mathrm{F}}(q) = (e^{2iPq} + e^{-2iPq}R(P))\Omega. \tag{119}$$

10.4. **Reflection in the potential**

We will now try to clarify the physical interpretation of $\mathsf{R}(P)$: We claim that $\mathsf{R}(P)$ represents the scattering operator that describes how a wave-packet coming in from $q \to -\infty$ for $t \to -\infty$ is reflected into another wave packet that is pushed out to $q \to -\infty$ for $t \to \infty$. The Liouville interaction acts like a perfectly reflecting potential "wall". To verify this statement let us consider a wave-packet

$$\psi(q,t) = e^{-i\mathsf{H}t}\psi(q) = \sum_{\mathrm{f}\in\mathrm{B}} \int_0^{\infty} \frac{dP}{2\pi}\, e^{-iE_{P,\mathrm{f}}t} \,\langle P,\mathrm{f}|\psi\rangle\, \psi_{P,\mathrm{f}}(q),$$

where we assume that the orthonormal basis B for $\mathcal{F}\otimes\mathcal{F}$ was chosen such that all $\mathrm{f} \in \mathrm{B}$ are eigenstates of the number operator N with eigenvalue $N(\mathrm{f})$, so that $\mathsf{H}|P,\mathrm{f}\rangle = E_{P,\mathrm{f}}|P,\mathrm{f}\rangle$ with $E_{P,\mathrm{f}} \equiv 2P^2 + \frac{Q^2}{2} + N(\mathrm{f})$. By the method of stationary phase it is possible to see that $\psi(q,t)$ will vanish for any finite q as $t \to -\infty$, the wave-packet is pushed out to large negative values of q: In order to pick out the mode with energy $2P^2 + Q^2/2$ as the dominant saddle point contribution one would have to consider $\psi(2Pt,t)$.

Since the wave packet is asymptotically supported for negative infinite values of q, one may approximate the $\psi_{P,\mathrm{f}}(q)$ by their asymptotic behavior $e^{2iPq}\mathrm{f}$ (the term with e^{-2iPq} gets suppressed in this limit). One is thereby led to the conclusion that the behavior of $\psi(q,t)$ for $t \to -\infty$ will be represented as time evolution according to the free Hamiltonian H^{F}, $\psi(q,t) \sim e^{-i\mathsf{H}^{\mathrm{F}}t}\psi^{\mathrm{in}}(q)$ with

$$\psi^{\mathrm{in}}(q) = \sum_{\mathrm{f}\in\mathrm{B}} \int_0^{\infty} \frac{dP}{2\pi}\, \langle P,\mathrm{f}|\psi\rangle\, e^{2iPq}\,\mathrm{f} \quad \in \quad \mathcal{H}^{\mathrm{F}} \equiv L^2(\mathbb{R})\otimes\mathcal{F}\otimes\mathcal{F} \tag{120}$$

In the other limit $t \to +\infty$ one would similarly find $\psi(q,t) \sim e^{-i\mathsf{H}^{\mathrm{F}}t}\psi^{\mathrm{out}}(q)$ with

$$\psi^{\mathrm{out}}(q) = \sum_{\mathrm{f}\in\mathrm{B}} \int_0^{\infty} \frac{dP}{2\pi}\, \langle P,\mathrm{f}|\psi\rangle\, e^{-2iPq}\,\mathsf{R}(P)\mathrm{f} \quad \in \quad \mathcal{H}^{\mathrm{F}} \tag{121}$$

One may then define *generalized wave operators* as

$$\mathsf{W}^{\pm}(\mathsf{H}_0,\mathsf{H},\mathsf{J}^{\pm}) = \lim_{t\to\pm\infty} e^{i\mathsf{H}_0t}\,\mathsf{J}^{\pm}\,e^{-i\mathsf{H}t},$$

where the identification maps $\mathsf{J}^{\pm} : \mathcal{H} \to \mathcal{H}^{\mathrm{F}}$ are defined by $\mathsf{J}^{\pm}\psi_{P,\mathrm{f}}(q) = e^{\mp 2iPq}\mathrm{f}$, and the corresponding scattering operator $\mathsf{S} = (\Omega^-)^{-1}\Omega^+ : \mathcal{H}^{\mathrm{F}} \to \mathcal{H}^{\mathrm{F}}$ is easily read off from (120) and (121) to be represented by our stationary reflection operator $\mathsf{R}(P)$. For this identification to be consistent we evidently need that $\mathsf{R}(P)$ is *unitary*.

10.5. **Conformal symmetry in the quantum theory**

Let us observe that the asymptotic correspondence between Liouville field and a free field that was discussed in Subsection 10.1 would suffice to conclude that quantum Liouville theory becomes a conformal field theory upon choosing the normal ordering constant N_b appropriately:

If the spectral decomposition of the Hamiltonian can be represented as in (118) one may always introduce an action of two commuting copies of the Virasoro algebra on $\mathcal{H}$ by using usual free field representations on the Fock-spaces $\mathcal{F}_P$. The action on the first tensor factor of $\mathcal{F}_P \otimes \mathcal{F}_P$ would be defined in terms of the generators

$$
(122) \qquad \begin{aligned} \mathsf{L}_n^{\mathrm{F}}(P) \; &= (2P + inQ)\mathsf{a}_n + \sum_{k\neq 0,n} \mathsf{a}_k \mathsf{a}_{n-k}, \qquad n \neq 0, \\ \mathsf{L}_0^{\mathrm{F}}(P) \; &= P^2 + \frac{Q^2}{4} + 2\sum_{k>0} \mathsf{a}_{-k}\mathsf{a}_k, \end{aligned}
$$

whereas the action on the second tensor factor is generated by operators $\bar{\mathsf{L}}_n$ that are obtained by replacing $\mathsf{a}_n \to \mathsf{b}_n$ in the expressions (122). The spectral decomposition (118) then allows us to define operators L_n, $\bar{\mathsf{L}}_n$ on $\mathcal{H}$ by means of the relations

$$
(123) \qquad \mathsf{L}_n \, |P, \mathrm{f}\rangle = |P, \mathsf{L}_n^{\mathrm{F}}(P)\mathrm{f}\rangle \qquad\qquad \bar{\mathsf{L}}_n \, |P, \mathrm{f}\rangle = |P, \bar{\mathsf{L}}_n^{\mathrm{F}}(P)\mathrm{f}\rangle.
$$

The operators L_n, $\bar{\mathsf{L}}_n$ satisfy the usual commutation relations of the Virasoro algebra with central charge c given in terms of the parameter b by the relations

$$
(124) \qquad c \;=\; 1 + 6Q^2, \qquad Q = b + b^{-1}.
$$

However, the crucial property for identifying conformal transformations as a symmetry of the theory is the fact that the Hamiltonian is recovered from L_0, $\bar{\mathsf{L}}_0$ as

$$
(125) \qquad \mathsf{H} \;=\; \mathsf{L}_0 + \bar{\mathsf{L}}_0.
$$

For this to be the case one just needs that the normal ordering constant N_b is equal to $\frac{Q^2}{2}$. Taking into account that the Fock space representation $\mathcal{F}_P$ is irreducible for real values of P [Fr], one may identify the spectral representation (118) of H with the decomposition of $\mathcal{H}$ into irreducible representations $\equiv$ Verma modules $\mathcal{V}_P \otimes \mathcal{V}_P$ of the Virasoro algebra. Under this correspondence one identifies the highest weight state $v \otimes v$ of $\mathcal{V}_P \otimes \mathcal{V}_P$ with $|P\rangle \equiv |P, \Omega\rangle \equiv |P, v \otimes v\rangle$ corresponding to the wave-function $\psi_P(q)$ with asymptotics (119).

10.6. **Consistency of conformal symmetry with reflection**

The next thing one needs to observe, however, is the fact that the requirement of having a consistent realization of conformal symmetry in the Schrödinger representation $\mathcal{H}^{\mathrm{Schr}}$ imposes strong constraints on the form of the reflection operator $\mathsf{R}(P)$. If one considers wave-packets supported for $q \to -\infty$, one finds that consistency of (123),(122) with the asymptotic form (117) of wave-functions requires that

$$
\text{(126)} \qquad \mathsf{L}_n \underset{q\to-\infty}{\sim} \mathsf{L}_n^{\mathrm{F}}, \quad \text{where} \qquad \begin{aligned} \mathsf{L}_n^{\mathrm{F}} &= (-i\partial_q + inQ)\mathsf{a}_n + \sum_{k\neq 0,n} \mathsf{a}_k \mathsf{a}_{n-k} \\ \mathsf{L}_0^{\mathrm{F}} &= \frac{1}{4}(-\partial_q^2 + Q^2) + 2\sum_{k>0} \mathsf{a}_{-k}\mathsf{a}_k. \end{aligned}
$$

However, the action of the Virasoro generators L_n on the second term in (117) may now be expressed in two ways: Either as $e^{-2iPq}\mathsf{R}(P)\mathsf{L}_n^{\mathrm{F}}(P)\mathrm{f}$ by using (123) or alternatively as $e^{-2iPq}\mathsf{L}_n^{\mathrm{F}}(-P)\mathsf{R}(P)\mathrm{f}$ when using (126) directly. We conclude that we must have

$$
\text{(127)} \qquad \mathsf{R}(P)\mathsf{L}_n^{\mathrm{F}}(P) \;=\; \mathsf{L}_n^{\mathrm{F}}(-P)\mathsf{R}(P) \qquad\qquad \mathsf{R}(P)\bar{\mathsf{L}}_n^{\mathrm{F}}(P) \;=\; \bar{\mathsf{L}}_n^{\mathrm{F}}(-P)\mathsf{R}(P),
$$

so the reflection operator $\mathsf{R}(P)$ must be an intertwining operator between the Fock-representations $\mathcal{F}_P$ and $\mathcal{F}_{-P}$.

Such an operator is uniquely determined by this intertwining property in terms of the *reflection amplitude* $R(P)$ which characterizes the action of $\mathsf{R}(P)$ on the Fock-vacuum Ω. This is easily seen by recalling that the Fock-space representation $\mathcal{F}_P \otimes \mathcal{F}_P$ is isomorphic to $\mathcal{V}_P \otimes \mathcal{V}_P$ for $P \in \mathbb{R}$. This means that any $\mathrm{f} \in \mathcal{F}_P \otimes \mathcal{F}_P$ can be uniquely written as the action of some polynomial $\mathcal{P}_{P,\mathrm{f}}$ in the variables $\mathsf{L}_n(P), \bar{\mathsf{L}}_n(P)$ on the Fock-vacuum Ω, i.e. $\mathrm{f} = \mathcal{P}_{P,\mathrm{f}}[\mathsf{L}_n^{\mathrm{F}}(P), \bar{\mathsf{L}}_n^{\mathrm{F}}(P)]\Omega$. The intertwining property (127) then implies that

$$
\mathsf{R}(P)\mathrm{f} \;=\; \mathsf{R}(P)\,\mathcal{P}_{P,\mathrm{f}}\big[\mathsf{L}_n^{\mathrm{F}}(P), \bar{\mathsf{L}}_n^{\mathrm{F}}(P)\big]\Omega \;=\; R(P)\,\mathcal{P}_{P,\mathrm{f}}\big[\mathsf{L}_n^{\mathrm{F}}(-P), \bar{\mathsf{L}}_n^{\mathrm{F}}(-P)\big]\Omega.
$$

Conformal symmetry therefore reduces the description of the scattering of wave-packets in the Liouville potential to the knowledge of a single function, the reflection amplitude $R(P)$. Let us note that $\mathsf{R}(P)$ will be unitary iff $|R(P)| = 1$.

10.7. **Macroscopic vs. microscopic states**

We had observed in Part I, Subsection 4.7 that one may naturally consider the analytic continuation of states $|P\rangle$ to complex values of P in some distributional sense. This should of course be reflected by the existence of an analytic continuation for the corresponding wave-function $\psi_P(q)$.

By means of analytic continuation one may in particular compare $\psi_P(q)$ and $\psi_{-P}(q)$: In view of the asymptotics (117) one would find $\psi_P(q) = R(P)\psi_{-P}(q)$. But this is to be compared to the reflection property $|P\rangle = S(\frac{Q}{2} + iP)|-P\rangle$ that follows from the DOZZ-proposal. We clearly must have $R(P) = S(\frac{Q}{2} + iP)$, so that

$$
\text{(128)} \qquad R(P) = -\big(\pi\mu\gamma(b^2)\big)^{-\frac{2iP}{b}} \frac{\Gamma(1+2ibP)\Gamma(1+2ib^{-1}P)}{\Gamma(1-2ibP)\Gamma(1-2ib^{-1}P)}.
$$

Our discussion in Subsection 4.7 can now easily be rephrased in terms of the asymptotic behavior of the wave-functions $\psi_P(q)$: In view of the asymptotic behavior (119) one would expect the $\psi_P(q)$ to be plane-wave normalizable for $P \in \mathbb{R}$, non-normalizable for $P \notin \mathbb{R}$. However, in the latter case one would still expect $\psi_P(q)$ to represent a distribution $|P\rangle$ that can be defined on the subspace of $\mathcal{H}$ which is represented by wave-functions $\psi(q)$ with sufficiently strong exponential decay. Let us adopt the terminology "macroscopic states" for the (plane-wave) normalizable states $|P, \mathrm{f}\rangle$ with $P \in \mathbb{R}$, and "microscopic states" for the $|P, \mathrm{f}\rangle$ with $P \notin \mathbb{R}$, as proposed by Seiberg in [Se].

The characterization of the domain of $\langle P|$, $P \notin \mathbb{R}$ in terms of the exponential decay properties of the wave-functions $\psi(q)$ can easily be translated into our previous characterization of the domain of $\langle P|$ in terms of analyticity of $\langle P|\psi\rangle$:

Consider a state $|\psi\rangle$ that is represented in the zero mode Schrödinger representation by a wave-function $\psi(q)$ which decays for $q \to \infty$ faster than $e^{2\lambda q}$ for $q \to -\infty$. The wave-functions of $|\psi\rangle$ in the spectral representation,

$$\langle P, \mathrm{f}|\psi\rangle \;=\; \int\limits_{-\infty}^{\infty} dq \; \langle \, \bar{\psi}_{P,\mathrm{f}}(q) \, , \, \psi(q) \, \rangle_{\mathcal{F}\otimes\mathcal{F}}, \tag{129}$$

will then be analytic in P as long as the integral in (129) converges. This will be the case for $|\Im(P)| < \lambda$. Conversely, if the wave-functions $\langle P, \mathrm{f}|\psi\rangle$ in the spectral representation are analytic in some strip around the real axis of width larger than λ, one may get the asymptotic behavior of

$$\psi(q) \;=\; \sum_{\mathrm{f}\in\mathrm{B}} \int\limits_{0}^{\infty} \frac{dP}{2\pi} \; \psi_{P,\mathrm{f}}(q) \; \langle P, \mathrm{f}|\psi\rangle, \tag{130}$$

by using unitarity of $\mathsf{R}(P)$ to rewrite the integral over $\mathbb{R}^+$ as an integral over the contour $\mathbb{R}$, and taking advantage of the analyticity of the integrand in (130) to shift that contour of integration to the axis $\mathbb{R} + i\lambda$: It follows that $\psi(q)$ decays faster as $e^{2\lambda q}$ for $q \to -\infty$.

10.8. **Next to leading order corrections to Virasoro generators**

Equations (126) describe the representation of the Virasoro algebra in (109) only to leading order for $q \to -\infty$. In next-to-leading order one will need corrections to (126): In order to have a consistent realization of conformal symmetry one needs to have generators $\mathsf{L}^{\mathrm{F}}_{n,\mu}$, $\bar{\mathsf{L}}^{\mathrm{F}}_{n,\mu}$ such that $\mathsf{H}^{\mathrm{F}}_{\mu} = \mathsf{L}^{\mathrm{F}}_{0,\mu} + \bar{\mathsf{L}}^{\mathrm{F}}_{0,\mu}$, where $\mathsf{H}^{\mathrm{F}}_{\mu}$ was defined in (114). And indeed, there exists such a one-parameter "deformation" of the representation (126) that preserves the commutation relations:

$$\mathsf{L}^{\mathrm{F}}_{n,\mu} \;=\; \mathsf{L}^{\mathrm{F}}_{n} + \frac{\mu}{2} \int\limits_{0}^{2\pi} d\sigma \; e^{in\sigma} \; :e^{2b\phi(\sigma)}:, \tag{131}$$

and similarly for $\bar{\mathsf{L}}^{\mathrm{F}}_{n,\mu}$. It was shown in [CT] that the modified generators $\mathsf{L}^{\mathrm{F}}_{n,\mu}$ indeed satisfy the same algebra as the $\mathsf{L}^{\mathrm{F}}_{n}$. It will be important to notice that there is a further deformation of the generators $\mathsf{L}^{\mathrm{F}}_{n,\mu}$, defined by

$$\mathsf{L}^{\mathrm{F}}_{n,\mu,\tilde{\mu}} \;=\; \mathsf{L}^{\mathrm{F}}_{n,\mu} + \frac{\tilde{\mu}}{2} \int\limits_{0}^{2\pi} d\sigma \; e^{in\sigma} \; : e^{2\tilde{b}\phi(\sigma)} :, \tag{132}$$

where $\tilde{b} = b^{-1}$, that still satisfies the same algebra as the $\mathsf{L}^{\mathrm{F}}_{n}$. This possibility is due to the fact that $: e^{2\tilde{b}\phi(\sigma)} :$ has conformal dimensions $(1,1)$ just like $: e^{2b\phi(\sigma)} :$. There is of course a very similar deformation $\bar{\mathsf{L}}^{\mathrm{F}}_{n,\mu,\tilde{\mu}}$ of the generators $\bar{\mathsf{L}}^{\mathrm{F}}_{n,\mu}$.

It seems natural to regard the term proportional to $\tilde{\mu}$ in (132) as a possible quantum correction in the definition of the Virasoro generators that preserves conformal symmetry. It would correspond to the following modification of the "perturbative" Hamiltonian $\mathsf{H}^{\mathrm{F}}_{\mu}$:

$$\mathsf{H}^{\mathrm{F}}_{\mu\tilde{\mu}} \equiv \mathsf{H}^{\mathrm{F}} + \mu\mathsf{U} + \tilde{\mu}\tilde{\mathsf{U}}, \tag{133}$$

where U and $\tilde{\mathsf{U}}$ are defined as

$$\mathsf{U} \;\equiv\; \int_0^{2\pi} d\sigma : e^{2b\phi(\sigma)} :, \qquad \tilde{\mathsf{U}} \;\equiv\; \int_0^{2\pi} d\sigma : e^{2b^{-1}\phi(\sigma)} : . \tag{134}$$

REMARK 6. — There do not seem to be any other operators that one could add to $\mathsf{L}^{\mathrm{F}}_{n,\mu,\tilde{\mu}}$ and $\bar{\mathsf{L}}^{\mathrm{F}}_{n,\mu,\tilde{\mu}}$ without destroying the Virasoro-algebra commutation relations. It is therefore tempting to identify $\mathsf{H} \equiv \mathsf{H}^{\mathrm{F}}_{\mu\tilde{\mu}}$, $\mathsf{L}_n \equiv \mathsf{L}^{\mathrm{F}}_{n,\mu,\tilde{\mu}}$ and $\bar{\mathsf{L}}_n \equiv \bar{\mathsf{L}}^{\mathrm{F}}_{n,\mu,\tilde{\mu}}$, with $\tilde{\mu}$ and μ related by the formula (32) required by the DOZZ-proposal. We will discuss later (Section 12) why such an identification appears to be problematic. However, these problems will *not* at all exclude the possibility that $\mathsf{H}^{\mathrm{F}}_{\mu\tilde{\mu}}$ represents the asymptotic behavior of H for $q \to -\infty$ up to terms that vanish faster than exponentially in that limit.

11. **EXPONENTIAL OPERATORS**

Let us now consider the exponential operators $[e^{2\alpha\phi}]_b(\sigma)$ in more detail. In the representation (109) it is of course natural to try the ansatz

$$[e^{2\alpha\phi}]_b(\sigma) \stackrel{?}{=} : e^{2\alpha\phi(\sigma)} : . \tag{135}$$

This would indeed yield local operators that transform covariantly under the Virasoro algebras generated by the $\mathsf{L}^{\mathrm{F}}_{n,\mu,\tilde{\mu}}$, $\bar{\mathsf{L}}^{\mathrm{F}}_{n,\mu,\tilde{\mu}}$,

$$\begin{aligned} \big[\mathsf{L}_{n,\mu,\tilde{\mu}}, [e^{2\alpha\phi}]_b(\sigma)\big] &= e^{+in\sigma}\big(-i\partial_\sigma + n\Delta_\alpha\big)\big)\, [e^{2\alpha\phi}]_b(\sigma), \\ \big[\bar{\mathsf{L}}_{n,\mu,\tilde{\mu}}, [e^{2\alpha\phi}]_b(\sigma)\big] &= e^{-in\sigma}\big(+i\partial_\sigma + n\Delta_\alpha\big)\big)\, [e^{2\alpha\phi}]_b(\sigma), \end{aligned} \tag{136}$$

as desired. However, as we are already expecting some trouble with the representation (109), it seems important to note that all that we will really be using in the following discussion is the assumption that

$$[e^{2\alpha\phi}]_b(\tau,\sigma) \underset{q\to-\infty}{\sim} : e^{2\alpha\phi(\tau,\sigma)} : \tag{137}$$

holds weakly (between wave-packets).

11.1. **State-operator correspondence**

We had previously (Section 4) discussed the correspondence between vertex operators V_α and microscopic states $|\alpha\rangle$ (in the distributional sense, cf. Subsection 4.7). It may be formulated as follows:

STATE-OPERATOR CORRESPONDENCE — *The vertex operators V_α are in one-to-one correspondence to the microscopic states $|\alpha\rangle$, and create these states via*

$$\begin{aligned} \langle\psi|\alpha\rangle &= \lim_{z\to 0} \langle\psi|[e^{2\alpha\phi}]_b(z,\bar{z})|0\rangle \\ \langle Q-\alpha|\psi\rangle &= \lim_{z\to\infty} \langle 0|[e^{2\alpha\phi}]_b(z,\bar{z})|\psi\rangle\, |z|^{4\Delta_\alpha} \end{aligned} \qquad \textit{for} \quad \psi \in \mathcal{D}_\alpha, \tag{138}$$

where $\mathcal{D}_\alpha \subset \mathcal{H}$ is the domain of $\langle\alpha|$.

