Notes about equivalence between the Sine -Gordon theory (free fermion point) and the free fermion theory.

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Abstract. The space of local integrals of motion for the Sine-Gordon theory (the free fermion point) and the theory of free fermions in the light cone coordinates is investigated. Some important differences between the spaces of local integrals of motion of these theories are obtaned. The equivalence is broken on the level of the integrals of motion between bosonic and fermionic theories (in the free fermion point). The integrals of motion are constructed without Quantum Inverse Scattering Method (QISM) and the additional quantum integrals of motion are obtained.So the QISM is not absolutely complete.

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1. Introduction

We have obtained equivalence between the Sine-Gordon theory (the free fermion point) and free fermions by comparing the expressions commuting with the Hamiltonians of these theories in a space of integrated expressions of local densities. Equivalence of these theories has been investigated by comparing the expressions commuting with the Hamiltonians in a space of non integrated expressions of local densities. We have obtained some important differences in this space. In the subsequent paper we will investigate this equivalence between arbitrary values of coupling constants. In our opinion, we must present quantum integrals of motion if we quantize the integrable models. Quantum equivalence of the theories must be checked also by using quantum integrals of motion. The procedure of quantization for an integrable theory, which has no quantum integrals of motion, must not be cousistent [1,2]. We have obtained that the S-G theory (the free fermions point) and the theory of massive fermions are not equivalent on the level of the quantum integrals of motion. So the question about massiveness of the S-G theory at this point is open. Our construction of quantum integrals of motion does not really use the QISM. Up to now in the literature (there are many articles and even textbook [5]) there is no evidence of that QISM gives all the integrals of motion.

2. Classical IM for free fermions

Let us consider the classical model of free fermions in the light -cone coordinates. The equations of motion have the form

$$i\partial_-\psi_1 = m\psi_2, \quad i\partial_+\psi_2 = m\psi_1, \tag{2.1}$$

$$-i\partial_-\psi_1^+ = m\psi_2^+, \quad -i\partial_+\psi_2^+ = m\psi_1^+,$$

where $\psi_i, and \psi_i^+$ are the anti-commutative fields. Now we see that some equations (2.1) are connections (second type in Dirac terminology). We choose the x^- coordinate to represent the evolution time. We will work in the Euclidean space. We should like to note that the functions, which are contained in the Hamiltonian and IM, are initial functions and do not depend on the time variable x^-. Obviously, they do not satisfy any equation of motion. The Poisson bracket is written for the initial functions. The operator of the time evolution (Hamiltonian) has the form

$$H = \frac{m}{2}\int_{-\infty}^{+\infty} dx(\psi_1(x)\psi_2^+(x) + \psi_2(x)\psi_1^+(x)). \tag{2.2}$$

Now we must resolve the connection (2.1) for the initial function $\psi_2(x)$.We will consider $\psi_1(x) = \psi(x)$ as a dynamic variable and $\psi_2(x)$ as a non dynamic variable. The Poisson bracket between $\psi(x)$ and $\psi^+(x)$ has the form

$$\{\psi(x), \psi^+(y)\} = \delta(x - y). \tag{2.3}$$

Now the Hamiltonian in these variables can be obtained as follows:

$$H = \frac{m^2}{2}\int_{-\infty}^{+\infty} dx[\psi(x)(\int_{-\infty}^{x} \psi^+(t)dt - \int_{x}^{+\infty} \psi^+(t))+$$

$$+\psi^+(x)(\int_{-\infty}^{x} \psi(t)dt - \int_{x}^{+\infty} \psi(t)dt)], \tag{2.4}$$

It is a simple exercise to check that the classical IM has the form

$$I_n = \int_{-\infty}^{+\infty} dx(\psi(x)\partial_x^n\psi^+(x) + h.c.), \quad n = 1, 2, 3... \tag{2.5}$$

We use the Poisson bracket (2.3) to check the following Poisson brackets:

$$\{H, I_n\} = 0, \quad \{I_n, I_m\} = 0. \tag{2.6}$$

3.Quantum IM for free fermions

The first step in the quantization of the IM is the following. We note that the functions ψ_i and ψ_i^+ (which are present in the IM and in the H) do not really depend on the evaluation time because they represent the initial data. In the light - cone coordinates this is equivalent to the holomorphic condition (in the 2D Euclidean space).The x^-- coordinate is determined by H, and cannot be defined before H is determined. So we have

$$\partial_-\psi_i(x^+, x^-) = 0, \quad i = 1, 2,$$

and the same condition for ψ^+. It is absolutely wrong to consider this condition as an equation of motion. The second step is to note that we can represent the initial data as Laurent series,

$$\psi_i(x) = \sum_n \psi_{in}x^{-n}, \quad \psi_i^+(x) = \sum_n \psi_{in}^+x^{-n-1},$$

and define H and IM like as ordering operators. The third step is to define the quantum commutators in our system. Let us obtain the connection for Laurent modes ψ_n. We introduce the following integrals:

$$\psi_n(l) = \int_{-\infty}^{+\infty} dx \psi(x)(x-l)^{n-1}, \quad \psi_n^+(l) = \int_{-\infty}^{+\infty} dx \psi^+(x)(x-l)^n,$$