We would now like to show that the above correspondence between operators and states holds if and only if $\Re(\alpha) < \frac{Q}{2}$ as a consequence of the fact that wave-packets get pushed out to $q \to -\infty$ for $t \to \pm\infty$:

For simplicity let us consider wave-packets of the form $\langle\psi| = \int_0^\infty dP\langle P|\psi(P)$. The generalization to descendants thereof is straightforward. The limit $z \to 0$ corresponds to $t \to -e^{-i\epsilon}\infty$ in the minkowskian formulation. One should therefore analyze

(139)
$$\lim_{t\to -e^{-i\epsilon}\infty} e^{2it\Delta_\alpha}\langle\psi|[e^{2\alpha\phi}]_b(t,\sigma)|0\rangle.$$

The limit (139) may be related to the asymptotic behavior of wave-packets for $t \to -\infty$ by changing from the Heisenberg to the Schrödinger picture, $\langle\psi|[e^{2\alpha\phi}]_b(t,\sigma)|0\rangle = \langle\psi(t)|[e^{2\alpha\phi}]_b(\sigma)|0\rangle$. As discussed in the previous section, one will for $t \to -e^{-i\epsilon}\infty$ find wave-packets to be supported far off the potential, i.e. moving off to $q \to -\infty$. On such wave-packets one may therefore represent the microscopic state $[e^{2\alpha\phi}]_b(\sigma)|0\rangle$ by its leading $q \to -\infty$ behavior $: \exp(2\alpha\phi(\sigma)) : \exp(-Qq)\Omega$. A non-vanishing result can indeed only be obtained for $\Re(\alpha) \leq Q/2$ since otherwise the wave function of $[e^{2\alpha\phi}]_b(\sigma)|0\rangle$ vanishes where the wave-packet $\langle\psi(t)|$ is supported for $t \to -\infty$. One may then represent the above matrix element in the limit $t \to -\infty$ by a matrix element in $\mathcal{H}$ between the states $|0\rangle_{\rm F} = e^{-Qq}\,\Omega$ and ${}_{\rm F}\langle\psi| = \int_0^\infty \frac{dP}{2\pi}\;\bar{\psi}(P)e^{-2iPq}\Omega^\dagger$:

$$\begin{aligned}
\lim_{t\to -e^{-i\epsilon}\infty} e^{2it\Delta_\alpha}\,\langle\psi(t)|[e^{2\alpha\phi}]_b(\sigma)|0\rangle \;&=\; \lim_{t\to -e^{-i\epsilon}\infty} e^{2it\Delta_\alpha}\;{}_{\rm F}\langle\psi(t)|: e^{2\alpha\phi(\sigma)} :|0\rangle_{\rm F}\\
&=\; \lim_{t\to -e^{-i\epsilon}\infty} e^{2it\Delta_\alpha}\;{}_{\rm F}\langle\psi|: e^{2\alpha\phi(t,\sigma)} :|0\rangle_{\rm F}\quad .\\
&=\; {}_{\rm F}\langle\psi|e^{2\alpha q}|0\rangle_{\rm F} \;=\; {}_{\rm F}\langle\psi|\alpha\rangle_{\rm F} \;=\; \langle\psi|\alpha\rangle
\end{aligned}$$

We conclude that only the operators $[e^{2\alpha\phi}]_b(\sigma)$ with $\Re(\alpha) \leq Q/2$ are in correspondence with microscopic/macroscopic states.

11.2. **Seiberg bound**

The operators V_α as characterized by the three point functions $C(\alpha_3,\alpha_2,\alpha_1)$ satisfy the remarkable reflection property $\mathsf{V}_\alpha = R(\alpha)\mathsf{V}_{Q-\alpha}$. We would like to identify the operators V_α with the (suitably quantum corrected) exponential operators $[e^{2\alpha\phi}]_b$. As identifying property for the latter we had required that these operators have leading asymptotics for $q \to -\infty$ given by the standard free field normal ordered exponential operators $: \exp(2\alpha\phi) :$, cf. (137). But the reflection property $\mathsf{V}_\alpha = R(\alpha)\mathsf{V}_{Q-\alpha}$ then produces a problem for identifying operators $[e^{2\alpha\phi}]_b$ for $\Re(\alpha) > \frac{Q}{2}$: In that case $: \exp(2(Q-\alpha)\phi) :$ would dominate over $: \exp(2\alpha\phi) :$ in the asymptotics for $q \to -\infty$, so that one would identify $\mathsf{V}_\alpha = R(\alpha)[e^{2(Q-\alpha)\phi}]_b$. If one insists on characterizing exponential operators $[e^{2\alpha\phi}]_b$ by having leading asymptotics (137), this simply means that one does not find any operators $[e^{2\alpha\phi}]_b$, $\Re(\alpha) > \frac{Q}{2}$ among the operators V_α, $\alpha \in \mathbb{C}$. So how about operators $[e^{2\alpha\phi}]_b$, $\Re(\alpha) > \frac{Q}{2}$: Do they escape our methods or don't they exist? An argument in favor of the second possibility was given by N. Seiberg in [Se], which is why this issue goes under the name of "Seiberg-bound".

Let us translate Seiberg's argument into the present framework: A slight modification of the argument used in the previous subsection gives

(140)
$$\langle\psi|[e^{2\alpha\phi}]_b(\sigma)|0\rangle \;=\; 0.$$

To see this, it suffices to write

$$\begin{aligned}\langle\psi|[e^{2\alpha\phi}]_b(\sigma)|0\rangle \;&=\; \lim_{t\to-\infty}\,\langle\psi(t)|e^{-iHt}[e^{2\alpha\phi}]_b(\sigma)e^{iHt}|0\rangle \\ &=\; \lim_{t\to-\infty}\,\langle\psi(t)|[e^{2\alpha\phi}]_b(\sigma,-t)|0\rangle.\end{aligned} \tag{141}$$

By again using that $[e^{2\alpha\phi}]_b(\sigma,-t)|0\rangle$ has zero mode dependence $\sim\exp((2\alpha-Q)q)$ and that the wave-function representing $\langle\psi(t)|$ will tend to zero for any finite value of q as $t\to\infty$, one concludes that the expression vanishes. But now it is easy to see that (140) indeed implies that

$$\langle\psi_2|[e^{2\alpha\phi}]_b(\sigma)|\psi_1\rangle \;=\; 0. \tag{142}$$

To this aim one only has to invoke state-operator correspondence as discussed in the previous subsection to represent $|\psi_2\rangle$ as the limit for $z\to 0$ of a state of the form

$$|\psi_2(z)\rangle \;=\; \int_0^\infty \frac{dP}{2\pi}\,\mathsf{V}_{\alpha_P}(\zeta_P|z)|0\rangle, \quad \zeta_P\in\mathcal{V}_P\otimes\mathcal{V}_P. \tag{143}$$

By using mutual locality of $V_{\alpha_P}(\zeta_P|z)$ and $[e^{2\alpha\phi}]_b(\sigma)$ one reduces (142) to (140). We conclude that the Seiberg-bound is a simple consequence of our potential scattering picture of Liouville dynamics.

11.3. **Asymptotics vs. analyticity**

We would finally like to understand the analytic properties of the three point functions (meromorphic continuation, location and residues of poles) from the point of view of the zero mode Schödinger representation.

Let us begin by noting that the smoothing properties of the operators V_α, $\Re(\alpha)>0$, that we noted in Section 4 are quite easily understood by considering the asymptotic behavior for $q\to-\infty$. Matrix elements of $[e^{2\alpha\phi}]_b(z,\bar{z})$, $|z|<1$, would in the zero mode Schrödinger representation (109) be represented as

$$\langle P_2,\mathrm{f}_2|[e^{2\alpha\phi}]_b(z,\bar{z})|P_1,\mathrm{f}_1\rangle \;=\; \int_{-\infty}^{\infty} dq\,\big(\,\psi_{P_2,\mathrm{f}_2}(q)\,,\,[e^{2\alpha\phi}]_b(z,\bar{z})\psi_{P_1,\mathrm{f}_1}(q)\,\big)_{\mathcal{F}\otimes\mathcal{F}}, \tag{144}$$

As the zero mode dependence of $[e^{2\alpha\phi}]_b(z,\bar{z})$ provides an exponential damping factor for $q\to-\infty$, one would expect the integration in (144) to converge and define a function that is analytic in

$$\{(P_2,P_1)\in\mathbb{C}^2\;;\;|\Im(P_2\pm P_1)|<\Re(\alpha)\}, \tag{145}$$

which fits our discussion of the DOZZ-proposal in Section 4.

11.4. **Asymptotic expansion**

In order to study the analytic properties of the matrix elements $\langle P_2,\mathrm{f}_2|[e^{2\alpha\phi}]_b(z,\bar{z})|P_1,\mathrm{f}_1\rangle$ in the case $\Re(\alpha)\leq 0$, one will need to take into account sub-leading contributions to the asymptotic behavior of wave-functions for $q\to-\infty$. On the basis of our discussion in Subsection 10.8, especially Remark 6, one expects that H may be approximated by $\mathsf{H}^{\mathrm{F}}_{\mu\tilde{\mu}}$ when studying the asymptotic behavior for $q\to-\infty$.

Standard perturbation theory for the Hamiltonian $\mathsf{H}^{\mathrm{F}}_{\mu\tilde{\mu}}$ yields a formal series expansion for eigenfunctions of that operator:

$$\psi_{P,\mathrm{f}}(q) \;=\; \sum_{m,n=0}^{\infty} \mu^n \tilde{\mu}^m \; \psi_{P,\mathrm{f}}^{(n,m)}(q), \tag{146}$$

where the initial term is given by $\psi_{P,\mathrm{f}}^{(0,0)}(q) \;=\; e^{2iPq}\mathrm{f} + e^{-2iPq}\mathsf{R}(P)\mathrm{f}$ and the higher terms can be expressed in terms of the operators $\mathsf{U}(t) \equiv e^{i\mathsf{H}^{\mathrm{F}}t}\mathsf{U}e^{-i\mathsf{H}^{\mathrm{F}}t}$, $\tilde{\mathsf{U}}(t) \equiv e^{i\mathsf{H}^{\mathrm{F}}t}\tilde{\mathsf{U}}e^{-i\mathsf{H}^{\mathrm{F}}t}$ as

$$\begin{aligned} \psi_{P,\mathrm{f}}^{(n,m)}(q) \;=\; & \frac{(-i)^n}{n!}\frac{(-i)^m}{m!} \int\limits_{-\infty}^{0} dt_1 \ldots dt_n d\tilde{t}_1 \ldots d\tilde{t}_m \times \\ & \times T\Big(\mathsf{U}(t_1)\ldots\mathsf{U}(t_n)\tilde{\mathsf{U}}(\tilde{t}_1)\ldots\tilde{\mathsf{U}}(\tilde{t}_m)\Big)\,\psi_{P,\mathrm{f}}^{(0,0)}(q), \end{aligned} \tag{147}$$

where $T(\ldots)$ denotes the usual time-ordered product of operators.

11.5. **Meromorphic continuation**

To start with, one may consider the case $-b < \Re(\alpha) < 0$. The natural ansatz for defining the analytic continuation of the representation (144) for the matrix element $\langle P_2, \mathrm{f}_2|[e^{2\alpha\phi}]_b(z,\bar{z})|P_1, \mathrm{f}_1\rangle$ would be

$$\begin{aligned} \langle P_2, \mathrm{f}_2|[e^{2\alpha\phi}]_b(z,\bar{z})|P_1, \mathrm{f}_1\rangle = \lim_{q_0\to-\infty} \bigg(& \int_{q_0}^{\infty} dq\; \langle\, \psi_{P_2,\mathrm{f}_2}(q)\,|\,[e^{2\alpha\phi}]_b(z,\bar{z})\,,\,\psi_{P_1,\mathrm{f}_1}(q)\,\rangle_{\mathcal{H}(q)} \\ & + \sum_{s_1,s_2=0,1} \frac{e^{2i((1-2s_1)P_1-(1-2s_2)P_2-i\alpha)q_0}}{2i((1-2s_1)P_1-(1-2s_2)P_2-i\alpha)} \times \\ & \times \big(\mathsf{R}^{s_2}(P_2)\mathrm{f}_2, :e^{2\alpha\bar{\phi}}(z,\bar{z}): \mathsf{R}^{s_1}(P_1)\mathrm{f}_1\big)_{\mathcal{F}\otimes\mathcal{F}} \bigg), \end{aligned} \tag{148}$$

where $\bar{\phi} \equiv \phi - q$. Poles and corresponding residues of the meromorphic continuation of the matrix elements $\langle P_2, \mathrm{f}_2|[e^{2\alpha\phi}]_b(z,\bar{z})|P_1, \mathrm{f}_1\rangle$ to $-b < \Re(\alpha) < 0$ are explicitly exhibited in (148).

In order to continue to values of $\Re(\alpha)$ smaller than $-b$ one needs to take into account sub-leading terms in the asymptotic expansion of the wave-functions $\psi_{P,\mathrm{f}}(q)$ as given in (146). One thereby finds poles for

$$\alpha + i(s_1P_1 - s_2P_2) = -nb - mb^{-1}, \quad s_1, s_2 = \pm 1; \quad n, m \in \mathbb{Z}^{\geq 0}, \tag{149}$$

in precise correspondence with the poles of the matrix element $\langle P_2, \mathrm{f}_2|[e^{2\alpha\phi}]_b(z,\bar{z})|P_1, \mathrm{f}_1\rangle$ as given by the DOZZ-proposal. Moreover, by some arguments that are familiar from standard derivations of quantum field theoretical perturbation theory one finds that the corresponding residues are precisely given by the Dotsenko-Fateev integrals that were discussed in Part I.

12. **DISCUSSION**

12.1. **The success**

Our attempt to understand Liouville theory in terms of the zero mode Schödinger representation (109) was in some respects amazingly successful:

(1) It gave a natural interpretation of the reflection property $|P\rangle = R(P)|-P\rangle$ in terms of reflection of wave-packets from a potential-"wall".

(2) The Seiberg-bound followed from the fact that wave-packets get pushed out to $q \to -\infty$ for time $t \to \pm\infty$: If the exponential $[e^{2\alpha\phi}]_b$ decays too strongly for $q \to -\infty$ it can not have any overlap with the asymptotic wave-packets.

(3) Considering the asymptotic expansion of eigenstates of the Hamiltonian H for $q \to -\infty$ allowed one to get a detailed understanding of the analytic properties of matrix elements such as $\langle P'|\mathsf{V}_\alpha(z)|P\rangle$ (location of poles, residues, reflection property), in perfect agreement with the DOZZ-proposal.

12.2. **Problems**

(1) Let us recall that the DOZZ-proposal requires that

$$\pi\tilde{\mu}\frac{\Gamma(b^{-2})}{\Gamma(1-b^{-2})} = \left(\pi\mu\frac{\Gamma(b^2)}{\Gamma(1-b^2)}\right)^{b^{-2}}.$$

For $0 < b < 1$ one may observe that there exist certain ranges for the values of b where not both of μ and $\tilde{\mu}$ can be positive: $\Gamma(1-b^{-2})$ may become negative. However, in such a case it seems impossible to have positivity of H: One could always find regions in q-space where the interaction term $\mu\mathsf{U} + \tilde{\mu}\tilde{\mathsf{U}}$ gives negative contributions to $\mathsf{H}^{\rm F}_{\mu\tilde{\mu}}$, which would contradict the positivity of H.

(2) The asymptotic expansion (146) can be rewritten in terms of $(\mathsf{H}^{\rm F} - E_{P,\rm f})^{-1}\mathsf{U}$, $(\mathsf{H}^{\rm F} - E_{P,\rm f})^{-1}\tilde{\mathsf{U}}$. If (146) would provide a valid representation for $\psi_{P,\rm f}(q)$ at finite values of q, one would not understand how $\psi_{P,\rm f}(q)$ could have the nice analytic properties in its P-dependence that are suggested by the correspondence with $|P,\rm f\rangle$: The operators $(\mathsf{H}^{\rm F} - E_{P,\rm f})^{-1}$ would introduce an awkward collection of poles for complex P.

(3) If $\mathsf{H} = \mathsf{H}^{\rm F}_{\mu\tilde{\mu}}$ one would of course also need to have $\mathsf{L}_n \equiv \mathsf{L}^{\rm F}_{n,\mu,\tilde{\mu}}$ and $\bar{\mathsf{L}}_n \equiv \bar{\mathsf{L}}^{\rm F}_{n,\mu,\tilde{\mu}}$. In this case nothing would prevent us to identify $[e^{2\alpha\phi}]_b(\sigma) \equiv :e^{2\alpha\phi(\sigma)}:$. But this would be inconsistent with the Seiberg bound as discussed in Subsection 11.2.

12.3. **What to conclude?**

The description of Liouville theory in a representation such as (109) where the zero mode q is diagonal works very well as long as only the asymptotics $q \to -\infty$ is considered. We conclude that one has $\mathcal{H} \sim \mathcal{H}^{\rm Schr}$ and $\mathsf{H} \sim \mathsf{H}^{\rm F}_{\mu\tilde{\mu}}$ up to corrections that vanish faster than any exponential for $q \to -\infty$. This is good enough to support the picture of Liouville dynamics as describing scattering of wave-packets off some perfectly reflecting "potential".

All the problems that we have mentioned have to do with the representation of Liouville theory "at finite q". The latter fact may not be too surprising in view of our remarks in Subsections 8.3 and 9.1. But what precisely goes wrong? Let us just point out two options that one might consider:

(1) There exists a self-adjoint zero mode operator q, but it has a spectral representation that is less trivial than (109):

$$\mathcal{H} \simeq \mathcal{H}^{\rm Schr} = \int_{\mathbb{R}} d\mu(q)\, \mathcal{H}(q),$$

where $\mathcal{H}(q)$ must not be a *Fock*-representation of the canonical commutation relations for a_n, b_n. This would of course also render the representation of H in $\mathcal{H}^{\rm Schr}$ nontrivial. However, making contact with the DOZZ-proposal would require that $\mathcal{H}(q)$ is at least weakly asymptotic to $\mathcal{F} \otimes \mathcal{F}$ for $q \to -\infty$. Moreover, possible q-dependent quantum corrections in the definition of $\mathcal{H}(q)$ and H would have to vanish faster than any exponential.

(2) The zero mode operator q does not exist, is not densely defined, or ceases to be selfadjoint for other reasons. In this case one would seem to loose any ground for describing Liouville theory in terms of wave-packets localized in target space. One could, however, look for a self-adjoint operator q' that approximates q in a suitable limit. There is in fact a natural and plausible candidate for such an operator q':

$$\mathsf{q}' = \tfrac{1}{2b} \log \mathsf{I}, \qquad \mathsf{I} \equiv \int_0^{2\pi} d\sigma\, \mathsf{V}_b(\sigma).$$

We consider the resolution of these issues as an important problem for future research.

Part III. OPERATOR-APPROACH

13. CLASSICAL INTEGRABILITY

Classically one standard way of exhibiting the complete integrability of the Liouville equation is based on the following observation:

13.1. Linear system

Claim: *The following statements are equivalent:*
(i) φ satisfies the classical Liouville equation of motion

$$\partial_+\partial_-\varphi = -2\pi\mu_c e^{\varphi}$$

(ii) $e^{-\frac{1}{2}\varphi}$ satisfies

$$\begin{aligned} \partial_+^2 e^{-\frac{1}{2}\varphi(x_+,x_-)} &= T_{++}(x_+)e^{-\frac{1}{2}\varphi(x_+,x_-)} \\ \partial_-^2 e^{-\frac{1}{2}\varphi(x_+,x_-)} &= T_{--}(x_-)e^{-\frac{1}{2}\varphi(x_+,x_-)}, \end{aligned} \tag{150}$$

where T_{++} (resp. T_{--}) depend on x_+ (resp. x_-) only.

Proof. — <u>(i)⇒(ii):</u>
Any $\varphi(x_+,x_-)$ will satisfy (150) with

$$T_{\pm\pm}(x_+,x_-) \;\equiv\; \tfrac{1}{4}(\partial_\pm\varphi)^2 - \tfrac{1}{2}\partial_\pm^2\varphi.$$

However, direct calculation shows that $\partial_\mp T_{\pm\pm} = 0$ if the Liouville equation of motion is satisfied.
<u>(ii)⇒(i):</u>
Equation (150) implies that

$$e^{-\frac{1}{2}\varphi(x_+,x_-)} = \sum_{i,j=1}^{2} f_i^+(x_+)C_{ij}f_j^-(x_-), \tag{151}$$

where $f_i^\pm(x_\pm)$, $i=1,2$ are two linearly independent real solutions of $\partial_\pm^2 f_i^\pm = T_{\pm\pm}f_i^\pm$, which can and will be normalized such that

$$f_2^\pm\partial_\pm f_1^\pm - f_1^\pm\partial_\pm f_2^\pm = 1. \tag{152}$$

It is no loss of generality to assume that $C_{ij} = C\delta_{ij}$ in (151) since this may always be achieved by forming suitable linear combinations of the $f_i^\pm(x_\pm)$, $i=1,2$. By direct calculation using (152) one may then verify that

$$\varphi(x_+,x_-) \;=\; -2\log\left(\sqrt{2\pi\mu_c}\sum_{i=1}^{2} f_i^+(x_+)f_i^-(x_-)\right) \tag{153}$$

satisfies the Liouville equation of motion. □

This means that the solution of the nonlinear Liouville equation of motion can be reduced to the integration of the linear equation (150), where $T_{\pm\pm}(x_\pm)$ can be considered as data, given in terms of either initial or asymptotic conditions.[5] From these one may then reconstruct $e^{-\frac{1}{2}\varphi(x_+,x_-)}$ by

[5] See [PP] for the solution of the initial value problem.

solving *linear* differential equations, and finally φ itself by taking the logarithm. Any solution of the Liouville equation is obtained in this way, and the data $T_{\pm\pm}(x_\pm)$ characterize a solution $\varphi(x_+, x_-)$ uniquely up to an additive constant.

13.2. **Boundary conditions**

We are interested in the case where $\varphi(\sigma, t)$ is periodic, $\varphi(\sigma + 2\pi, t) = \varphi(\sigma, t)$. It follows that $T_{\pm\pm}(x_\pm)$ must also be periodic. However, the solutions $f_i^\pm$ will only be *quasi*-periodic

(154) $$f^\pm(x_\pm \pm 2\pi) \;=\; M^\pm \cdot f^\pm(x_\pm),$$

where we used matrix notation, $f^\pm$ being the transpose of the row vector $(f_1^\pm, f_2^\pm)$. The *monodromy* matrices $M^\pm$ must be elements of $SL(2,\mathbb{R})$ in order for (154) to be consistent with (152). Periodicity of $\varphi(\sigma, t)$ therefore requires that $(M^+)^t \cdot M^- = 1$. The Liouville field (153) is clearly unchanged under the transformations

(155) $$f^\pm \to A^\pm f^\pm, \qquad (A^+)^t \cdot A^- = 1,$$

under which M transforms as $M^\pm \to A^\pm \cdot M^\pm \cdot (A^\pm)^{-1}$. The Liouville field therefore only depends on the conjugacy class of the matrix M. It can be shown [PP] that the Liouville field is regular if and only if $\mathrm{Tr} M > 2$, corresponding to the so-called *hyperbolic* conjugacy classes of $SL(2,\mathbb{R})$. By means of (155) one may then always bring M into the form $M = \mathrm{diag}(e^{\frac{p}{2}}, e^{-\frac{p}{2}})$. We will henceforth assume M to be of that form.

13.3. **Map to free field**

Another useful representation of the general solution to the Liouville equation is obtained by introducing

(156) $$A_\pm(x_\pm) \equiv f_2(x_\pm)/f_1(x_\pm).$$

Note that the normalization conditions (152) imply that $A_\pm$ are monotonic, i.e. $\partial_\pm A_\pm = (f_1^\pm)^{-2} > 0$. It is therefore possible to recover $f_i^\pm$ via

(157) $$f_1^\pm = (\partial_\pm A_\pm)^{-\frac{1}{2}} \qquad f_2^\pm = (\partial_\pm A_\pm)^{-\frac{1}{2}} A_\pm,$$

which leads to the following classical representation for the Liouville field:

(158) $$\varphi(x_+, x_-) = \log\left(\sqrt{2\pi\mu_c}\,\frac{\partial_+ A_+\, \partial_- A_-}{(1 + A_+ A_-)^2}\right).$$

However, it turns out that the data $A_\pm$ are not very convenient as starting point for quantization: Their Poisson brackets are complicated [PP] and it is not easy to realize the condition of monotonicity on the quantum level. The following variables appear to be better suited: Define $\varphi_\pm^{\mathrm{F}}(x_\pm)$ by

(159) $$e^{\varphi_\pm^{\mathrm{F}}(x_\pm)} \;\equiv\; \partial_\pm A_\pm.$$

The quasi-periodicity of $A_\pm$ implies that $\varphi^{\rm F}(x_+,x_-)\equiv\varphi^{\rm F}_+(x_+)+\varphi^{\rm F}_-(x_-)$ is a solution of the free wave equation $\partial_+\partial_-\varphi^{\rm F}=0$ on the *cylinder*. It is easy to see that $A_\pm$ (and therefore the Liouville field itself) can be recovered from $\varphi^{\rm F}_\pm(x_\pm)$ by means of

$$A_\pm[\varphi^{\rm F}](x_\pm)=\frac{1}{e^{\pm p}-1}\int_0^{2\pi}dy_\pm\; e^{\varphi^{\rm F}_\pm(y_\pm+x_\pm)}. \tag{160}$$

These considerations yield a map from the phase space $\mathcal{P}^{\rm F}$ into the phase space $\mathcal{P}$ of Liouville theory.

There is a problem, though: The map $\mathcal{P}^{\rm F}\to\mathcal{P}$ is not one-to-one but two-to-one. For any given solution $\varphi^{\rm F}_1(x_+,x_-)$ of the free field wave equation there exists a second one $\varphi^{\rm F}_2(x_+,x_-)$ that maps to the same solution of the Liouville equation. This is most easily verified by noting that (158) is unchanged if one replaces $A_\pm\to 1/A_\pm$, which corresponds to the exchange of $f_1^\pm$ and $f_2^\pm$. The second free field configuration $\varphi^{\rm F}_2(x_+,x_-)$ is therefore given in terms of $\varphi^{\rm F}_1(x_+,x_-)\equiv\varphi^{\rm F}_{1,+}(x_+)+\varphi^{\rm F}_{1,-}(x_-)$ via

$$\varphi^{\rm F}_2(x_+,x_-)\;=\;\log\biggl(\partial_+\frac{1}{A_+[\varphi_1]}\partial_-\frac{1}{A_-[\varphi_1]}\biggr). \tag{161}$$

The resulting map $\mathcal{P}^{\rm F}\to\mathcal{P}^{\rm F}$ will be denoted by S.

In order to get a unique parametrization of the Liouville phase space in terms of free field variables one therefore has two obvious options: Either one may note that S maps the zero mode p which is recovered from $\varphi^{\rm F}$ as

$$p=\frac{1}{2\pi}\int_0^{2\pi}d\sigma\;\partial_t\varphi^{\rm F}$$

into its negative. The map $\mathcal{P}^{\rm F}\to\mathcal{P}$ is therefore invertible when restricted e.g. to the subspaces $\mathcal{P}^{\rm F}_\pm$ defined by the condition $\pm p>0$. The corresponding maps $\mathcal{P}\to\mathcal{P}^{\rm F}_\pm$ will be denoted $W_\pm$. But it may sometimes be more convenient to think of $\mathcal{P}^{\rm F}_+$ as $\mathcal{P}^{\rm F}/\!\sim$, where two configurations $\varphi^{\rm F}_i(x_+,x_-)$ $i=1,2$ are considered as equivalent (notation: $\varphi^{\rm F}_1\sim\varphi^{\rm F}_2$) iff $S[\varphi_1]=\varphi_2$.

The importance of this “gauge symmetry” represented by the map S was emphasized by Gervais and Neveu. We will identify its proper quantum counterpart below.

13.4. **Canonical formalism**

We had introduced the canonical formalism for Liouville theory in section 8.1. Its counterpart for the free field $\varphi^{\rm F}$ is obviously obtained by replacing $\varphi\to\varphi^{\rm F}$ and setting $\mu_c=0$. A basic result that represents important motivation for the program of constructing quantum Liouville theory in terms of the quantized free field is the following:

The maps $W_\pm:\mathcal{P}\to\mathcal{P}^{\rm F}_\pm$ are canonical. More precisely: The canonical Poisson bracket relations

$$\{\Pi_\varphi(\sigma),\varphi(\sigma')\}\;=\;\delta(\sigma-\sigma') \tag{162}$$

imply the same commutation relations for the images $\varphi^{\rm F}$, $\Pi^{\rm F}_\varphi$ of φ, Π_φ under $W_\pm$. Conversely: Canonical Poisson bracket relations

$$\{\Pi^{\rm F}_\varphi(\sigma),\varphi^{\rm F}(\sigma')\}\;=\;\delta(\sigma-\sigma') \tag{163}$$

imply (162).

The inverse direction (163) ⇒ (162) can be shown by direct, but tedious calculation (see e.g. [KN] for details). To go from (162) ⇒ to (163) is more difficult, see [PP]. It also follows that the map $S:\mathcal{P}^{\rm F}\rightarrow\mathcal{P}^{\rm F}$ will be canonical.

13.5. **Conformal symmetry**

It is quite important to note that the integrable structure of Liouville theory (as represented by the maps $W_{\pm}:\mathcal{P}\rightarrow\mathcal{P}^{\rm F}_{\pm}$) is compatible with conformal symmetry: Let us first remark that canonical generators of conformal transformations in the free field theory are easily identified as

$$T^{\rm F}_{\pm\pm}=\tfrac{1}{4}(\partial_{\pm}\varphi^{\rm F})^2-\tfrac{1}{2}\partial^2_{\pm}\varphi^{\rm F}. \tag{164}$$

The modes $l_n^{\pm}=\frac{1}{2\pi}\int_0^{2\pi}d\sigma e^{in\sigma}T^{\rm F}_{\pm\pm}(\sigma)$ will then satisfy a Poisson-counterpart of the Virasoro algebra:

$$\{l_n^{\pm},l_m^{\pm}\}\;=\;i(n-m)l_{n+m}^{\pm}+\tfrac{i}{2}n^3\delta_{n+m}. \tag{165}$$

Compatibility of conformal symmetry with the integrable structure of Liouville theory is a consequence of the fact that the maps $W_{\pm}:\mathcal{P}\rightarrow\mathcal{P}^{\rm F}_{\pm}$ indeed transform $T_{\pm\pm}[\varphi]$ into $T^{\rm F}_{\pm\pm}[\varphi^{\rm F}]$ [PP]. It follows in particular that the maps $W_{\pm}$ intertwine the actions of conformal symmetry generated by $T^{\rm F}_{\pm\pm}$ and $T_{\pm\pm}$ respectively. A check of this statement may be based on the observation that $f_i^{\pm}$ $i=1,2$ transform under the conformal transformations generated by $T^{\rm F}_{\pm\pm}$ as tensors of weight $\frac{1}{2}$:

$$\{T^{\rm F}_{\pm\pm},f_i^{\pm}(x_{\pm})\}\;=\;e^{inx_{\pm}}(\partial_{\pm}-in\tfrac{1}{2})f_i^{\pm}(x_{\pm}). \tag{166}$$

Closely related is the fact that the map $S:\mathcal{P}^{\rm F}\rightarrow\mathcal{P}^{\rm F}$ commutes with the action of conformal symmetry as generated by $T^{\rm F}_{\pm\pm}$. Very similar considerations for the case of Liouville theory on the strip have first appeared in [GN0].

14. **QUANTIZATION OF THE FREE FIELD THEORY**

Given the possibility to map classical Liouville theory to a free field theory it is of course natural to approach quantization of Liouville theory by first quantizing the free field theory and then trying to reconstruct the Liouville field operators in terms of operators in the free field theory.

So let us introduce the free field $\phi^{\rm F}(\sigma,t)$ with canonical commutation relations

$$[\phi^{\rm F}(\sigma),\partial_t\phi^{\rm F}(\sigma)]=2\pi i\delta(\sigma-\sigma') \tag{167}$$

at time $t=0$. Modes are introduced via

$$\phi^{\rm F}(\sigma,t)\;=\;\mathsf{q}^{\rm F}+2\mathsf{p}t+\sum_{n\neq 0}\frac{1}{n}\big(\mathsf{a}_n^{\rm F}e^{-inx_+}+\mathsf{b}_n^{\rm F}e^{-inx_-}\big), \tag{168}$$

and the Hilbert-space will be defined as

$$\mathcal{H}^{\rm F}\;\equiv\;L^2(\mathbb{R})\otimes\mathcal{F}\otimes\mathcal{F}, \tag{169}$$

where $\mathcal{F}\otimes\mathcal{F}$ is the Fock-space generated by action of the modes $\mathsf{a}_n^{\rm F}$, $\mathsf{b}_n^{\rm F}$, $n<0$ on the Fock-vacuum Ω that satisfies $\mathsf{a}_n^{\rm F}\Omega=0$ and $\mathsf{b}_n^{\rm F}\bar{\Omega}=0$, $n>0$. We will work in a representation where P is diagonal.

14.1. **Conformal symmetry**

Conformal symmetry is realized on $\mathcal{H}^{\rm F}$ by means of

$$\mathsf{T}_{++}(x_+) \;=\; \sum_{n\in\mathbb{Z}} \mathsf{L}_n^{\rm F} e^{-inx_+} \qquad \mathsf{T}_{--}(x_-) \;=\; \sum_{n\in\mathbb{Z}} \bar{\mathsf{L}}_n^{\rm F} e^{-inx_-}, \tag{170}$$

where the expressions for $\mathsf{L}_n^{\rm F}$, $\bar{\mathsf{L}}_n^{\rm F}$ are obtained from the formulae for $\mathsf{L}_n^{\rm F}(P)$, $\bar{\mathsf{L}}_n^{\rm F}(P)$ (cf. eqn. (122)) by replacing $\mathsf{a}_n \to \mathsf{a}_n^{\rm F}$, $\mathsf{b}_n \to \mathsf{b}_n^{\rm F}$ and $P \to \mathsf{p}$.

This means that $\mathcal{H}^{\rm F}$ decomposes as the direct integral of Fock-space representations of the Virasoro algebra:

$$\mathcal{H}^{\rm F} \;\simeq\; \int\limits_{-\infty}^{\infty} dP\; \mathcal{F}_P \otimes \bar{\mathcal{F}}_P, \tag{171}$$

where $\mathcal{F}_P$ (resp. $\bar{\mathcal{F}}_P$) denote the Virasoro representations defined on $\mathcal{F}$ by means of the generators $\mathsf{L}_n^{\rm F}$ (resp. $\bar{\mathsf{L}}_n^{\rm F}$) defined in (170).

14.2. **Building blocks**

The basic building blocks of all constructions will be the following operators:
NORMAL ORDERED EXPONENTIALS:

$$\begin{aligned} \mathsf{E}^{\alpha}(x_+) \;&\equiv e^{\alpha \mathsf{q}^{\rm F}} \exp\left(\sum_{n<0} \frac{2\alpha}{n} \mathsf{a}_n^{\rm F} e^{-inx_+}\right) e^{2\alpha x_+ \mathsf{p}} \exp\left(\sum_{n>0} \frac{2\alpha}{n} \mathsf{a}_n^{\rm F} e^{-inx_+}\right) e^{\alpha \mathsf{q}^{\rm F}} \\ \bar{\mathsf{E}}^{\alpha}(x_-) \;&\equiv e^{\alpha \mathsf{q}^{\rm F}} \exp\left(\sum_{n<0} \frac{2\alpha}{n} \mathsf{b}_n^{\rm F} e^{-inx_-}\right) e^{2\alpha x_- \mathsf{p}} \exp\left(\sum_{n>0} \frac{2\alpha}{n} \mathsf{b}_n^{\rm F} e^{-inx_-}\right) e^{\alpha \mathsf{q}^{\rm F}}. \end{aligned} \tag{172}$$

SCREENING CHARGES:

$$\mathsf{Q}(x_+) \;\equiv\; \int\limits_0^{2\pi} dy_+\; \mathsf{E}^b(x_+ + y_+), \qquad \bar{\mathsf{Q}}(x_-) \;\equiv\; \int\limits_0^{2\pi} dy_+\; \bar{\mathsf{E}}^b(x_- + y_-). \tag{173}$$

The normal ordered exponentials can be understood as true (unbounded) operators if $\Im(x_\pm) > 0$ (negative euclidean time), as operator valued distributions for real values of $x_\pm$. This being understood we will often call the normal ordered exponentials and functions thereof "operators" in the following.

The screening charges on the contrary represent densely defined unbounded operators even for $x_\pm \in \mathbb{R}$. They can be seen to have a canonical self-adjoint extension due to their property of positivity. This allows to take arbitrary powers of these operators.

The behavior of these operators under conformal transformations can be summarized by

$$\begin{aligned} \big[\mathsf{L}_n, \mathsf{E}^{\alpha}(x_+)\big] &= e^{inx_+}\big(-i\partial_+ + n\Delta_\alpha)\big)\; \mathsf{E}^{\alpha}(x_+), \\ \big[\bar{\mathsf{L}}_n, \bar{\mathsf{E}}^{\alpha}(x_-)\big] &= e^{inx_-}\big(-i\partial_- + n\Delta_\alpha)\big)\; \bar{\mathsf{E}}^{\alpha}(x_-), \end{aligned} \tag{174}$$

whereas the screening charges transform as

$$\big[\mathsf{L}_n, \mathsf{Q}(x_+)\big] = -ie^{inx_+}\partial_+ \mathsf{Q}(x_+), \qquad \big[\bar{\mathsf{L}}_n, \bar{\mathsf{Q}}(x_-)\big] = -ie^{inx_-}\partial_- \bar{\mathsf{Q}}(x_-), \tag{175}$$

14.3. **Quantum counterparts of** $f_i^{\pm}$

Let us define the following set of operators:

$$\begin{aligned} \mathsf{f}_1(x_+) &= \mathsf{E}^{-\frac{b}{2}}(x_+), & \mathsf{f}_2(x_+) &= \frac{e^{-2\pi b(\mathsf{p}+i\frac{b}{2})}}{\sin(\pi b(Q-2i\mathsf{p}))}\mathsf{Q}(x_+)\mathsf{f}_1(x_+), \\ \bar{\mathsf{f}}_1(x_-) &= \bar{\mathsf{E}}^{-\frac{b}{2}}(x_-), & \bar{\mathsf{f}}_2(x_-) &= \bar{\mathsf{f}}_1(x_-)\bar{\mathsf{Q}}(x_-)\frac{e^{-2\pi b(\mathsf{p}-i\frac{b}{2})}}{2\sin(\pi b(Q+2i\mathsf{p}))}. \end{aligned} \tag{176}$$

$\mathsf{f}_i, \bar{\mathsf{f}}_i$ should be considered as the quantum counterparts of f_i^+, f_i^- respectively. This correspondence is quite obvious when comparing the expressions (176) to the expressions for f_i^+, f_i^- that follow from eqns. (157), (159) and (160) (The shifts of p in the p-dependent pre-factors could be absorbed by choosing a different ordering).

It is encouraging to note that the operators f_i $i = 1, 2$ satisfy a second order differential equation of a very similar form as their classical counterparts:

$$\partial_+^2\, \mathsf{f}_i \;=\; -b^2 : \mathsf{T}\, \mathsf{f}_i :, \quad i = 1, 2, \tag{177}$$

where the normal ordering of the expression on the right hand side is defined as follows:

$$\begin{aligned} : \mathsf{T}\, \mathsf{f}_i : \;=\; & \sum_{n<0} \mathsf{L}_n e^{-inx_+}\, \mathsf{f}_i + \sum_{n>0} \mathsf{f}_1\; e^{-inx_+}\mathsf{L}_n \\ & + \frac{1}{2}\Big(\mathsf{L}_0\mathsf{f}_i^+ + \mathsf{f}_i^+\mathsf{L}_0\Big) - \Big(\frac{c-1}{24} - \frac{b^2}{16}\Big)\mathsf{f}_i. \end{aligned} \tag{178}$$

The operators $\bar{\mathsf{f}}_i$ $i = 1, 2$ satisfy a second order differential equation that is obtained by obvious replacements.

14.4. **Quantum counterpart of** $e^{-\frac{1}{2}\varphi}$

Let us consider the following operator:

$$\mathsf{V}(\sigma, t) \;=\; \mathsf{f}_1(x_+)\, e^{b\mathsf{q}}\, \bar{\mathsf{f}}_1(x_-) + \mu_e\, \mathsf{f}_2(x_+)\, e^{-b\mathsf{q}}\, \bar{\mathsf{f}}_2(x_-). \tag{179}$$

It can be shown to satisfy the following properties:

a) CONFORMAL COVARIANCE

$$\begin{aligned} \big[\mathsf{L}_n, \mathsf{V}(\sigma,t)\big] &= e^{+inx_+}\big(-i\partial_+ + n\Delta_b)\big)\; \mathsf{V}(\sigma,t), \\ \big[\bar{\mathsf{L}}_n, \mathsf{V}(\sigma,t)\big] &= e^{-inx_-}\big(-i\partial_- + n\Delta_b)\big)\; \mathsf{V}(\sigma,t) \end{aligned}$$

b) EQUATION OF MOTION

$$\begin{aligned} \partial_+^2\; \mathsf{V} \;&=\; -\, b^2 : \mathsf{T}(x_+)\, \mathsf{V} :, \\ \partial_-^2\; \mathsf{V} \;&=\; -\, b^2 : \bar{\mathsf{T}}(x_-)\, \mathsf{V} :, \end{aligned}$$

c) LOCALITY

$$\big\langle \psi_2 \big|\, \big[\; \mathsf{V}(f)\;,\; \mathsf{V}(f')\; \big]\, \big|\psi_1\big\rangle \;=\; 0,$$

d) POSITIVITY

$$\langle\psi|\, \mathsf{V}(f)\, |\psi\rangle \;>\; 0,$$

where $|\psi\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle$ are taken from a dense subset of $\mathcal{H}^{\mathrm{F}}$ and the *smeared* operator $\mathsf{V}(f)$ is defined as

$$\mathsf{V}(f) \;\equiv\; \int\limits_{S^1} d\sigma\;\; f(\sigma)\,\mathsf{V}(\sigma). \tag{180}$$

Properties a) and b) are obvious. Locality is proved by using the exchange relations of the operators f_i, $\bar{\mathsf{f}}_i$ [GN2], and property d) is easy to prove by means of the reflection operator introduced in the next subsection.

14.5. **Reflection operator**

We had found in our discussion of the classical integrability of the Liouville equation that the free field with unrestricted zero mode p parameterizes the Liouville field in an ambiguous way: There exists a transformation S (defined in (161)) of free field configurations $\varphi^{\mathrm{F}}(t,\sigma)$ that leaves the Liouville field unchanged. When using the free field phase space $\mathcal{P}^{\mathrm{F}}$ as a parameterization of the space of solutions to the Liouville equation one should therefore identify any two configurations of the free field that are related by S: $\varphi_1^{\mathrm{F}} \sim \varphi_2^{\mathrm{F}}$ iff $S[\varphi_1^{\mathrm{F}}] = \varphi_2^{\mathrm{F}}$.