($\psi_1(x) = \psi(x)$ for simplicity), where l is an arbitrary coordinate. The contour of integration can be chosen as in Fig. 1. We must deform the integration contour

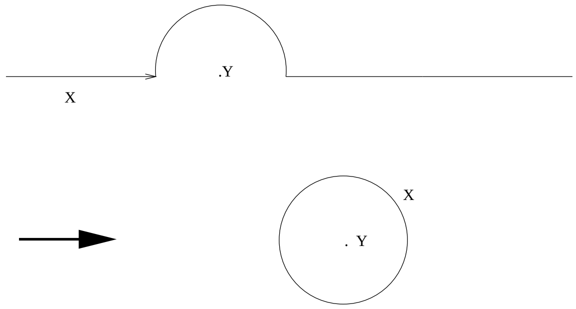

Fig.1 Contour of the integration for the expandsion.

like in Fig.1 and obtain the following:

$$\psi_n(l) = \oint_l dx \psi(x)(x-l)^n, \tag{3.1}$$

i.e., the ordinary coefficients of the Lourent expansion. Now we have

$$\{\psi_n(l), \psi_m^+(l)\} = \int_{-\infty}^{+\infty} dx (x-l)^{n-1} \int_{-\infty}^{+\infty} dt (t-l)^m \delta(x-t) =$$

$$= \int_{-\infty}^{+\infty} dx (x-l)^{n+m-1} = \oint_l dx \frac{1}{(x-l)^{-n-m+1}} = \delta_{n+m,0}.$$

We can always take $l = 0$ and obtain

$$\{\psi_n, \psi_m^+\} = \delta_{n+m,0}.$$

Now we will quantize the bracket replacing the Poisson bracket with the anticommutator of the fields. Let us introduce ordering of the operators ψ_n as follows:

$$\psi_n \psi_{-n}^+ =: \psi_n \psi_{-n}^+ : +1, \quad n > 0 \quad \psi_{-n}^+ \psi_n =: \psi_{-n}^+ \psi_n : .$$

$$\psi_{-n} \psi_n^+ =: \psi_n^+ \psi_{-n} :, \quad \psi_n^+ \psi_{-n} =: \psi_{-n} \psi_n^+ : +1$$

It is a very simple exercise to check that

$$\psi(x)\psi^+(y) = \frac{1}{(x-y)} + : \psi(x)\psi^+(y) :, \quad \psi^+(x)\psi(y) = \frac{1}{(x-y)} + : \psi^+(x)\psi(y) : .$$

Of course, there are many ways of ordering these operators, but if we want to preserve quantum integrability we must choose the one above. Probably there are other possibilities.

Now let us determine the quantum Hamiltonian and the quantum IM. It is not a problem to write the quantum Hamiltonian

$$H = \frac{m^2}{2}\int_{-\infty}^{+\infty} dx(:\psi(x)(\int_{-\infty}^{x}\psi^+(t): - \int_{x}^{+\infty}\psi^+(t):)dt + :\psi(x)^+(\int_{-\infty}^{x}\psi(t): - \int_{x}^{\infty}\psi(t):)dt \tag{3.2}$$

and

$$I_n = \int_{-\infty}^{+\infty} P_n[\psi(x), \psi^+(x)]dx,$$

where P_n are the ordering differential polynomials of $\psi(x)$ and $\psi^+(x)$. The coefficients of these polynomials are determined by the conditions

$$[H, I_n] = 0. \tag{3.3}$$

The quantum field $\psi(x^+, x^-)$ (solution of the quantum equation of motion) has the form

$$\psi(x^+, x^-) = \exp iHx^- \psi(x^+) \exp -iHx^-,$$

and this representation leads to the ordinary equal time anticommutator. Now we must determine the way of calculating the commutator. Let us to choose the contour of integration in (3.4) as in Fig.2.

$$[\int_{-\infty}^{+\infty} h(x)dx; i_n(y)] = \int_{-\infty}^{+\infty} h(x)i_n(y)dx - \int_{-\infty}^{+\infty} i_n(y)h(x)dx, \tag{3.4}$$

where $h(x)$ and $i_n(x)$ are the densities of H and I_n.