In the quantization scheme that we have discussed so far, we have started from a realization of the zero mode p with spectrum being the entire real line. The quantum analogue of S should be an operator S on $\mathcal{H}^{\mathrm{F}}$ that maps p to $-\mathsf{p}$ but leaves $[e^{-b\phi}]_b$ unchanged,

$$\mathsf{S}^{-1}\cdot[e^{-b\phi}]_b\cdot\mathsf{S} = [e^{-b\phi}]_b. \tag{181}$$

This operator can be considered as expressing the ambiguity in the parameterization of Liouville states by free field variables on the quantum level. The true Liouville Hilbert $\mathcal{H}$ space should then be identified with the subspace in $\mathcal{H}^{\mathrm{F}}$ of S-invariant vectors. For the identification of $\mathcal{H}$ with a subspace in $\mathcal{H}^{\mathrm{F}}$ to be compatible with conformal symmetry one evidently needs that S commutes with the Virasoro generators,

$$\mathsf{S}\cdot\mathsf{L}_n^{\mathrm{F}}(\mathsf{p}) \;=\; \mathsf{L}_n^{\mathrm{F}}(-\mathsf{p})\cdot\mathsf{S}, \qquad \mathsf{S}\cdot\bar{\mathsf{L}}_n^{\mathrm{F}}(\mathsf{p}) \;=\; \bar{\mathsf{L}}_n^{\mathrm{F}}(-\mathsf{p})\cdot\mathsf{S}. \tag{182}$$

Such an operator can be constructed as follows: Since the Fock-space representations $\mathcal{F}_P$ and $\mathcal{F}_{-P}$ are unitarily equivalent (cf. Subsection 10.6) one has a unique operator $\mathsf{S}(P):\mathcal{F}\otimes\mathcal{F}\to\mathcal{F}\otimes\mathcal{F}$ that satisfies (182) and $\mathsf{S}(P)\Omega=\Omega$. The sought-for operator S must therefore be of the form

$$\mathsf{S} \;=\; \mathsf{P}\cdot R(\mathsf{p})\mathsf{S}(\mathsf{p}), \tag{183}$$

where P denotes the parity operation in $L^2(\mathbb{R})\otimes\mathcal{F}\otimes\mathcal{F}$, $\mathsf{P}\psi(P)=\psi(-P)$. One may then show that there exists a unique choice of the function $R(P)$ that was introduced in (183) such that (181) is satisfied as well. This choice is given as

$$R(P) = -\big(\mu_{\mathrm{e}}\Gamma^2(b^2)\big)^{-\frac{2iP}{b}}\frac{\Gamma(1+2ibP)\Gamma(1+2ib^{-1}P)}{\Gamma(1-2ibP)\Gamma(1-2ib^{-1}P)}, \tag{184}$$

which coincides with the reflection amplitude $R(P)$ obtained from the DOZZ-proposal if we identify the constant μ_{e} introduced in the definition (179) of V with the constant μ that appears in the DOZZ-formula via $\mu_{\mathrm{e}}=\mu\sin(\pi b^2)$.

By means of S we may now identify the Liouville Hilbert space $\mathcal{H}$ as

$$\mathcal{H} \;=\; \{|\psi\rangle\in\mathcal{H}^{\mathrm{F}};(1-\mathsf{S})\Psi=0\}. \tag{185}$$

15. GENERAL EXPONENTIAL OPERATORS

Let us start by reconsidering the classical expression for $e^{\lambda\varphi}$ which may be written in terms of $f_s^{\pm}$ as (cf. (153))

$$e^{\lambda\varphi(\sigma,t)} = \left(f_1^+ f_1^- + 2\pi\mu_c f_2^+ f_2^-\right)^{-2\lambda}. \tag{186}$$

Let us note that at least for $\lambda > 0$ one has a useful representation of (186) as sum of *imaginary* powers of $f_i^+ f_i^-$:

$$e^{\lambda\varphi(\sigma,t)} = \frac{i}{2\pi}\int_{i\mathbb{R}} ds\, (2\pi\mu_c)^s \frac{\Gamma(s+2\lambda)\Gamma(-s)}{\Gamma(2\lambda)} \left(f_2^+ f_2^-\right)^s \left(f_1^+ f_1^-\right)^{-2\lambda-s}. \tag{187}$$

This expansion is related to the binomial expansion by writing the integral as sum over residues. However, it has the advantage to be valid both for $f_1^+ f_1^- > f_2^+ f_2^-$ and $f_1^+ f_1^- < f_2^+ f_2^-$, which is important for quantization:

The essential point is captured by the following example: Consider the operator v_λ of multiplication by the function $h_\lambda(q) \equiv (e^{\frac{q}{2}} + e^{-\frac{q}{2}})^{-2\lambda}$ on $L^2(\mathbb{R})$ in the case that $\Re(\lambda) > 0$. v_λ is a nice, bounded operator on $L^2(\mathbb{R})$ since $h_\lambda(q)$ behaves asymptotically as $h_\lambda(q) \sim e^{-\lambda|q|}$. Now let us try to represent this operator by using the usual binomial expansion:

$$\mathsf{v}_\lambda = e^{\lambda\mathsf{q}} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+1)} e^{n\mathsf{q}} \tag{188}$$

which clearly can represent the operator on wave-functions with support on the negative half-axis only. The domain in which (188) serves to represent the operator v_λ is not even dense in $L^2(\mathbb{R})$! This problem does not occur if one uses the integral version (187) of the binomial expansion. By means of a shift of the contour of integration one may even write it as an expansion over the unitary operators $e^{itb\mathsf{q}}$.

15.1. Definition

We would like to define $[e^{2\alpha\phi}]_b(\sigma)$ as something like $[e^{-b\phi}]_b \equiv \mathsf{V}$ raised to the power -2α. This is of course a nontrivial thing to do in the case of local operators. One may, however, recall that the usual exponential function is the unique solution to the functional equation $\exp(x)\exp(y) = \exp(x+y)$ within the class of continuous functions. This motivates the following definition:

DEFINITION 1. — *Let* $\mathsf{V}_\alpha(\sigma)$ *be a family of operators that has the properties*

i) *There exists a complex number* ρ *such that*

$$\big(\mathsf{V}_\alpha \star \mathsf{V}_\beta\big)(\sigma) \equiv \lim_{\sigma'\to\sigma} \left|\sigma' - \sigma\right|^\rho \mathsf{V}_\alpha(\sigma')\mathsf{V}_\beta(\sigma)$$

exists and satisfies

$$\big(\mathsf{V}_\alpha \star \mathsf{V}_\beta\big)(\sigma) = \mathsf{V}_{\alpha+\beta}(\sigma).$$

ii) *Each two operators* $\mathsf{V}_\alpha(\sigma)$ *and* $\mathsf{V}_\beta(\sigma)$ *are mutually local:*

$$\big[\, \mathsf{V}_\alpha(\sigma')\, ,\, \mathsf{V}_\beta(\sigma)\, \big] = 0.$$

iii) $\mathsf{V}_\alpha(\sigma)$ *reduces to* $\mathsf{V}(\sigma)$ *for* $\alpha = -\frac{b}{2}$.

We then call $\mathsf{V}_\alpha(\sigma)$ *a renormalized power function of* V*, in symbols* $\mathsf{V}_\alpha(\sigma) \equiv \mathsf{V}^{-2\alpha}(\sigma)$.

It turns out that there there exists a family of operators $\mathsf{V}_\alpha(\sigma)$ that fulfills these properties. This family $\mathsf{V}_\alpha(\sigma)$ can be shown to be unique at least for irrational $b \in (0,1)$. It will be constructed by an expression similar to (187) out of building blocks that can be considered as quantum analogs of $(f_2^+)^{is}(f_1^+)^{-2\lambda-is}$ and $(f_2^-)^{is}(f_1^-)^{-2\lambda-is}$, which will be introduced in the next subsection.

15.2. **Covariant chiral operators**

We will define operators $\mathsf{f}_s^\alpha(x_+), \bar{\mathsf{f}}_s^\alpha(x_-)$ in the spirit of Definition 1 such that

$$\begin{aligned} \big(\mathsf{f}_s^\alpha \star \mathsf{f}_1\big)(x_+) &= \mathsf{f}_s^{\alpha-\frac{b}{2}}(x_+), \qquad & \big(\bar{\mathsf{f}}_1 \star \bar{\mathsf{f}}_s^\alpha\big)(x_-) &= \bar{\mathsf{f}}_s^{\alpha-\frac{b}{2}}(x_-), \\ \big(\mathsf{f}_2 \star \mathsf{f}_s^\alpha\big)(x_+) &= \mathsf{f}_{s+1}^{\alpha-\frac{b}{2}}(x_+), \qquad & \big(\bar{\mathsf{f}}_s^\alpha \star \bar{\mathsf{f}}_2\big)(x_-) &= \bar{\mathsf{f}}_{s+1}^{\alpha-\frac{b}{2}}(x_-). \end{aligned} \tag{189}$$

The operators $\mathsf{f}_s^\alpha(x_+), \bar{\mathsf{f}}_s^\alpha(x_-)$ can be represented explicitly as follows:

$$\begin{aligned} \mathsf{f}_s^\alpha(x_+) \; &\equiv e^{-2\pi b s(\mathsf{p}+i\frac{b}{2})} \frac{S_b(Q-2i\mathsf{p})}{S_b(Q-2i\mathsf{p}+bs)} \big(\mathsf{Q}(x_+)\big)^s \mathsf{E}^\alpha(x_+), \\ \bar{\mathsf{f}}_s^\alpha(x_-) \; &\equiv \bar{\mathsf{E}}^\alpha(x_-)\big(\bar{\mathsf{Q}}(x_-)\big)^s \frac{S_b(Q+2i\mathsf{p})}{S_b(Q+2i\mathsf{p}+bs)} e^{-2\pi b s(\mathsf{p}-i\frac{b}{2})}, \end{aligned} \tag{190}$$

where the special function $S_b(x)$ is defined as

$$S_b(x) = \Gamma_b(x)/\Gamma_b(Q-x). \tag{191}$$

Let us observe that the definition (190) makes sense for complex values of s due to the positivity of Q. The operators $\mathsf{f}_s^\alpha(x_+)$, $\bar{\mathsf{f}}_s^\alpha(x_-)$ transform under conformal symmetry the same way as the operators $\mathsf{E}^\alpha(x_+)$, $\bar{\mathsf{E}}^\alpha(x_-)$, cf. (174).

It is technically often more convenient to work with the operators $\mathsf{g}_s^\alpha(x_+) = \mathsf{E}^\alpha(x_+)\big(\mathsf{Q}(x_+)\big)^s$, $\bar{\mathsf{g}}_s^\alpha(x_-) = \bar{\mathsf{E}}^\alpha(x_-)\big(\bar{\mathsf{Q}}(x_-)\big)^s$, which are related to $\mathsf{f}_s^\alpha(x_+)$, $\bar{\mathsf{f}}_s^\alpha(x_-)$ by multiplication with a p-dependent factor. The following two results are the main technical ingredients to our proof of the DOZZ-proposal, the details of which will appear in [TO].

NORMALIZATION — The matrix elements of operators g_s^α between primary states for the Virasoro algebra are given as

$$\langle P'|\mathsf{g}_s^\alpha(x_+)|P\rangle \; = \; \delta\big(P_s^\alpha - P'\big)\, e^{ix_+(\Delta(P')-\Delta(P))}\, G_s^\alpha(P), \tag{192}$$

where $P_s^\alpha \equiv P - i\alpha - ibs$ and

$$\begin{aligned} G_s^\alpha(P) \; = \; &\Big(\Gamma(1+b^2)b^{-1-b^2}\Big)^s e^{2\pi b s P} e^{-\pi i b^2 s^2} \\ &\times \frac{\Gamma_b(Q-2iP-2\alpha-sb)\Gamma_b(Q+2iP+sb)}{\Gamma_b(Q-2iP-2\alpha-2sb)\Gamma_b(Q+2iP)} \frac{\Gamma_b(Q-2\alpha-sb)\Gamma_b(Q+sb)}{\Gamma_b(Q-2\alpha)\Gamma_b(Q)}. \end{aligned} \tag{193}$$

The corresponding formula for $\bar{\mathsf{g}}_s^\alpha(x_-)$ is obtained by simply replacing $\mathsf{g}_s^\alpha \to \bar{\mathsf{g}}_s^\alpha$ and $x_+ \to x_-$.

BRAID RELATIONS — The operators g_s^α satisfy braid relations of the form

$$\mathsf{g}_{s_2}^{\alpha_2}(\sigma_2)\mathsf{g}_{s_1}^{\alpha_1}(\sigma_1) \; = \; \frac{1}{4}\int_{\mathbb{T}} dt_2 dt_1 \;\, \mathsf{g}_{t_1}^{\alpha_1}(\sigma_1)\mathsf{g}_{t_2}^{\alpha_2}(\sigma_2)\; B_\epsilon(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1}, \tag{194}$$

where $\mathbb{T} \equiv -\frac{Q}{2} + i\mathbb{R}$ and $\epsilon \equiv \mathrm{sgn}(\sigma_2 - \sigma_1)$. The distributional kernel $B_\epsilon(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1}$ has support for $t_2 + t_1 = s_2 + s_1$ only.

15.3. **Powers of** V

Let us now consider the operator

(195) $$\mathsf{V}_\alpha(t,\sigma) \;\equiv\; \int\limits_{i\mathbb{R}} ds\; \mu_{\rm e}^s\; B_b(\alpha,s)\; \mathsf{f}_s^\alpha(x_+)\, \mathsf{T}_s^\alpha\; \bar{\mathsf{f}}_s^\alpha(x_-),$$

where $\mathsf{T}_s^\alpha \equiv e^{-2(\alpha+bs)\mathsf{q}}$ and the b-binomial coefficients $B_b(\alpha,s)$ are given by

(196) $$B_b(\alpha,s) \;=\; \frac{S_b(-bs)S_b(2\alpha+bs)}{S_b(2\alpha)}.$$

$\mathsf{V}_\alpha(t,\sigma)$ is clearly a primary field for the left- and right Virasoro algebras generated by L_n, $\bar{\mathsf{L}}_n$. The main result of [TO] will be that $\mathsf{V}_\alpha(\sigma) \equiv \mathsf{V}_\alpha(t=0,\sigma)$ is indeed a renormalized power of V in the sense of Definition 1. The least trivial part of this statement is of course the verification of mutual locality, which becomes possible thanks to the existence of the braid relations (194). We will outline in the following Part IV how this verification can be done using the explicit calculation of the kernel $B_\epsilon(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1}$ in Section 16 below.

The matrix elements of V_α can be assembled from (193), (189) and (195). By identifying $\mu_{\rm e} = \mu\sin(\pi b^2)$ one finds that

(197) $$\langle P_2|\mathsf{V}_\alpha(0)|P_1\rangle \;=\; C\big(\tfrac{Q}{2}-iP_2,\alpha,\tfrac{Q}{2}+iP_1\big),$$

where $C(\alpha_3,\alpha_2,\alpha_1)$ is the DOZZ three point function. It follows that the operators V_α are indeed identical to those discussed in Part I.

Let us also note that $\mathsf{S}\cdot\mathsf{V}_\alpha(x)\cdot\mathsf{S}^{-1} = \mathsf{V}_\alpha(x)$ as a consequence of the reflection property of the DOZZ three point function. This means in particular that $\mathsf{V}_\alpha(x)$ leaves the Liouville Hilbert-space $\mathcal{H}$ invariant.

15.4. **Related work**

The operator approach to Liouville theory based on the quantization of the classical map to a free field has a long tradition going back to [BCT1][BCT2],[GN1][GN2] and [OW]. We should therefore discuss the relations between the treatment given here to the results of these works.

The approach taken in this paper is close in spirit to the approach of Gervais and Neveu, which was originally developed for Liouville theory on a strip instead of the cylinder. Subsequent work aimed at the development of this approach for Liouville theory on the cylinder includes [LS][GS3] and references therein. These works are concerned with the construction of field operators in a space of states called the "elliptic sector" which in our notation would be generated by states $|P\rangle$ with purely *imaginary* values of P. One needs to note, however, that the elliptic sector unavoidably contains non-unitary representations of the Virasoro algebra [LS] and that exponential field operators must have unusual hermiticity properties if such a spectrum is assumed [LS][BP][GS3]. Unfortunately we could not find a treatment of the so-called hyperbolic sector (generated by $|P\rangle$ with $P\in\mathbb{R}$) in the spirit of Gervais and Neveu. Moreover, we do not know how to employ the results of [GS3] for the construction of general exponential operators in the hyperbolic sector that is studied in this paper.

The approaches of [BCT1][BCT2] and [OW] on the other hand were in fact devoted to the hyperbolic sector. However, it was not obvious to us how to define and calculate the matrix elements of general exponential operators from the results of these references. The points where it is possible to compare the results of [BCT1][BCT2] and [OW] to the consequences of the DOZZ-proposal as discussed in Part I are therefore somewhat limited.[6] We would like to mention, however, that the necessity to restrict to "half" of the free field space of states in the scheme of [BCT1][BCT2] was also observed in [BCGT], but the relation with the reflection operator $\mathsf{R}(P)$ was not discussed there.

16. APPENDIX: BRAIDING OF COVARIANT CHIRAL OPERATORS

We would like to outline the derivation of the key fact that the operators $\mathsf{g}_s^\alpha(x_+)$, $\bar{\mathsf{g}}_s^\alpha(x_-)$ satisfy braid relations of the form (194). More details will be given in [TO]. In the following we will mainly consider the case $\alpha_i \in \frac{Q}{2} + i\mathbb{R}$, $s_i \in -\frac{Q}{2} + i\mathbb{R}$, $t_i \in -\frac{Q}{2} + i\mathbb{R}$, $i = 1, 2$. The braid relations in the general case can be obtained by analytic continuation.

16.1. Reformulation in terms of Weyl-type operators

The starting point of our calculation will be the following nice trick introduced in [GS1]. We will only consider the case $\sigma_2 < \sigma_1 \leftrightarrow \epsilon = -1$ explicitly in the following, the other case being completely analogous. Let us split $\mathsf{Q}(\sigma) = \mathsf{Q}_I^c + \mathsf{Q}_I$, $\mathsf{Q}(\sigma') = \mathsf{Q}_I^c + \mathsf{Q}_I'$, where

$$\mathsf{Q}_I^c \equiv \int_{\sigma'}^{\sigma+2\pi} d\varphi\, \mathsf{E}^b(\varphi), \quad \mathsf{Q}_I \equiv \int_{\sigma}^{\sigma'} d\varphi\, \mathsf{E}^b(\varphi), \quad \mathsf{Q}_I' \equiv \int_{\sigma+2\pi}^{\sigma'+2\pi} d\varphi\, \mathsf{E}^b(\varphi). \tag{198}$$

It follows from the exchange relations

$$\mathsf{E}^{\alpha_2}(\sigma_2)\mathsf{E}^{\alpha_1}(\sigma_1) = e^{-2\pi i\alpha_2\alpha_1\epsilon(\sigma_2-\sigma_1)}\mathsf{E}^{\alpha_1}(\sigma_1)\mathsf{E}^{\alpha_2}(\sigma_2), \tag{199}$$

where $\epsilon(\sigma) = 1$ if $\sigma > 0$ and $\epsilon(\sigma) = -1$ if $\sigma < 0$, that (194) is equivalent to the following identity:

$$\begin{aligned} &e^{4\pi i\alpha_2\alpha_1}\big(e^{-2\pi ib\alpha_1}\mathsf{Q}_I^c + e^{2\pi ib\alpha_1}\mathsf{Q}_I\big)^{s_2}\big(\mathsf{Q}_I^c + \mathsf{Q}_I'\big)^{s_1} = \\ &\qquad = \int_{\mathbb{T}} dt_2 dt_1\, \big(e^{-2\pi ib\alpha_2}\mathsf{Q}_I^c + e^{-6\pi ib\alpha_2}\mathsf{Q}_I'\big)^{t_1}\big(\mathsf{Q}_I^c + \mathsf{Q}_I\big)^{t_2}\, B_-(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1}. \end{aligned} \tag{200}$$

The operators Q_I, Q_I', Q_I^c satisfy the following algebra:

$$\begin{aligned} \mathsf{Q}_I^c\mathsf{Q}_I &= q^{-2}\mathsf{Q}_I\mathsf{Q}_I^c, \\ \mathsf{Q}_I^c\mathsf{Q}_I' &= q^{+2}\mathsf{Q}_I'\mathsf{Q}_I^c, \end{aligned} \qquad \mathsf{Q}_I\mathsf{Q}_I' = q^4\mathsf{Q}_I'\mathsf{Q}_I, \tag{201}$$

where $q = e^{\pi ib^2}$. It follows that these operators can be realized as

$$\mathsf{Q}_I^c = e^{2b\mathsf{x}}e^{-\pi ibt}, \qquad \mathsf{Q}_I = e^{b\mathsf{x}}e^{-2\pi b\mathsf{p}}e^{b\mathsf{x}}e^{\pi ibt}, \qquad \mathsf{Q}_I' = e^{b\mathsf{x}}e^{2\pi b\mathsf{p}}e^{b\mathsf{x}}e^{\pi ibt}, \tag{202}$$

where x and p satisfy the commutation relations $[\mathsf{x},\mathsf{p}] = \frac{i}{2}$ and $t \in i\mathbb{R}$ commutes with x and p.

---

[6] These include comparison of certain structure functions related to the cases for the three point functions that are calculable in terms of Dotsenko-Fateev integrals (cf. part I). A detailed comparison of such data as obtained from [BCT1][BCT2], [GN1][GN2] and [OW] was carried out in [GS2], where mutual consistency of these results was found.