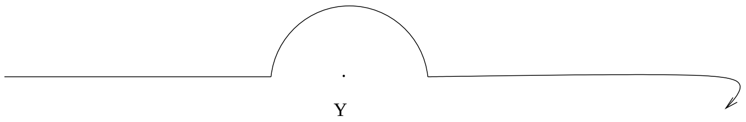

Fig.2a Contour of integration for the commutator

The contour of integration of the fist integral (3.4) is shown in Fig. 2a and the contour of integration of the second integral (3.4) is shown in Fig 2b. Now we can add C_R, or continue the integration contour analytically. Therefore, we

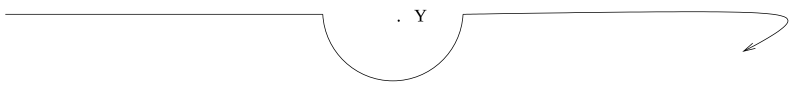

Fig.2b Contour of integration for the commutator

have

$$[H, i_n(y)] = \oint_y dx h(x) i_n(y). \tag{3.5}$$

Of course, there is another singular point l which is the center of expansion for the field $\psi(x)$ (see (3.1)) which does not have any physical meaning. After implementing the procedure (3.5), we must obtain the full difference. Now we represent IM in the form

$$I_n = \int_{-\infty}^{+\infty} dx(:\psi(x)\partial^n\psi^+(x): +h.c.) \quad n = 1, 2, 3.... \tag{3.6}$$

They (IM) are very similar to the classical ones, because we use only one contraction (Poisson bracket) for calculating I_n. For Massive Thirring model we will have more complicated contractions, and I_n will be different from classical ones. Now we can obtain the involution

$$[I_n, I_m] = 0,$$

where the rule of calculation of the commutator has been introduced above.

4. Quantum IM for Sine-Gordon equation

Quantum IM for the Sine -Gordon theory can be obtained in a similar way. We have holomorphy condition for the initial function

$$\partial_-\varphi(x^+, x^-) = 0.$$

The holomorphic function can be expanded as $(x^+ = x)$

$$\varphi(x) = q + p\log x - \sum_{n\neq 0} \frac{a_n x^{-n}}{n}.$$

We consider the boundary condition $\partial\varphi(x) \to 0$ for $x \to \pm\infty$.The Poisson bracket and the Hamiltonian have the form

$$\{\varphi(x); \varphi(y)\} = \pi\varepsilon(x-y), \quad H = \int_{-\infty}^{+\infty} \exp\alpha\varphi(x) + \exp -\alpha\varphi(x)dx.$$

Let us introduce the following commutation relations

$$[\hat{p}, \hat{q}] = 1, \quad [\hat{a}_n, \hat{a}_m] = n\delta_{n+m,0}.$$

The ordering of these operators can be determined by

$$\hat{p}\hat{q} =: \hat{p}\hat{q}: +1, \quad \hat{q}\hat{p} =: \hat{p}\hat{q}:,$$

$$\hat{a}_n\hat{a}_{-n} =: \hat{a}_n\hat{a}_{-n}: +n, \quad \hat{a}_{-n}\hat{a}_n =: \hat{a}_{-n}\hat{a}_n:, \quad n > 0.$$

So we have

$$\varphi(x)\varphi(y) =: \varphi(x)\varphi(y): + \log(x-y). \tag{4.1}$$

From (4.1) we can obtain

$$[\varphi(x); \varphi(y)] = i\pi\varepsilon(x-y).$$

So we have quantized the above Poisson bracket, and the quantum Hamiltonian has the form

$$H = \int_{-\infty}^{+\infty} : \exp \alpha\varphi(x) : + : \exp -\alpha\varphi(x) : dx.$$

The rules of calculation for the commutators are the same as for the fermion theory. Now we can obtain the simplest IM for an arbitrary position of the coupling constant

$$\begin{aligned} i_2(x) &=: (\partial\varphi(x))^2 :, \\ i_4(x) &=: (\partial\varphi(x))^4 : + \frac{(\alpha^4 - 6\alpha^2 + 4)}{\alpha^2} : (\partial^2\varphi(x))^2 :, \\ i_6(x) &=: (\partial\varphi(x))^6 : + \frac{5(-\alpha^4 + 8\alpha^2 - 4)}{3\alpha^2} : (\partial\varphi(x))^3\partial^3\varphi(x) : + \\ &+ \frac{(3\alpha^8 - 40\alpha^6 + 155\alpha^4 - 160\alpha^2 + 48)}{6\alpha^4} : (\partial^3\varphi(x))^2 : . \end{aligned} \tag{4.2}$$