16.2. **Ordering the operators**

It turns out to be possible to translate (200) into a relation that only contains commuting quantities: This becomes possible thanks to identities such as[7]

$$\big(e^{-2\pi i b\alpha_1}\mathsf{Q}_I^c + e^{2\pi i b\alpha_1}\mathsf{Q}_I\big)^{s_2} \;=\; e^{2bs_2\mathsf{x}}e^{-\pi b s_2(\mathsf{p}-is_2\frac{b}{2})}\,\frac{S_b\big(\frac{Q}{2}+2\alpha_1+i\mathsf{p}+s_2b+t\big)}{S_b\big(\frac{Q}{2}+2\alpha_1+i\mathsf{p}+t\big)},$$

which follow quite easily from the functional equation $S_b(x+b) = 2\sin(\pi b x)S_b(x)$. These identities allow one to collect all x-dependent factors to the left of the terms containing p. It follows that (200) is equivalent to[8]

$$e^{\frac{\pi i}{2}b^2(s_2^2-s_1^2)}e^{\pi i b^2 s_1 s_2}e^{\pi b(s_1-s_2)\mathsf{p}}\,\frac{S_b\big(\frac{Q}{2}+i\mathsf{p}+2\alpha_1+(s_2+s_1)b+t\big)S_b\big(\frac{Q}{2}-i\mathsf{p}+t\big)}{S_b\big(\frac{Q}{2}+i\mathsf{p}+2\alpha_1+bs_1+t\big)S_b\big(\frac{Q}{2}-i\mathsf{p}-bs_1+t\big)} =$$

$$(203)\qquad = \int_{\mathbb{T}} dt_2dt_1\; e^{-4\pi i(\alpha_1+bt_1)\alpha_2}e^{\frac{\pi i}{2}b^2(t_2^2-t_1^2)}e^{-\pi i b^2 t_1t_2}e^{\pi b(t_1-t_2)\mathsf{p}}B_-(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1} \times\frac{S_b\big(\frac{Q}{2}+i\mathsf{p}+2\alpha_2+(t_2+t_1)b-t\big)S_b\big(\frac{Q}{2}-i\mathsf{p}-t\big)}{S_b\big(\frac{Q}{2}+i\mathsf{p}+2\alpha_2+bt_2-t\big)S_b\big(\frac{Q}{2}-i\mathsf{p}-bt_2-t\big)}.$$

It is furthermore convenient to take the Fourier-transformation w.r.t. the variable t of this identity. Let us introduce the following notation:

$$(204)\qquad \begin{aligned}\Phi_\lambda(A,B,C;y) \;&=\; \int_{\mathbb{R}} d\tau\; e^{2\pi i\tau y}\,\tilde{\Phi}_\lambda(A,B,C;\tau),\\ \tilde{\Phi}_\lambda(A,B,C;\tau) \;&\equiv\; \frac{S_b\big(\frac{Q}{2}+i(\tau+A)\big)S_b\big(\frac{Q}{2}+i(\tau+B)\big)}{S_b\big(Q+i(\tau-C+\lambda+i0)\big)S_b\big(Q+i(\tau-C-\lambda+i0)\big)},\end{aligned}$$

where A,B,C are real parameters. The identity obtained by Fourier-transformation of (203) takes the form

$$e^{4\pi i(\alpha_1+bs)\alpha_2}e^{\pi i b^2(s^2-s_1^2)}e^{2\pi b(s_1-s)\mathsf{p}}\Phi_\lambda(A_1,B_1;C_1;y) =$$

$$(205)\qquad = \int_{\mathbb{T}} dt_2dt_1\; e^{4\pi i b t_2\alpha_2}e^{\pi i b^2t_2^2}e^{-2\pi b t_2\mathsf{p}}B_-(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1}\,\Phi_\lambda(A_2,B_2;C_2;-y),$$

with parameters A_1,B_1,C_1 and A_2,B_2,C_2 that can be easily read off from (203), and $s=s_1+s_2=t_1+t_2$.

16.3. **Exploiting the completeness of the Φ_λ**

[PT2, Theorem 4] is equivalent to the statement that for any choice of $A,B,C\in\mathbb{R}$ the set $\{\Phi_\lambda(A,B;C;y);\lambda\in\mathbb{R}^+\}$ forms a basis for $L^2(\mathbb{R})$ with normalization given by

$$\int_{\mathbb{R}} dy\;\big(\Phi_\lambda(A,B;C;y)\big)^*\Phi_{\lambda'}(A,B;C;y) = |S_b(Q+2i\lambda)|^{-2}\delta(\lambda-\lambda').$$

[7] Some care is needed in applying these identities in the present case: They represent relations between unitary operators as long as $\alpha_k\in i\mathbb{R}$ and $s_k\in i\mathbb{R}$, $k=1,2$. For the case we are presently interested in ($\alpha_i\in\frac{Q}{2}+i\mathbb{R}$, $s_i\in-\frac{Q}{2}+i\mathbb{R}$, $t_i\in-\frac{Q}{2}+i\mathbb{R}, i=1,2$) one needs to interpret these relations by analytic continuation from the unitary case.

[8] In the presently considered case $\alpha_i\in\frac{Q}{2}+i\mathbb{R}, s_i\in-\frac{Q}{2}+i\mathbb{R}, t_i\in-\frac{Q}{2}+i\mathbb{R}, i=1,2$ one needs to interpret (203) as relation between tempered distributions defined by replacing $\pm i\mathsf{p}$ by $\pm i(\mathsf{p}\pm i0)$, cf. the previous footnote.

This means that in the case $\alpha_i \in \frac{Q}{2} + i\mathbb{R}$, $s_i \in -\frac{Q}{2} + i\mathbb{R}$, $t_i \in -\frac{Q}{2} + i\mathbb{R}$, $i = 1, 2$, the sought-for relation (203) is nothing but a representation for the unitary transformation relating two such bases $\{\Phi_\lambda(A_1, B_1; C_1; y); \lambda \in \mathbb{R}^+\}$ and $\{\Phi_\lambda(A_2, B_2; C_2; -y); \lambda \in \mathbb{R}^+\}$, with kernel $B_\epsilon(\alpha_2, \alpha_1|\mathsf{p})^{s_2 s_1}_{t_2 t_1}$ given in terms of the overlap

$$\int_{\mathbb{R}} dy\ \big(\Phi_\lambda(A_1, B_1; C_1; y)\big)^* \Phi_{\lambda'}(A_2, B_2; C_2; -y).$$

The rest of the calculation is straightforward. The result can be written as

$$B_\epsilon(\alpha_2, \alpha_1|P)^{s_2 s_1}_{t_2 t_1} \ = \ \delta_{st}\ \frac{e^{\epsilon A} e^{B}}{|S_b(2C)|^2} \int_{i\mathbb{R}} ds\ \prod_{i=1}^{4} \frac{S_b(s + R_i)}{S_b(s + S_i)}, \tag{206}$$

where $\delta_{st} \equiv \delta(s_1 + s_2 - t_1 + t_2)$, A, B, and C are given as

$$\begin{aligned} A \ &= \pi i b^2(s_1^2 + t_2^2 - s^2) - 2\pi b P(s_1 + t_2 - s) - 2\pi i b\big((s - t_2)\alpha_2 + (s - s_1)\alpha_1\big), \\ B \ &= 2\pi i b\big((s - t_2)\alpha_2 - (s - s_1)\alpha_1\big), \\ C \ &= \ - iP - \alpha_2 - bt_2, \end{aligned} \tag{207}$$

and the coefficients R_i and S_i, $i = 1, \ldots, 4$ are defined by

$$\begin{aligned} R_1 &= \tfrac{Q}{2} - 2\alpha_2 - iP - b(t_1 + t_2), & S_1 &= \tfrac{3Q}{2} - 2\alpha_2 - iP - bt_2, \\ R_2 &= \tfrac{Q}{2} + 2\alpha_1 + iP + b(s_1 + s_2), & S_2 &= \tfrac{Q}{2} + 2\alpha_1 + iP + bs_1, \\ R_3 &= \tfrac{Q}{2} + iP, & S_3 &= \tfrac{3Q}{2} + iP + bt_2, \\ R_4 &= \tfrac{Q}{2} - iP, & S_4 &= \tfrac{Q}{2} - iP - bs_1. \end{aligned} \tag{208}$$

Part IV. CHIRAL BOOTSTRAP

In this part we would like to explain how Liouville theory fits into a "chiral bootstrap" formalism similar to the one developed by Moore and Seiberg [MS] for rational conformal field theories (see [FFK] for a very similar formalism). Such a formalism is useful to gain further insight into the mathematics that is behind the consistency of Liouville theory, like fusion of unitary Virasoro representations, and the connection with quantum group representation theory [PT1]. It furthermore provides a convenient framework for completing the proof of locality and crossing symmetry of operators V_α that are characterized by the DOZZ three-point function. Let us finally note that having such a formulation is important when trying to construct Liouville theory with conformally invariant boundary conditions by means of a formalism that resembles the one introduced in [BPPZ][FFFS] for rational conformal field theories in the presence of boundaries.

17. FORMAL CHIRAL VERTEX OPERATORS

17.1. Definition

Chiral vertex operators $\mathsf{V}^{\mathrm{N}}_{\mathbb{A}}(z)$, $\mathbb{A} \equiv \big({}_{\alpha_3}{}^{\alpha_2}{}_{\alpha_1}\big)$ may be introduced as maps from the Verma module $\mathcal{V}_{\alpha_1}$ (cf. section 6 for notation) to $\mathcal{V}_{\alpha_3}$ with the properties

$$
(209)\qquad \begin{aligned} &(\mathrm{i}) \quad \big[L_n, \mathsf{V}^{\mathrm{N}}_{\mathbb{A}}(z)\big] \;=\; z^n(z\partial_z + \Delta_{\alpha_2}(n+1))\mathsf{V}^{\mathrm{N}}_{\mathbb{A}}(z),\\ &(\mathrm{ii}) \quad \mathsf{V}^{\mathrm{N}}_{\mathbb{A}}(z)v_{\alpha_1} \;=\; z^{\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1}}\big(\mathrm{N}(\alpha_3,\alpha_2,\alpha_1)v_{\alpha_3} + \mathcal{O}(z)\big), \end{aligned}
$$

where v_α denotes the highest weight vector in $\mathcal{V}_\alpha$. We will denote $\mathsf{V}_{\mathbb{A}}(z) \equiv \mathsf{V}^{N\equiv 1}_{\mathbb{A}}(z)$. Knowing $\mathsf{V}_{\mathbb{A}}(z)$ is good enough since $\mathsf{V}^{\mathrm{N}}_{\mathbb{A}}(z) = \mathrm{N}(\alpha_3,\alpha_2,\alpha_1)\mathsf{V}_{\mathbb{A}}(z)$.

REMARK 7. — The variable z is to be considered as a formal variable for the moment: $\mathsf{V}_{\mathbb{A}}(z)$ is considered as a generating device for a collection of maps $\mathsf{V}_{\mathbb{A}}(k) : \mathcal{V}_{\alpha_1}[n] \to \mathcal{V}_{\alpha_3}[n+k]$ via

$$
(210)\qquad \mathsf{V}_{\mathbb{A}}(z) \;=\; z^{\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1}} \sum_{k\in\mathbb{Z}} z^k\, \mathsf{V}_{\mathbb{A}}(k).
$$

Similar remarks will apply to the generalizations of the operators $\mathsf{V}_{\mathbb{A}}(z)$ that we will introduce in the rest of this section. This should be kept in mind as we will not present a formalism that makes the "formal" interpretation of chiral vertex operators mathematically precise.

It is often convenient to generalize the $\mathsf{V}_{\mathbb{A}}(z)$ by introducing a family of operators $\mathsf{V}_{\mathbb{A}}(\xi|z)$ called *descendants* of $\mathsf{V}_{\mathbb{A}}(z)$ that are indexed by an element $\xi \in \mathcal{V}_{\alpha_2}$. The operators $\mathsf{V}_{\mathbb{A}}(\xi|z)$ are defined in terms of $\mathsf{V}_{\mathbb{A}}(z)$ by means of $\mathsf{V}_{\mathbb{A}}(v_{\alpha_2}|z) \equiv \mathsf{V}_{\mathbb{A}}(z)$, $\mathsf{V}_{\mathbb{A}}(L_{-1}\zeta|z) = \partial_z \mathsf{V}_{\mathbb{A}}(\zeta|z)$ and

$$
(211)\qquad \begin{aligned} \mathsf{V}_{\mathbb{A}}(L_{-n}\zeta|z) \;&=\; \frac{1}{(n-2)!} : (\partial_z^{n-2}T(z))\mathsf{V}_{\mathbb{A}}(\zeta|z) :,\\ : T(w)\mathsf{V}_{\mathbb{A}}(\zeta|z) : \;&\equiv\; \sum_{n<-1} w^{-n-2}L_n\mathsf{V}_{\mathbb{A}}(\zeta|z) + \mathsf{V}_{\mathbb{A}}(\zeta|z)\sum_{n>-2} w^{-n-2}L_n\,. \end{aligned}
$$

The operators $\mathsf{V}_{\mathbb{A}}(\zeta|z)$, $\zeta \in \mathcal{V}_{\alpha_2}$ satisfy the commutation relations

$$[L_n, \mathsf{V}_{\mathbb{A}}(\zeta|z)] = \sum_{k=-1}^{l(n)} z^{n-k} \binom{n+1}{k+1} \mathsf{V}_{\mathbb{A}}(L_k \zeta|z), \tag{212}$$

where $l(n) = n$ if $n > -2$ and $l(n) = \infty$ otherwise. It is not difficult to show that chiral vertex operators are in fact uniquely characterized by the properties (209): Transformation property (212) and definition (211) of the operators $\mathsf{V}_{\mathbb{A}}(\xi|z)$ are closely related to the conformal Ward identities in the case of the three-punctured Riemann sphere. It follows that the matrix elements of $\mathsf{V}_{\mathbb{A}}(\xi|z)$ can be expressed in terms of the trilinear form $\rho^{\alpha_3,\alpha_2,\alpha_1}_{\infty,z,0}$ as follows (cf. Part I, Section 6 for the notation):

$$\big(\xi_3\,,\, \mathsf{V}_{\mathbb{A}}(\xi_2|z)\xi_1\big)_{\alpha_3} \;=\; \rho^{\alpha_3,\alpha_2,\alpha_1}_{\infty,z,0}(\xi_3,\xi_2,\xi_1), \tag{213}$$

which is equivalent to the following expression for the image of $\xi_1 \in \mathcal{V}_{\alpha_1}$ under $\mathsf{V}_{\mathbb{A}}(\xi_2|z)$:

$$\mathsf{V}_{\mathbb{A}}(\xi_2|z)\xi_1 \;=\; \sum_{\nu\in\mathcal{T}} z^{\Delta(v_\nu)-\Delta(\xi_2)-\Delta(\xi_1)}\, v^t_{\nu,\alpha_3}\, \rho(v_\nu,\xi_2,\xi_1). \tag{214}$$

17.2. **Generalized chiral vertex operators**

It is often useful to further generalize the notion of chiral vertex operators [MS]: One considers them as describing the "fusion" of two "in-going" representations $\mathcal{V}_{\alpha_2}$ and $\mathcal{V}_{\alpha_1}$ associated to points z_2 and z_1 on the Riemann sphere into a third representation $\mathcal{V}_{\alpha_3}$. A generalized vertex operator $\mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;z_2\,z_1}$, will then represent a map $\mathcal{V}_{\alpha_2}\otimes\mathcal{V}_{\alpha_1}\to\mathcal{V}_{\alpha_3}$ which is defined by

$$\mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;z_2\,z_1}(\xi_2\otimes\xi_1) \;=\; \sum_{\nu\in\mathcal{T}} v^t_{\nu,\alpha_3}\, \rho^{\alpha_3,\alpha_2,\alpha_1}_{\infty,z_2,z_1}(v_\nu,\xi_2,\xi_1). \tag{215}$$

These objects are related to the operators $\mathsf{V}_{\mathbb{A}}(\xi|z)$ as follows:

$$\mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;z_2\,z_1}(\xi_2\otimes\xi_1) \;=\; \mathsf{V}^{\;\;\alpha_3}_{\alpha_2\alpha_1}(\xi_2|z_2)\mathsf{V}^{\;\;\alpha_1}_{\alpha_1 0}(\xi_1|z_1). \tag{216}$$

Introducing the operators $\mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;z_2\,z_1}$ is useful to make a profound analogy between conformal field theory and representation theory more manifest: Let us define an action of the Virasoro algebra on the tensor product $\mathcal{V}_{\alpha_2}\otimes\mathcal{V}_{\alpha_1}$ by means of the co-product

$$\Delta_{z_2,z_1}(L_n) \equiv \sum_{k=-1}^{l(n)} \binom{n+1}{k+1} \big(z_2^{n-k} L_k\otimes \mathrm{id} + z_1^{n-k}\mathrm{id}\otimes L_k\big). \tag{217}$$

The definition of $\mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;z_2\,z_1}$ in terms of the conformal Ward identities (via ρ in (215)) is then equivalent to the following intertwining property:

$$L_n\, \mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;z_2\,z_1}(\xi_2\otimes\xi_1) \;=\; \mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;z_2\,z_1}(\Delta_{z_2,z_1}(L_n)\cdot\xi_2\otimes\xi_1). \tag{218}$$

The operators $\mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;z_2\,z_1}$ are analogous to the Clebsch-Gordon maps that describe the projection of the tensor product of two representations onto a third one. They satisfy the following simple relation w.r.t. the exchange of the two tensor factors in $\mathcal{V}_{\alpha_2}\otimes\mathcal{V}_{\alpha_1}$:

$$\mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;z_2\,z_1}(\xi_2\otimes\xi_1) \;=\; O_\epsilon\big({}_{\alpha_3}{}^{\alpha_2}{}_{\alpha_1}\big)\mathsf{V}^{\alpha_3;\alpha_1\alpha_2}_{\;\;\;\;z_1\,z_2}(\xi_1\otimes\xi_2), \tag{219}$$

where $O_\epsilon\big({}_{\alpha_3}{}^{\alpha_2}{}_{\alpha_1}\big) \;\equiv\; e^{\epsilon\pi i(\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1})}$ and $\epsilon = 1$ if $\arg(z_2-z_1)\in(0,\pi]$ and $\epsilon=-1$ if $\arg(z_2-z_1)\in(-\pi,0]$.

REMARK 8. — In order to make the interpretation of the generalized chiral vertex operators as generalizations of Clebsch-Gordon maps more precise, one would need to show that a certain set $\{\mathcal{V}_\alpha; \alpha \in \mathbb{F}\}$ of Virasoro representations is indeed closed under the tensor product operation defined by means of the co-product Δ_{z_2,z_1}, i.e. that there exists a canonical decomposition of the representation defined on $\mathcal{V}_{\alpha_2} \otimes \mathcal{V}_{\alpha_1}$ by means of Δ_{z_2,z_1} into irreducible representations $\mathcal{V}_\alpha$ with $\alpha \in \mathbb{F}$. We will outline an approach to do so (based on "Connes-fusion") in Section 20.

17.3. Composition of chiral vertex operators

By forming compositions of chiral vertex operators it is possible to construct maps $\mathcal{V}_{\alpha_{n-1}} \otimes \ldots \otimes \mathcal{V}_{\alpha_1} \to \mathcal{V}_{\alpha_n}$. Let us consider the example $n = 4$: There are two natural types of compositions of chiral vertex operators: On the one hand ("s-channel"):

$$\mathsf{V}^{\alpha_4;\alpha_3\alpha_s}_{\;z_3\;z_1}\big(\xi_3 \otimes \mathsf{V}^{\alpha_s;\alpha_2\;\alpha_1}_{\;z_{21}\;0}(\xi_2 \otimes \xi_1)\big), \tag{220}$$

where $z_{21} = z_2 - z_1$, and on the other hand ("t-channel"):

$$\mathsf{V}^{\alpha_4;\alpha_t\alpha_1}_{\;z_2\;z_1}\big(\mathsf{V}^{\alpha_t;\alpha_3\;\alpha_2}_{\;z_{32}\;0}(\xi_3 \otimes \xi_2) \otimes \xi_1\big), \tag{221}$$

where $z_{32} = z_3 - z_2$. The corresponding conformal blocks are then recovered as the matrix elements of compositions of generalized chiral vertex operators, for example:

$$\begin{aligned}
\mathcal{F}^s_{\alpha_s}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big](\mathsf{z}) \;&= \big\langle v\,,\,\mathsf{V}^{\alpha_4;\alpha_3\alpha_s}_{\;z_3\;z_1}\big(v \otimes \mathsf{V}^{\alpha_s;\alpha_2\;\alpha_1}_{\;z_{21}\;0}(v \otimes v)\big)\big\rangle_{\alpha_4}, \\
\mathcal{F}^t_{\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big](\mathsf{z}) \;&= \big\langle v\,,\,\mathsf{V}^{\alpha_4;\alpha_t\alpha_1}_{\;z_2\;z_1}\big(\mathsf{V}^{\alpha_t;\alpha_3\;\alpha_2}_{\;z_{32}\;0}(v \otimes v) \otimes v\big)\big\rangle_{\alpha_4},
\end{aligned} \tag{222}$$

where $\mathsf{z} = (z_3, z_2, z_1)$.

18. FUSION AND BRAIDING

So far we have considered chiral vertex operators and conformal blocks in the sense of formal power series only. It appears to be difficult to get information on the analytical properties of these objects from the above definition in terms of representation theory of the Virasoro algebra. It will therefore be important to have an alternative realization that allows one to prove convergence of the power series defining the conformal blocks and to study their analytic properties (monodromies, braiding...).

18.1. Free field realization

A useful realization of chiral vertex operators is furnished by the operators $\mathsf{g}^\alpha_s(x_+)$ that were introduced in Part III, Subsection 15.2: Let us recall that these were well-defined as operators for negative euclidean time $\tau = it$. The corresponding objects on the punctured Riemann sphere are obtained by introducing the variable $z = e^{ix_+}$ and letting $\mathsf{g}^\alpha_s(z) \equiv z^{-\Delta_\alpha}\mathsf{g}^\alpha_s(x_+)$.

We need to explain how to go from such operators on $\mathcal{H}^F$ to operators between Verma modules. To simplify slightly one may assume $\alpha + bs \in i\mathbb{R}$, from which one may get the general case by analytic continuation. We will furthermore assume having chosen a Gelfand triple $\mathcal{T} \subset \mathcal{H}^{\mathrm{F}} \subset \mathcal{T}^\dagger$ (cf. the discussion in Part I, Subsection 4.7). The direct integral representation

$$\mathcal{H}^{\mathrm{F}} \simeq \int_{\mathbb{S}\cup\bar{\mathbb{S}}}^{\oplus} d\alpha\, \mathcal{V}_\alpha \otimes \mathcal{V}_\alpha \tag{223}$$

allows to identify $\mathcal{V}_\alpha \simeq \mathcal{V}_\alpha \otimes v$ with a subspace $\mathcal{T}_\alpha^\dagger \subset \mathcal{T}^\dagger$.

Due to the smooth dependence of $G_s^\alpha(P)$ on P it is for $|z| < 1$ possible to define $\mathsf{g}_s^{\alpha_2}(z)|P_1, \xi\rangle$ as an element of $\mathcal{T}^\dagger$. Since

$$\mathsf{p}\, \mathsf{g}_s^{\alpha_2}(z)|P_1,\xi\rangle = \big(P_1 - i(\alpha_2 + bs)\big)\mathsf{g}_s^{\alpha_2}(z)|P_1,\xi\rangle,$$

we may identify $\mathsf{g}_s^{\alpha_2}(z)|P_1,\xi\rangle$ as an element of $\mathcal{T}^\dagger_{\alpha_1+\alpha_2+bs}$. Let us denote the resulting operator $\mathcal{T}^\dagger_{\alpha_1} \to \mathcal{T}^\dagger_{\alpha_1+\alpha_2+bs}$ by $\mathsf{g}_s^{\alpha_2}(P_1|z)$. We have

$$\begin{aligned}\mathsf{L}_n(P_1 - i(\alpha_2+bs))\mathsf{g}_s^{\alpha_2}(P_1|z)|P_1,\xi\rangle - \mathsf{g}_s^{\alpha_2}(P_1|z)\mathsf{L}_n(P_1)|P_1,\xi\rangle = &\\ = z^n(z\partial_z + \Delta_{\alpha_2}(n+1))\mathsf{g}_s^{\alpha_2}(P_1|z)|P_1,\xi\rangle,&\end{aligned}$$

so that the identification (223) induces

$$\mathsf{g}_s^{\alpha_2}(P_1|z) \simeq \mathsf{V}^{\mathrm{G}}\big(\,{}_{\alpha_3}{}^{\alpha_2}{}_{\alpha_1}\big)(z) \equiv \mathrm{G}(\alpha_3,\alpha_2,\alpha_1)\mathsf{V}\big(\,{}_{\alpha_3}{}^{\alpha_2}{}_{\alpha_1}\big)(z), \tag{224}$$

where $\alpha_1 = \frac{Q}{2} + iP_1$, $\alpha_3 = \alpha_1 + \alpha_2 + bs$ and $\mathrm{G}(\alpha_3,\alpha_2,\alpha_1) \equiv G_s^{\alpha_2}(P_1)$. It now follows from the corresponding properties of the $\mathsf{g}_s^{\alpha_2}(z)$ that the chiral vertex operators $\mathsf{V}_{\mathbb{A}}(z)$ indeed make sense as operators for $|z| < 1$ and as operator-valued distributions for $|z| = 1$.

18.2. **Generalized braiding**

It is then straightforward to translate the braid relations (194) for the covariant chiral operators $\mathsf{g}_s^{\alpha_2}(\sigma)$ into the corresponding relations for the chiral vertex operators $\mathsf{V}^{\mathrm{G}}\big(\,{}_{\alpha_3}{}^{\alpha_2}{}_{\alpha_1}\big)(z)$. They take the following form:

$$\begin{aligned}&\mathsf{V}^{\mathrm{G}}\big(\,{}_{\alpha_4}{}^{\alpha_3}{}_{\alpha_s}\big)(\sigma_2)\mathsf{V}^{\mathrm{G}}\big(\,{}_{\alpha_s}{}^{\alpha_2}{}_{\alpha_1}\big)(\sigma_1) = \\ &\quad = \int_{\mathbb{S}} d\alpha_u\; B^{\mathrm{G},\epsilon}_{\alpha_s\alpha_u}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]\, \mathsf{V}^{\mathrm{G}}\big(\,{}_{\alpha_4}{}^{\alpha_2}{}_{\alpha_u}\big)(\sigma_1)\mathsf{V}^{\mathrm{G}}\big(\,{}_{\alpha_u}{}^{\alpha_3}{}_{\alpha_1}\big)(\sigma_2)\;.\end{aligned} \tag{225}$$

It will be useful to write the braiding coefficients B as follows:

$$B^{\mathrm{G},\epsilon}_{\alpha_s\alpha_u}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big] = e^{\epsilon A}e^{B}\frac{S_b(\alpha_2+\bar{\alpha}_u-\bar{\alpha}_4)S_b(\alpha_s+\bar{\alpha}_4-\alpha_3)}{S_b(\alpha_2+\alpha_s-\alpha_1)S_b(\bar{\alpha}_u+\alpha_1-\alpha_3)}\big\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \bar{\alpha}_4 & \bar{\alpha}_3\end{smallmatrix}\big|\begin{smallmatrix}\alpha_s\\ \bar{\alpha}_u\end{smallmatrix}\big\}_b, \tag{226}$$

where $\bar{\alpha} \equiv Q - \alpha$, A and B are given as

$$\begin{aligned} A &= \pi i b^2(s_1^2 + t_2^2 - s^2) - 2\pi b P(s_1 + t_2 - s) - 2\pi i b\big((s-t_2)\alpha_2 + (s-s_1)\alpha_1\big)\\ V &= 2\pi i b\big((s-t_2)\alpha_2 - (s-s_1)\alpha_1\big),\end{aligned} \tag{227}$$

and $\{\dots\}_b$ are the so-called b-Racah-Wigner symbols [PT2],

$$\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \bar{\alpha}_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b = \frac{S_b(\alpha_2+\alpha_s-\alpha_1)S_b(\alpha_t+\alpha_1-\alpha_4)}{S_b(\alpha_2+\alpha_t-\alpha_3)S_b(\alpha_s+\alpha_3-\alpha_4)} \cdot \\ \cdot |S_b(2\alpha_t)|^2 \int\limits_{-i\infty}^{i\infty} ds \;\frac{S_b(U_1+s)S_b(U_2+s)S_b(U_3+s)S_b(U_4+s)}{S_b(V_1+s)S_b(V_2+s)S_b(V_3+s)S_b(V_4+s)}, \tag{228}$$

with coefficients U_i and V_i, $i=1,\dots,4$ given by

$$\begin{aligned} U_1 &= \alpha_s+\alpha_1-\alpha_2 & V_1 &= 2Q+\alpha_s-\alpha_t-\alpha_2-\alpha_4 \\ U_2 &= Q+\alpha_s-\alpha_2-\alpha_1 & V_2 &= Q+\alpha_s+\alpha_t-\alpha_4-\alpha_2 \\ U_3 &= \alpha_s+\alpha_3-\alpha_4 & V_3 &= 2\alpha_s \\ U_4 &= Q+\alpha_s-\alpha_3-\alpha_4 & V_4 &= Q. \end{aligned} \tag{229}$$

Taking into account the change of normalization (224) it is straightforward to obtain the corresponding relations for the generalized chiral vertex operators $\mathsf{V}^{\alpha_3;\alpha_2\alpha_1}_{\;\;\;\;\;\; z_2 z_1}$ from (225). In particular it is possible to show by direct calculation that (225) simplifies for $\alpha_1 \to 0$ to

$$\mathsf{V}\big({}_{\alpha_4}{}^{\alpha_3}{}_{\alpha_2}\big)(\sigma_2)\mathsf{V}\big({}_{\alpha_2}{}^{\alpha_2}{}_{0}\big)(\sigma_1) \;=\; O_\epsilon\big({}_{\alpha_4}{}^{\alpha_3}{}_{\alpha_2}\big)\mathsf{V}\big({}_{\alpha_4}{}^{\alpha_2}{}_{\alpha_3}\big)(\sigma_1)\mathsf{V}\big({}_{\alpha_3}{}^{\alpha_3}{}_{0}\big)(\sigma_2)\;, \tag{230}$$

showing that the generalized braid relations (225) are indeed consistent with the elementary one (219). One thereby concludes that the Liouville conformal blocks form a representation of the braid group.

18.3. **Fusion**

The generalized braid relations (225) also allow one to construct a kind of associativity relation between s- and t-channel compositions of chiral vertex operators that is analogous to the Moore-Seiberg "fusion move": It is a relation of the following form:

$$\mathsf{V}^{\alpha_4;\alpha_3\alpha_s}_{\;\;\;\;\;\; z_3 z_1}\big(\xi_3 \otimes \mathsf{V}^{\alpha_s;\alpha_2\;\alpha_1}_{\;\;\;\;\;\; z_{21}\, 0}(\xi_2\otimes\xi_1)\big) \;= \\ = \int\limits_{\mathbb{S}} d\alpha_t \;\; F_{\alpha_s\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big]\, \mathsf{V}^{\alpha_4;\alpha_t\alpha_1}_{\;\;\;\;\;\; z_2 z_1}\big(\mathsf{V}^{\alpha_t;\alpha_3\;\alpha_2}_{\;\;\;\;\;\; z_{32}\, 0}(\xi_3\otimes\xi_2)\otimes\xi_1\big). \tag{231}$$

Such relations can be read as expressing some sort of associativity of the "fusion products" of representations defined by means of Δ_{z_2,z_1}, with fusion coefficients $F_{\alpha_s\alpha_t}$ playing the role of the Racah-Wigner coefficients from angular momentum theory. The associativity relation (231) is essentially equivalent to relations for the corresponding conformal blocks like

$$\mathcal{F}^s_{\alpha_s}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big](\mathsf{z}) \;=\; \int\limits_{\mathbb{S}} d\alpha_t \;\; F_{\alpha_s\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big]\, \mathcal{F}^t_{\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big](\mathsf{z}). \tag{232}$$

Relations of the form (231) can indeed be constructed in terms of the braid relations (225) and (219). This is done by *defining* fusion as the result of the sequence of braid-moves that may be schematically indicated by $3(21) \to 3(12) \to 1(32) \to (32)1$, cf. [MS]. The first and the last of these moves are represented by (219), the middle one by (225). By means of the explicit expressions for O_ϵ and $B^\epsilon_{\alpha_s\alpha_u}$ it is possible to verify that the resulting relation (231) does not depend on the sign choices made to define the generalized chiral vertex operators involved in (231).

Taking into account the explicit expression for $G(\alpha_3, \alpha_2, \alpha_1)$ that follows from (193), it is straightforward to deduce the following explicit expression for $F_{\alpha_s \alpha_t}$ from (226):

$$F_{\alpha_s\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big] \;\equiv\; \frac{\mathrm{N}(\alpha_4, \alpha_3, \alpha_s)\mathrm{N}(\alpha_s, \alpha_2, \alpha_1)}{\mathrm{N}(\alpha_4, \alpha_t, \alpha_1)\mathrm{N}(\alpha_t, \alpha_3, \alpha_2)} \left\{ \begin{smallmatrix} \alpha_2 & \alpha_1 \\ \bar{\alpha}_4 & \bar{\alpha}_3 \end{smallmatrix} \Big| \begin{smallmatrix} \alpha_s \\ \bar{\alpha}_t \end{smallmatrix} \right\}_b, \tag{233}$$

with $\mathrm{N}(\alpha_3, \alpha_2, \alpha_1)$ being defined by the expression

$$\begin{aligned} &\mathrm{N}(\alpha_3, \alpha_2, \alpha_1) = \\ &= \frac{\Gamma_b(2\alpha_1)\Gamma_b(2\alpha_2)\Gamma_b(2Q - 2\alpha_3)}{\Gamma_b(2Q - \alpha_1 - \alpha_2 - \alpha_3)\Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_3)\Gamma_b(\alpha_1 + \alpha_3 - \alpha_2)\Gamma_b(\alpha_2 + \alpha_3 - \alpha_1)}. \end{aligned} \tag{234}$$

We will discuss the relation between the b-Racah-Wigner coefficients that appear in (233) and the representation theory of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ in Section 19.

18.4. **Consistency conditions**

Different ways to construct conformal blocks by composing generalized chiral vertex operators can be associated with the tree-level Feynman graphs in a φ^3 theory, where all lines are "colored" with labels $\alpha \in \mathbb{S}$ of Virasoro Verma modules. The corresponding graphs with colors only on the external lines parameterize *sets* of conformal blocks which have elements distinguished by the coloring of the internal lines. One may view the elementary braiding transformation (219) and the fusion relation (231) as elementary moves that allow one to relate sets of conformal blocks which have graphs with the same number and coloring of the external lines. The relation between two such graphs can generically be decomposed into elementary braid- and fusion transformations in more than one way, but the resulting relation between the two sets of conformal blocks must be identical. This leads to a bunch of identities that fusion and braid coefficients have to satisfy [MS][FFK]. These identities include:

PENTAGON: —

$$\begin{aligned} &\int_{\mathbb{S}} d\delta_1 \; F_{\beta_1\delta_1}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \beta_2 & \alpha_1 \end{smallmatrix}\big] F_{\beta_2\gamma_2}\big[\begin{smallmatrix} \alpha_4 & \delta_1 \\ \alpha_5 & \alpha_1 \end{smallmatrix}\big] F_{\delta_1\gamma_1}\big[\begin{smallmatrix} \alpha_4 & \alpha_3 \\ \gamma_2 & \alpha_2 \end{smallmatrix}\big] = \\ &\qquad = F_{\beta_2\gamma_1}\big[\begin{smallmatrix} \alpha_4 & \alpha_3 \\ \alpha_5 & \beta_1 \end{smallmatrix}\big] F_{\beta_1\gamma_2}\big[\begin{smallmatrix} \gamma_1 & \alpha_2 \\ \alpha_5 & \alpha_1 \end{smallmatrix}\big], \end{aligned} \tag{235}$$

HEXAGON: —

$$\begin{aligned} &\int_{\mathbb{S}} d\alpha_{32} \; F_{\alpha_{21}\alpha_{32}}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big] O_{\pm}\big(\alpha_{32} {}^{\alpha_4} \alpha_1\big) F_{\alpha_{32}\alpha_{31}}\big[\begin{smallmatrix} \alpha_1 & \alpha_3 \\ \alpha_4 & \alpha_2 \end{smallmatrix}\big] \;= \\ &\qquad = \; O_{\pm}\big(\alpha_{21} {}^{\alpha_2} \alpha_1\big) F_{\alpha_{21}\alpha_{31}}\big[\begin{smallmatrix} \alpha_3 & \alpha_1 \\ \alpha_4 & \alpha_2 \end{smallmatrix}\big] O_{\pm}\big(\alpha_{31} {}^{\alpha_3} \alpha_1\big) \end{aligned}$$

It can be shown [MS] that the pentagon and hexagon identities suffice to ensure consistency of fusion and braiding in general.

19. RELATION TO TENSOR CATEGORY OF QUANTUM GROUP REPRESENTATIONS

We had expressed the fusion- and braid coefficients of Liouville theory in terms of some object that we called "b-Racah-Wigner coefficients". These objects were first constructed as "re-coupling" coefficients that describe the relation between two canonical ways to reduce the triple tensor products of certain quantum group representations into irreducible representations [PT2]. Their role within quantum group representation theory is therefore analogous to the role of the fusion coefficients for the representation theory of the Virasoro algebra. We will see that fusion- and b-Racah-Wigner coefficients become *identical* by a suitable choice of normalization of the chiral vertex operators. We therefore observe a deep relationship between representation theory of the Virasoro algebra and quantum group representation theory, for which a more direct explanation remains to be found.

19.1. A class of representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Let us recall the definition of the quantum group in question: $\mathcal{U}_q(\mathfrak{sl}_2)$ is a Hopf-algebra with

(236)
$$\begin{aligned} &\text{generators:}\quad E,\quad F,\quad K,\quad K^{-1}\\ &\text{relations:}\quad KE = qEK \qquad KF = q^{-1}FK \qquad [E,F] = -\frac{K^2-K^{-2}}{q-q^{-1}}\\ &\text{co-product:}\quad \Delta(K) = K\otimes K \qquad \begin{aligned}\Delta(E) &= E\otimes K + K^{-1}\otimes E\\ \Delta(F) &= F\otimes K + K^{-1}\otimes F\end{aligned}\\ &\text{q-Casimir:}\quad C = FE - (q-q^{-1})^{-2}\big(qK^2 + q^{-1}K^{-2} - 2\big) \end{aligned}$$

We will use the notation $U_q(\mathfrak{sl}(2,\mathbb{R}))$ for $\mathcal{U}_q(\mathfrak{sl}_2)$ supplemented by the following star-structure

(237)
$$\text{star-structure:}\quad K^* = K \qquad E^* = E \qquad F^* = F,$$

which defines the hermiticity assignments for what is called a unitary representation of $U_q(\mathfrak{sl}(2,\mathbb{R}))$.

The following set of representations $\mathcal{P}_\alpha$ by unbounded operators on the Hilbert-space $L^2(\mathbb{R})$ was considered in [PT1][PT2]:

(238)
$$\begin{aligned} \mathsf{E}_\alpha &= \mathsf{U}^{+1}\,\frac{e^{\pi ib(Q-\alpha)}\mathsf{V} - e^{-\pi ib(Q-\alpha)}\mathsf{V}^{-1}}{e^{\pi ib^2} - e^{-\pi ib^2}}\\ \mathsf{F}_\alpha &= \mathsf{U}^{-1}\,\frac{e^{-\pi ib(Q-\alpha)}\mathsf{V} - e^{\pi ib(Q-\alpha)}\mathsf{V}^{-1}}{e^{\pi ib^2} - e^{-\pi ib^2}} \end{aligned} \qquad \mathsf{K}_\alpha = \mathsf{V},$$

where U is the operator of multiplication by $e^{2\pi bx}$, and V acts on $f(x)$ as $\mathsf{V}f(x) = f(x + i\frac{b}{2})$. These representations appear in the list of "well-behaved" unitary irreducible representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that was obtained in [S1] when $\alpha \in \frac{Q}{2} + i\mathbb{R}$. However, these representations do not reproduce representations of the classical Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ in the limit $b \to 0$, which is why we will call this series of representations the "strange series" of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.

19.2. Why to consider the strange series?

The proposal [PT1] to consider the representations introduced in the previous subsection was motivated by previous proposals concerning the appearance of objects from the representation theory of quantum groups in Liouville theory, going back to [FT, B1, G1] (see [CGR, GS1] for more recent developments). It was shown in particular in [CGR, GS1] that chiral vertex operators which correspond to triples $(\alpha_3, \alpha_2, \alpha_1)$ subject to a certain integrality constraint have fusion coefficients that can be expressed in terms of q-6j symbols associated to the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$. It is then natural to expect that fusion and braiding transformations of general chiral vertex operators can also be expressed in terms of q-6j symbols associated to $\mathcal{U}_q(\mathfrak{sl}_2)$.

The problem is to choose the right set of representations of $\mathcal{U}_q(\mathfrak{sl}_2)$. Let us present an a-posteriori line of argument that leads to the proposal of [PT1]. This proposal was originally made for the "weak-coupling" case of real b. The first question is: What hermiticity relations should the generators of $\mathcal{U}_q(\mathfrak{sl}_2)$ satisfy in the representation (choice of star-structure for $\mathcal{U}_q(\mathfrak{sl}_2)$)? The results of [CGR, GS1] indicate that the deformation parameter q is related to the coupling constant b of Liouville theory via

$$q = e^{\pi i b^2}. \tag{239}$$

It is shown in [MMNNSU] that the only star-structure on $\mathcal{U}_q(\mathfrak{sl}_2)$ that is compatible with (239) is (237). This motivates us to look for unitary representations of $U_q(\mathfrak{sl}(2,\mathbb{R}))$. A classification result for such representations can be found in [S1]. But which of these representations should we choose?

The representation introduced in (238) is distinguished by having a certain remarkable self-duality property under the replacement $b \to b^{-1}$: These representations have the property that replacing $b \to b^{-1}$ in the expressions for the generators E_α, F_α, E_α above yields operators $\tilde{\mathsf{E}}_\alpha$ $\tilde{\mathsf{F}}_\alpha$, $\tilde{\mathsf{K}}_\alpha$ that generate a representation of the "dual" algebra $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ that commutes with the $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-representation generated by E_α, F_α, K_α [9].

REMARK 9. — An alternative point of view on this self-duality phenomenon is to regard the strange series representations as representations of the *modular double* of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ introduced by Faddeev[F], see [KLS, Section 1] for a nice discussion of this concept.

It is this self-duality that may be taken as a hint that this class of representations is well-suited for making contact with the DOZZ-proposal on the one hand (where we had observed such a self-duality earlier), and with the results of [CGR, GS1] on the other hand. In these latter references it was in particular found that the fusion coefficients for the special class of chiral vertex operators considered therein show a factorization into q-6j symbols for $\mathcal{U}_q(\mathfrak{sl}_2)$ times $\tilde{q}$-6j symbols for $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}_2))$ where $\tilde{q} = e^{\pi i/b^2}$, expressing a form of $b \to b^{-1}$ duality for that particular class of operators.

19.3. **Generalized Clebsch-Gordan coefficients**

We will now briefly review the results of [PT2] which is devoted to the construction of the b-Racah-Wigner coefficients of the strange series representations.