We can check the classical limit ($\alpha \to 0$) [4] and the commutativity of these operators I_n with the densities (4.2). For special position of the coupling constant $\alpha = 1$ (the free fermion point) we have more integrals of motion

$$\begin{aligned} i_2(x) &=: (\partial\varphi(x))^2 :, \\ i_3(x) &=: (\partial\varphi(x))^3 :, \\ i_4(x) &=: (\partial\varphi(x))^4 : -6 : (\partial\varphi(x))^2\partial^2\varphi(x) : +4 : \partial\varphi(x)\partial^3\varphi(x) : + \\ &+3 : (\partial^2\varphi(x)^2) : -\partial^4\varphi(x), \\ i_5(x) &=: (\partial\varphi(x))^5 : -10 : (\partial\varphi(x))^3\partial^2\varphi(x) : + \\ &+10 : (\partial\varphi(x))^2\partial^3\varphi(x) : -10 : \partial^2\varphi(x)\partial^3\varphi(x) : + \\ &+\partial^5\varphi(x) + 5 : (3(\partial^2\varphi(x))^2 - \partial^4\varphi(x))\partial\varphi(x) : . \end{aligned} \tag{4.3}$$

We can check the commutativity of the I_n operators with the densities (4.3) [7]. Now we see some connection between IM (3.6) and (4.3). Let us introduce some equivalence between $\psi(x)$ and $: \exp\varphi(x) :$ ($\psi(x) \sim: \exp\varphi(x) :$ and $\psi^+(x) \sim: \exp -\varphi(x) :$) where $\psi(x)$, $\psi^+(x)$ are the initial functions in the fermion theory and $\varphi(x)$ is the initial function in the Sine-Gordon theory. After some algebra we see that IM (3.6) and (4.3) are the same. So we must consider that $\alpha = 1$ is the free fermion point in the Sine-Gordon theory. We see that i_3 and i_5 are new integrals of motion, which cannot be obtained by QISM, so we can conclude that the QISM is not a complete method. Another interesting note consists in the following. Where the completeness of the QISM is broken,we have a new infinite dimensional symmetry that is very similar to some reduction of the $W_{1+\infty}$ algebras obtained in [6]. If we have $\alpha = 1$, we can obtain local conservation currents in the Sine-Gordon theory

$$[H, J^i(x)] = 0,$$

where

$$J^i(x) = \frac{1}{(i+1)} : (\partial^{i+1} \exp \varphi(x)) \partial \exp -\varphi(x) : + \frac{1}{(i+1)(i+2)} : (\partial^{i+2} \exp \varphi(x)) \exp -\varphi(x) :,$$

where $i \in N$.We have holomorphic operators $J^i(x)$. Now we can calculate the commutators of Laurent modes of these currents and obtain

$$[J_n^k, J_m^p] = (k+1)! \sum_{l'=0}^{k} \frac{J_{n+m}^{p+l'}}{l'!(k+1-l')!} \prod_{j'=0}^{k-l'} (m+p+1-j') -$$

$$-(p+1)! \sum_{l=0}^{p} \frac{J_{n+m}^{k+l}}{l!(p+1-l)!} \prod_{j=0}^{p-l} (n+k+1-j)$$

$$+\delta_{n+m,0} \frac{(p+1)!(k+1)!}{2(k+p+3)!} \times$$

$$\times((-1)^{k+1} \prod_{j=0}^{2+k+p} (n+k+1-j) - (-1)^{p+1} \prod_{j=0}^{2+k+p} (m+p+1-j)),$$

where we have $J_n^i = \frac{1}{2\pi i} \oint_0 J^i(x) x^{i+1+n} dx$.

The terms in H for free fermions prohibit the existence of the currents (4.4) in the fermion theory. So we see the absence of equivalence between the massive fermion theory and the free fermion point for the Sine-Gordon theory.

5. Conclusion

If we do not obtain equivalence between the theory of free fermions and the Sine- Gordon theory (in free fermions point), so we can not consider Sine- Gordon theory as a massive theory. The existence of local conservation currents also confirms this fact. It is very interesting to solve this model in the "free fermions" point using the new infinite dimensional symmetry which is very similar to conformal symmetry.

Another important note of the article consists in the following. We have constructed the example where the QISM does not give all the integrals of motion. There are many articles about QISM, but the problem of completeness of this method has not been discussed, so our example appears very important.

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