[9] The precise sense of this term is subtle in this context involving unbounded operators. In fact, these two sets of operators do not commute in the strict sense (commutativity of spectral projections). The statement rather is that there exists a natural domain $\mathcal{P}_\alpha$ such that the representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ that are generated by E_α, F_α, K_α and $\tilde{\mathsf{E}}_\alpha$ $\tilde{\mathsf{F}}_\alpha$, $\tilde{\mathsf{K}}_\alpha$ respectively commute on $\mathcal{P}_\alpha$. The domain $\mathcal{P}_\alpha$ may be considered as an analogue of the Schwartz-space for the representations of the two Hopf-algebras

The first main result of [PT2] is the closure of the strange series representations under tensor products:

(240) $$\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \;\simeq\; \int_{\mathbb{S}}^{\oplus} d\alpha\; \mathcal{P}_{\alpha,} \qquad \mathbb{S} \equiv \frac{Q}{2} + i\mathbb{R}^{+}.$$

This result forms the basis for an associated calculus of Clebsch-Gordan and Racah-Wigner coefficients that strongly resembles standard angular momentum theory:

To begin with, one needs to note that the projection operators $\mathsf{C}(\alpha_3|\alpha_2,\alpha_1) : \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \to \mathcal{P}_{\alpha_3}$ may be explicitly represented by an integral transform

(241) $$\mathsf{C}(\alpha_3|\alpha_2,\alpha_1) : f(x_2,x_1) \longrightarrow F[f](\alpha_3|x_3) \equiv \int_{\mathbb{R}} dx_2 dx_1 \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix} \right] f(x_2,x_1).$$

Explicit expressions for the distributional kernel $[\ldots]$ (the "Clebsch-Gordan coefficients") can be found in [PT2]. The kernel $[\ldots]$ can be shown to satisfy orthogonality and completeness relations of the form:

(242) $$\begin{aligned} &\int_{\mathbb{R}} dx_1 dx_2 \; \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix} \right]^* \left[\begin{smallmatrix} \beta_3 & \alpha_2 & \alpha_1 \\ y_3 & x_2 & x_1 \end{smallmatrix} \right] = |S_b(2\alpha_3)|^{-2} \delta(\alpha_3 - \beta_3)\delta(x_3 - y_3). \\ &\int_{\mathbb{S}} d\alpha_3 \; |S_b(2\alpha_3)|^2 \int_{\mathbb{R}} dx_3 \; \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix} \right]^* \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & y_2 & y_1 \end{smallmatrix} \right] = \delta(x_2 - y_2)\delta(x_1 - y_1). \end{aligned}$$

19.4. **Generalized Racah-Wigner coefficients**

Triple tensor products $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ carry a representation π_{321} of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ given by $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$. The projections affecting the decomposition of this representation into irreducibles can be constructed by iterating Clebsch-Gordan maps. One thereby obtains two canonical bases in the sense of generalized eigenfunctions for $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ given by the sets of distributions ($\mathfrak{x} = (x_4, \ldots, x_1)$)

(243) $$\begin{aligned} \Phi^s_{\alpha_s}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right](\mathfrak{x}) &= \int_{\mathbb{R}} dx_s \; \left[\begin{smallmatrix} \alpha_4 & \alpha_3 & \alpha_s \\ x_4 & x_3 & x_s \end{smallmatrix} \right] \left[\begin{smallmatrix} \alpha_s & \alpha_2 & \alpha_1 \\ x_s & x_2 & x_1 \end{smallmatrix} \right] \quad \alpha_4, \alpha_s \in \mathbb{S}, x_4 \in \mathbb{R} \\ \Phi^t_{\alpha_t}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right](\mathfrak{x}) &= \int_{\mathbb{R}} dx_t \; \left[\begin{smallmatrix} \alpha_4 & \alpha_t & \alpha_1 \\ x_4 & x_t & x_1 \end{smallmatrix} \right] \left[\begin{smallmatrix} \alpha_t & \alpha_3 & \alpha_2 \\ x_t & x_3 & x_2 \end{smallmatrix} \right]. \quad \alpha_4, \alpha_t \in \mathbb{S}, x_4 \in \mathbb{R} \end{aligned} .$$

It is possible to show that the resulting spectral decompositions for the operators $\pi_{321}(C)$ and $\pi_{321}(K)$ do not depend on the order in which the Clebsch-Gordan decompositions were performed:

(244) $$\mathcal{P}_{a_3} \otimes \mathcal{P}_{a_2} \otimes \mathcal{P}_{a_1} \simeq \int_{\mathbb{S}}^{\oplus} d\alpha \int_{\mathbb{R}}^{\oplus} dk \;\; \mathcal{H}_{\alpha,k}.$$

It then follows from completeness of the bases $\mathfrak{B}^s_{321}$ and $\mathfrak{B}^t_{321}$ and orthogonality of the eigenspaces $\mathcal{H}_{\alpha,k}$ that the bases Φ^s and Φ^t must be related by a transformation of the form

(245) $$\Phi^s_{\alpha_s}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right](\mathfrak{x}) \;=\; \int_{\mathbb{S}} d\alpha_t \;\; \left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b \; \Phi^t_{\alpha_t}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right](\mathfrak{x})$$

thereby defining the b-Racah-Wigner symbols $\left\{ \begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix} \middle| \begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix} \right\}_b$. Their explicit expression was found in [PT2] to be given by equation (228).

19.5. **Equivalence to Liouville fusion coefficients**

Let us reconsider the expression (233) that we had found for the fusion coefficients. It is natural to introduce

$$\mathsf{C}^{\alpha_3;\alpha_2\,\alpha_1}_{\;\;\;\;z_2\,z_1}(\xi_2\otimes\xi_1) \;=\; \mathrm{N}^{-1}(\alpha_3,\alpha_2,\alpha_1)\mathsf{V}^{\alpha_3;\alpha_2\,\alpha_1}_{\;\;\;\;z_2\,z_1}(\xi_2\otimes\xi_1), \tag{246}$$

since the chiral vertex operators C will then satisfy a fusion relation like (231) that involves only the b-Racah-Wigner symbols:

$$\begin{aligned} &\mathsf{C}^{\alpha_4;\alpha_3\,\alpha_s}_{\;\;\;\;z_3\,z_1}\big(\xi_3\otimes\mathsf{C}^{\alpha_s;\alpha_2\;\alpha_1}_{\;\;\;\;z_{21}\;0}(\xi_2\otimes\xi_1)\big) \;= \\ &\qquad = \int_{\mathbb{S}} d\alpha_t\;\; \left\{ \begin{smallmatrix} \alpha_2 & \alpha_1 \\ \bar\alpha_4 & \bar\alpha_3 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \bar\alpha_t \end{smallmatrix} \right\}_b \mathsf{C}^{\alpha_4;\alpha_t\,\alpha_1}_{\;\;\;\;z_2\,z_1}\big(\mathsf{C}^{\alpha_t;\alpha_3\;\alpha_2}_{\;\;\;\;z_{32}\;0}(\xi_3\otimes\xi_2)\otimes\xi_1\big). \end{aligned} \tag{247}$$

It remains to observe that the b-Racah-Wigner symbols satisfy a symmetry relation of the following form:

$$\left\{ \begin{smallmatrix} \alpha_2 & \alpha_1 \\ \bar\alpha_4 & \bar\alpha_3 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \bar\alpha_t \end{smallmatrix} \right\}_b \;=\; \left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b. \tag{248}$$

The proof of this relation involves the behavior of the kernel $[\dots]$ under complex conjugation, reflection identities that express the equivalence of representations $\mathcal{P}_\alpha$ and $\mathcal{P}_{Q-\alpha}$ [PT2], together with some simple symmetry properties of the integral representation (228). Details will be given elsewhere.

By inserting (248) into (247) one has finally brought the fusion transformations of conformal blocks into a form that makes the analogy with the definition (245) of the b-Racah-Wigner symbols perfect.

19.6. **Locality and crossing symmetry**

The relation between fusion coefficients and b-Racah-Wigner symbols is quite useful: Having established the existence of fusion transformations of the form (231) allows one to translate the condition of crossing symmetry into a relation involving the three point functions $C(\alpha_3,\alpha_2,\alpha_1)$ together with the fusion coefficients:

$$\begin{aligned} &\int_{\mathbb{S}} d\alpha_s\; C(\alpha_4,\alpha_3,\alpha_s)C(\bar\alpha_s,\alpha_2,\alpha_1)F_{\alpha_s\beta_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big]F_{\bar\alpha_s\bar\alpha_t}\big[\begin{smallmatrix} \bar\alpha_3 & \bar\alpha_2 \\ \bar\alpha_4 & \bar\alpha_1 \end{smallmatrix}\big] = \\ &\qquad\qquad\qquad\qquad = \delta_{\mathbb{S}}(\alpha_t,\beta_t)C(\alpha_4,\alpha_t,\alpha_1)C(\bar\alpha_t,\alpha_3,\alpha_2), \end{aligned} \tag{249}$$

where $\delta_{\mathbb{S}}(\alpha_t,\beta_t)=\delta(P-P')$ if $\alpha_t=\frac{Q}{2}+iP$, $\beta_t=\frac{Q}{2}+iP'$. We have used that $F_{\alpha_s\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big] = F_{\bar\alpha_s\bar\alpha_t}\big[\begin{smallmatrix} \bar\alpha_3 & \bar\alpha_2 \\ \bar\alpha_4 & \bar\alpha_1 \end{smallmatrix}\big]$, which trivially follows from the facts that the conformal blocks depend on the conformal dimensions only and $\Delta_\alpha=\Delta_{\bar\alpha}$.

But on the other hand one may observe that the construction of the b-Racah-Wigner symbols in terms of representation theory of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ implies the following unitarity relation [PT2] :

$$\int_{\mathbb{S}} d\alpha_s \; M_b(\alpha_s) \; \left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b \left\{ \begin{smallmatrix} \bar{\alpha}_1 & \bar{\alpha}_2 \\ \bar{\alpha}_3 & \bar{\alpha}_4 \end{smallmatrix} \middle| \begin{smallmatrix} \bar{\alpha}_s \\ \bar{\alpha}'_t \end{smallmatrix} \right\}_b \;=\; M_b(\alpha_t)\; \delta_{\mathbb{S}}(\alpha_t,\alpha'_t), \tag{250}$$

where $\bar{\alpha}_i = Q - \alpha_i$, $\alpha_t, \beta_t \in \mathbb{S}$, and the measure $M_b(\alpha)$ is given by

$$M_b(\alpha) \;=\; |S_b(2\alpha)|^2 = \; -4\sin(\pi b(2\alpha - Q))\sin(\pi b^{-1}(2\alpha - Q)). \tag{251}$$

It now suffices to observe (250) ensures validity of (249) for all $C(\alpha_3,\alpha_2,\alpha_1)$ of the form

$$C(\alpha_3,\alpha_2,\alpha_1) \;=\; N_b \prod_{i=1}^{3} \kappa(\alpha_i) \mathrm{N}^{-1}(\alpha_3,\alpha_2,\alpha_1) \mathrm{N}^{-1}(\bar{\alpha}_3,\bar{\alpha}_2,\bar{\alpha}_1), \tag{252}$$

with arbitrary function $\kappa(\alpha)$ and constant N_b. The DOZZ-formula for $C(\alpha_3,\alpha_2,\alpha_1)$ is easily found to be of the form (252) for suitable choice of $\kappa(\alpha)$, N_b.

This establishes crossing symmetry for the four-point functions of operators V_α that are characterized by the DOZZ-formula for $C(\alpha_3,\alpha_2,\alpha_1)$. The proof of locality is almost identical.

REMARK 10. — Up to now we have mostly assumed that the relevant representations $\mathcal{V}_\alpha$ all correspond to the unitary representations in the spectrum, $\alpha \in \mathbb{S} \equiv \frac{Q}{2} + i\mathbb{R}$. However, fusion-coefficients F and normalization coefficients $N(\alpha_3,\alpha_2,\alpha_1)$ all possess a meromorphic continuation to generic complex values of the representation labels. Due to the analytic properties of the conformal blocks w.r.t. the representation labels (cf. Part I, Subsection 7.1) one may study the analytic continuation of the fusion relations (231) in a way that is very similar to our discussion in Part I, Subsection 7.2.

19.7. **Remark on the strong coupling regime**

Our derivation of braiding and fusion transformations of the conformal blocks in Liouville theory was based on the results of [PT2]. Strictly speaking, it is therefore not directly applicable to the regime of strong coupling ($b = e^{i\varphi}$, $\varphi \in [0, \frac{\pi}{2})$). However, it seems to us that the arguments of [PT2] can be modified to become applicable to the strong coupling case as well. Moreover, if one considers the relations on the level of conformal blocks, cf. e.g. (232), one may note that all the appearing objects can be analytically continued w.r.t. the parameter b from the weak- to the strong coupling regime. We consider it therefore as very likely that all of our discussion applies to the strong coupling case with hardly any change.

Let us note, however, that there exists an alternative proposal for fusion and braid relations of chiral vertex operators at special values of b [G2] (see also [GR1][GR2]). This proposal is based on the construction of a solution to the consistency conditions (cf. Subsection 18.4) that fusion and braiding coefficients must satisfy, which involves only a *discrete* set of representation labels corresponding to unitary representations of the Virasoro algebra.

So far it seems difficult to decide whether this alternative proposal is actually realized. On the one hand it is not clear whether a given solution of the consistency conditions discussed in Subsection 18.4 must in fact be realized by the chiral vertex operators that are uniquely defined in terms of the representation theory of the Virasoro algebra. To the author's knowledge there does not exist a calculation like the one given in Section 16 which would prove that the chiral vertex operators for irreducible representation of the Virasoro algebra satisfy the braid relations proposed in [G2][GR1][GR2].

On the other hand, it is also not obvious that the proposal of [G2] [GR1][GR2] contradicts our results. So far we do not see any reason why there should not exist two different representations of fusion- and braid relations, at least for special values of b and under suitable restrictions on the representation labels.

20. UNITARY FUSION?

As an outlook, we would like to discuss a notion of "fusion" for unitary representations of the Virasoro algebra that should allow to make the deeper mathematical reasons for the consistency of Liouville theory more transparent.

Viewing the chiral vertex operators as Clebsch-Gordan maps for a modified tensor product ("fusion product") of Virasoro representations naturally leads to the question of compatibility of the fusion product with unitarity of the representation. More precisely: Is it possible to relate the above notion of fusion product to a concept of fusion which manifestly creates a unitary "fused" representation as product of two given unitary representations of the Virasoro algebra? In more physical terms this is of course closely related to the natural question whether the set of vertex operators creating *unitary* representations of the Virasoro algebra is closed under operator product expansion. A concept of fusion ("Connes-fusion") that fulfills such a task is known for the WZNW-models associated to the group $SU(N)$ [Wa]. We will try to outline how a similar treatment should look like in the case of unitary Virasoro representations. But before let us indicate how Connes-fusion is related to the more standard picture of fusion that goes back to [BPZ]:

20.1. Heuristics: Fusion on the unit circle

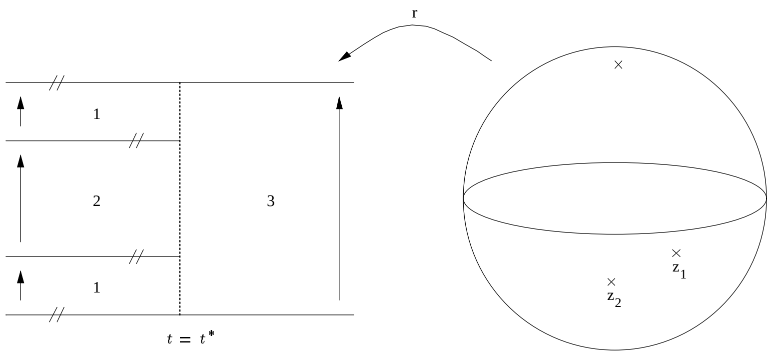
Let us briefly recall (from e.g. [BPZ][MS]) the complex analytic picture of fusion and the associated notion of chiral vertex operators: A chiral vertex operator $\mathsf{V}^{\alpha_3;}{}^{\alpha_2}_{z_2}{}^{\alpha_1}_{z_1}$ is thought of as being associated with a Riemann sphere with three marked points, one of which was chosen to coincide with infinity. The representations $\mathcal{V}_{\alpha_3}$, $\mathcal{V}_{\alpha_2}$ and $\mathcal{V}_{\alpha_1}$ are assigned to the marked points at ∞, z_2 and z_1 respectively. A Virasoro generator $T[v] = \sum_{n\in\mathbb{Z}} v_n L_n$ that "goes out to" ∞ would then be represented by the contour integral $\int_{\mathcal{C}_\infty} dz T(z)v(z)$, $v(z) = \sum_{n\in\mathbb{Z}} z^{n+1}v_n$, where $\mathcal{C}_\infty$ is a small circle around the point ∞. Its action on the "ingoing" representation $\mathcal{V}_{\alpha_2}$ and $\mathcal{V}_{\alpha_1}$ is obtained by deforming the contour $\mathcal{C}_\infty$ into two small circles around the points z_2 and z_1 respectively. This prescription directly leads to the formula (217) for the co-product.

This prescription encodes the analyticity of $T(z)$ within euclidean vacuum expectation values. It is clear that the euclidean picture is not well-suited for questions of unitarity: The euclidean time evolution is not unitary. One should therefore try to change ("Wick-rotate") to a Minkowskian picture. Let us observe that in the case of the cylinder the possibility of deforming the contour $\mathcal{C}$ that enters the definition of $T[v] = \int_{\mathcal{C}} dz T(z)v(z)$ would correspond to conservation of the charge $T[v] \equiv \int_0^{2\pi} d\sigma v(\sigma)T(\sigma)$, $\partial_t T[v] = 0$ in a minkowskian framework.

In order to change to a Minkowskian picture in the case of the sphere with three marked points, one needs a global notion of time on that geometry. A convenient choice is given by the mapping

$$r(z) \;=\; d\ln(z-z_2)+(1-d)\ln(z-z_1). \tag{253}$$

The global (euclidean) time-coordinate is given by $t = \Re(r)$. Lines of constant t may on the Riemann sphere be visualized as electrostatic “equipotential lines” for the potential around the charges at points z_2 and z_1. $r(z)$ maps the complex plane to a diagram that is known as Mandelstam diagram in light-cone string theory:



The markings on the lines in the left part of the diagram are supposed to indicate identifications. The regions marked by 1 and 2 therefore represent (part of) “ingoing” semi-infinite cylinders, whereas the region 3 represents an “outgoing” semi-infinite cylinder. These cylinders are joined at time $t = t^*$, where t^* is a function of z_1, z_2 and d that can be figured out from the definition of $r(z)$. The time-slice at $t = t^*$ looks like the unit circle S^1 divided into two intervals I and I^c, where $S(I)$ (I with end-points identified) represents the ending of the semi-infinite cylinder 2, $S(I^c)$ the ending of cylinder 1.

By means of $r(z)$ one therefore maps the previous contour-deformation picture for the definition of the co-product into a picture with obvious Minkowskian counterpart: On each of the cylinders one may use charge conservation to shift $T[v]$ to the left or right, so everything boils down to the splitting at $t = t^*$. This splitting should clearly be purely geometrical: If $T[f] = \int_{S^1} d\sigma f(\sigma)T(\sigma)$ is the generator of an infinitesimal transformation, it should split into $T[f\Theta_I]$ and $T[f\Theta_{I^c}]$, where Θ_I denotes the characteristic function of the interval I.

It is not a trivial task to turn this heuristic idea into a rigorous definition of fusion that makes the issue of unitary more transparent: First, it is not clear which scalar product to put on $\mathcal{V}_{\alpha_2} \otimes \mathcal{V}_{\alpha_1}$ such that the action defined by the geometric splitting of the unit circle into two subintervals is well-defined and unitary. If one would just take the canonical scalar product on the tensor product of representations, one would get the following problem: $f\Theta_I$ and $f\Theta_{I^c}$, when considered as functions on the circles $S(I)$ and $S(I^c)$, will generically have jumps at the points corresponding to the end-points of I. This leads to the problem that the vector $T[f\Theta_I]v_2 \otimes T[f\Theta_{I^c}]v_1$ will generically not be normalizable when the standard norm in $\mathcal{V}_{\alpha_2} \otimes \mathcal{V}_{\alpha_1}$ is taken.

It seems to us that these problems are just what is overcome by the so-called “Connes-fusion”, more precisely its more explicit version that was developed in [Wa] in the case of loop groups. We would next like to give an idea of such a formulation for the case of unitary representations of the Virasoro algebra.

20.2. **Connes-fusion for unitary Virasoro-representations?**

First, it is technically better to consider the group of diffeomorphisms of the unit circle and its (projective) unitary representations instead of its Lie algebra (the Virasoro) algebra and their representations. It is known [GW] that for $c > 1$ the Virasoro algebra representation $\mathcal{V}_\alpha$ indeed exponentiates to a projective unitary representation of $\mathrm{Diff}(S^1)$ if $\Delta > 0$, i.e. for all unitarizable representations of the Virasoro algebra.

It should be possible to "restrict" $\mathrm{Diff}(S^1)$ to an interval I and its complement I^c in a suitable sense, e.g. by considering the subgroups $\mathrm{Diff}(I)$ and $\mathrm{Diff}(I^c)$ generated by elements of the algebra like $\int_{S^1} d\sigma T(\sigma) f(\sigma)$ and $\int_{S^1} d\sigma T(\sigma) g(\sigma)$, where f, g have support only in I, I^c respectively.

Central objects are then the intertwining operators V_α that "create" the representation $\mathcal{V}_\alpha$ from the vacuum. Here one would consider in particular maps $\mathsf{V}_\alpha \in \mathrm{Hom}_{\mathrm{Diff}(I)}(\mathcal{V}_0, \mathcal{V}_\alpha)$ that intertwine the respective actions of $\mathrm{Diff}(I)$ according to

$$\pi_\alpha(g)\mathsf{V}_\alpha \;=\; \mathsf{V}_\alpha \pi_0(g), \qquad g \in \mathrm{Diff}(I), \tag{254}$$

as well as their counterparts for I^c. Such maps should be given by smeared chiral vertex operators:

$$\mathsf{V}_\alpha(\xi|f) \;\equiv\; \int_{S^1} d\sigma\; f(\sigma)\; \mathsf{V}\big({}_{\alpha}{}^{\alpha}{}_{0}\big)(\xi|\sigma). \tag{255}$$

where $f(\sigma)$ has support only in I^c. It is important that f vanishes in I in order for the ordinary intertwining property (212) to translate into (254).

Let $\mathcal{O}^I_{\alpha_2}$, $\mathcal{O}^{I^c}_{\alpha_1}$ be the spaces of operators $\mathrm{Hom}_{\mathrm{Diff}(I)}(\mathcal{V}_0, \mathcal{V}_\alpha)$ and $\mathrm{Hom}_{\mathrm{Diff}(I^c)}(\mathcal{V}_0, \mathcal{V}_\alpha)$ respectively. One has a correspondence between elements $\mathsf{V}_\alpha \in \mathrm{Hom}_{\mathrm{Diff}(I)}(\mathcal{V}_0, \mathcal{V}_\alpha)$ and the state $\mathsf{V}_\alpha v_0$ that they create when acting on the vacuum. In view of that correspondence one may define fusion for elements of $\mathcal{O}^I_{\alpha_2}, \mathcal{O}^{I^c}_{\alpha_1}$ instead of $\mathcal{V}_{\alpha_2}$, $\mathcal{V}_{\alpha_1}$.

First define a scalar product on $\mathcal{O}^I_{\alpha_2} \otimes \mathcal{O}^{I^c}_{\alpha_1}$ via the four-point conformal block

$$\begin{aligned} \big(\mathsf{V}_{\alpha_2}(\zeta_2|g_2) \otimes \mathsf{V}_{\alpha_1}(\xi_2|f_2)\;,\; \mathsf{V}_{\alpha_2}(\zeta_1|g_1) \otimes \mathsf{V}_{\alpha_1}(\xi_1|f_1)\big)_{\mathcal{O}^I_{\alpha_2}\boxtimes\mathcal{O}^{I^c}_{\alpha_1}} \;\equiv \\ \equiv\; \big\langle v_0\,,\, \mathsf{V}^\dagger_{\alpha_2}(\zeta_2|g_2)\mathsf{V}_{\alpha_2}(\zeta_1|g_1)\, \mathsf{V}^\dagger_{\alpha_1}(\xi_2|f_2)\mathsf{V}_{\alpha_1}(\xi_1|f_1)\, v_0\big\rangle_{\mathcal{V}_0}, \end{aligned} \tag{256}$$

where g_i, f_i, $i = 1, 2$ have support in I^c, I respectively, and $\mathsf{V}^\dagger_\alpha(\zeta|f)$ is the adjoint of $\mathsf{V}^\dagger_\alpha(\zeta|f)$ that may be expressed as

$$\mathsf{V}^\dagger_\alpha(\xi|f) \;\equiv\; \int_{S^1} d\sigma\; f^*(\sigma)\; \mathsf{V}\big({}_{0}{}^{\alpha}{}_{\alpha}\big)(\xi^*|\sigma). \tag{257}$$

The fusion $\mathcal{O}^I_{\alpha_2} \boxtimes \mathcal{O}^{I^c}_{\alpha_1}$ is defined as the completion of $\mathcal{O}^I_{\alpha_2} \otimes \mathcal{O}^{I^c}_{\alpha_1}$ w.r.t. the scalar product introduced in (256).

$\mathcal{O}^I_{\alpha_2} \otimes \mathcal{O}^{I^c}_{\alpha_1}$ carries a natural action of $\mathrm{Diff}(I) \times \mathrm{Diff}(I^c)$. It is easy to see that this action is unitary w.r.t. the scalar product (256). The crucial question now is whether the action of $\mathrm{Diff}(I) \times \mathrm{Diff}(I^c)$ uniquely extends to a unitary action of $\mathrm{Diff}(S^1)$ on $\mathcal{O}^I_{\alpha_2} \boxtimes \mathcal{O}^{I^c}_{\alpha_1}$. This would be the desired representation of $\mathrm{Diff}(S^1)$ obtained as fusion of representations $\mathcal{V}_{\alpha_2}$ and $\mathcal{V}_{\alpha_1}$ associated to the intervals I and I^c respectively. The unique extension of the $\mathrm{Diff}(I) \times \mathrm{Diff}(I^c)$-action looks intuitively plausible: As $\mathrm{Diff}(I)$ and $\mathrm{Diff}(I^c)$ contain diffeomorphisms that are nonzero arbitrarily close to the end points of the respective intervals, it seems unlikely that there is much freedom in “defining the action at the touching points”.

Let us note that the proof of the corresponding unique extension property for loop group representations given in [Wa] makes essential use of the braiding relations for chiral vertex operators. In our case this would be the relations of the form

$$\begin{aligned}
&\big\langle v_0\,,\,\mathsf{V}^{\dagger}_{\alpha_2}(\zeta_2|g_2)\mathsf{V}_{\alpha_2}(\zeta_1|g_1)\,\mathsf{V}^{\dagger}_{\alpha_1}(\xi_2|f_2)\mathsf{V}_{\alpha_1}(\xi_1|f_1)\,v_0\big\rangle_{\mathcal{V}_0} \;=\\
&\quad= \int\limits_{\mathbb{S}} d\alpha\; O_\epsilon\big({}_{\alpha}{}^{\alpha_2}{}_{\alpha_1}\big)\,B^{-\epsilon}_{0\alpha}\big[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_2 & \alpha_1\end{smallmatrix}\big]\,\big\langle v_0\,,\,\mathsf{V}\big({}_{0}{}^{\alpha_1}{}_{\alpha_1}\big)(\xi_2^*|f_2^*)\mathsf{V}\big({}_{\alpha_1}{}^{\alpha_2}{}_{\alpha}\big)(\zeta_2^*|g_2^*)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\mathsf{V}\big({}_{\alpha}{}^{\alpha_2}{}_{\alpha_1}\big)(\zeta_1|g_1)\mathsf{V}\big({}_{\alpha_1}{}^{\alpha_1}{}_{0}\big)(\xi_1|f_1)\,v_0\big\rangle_{\mathcal{V}_0}.
\end{aligned} \tag{258}$$

that we had found previously. These relations would imply that

$$\begin{aligned}
&\big\|\mathsf{V}_{\alpha_2}(\zeta_1|g_1)\otimes\mathsf{V}_{\alpha_1}(\xi_1|f_1)\big\|^2_{\mathcal{O}^I_{\alpha_2}\boxtimes\mathcal{O}^{I^c}_{\alpha_1}} \;=\\
&\quad= \int\limits_{\mathbb{S}} d\mu_{\alpha_2\alpha_1}(\alpha)\;\big\|\mathsf{V}\big({}_{\alpha}{}^{\alpha_2}{}_{\alpha_1}\big)(\zeta_1|g_1)\mathsf{V}\big({}_{\alpha_1}{}^{\alpha_1}{}_{0}\big)(\xi_1|f_1)\,v_0\big\|^2_{\mathcal{V}_\alpha},
\end{aligned} \tag{259}$$

where $d\mu_{\alpha_2\alpha_1}(\alpha_3)$ can be worked out from the explicit expression for the braiding coefficients given in (226) as

$$\begin{aligned}
d\mu_{\alpha_2,\alpha_1}(\alpha_3) \;=\; &\frac{\Gamma_b(2Q)}{\Gamma_b(Q)}\;d\alpha_3\;|S_b(2\alpha_3)|^2\;\times\\
&\times\left|\frac{\Gamma_b(\alpha_1+\alpha_2+\alpha_3-Q)\Gamma_b(\alpha_1+\alpha_2-\alpha_3)\Gamma_b(\alpha_1+\alpha_3-\alpha_2)\Gamma_b(\alpha_2+\alpha_3-\alpha_1)}{\Gamma_b(2\alpha_1)\Gamma_b(2\alpha_2)\Gamma_b(2\alpha_3)}\right|^2.
\end{aligned} \tag{260}$$

Relation (259) can be read as an expression for the unitary equivalence

$$\mathcal{O}^I_{\alpha_2}\boxtimes\mathcal{O}^{I^c}_{\alpha_1} \;\simeq\; \int\limits_{\mathbb{S}}^{\oplus} d\mu_{\alpha_2\alpha_1}(\alpha)\;\mathcal{V}_\alpha. \tag{261}$$

The unique extension of the $\mathrm{Diff}(I)\times\mathrm{Diff}(I^c)$-action to an action of $\mathrm{Diff}(S^1)$ should then follow as in [Wa] from similar statements concerning the respective actions on the irreducible representations $\mathcal{V}_\alpha$ that appear in (261).

Having established the braid relations (258) motivates our hope that a treatment along such lines is within reach. We find it particularly satisfactory to observe that the change of normalization of the chiral vertex operators that established the relation between fusion coefficients and b-Racah-Wigner symbols is precisely such that the fusion density $d\mu_{\alpha_2\alpha_1}(\alpha)$ would become proportional to the canonical measure $d\alpha|S_b(2\alpha)|^2$ if we had used the chiral vertex operators $\mathsf{C}^{\alpha_3;}{}^{\alpha_2}_{z_2}{}^{\alpha_1}_{z_1}$ instead of the $\mathsf{V}^{\alpha_3;}{}^{\alpha_2}_{z_2}{}^{\alpha_1}_{z_1}$. This means that the normalization of chiral vertex operators that makes the relation to quantum group representation theory manifest is simultaneously a natural one from the point of view of Connes-Wassermann fusion. We believe that these ideas should make up for a rather beautiful story when being worked out.

REFERENCES

[ABKS] O. Aharony, M. Berkooz, D. Kutasov, N. Seiberg: Linear Dilatons, NS5-branes and Holography, JHEP 9810 (1998) 004 (hep-th/9808149)

[B1] O. Babelon: Extended conformal algebra and Yang-Baxter equation, *Phys. Lett.* **B215** (1988) 523-529

[B2] O. Babelon: Universal exchange algebra for Bloch waves and Liouville theory, *Comm. Math. Phys.* **139** (1991) 619-643

[Ba] E.W. Barnes: Theory of the double gamma function, Phil. Trans. Roy. Soc. A **196** (1901) 265-388

[Be] J. Bernstein: On the support of Plancherel measure, J. Geom. Phys. **5** (1988) 663-710 .

[Bz] F. Berezin: The method of Second Quantization, Academic Press, New York, 1966

[BCGT] E. Braaten, T. Curtright, G. Ghandour, C. Thorn: Nonperturbative Weak-Coupling Analysis of the Quantum Liouville field theory, *Ann. Phys. (N.Y.)* **153** (1984) 147

[BCT1] E. Braaten, T. Curtright, C. Thorn: Quantum Bäcklund transformation for the Liouville theory, *Phys. Lett.* **B118** (1982) 115-120

[BCT2] E. Braaten, T. Curtright, C. Thorn: *Ann. Phys. (N.Y.)* **147** (1983) 365;

[BDIZ] M. Bauer, Ph. Di Francesco, C. Itzykson, J.-B. Zuber: Covariant Differential equations and singular vectors in Virasoro representations, *Nucl. Phys.* **B362** (1991) 515-562

[BP] J. Balog, L. Palla: Locality problem for the Liouville field, *Phys. Lett.* **B274** (1992) 232

[BPPZ] R. Behrend, P. Pearce, V. Petkova, J.-B. Zuber: Boundary conditions in rational conformal field theories, *Nucl. Phys.* **B570** (2000) 525-589

[BPZ] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov: Infinite conformal symmetry in 2D quantum field theory, *Nucl. Phys.* **B241** (1984) 333

[CF] L. Chekhov, V. V. Fock: Quantum Teichmüller space, math/9908165

[CG] E. Cremmer, J.-L. Gervais: The Quantum Strip: Liouville Theory for Open Strings, *Comm. Math. Phys.* **144** (1992) 279-301

[CGR] E. Cremmer, J.-L. Gervais, J.-F. Roussel: The quantum group structure of 2D gravity and the minimal models II: The genus-zero chiral bootstrap, *Comm. Math. Phys.* **161** (1994) 597-630

[CGR2] E. Cremmer, J.-L. Gervais, J.-F. Roussel: The genus zero bootstrap of chiral vertex operators in Liouville theory, *Nucl. Phys.* **B413** (1994) 244-277

[CT] T. Curtright, C. Thorn: *Phys. Rev. Lett.* **48** (1982) 1309;

[dHJ] E. D'Hoker, R. Jackiw: Classical and quantal Liouville field theory, *Phys. Rev.***D26** (1982) 3517

[D1] V. Dotsenko: Lectures on conformal field theory, Adv. Stud. Pure Math. **16**(1988) 123-170

[D2] V. Dotsenko: Three-point Correlation functions of the Minimal Conformal Field Theories coupled to 2D Gravity, *Mod. Phys. Lett.* **A6** (1991) 3601-3612

[DF1] V.S. Dotsenko, V. Fateev: Conformal algebra and multipoint correlation functions in two-dimensional statistical models, *Nucl. Phys.* **B240** (1984) 312

[DF2] S. Dotsenko, V. Fateev: Four point correlation functions and the operator algebra in two-dimensional conformal invariant theories with the central charge $C < 1$, *Nucl. Phys.* **B251** (1985) 691

[DK] P. Di Francesco, D. Kutasov: World-sheet and space-time physics in two-dimensional (super)string theory, *Nucl. Phys.* **B375** (1992) 119-170

[DO] H. Dorn, H.J. Otto: Two and three point functions in Liouville theory, *Nucl. Phys.* **B429** (1994) 375-388

[F] L. Faddeev: Modular Double of Quantum Group, preprint math.QA/9912078

[Fr] E. Frenkel: Determinant formulas for the free field representation of Virasoro and Kac-Moody algebras, *Phys. Lett.* **B286** (1992) 71-77

[FF] B.L. Feigin, D.B. Fuchs: Representations of the Virasoro algebra, in: Representations of Lie groups and related topics, eds. A.M. Vershik, D.P. Zhelobenko (Gordon and Breach, London 1990)

[FFFS] G. Felder, J. Fröhlich, J. Fuchs, C. Schweigert: Correlation functions and boundary conditions in RCFT abd three-dimensional topology, hep-th/9912239

[FFK] G. Felder, J. Fröhlich, G. Keller: On the structure of unitary conformal field theory, I, *Comm. Math. Phys.* **124** (1989) 417-463, II, *Comm. Math. Phys.* **130** (1990) 1-49

[FK] L. Faddeev, R. Kashaev: Private communication; paper in preparation

[FKV] L. Faddeev, R. Kashaev, V. Volkov: Strongly coupled quantum discrete Liouville theory. I: Algebraic approach and duality, hep-th/0006156

[FT] L.D. Faddeev, L.A. Takhtajan: Liouville model on the lattice, Lect. Notes in Physics. **246**, Springer-Verlag, Berlin, 1986, pp. 166-179

[FZZ] V. Fateev, A.B. Zamolodchikov, Al.B. Zamolodchikov: Boundary Liouville Field Theory I, hep-th/0001012

[G1] J.-L. Gervais: The quantum group structure of 2D gravity and the minimal models I, *Comm. Math. Phys.* **130** (1990) 257-280

[G2] J.-L. Gervais: Solving the strongly coupled 2D gravity: 1. Unitary truncation and Quantum Group Structure, *Comm. Math. Phys.* **138** (1991) 301-338

[GL] M. Goulian, M. Li: Correlation functions in Liouville theory, *Phys. Rev. Lett.* **66** (1991) 2051

[GM] P. Ginsparg, G. Moore: Lectures on 2D gravity and 2D string theory, Proceedings of the TASI summer school 1992, eds. J. Harvey and J. Polchinski. World Scientific, 1993 (hep-th/9304011)

[GN0] J.-L. Gervais, A. Neveu: Dual string spectrum in Polyakov's quantization (II). Mode separation, *Nucl. Phys.* **B209** (1982) 125-145

[GN1] J.-L. Gervais, A. Neveu: New quantum treatment of Liouville field theory, *Nucl. Phys.* **B224** (1982) 329-348, and:

[GN2] J.-L. Gervais, A. Neveu: Novel triangle relation and absence of tachyons in Liouville string field theory, *Nucl. Phys.* **B238** (1984) 125-141

[GN3] J.-L. Gervais, A. Neveu: Locality in strong coupling Liouville field theory and string models for seven-dimensions, thirteen-dimensions anbd nineteen-dimensions, *Phys. Lett.* **B151** (1985) 271

[GR1] J.-L. Gervais, J.-F. Roussel: Solving the strongly coupled 2D gravity: 2. Fractional-spin operators and topological three point functions, *Nucl. Phys.* **B426** (1994) 140

[GR2] J.-L. Gervais, J.-F. Roussel: Solving the strongly coupled 2D gravity: 3. String susceptibility and topological N-point functions, *Nucl. Phys.* **B478** (1996) 245-272

[GS] I.M. Gel'fand, G.E. Shilov: Generalized functions. Vol. I: Properties and operation, Academic Press, New York and London 1964

[GS1] J.-L. Gervais, J. Schnittger: The braiding of chiral vertex operators with continuous spins in 2D gravity, Phys. Lett. **B315**(1993) 258

[GS2] J.-L. Gervais, J. Schnittger: The Many Faces of the Quantum Liouville Exponentials, *Nucl. Phys.* **B413** (1994) 433-460

[GS3] J.-L. Gervais, J. Schnittger: Continous Spins in 2D Gravity: Chiral Vertex Operators and Local Fields, *Nucl. Phys.* **B431** (1994) 273

[GW] R. Goodman, N.R. Wallach: Projective Unitary Positive-Energy representations of Diff(S^1), J. Funct. Anal. **63** (1985)299-321

[Ha] R. Haag: Local quantum physics, 2nd ed., Springer 1996

[K1] V. Kac: Contravariant form for infinite-dimensional Lie algebras and superalgebras, Lect. Notes in Phys. **94**(1979)441-445

[K2] V. Kac: Some problems on infinite-dimensional Lie algebras and their representations, in: Lect. Notes in Math. **933**(1982)117-126

[Ka1] R.M. Kashaev: Quantization of Teichmüller spaces and the quantum dilogarithm, *Lett. Math. Phys.* **43** (1998) 105-115, q-alg/9705021, and: Liouville central charge in quantum Teichmuller theory Proc. of the Steklov Inst. of Math. Vol.**226**(1999)63-71, hep-th/9811203

[Ka2] R.M. Kashaev: On the spectrum of Dehn twists in quantum Teichmüller theory, math.QA/0008148

[KLS] S. Kharchev, D. Lebedev, M. Semenov-Tian-Shansky: Unitary representations of $U_q(\mathfrak{sl}(2,\mathbb{R}))$, the modular double, and the multiparticle q-deformed Toda chains, preprint hep-th/0102180

[KN] Y. Kazama, H. Nicolai: On the Exact Operator Formalism of Two-Dimensional Liouville Quantum Gravity in Minkowski Spacetime, *Int. J. Mod. Phys.* **A9** (1994) 667-710

[KS] D. Kutasov, N. Seiberg: Non-Critical superstrings, *Phys. Lett.* **B251** (1990) 67-72

[LS] D. Lüst, J. Schnittger: The Closed String Gervais-Neveu Liouville theory, *Int. J. Mod. Phys.* **A6** (1991) 3625-3642

[MMNNSU] T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi, Y. Saburi, K. Ueno: Unitary representations of the quantum group $SU_q(1,1)$. I: Structure of the dual space of $\mathcal{U}_q(sl(2))$, *Lett. Math. Phys.* **19** (1990) 187-194

[MS] G. Moore, N. Seiberg: Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989) 177-254

[OPS1] L. O'Raifeartaigh, J. M. Pawlowski, V. V. Sreedhar: Duality in Quantum Liouville Theory, *Ann. Phys. (N.Y.)* **277** (1999) 117-143

[OPS2] L. O'Raifeartaigh, J. M. Pawlowski, V. V. Sreedhar: The Two-exponential Liouville Theory and the Uniqueness of the Three-point Function, *Phys. Lett.* **B481** (2000) 436-444

[OW] H.J. Otto, G. Weigt: Construction of Exponential Liouville Field Operators for Closed String Models, Z. Phys. C **31**(1986)219-228

[Po] R.H. Poghossian: Structure constants in the N=1 super-Liouville field theory, *Nucl. Phys.* **B496** (1997) 451-464

[PP] A.K. Pogrebkov, M.K. Polivanov: The Liouville and Sinh-Gordon equations. Sov. Sci. Rev. C, Math. Phys. 5(1985)197-272

[PT1] B. Ponsot, J. Teschner: Liouville bootstrap via harmonic analysis on a noncompact quantum group, hep-th/9911110

[PT2] B. Ponsot, J. Teschner: Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, math.QA/0007097

[RS] R.C. Rashkov, M. Stanishkov: Three-point correlation functions in $N = 1$ Super Lioville Theory, *Phys. Lett.* **B380** (1996) 49-58

[RS1] M. Reed, B. Simon: Methods of Modern Mathematical Physics I: Functional Analysis; Academic Press 1980 (revised ed.)

[RS2] M. Reed, B. Simon: Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness; Academic Press 1975

[Se] N. Seiberg: Notes on Quantum Liouville Theory and Quantum Gravity, *Progr. Theor. Phys. Suppl.* **102** (1990) 319

[Sh] T. Shintani: On a Kronecker limit formula for real quadratic fields, J. Fac. Sci. Univ. Tokyo Sect.1A **24**(1977)167-199

[S1] K. Schmüdgen: An operator-theoretic approach to a cocycle problem in the complex plane, Bull London Math. Soc. **27** (1995)341-346

[S2] K. Schmüdgen: Operator representations of $\mathcal{U}_q(sl(2,\mathbb{R}))$, *Lett. Math. Phys.* **37** (1996) 211-222

[SW] N. Seiberg, E. Witten: The D1/D5 System And Singular CFT, JHEP **9904** (1999) 017 (hep-th/9903224)

[Ta] L. Takhtajan: Topics in quantum geometry of Riemann surfaces: Two-dimensional gravity, in: Quantum Groups and their applications in physics, Proceedings of Enrico Fermi summer school, Como 1994. Edited by L. Castellani and J. Wess. IOS Press Amsterdam, 1996 (hep-th/9409088)

[TB] J. Teschner: Remarks on Liouville theory with boundary, in: Proceedings of "Non-perturbative Quantum Effects 2000", 7-13 September 2000, Paris, http://jhep.sissa.it/cgi-bin/PrHEP/cgi/reader/list.cgi?confid=6, (hep-th/0009138)

[Te1] J. Teschner: On the Liouville three-point function, *Phys. Lett.* **B363** (1995) 65

[Te2] J. Teschner: On structure constants and fusion rules in the $SL(2,\mathbb{C})/SU(2)$-WZNW model, *Nucl. Phys.* **B546** (1999) 390-422

[Te3] J. Teschner: Crossing Symmetry in the H_3^+ WZNW model, Preprint hep-th/0108121, to appear in Physics Letters B

[TO] J. Teschner: In preparation.

[V] H. Verlinde: Conformal field theory, two-dimensional quantum gravity and quantization of Teichmüller space, *Nucl. Phys.* **B337** (1990) 652-680

[Wa] A. Wassermann: Operator algebra and conformal field theory III. Fusion of positive energy representations of $LSU(N)$ using bounded operators, Invent. Math. **133** (1998)467-538

[Wo] S.L. Woronowicz: Quantum Exponential Function, Rev. Math. Phys. **12**(2000)873-920

[ZZ] A.B. Zamolodchikov, Al.B. Zamolodchikov: Structure constants and conformal bootstrap in Liouville field theory, *Nucl. Phys.* **B477** (1996) 577-605

[ZZ2] A.B. Zamolodchikov, Al.B. Zamolodchikov: Liouville Field Theory on a Pseudosphere, preprint hep-th/0101152

J. Teschner

Inst. für theoretische Physik,
Freie Universität Berlin,
Arnimallee 14,
14195 Berlin,
Germany