Sine-Gordon breather form factors and quantum field equations

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Abstract

Using the results of previous investigations on sine-Gordon form factors exact expressions of all breather matrix elements are obtained for several operators: all powers of the fundamental bose field, general exponentials of it, the energy momentum tensor and all higher currents. Formulae for the asymptotic behavior of bosonic form factors are presented which are motivated by Weinberg's power counting theorem in perturbation theory. It is found that the quantum sine-Gordon field equation holds and an exact relation between the "bare" mass and the renormalized mass is obtained. Also a quantum version of a classical relation for the trace of the energy momentum is proven. The eigenvalue problem for all higher conserved charges is solved. All results are compared with Feynman graph expansions and full agreement is found.

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1 Introduction

This work continuities previous investigations [1, 2] on exact form factors for the sine-Gordon alias the massive Thirring model. Some results of the present paper have been published previously [3]. The classical sine-Gordon model is given by the wave equation

$$\Box\varphi(t,x)+\frac{\alpha}{\beta}\sin\beta\varphi(t,x)=0.$$

Since Coleman [4] found the wonderful duality between the quantum sine-Gordon and the massive Thirring model a lot of effort has been made to understand this quantum field theoretic model. A further contribution in this direction is the present paper. The

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sine-Gordon model alias massive Thirring model describes the interaction of several types of particles: solitons, anti-solitons alias fermions and anti-fermions and a finite number of charge-less breathers, which may be considered as bound states of solitons and anti-solitons. Integrability of the model implies that the n-particle S-matrix factorizes into two particle S-matrices.

The "bootstrap" program for integrable quantum field theoretical models in 1+1 dimensions starts as the first step with the calculation of the S-matrix. Here (see e.g. [5, 6]) our starting point is the two particle sine-Gordon S-matrix for the scattering of fundamental bosons (lowest breathers) [7][1]

$$S(\theta) = \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu}\,.$$

The pole of $S(\theta)$ at $\theta = i\pi\nu$ belongs to the second breather b_2 as a breather-breather bound state. The parameter ν is related to the sine-Gordon and the massive Thirring model coupling constant by

$$\nu = \frac{\beta^2}{8\pi - \beta^2} = \frac{\pi}{\pi + 2g}$$

where the second equation is due to Coleman [4].

As a second step of the "bootstrap" program off-shell quantities as arbitrary matrix elements of local operators

$$^{out}\langle\, p'_m, \dots, p'_1\,|\mathcal{O}(x)|\, p_1, \dots, p_n\,\rangle^{in}$$

are obtained by means of the "form factor program" from the S-matrix as an input. Form factors for integrable model in 1+1 dimensions were first investigated by Vergeles and Gryanik [8] for the sinh-Gordon model and by Weisz [10][2] for the sine-Gordon model. The 'form factor program' was formulated in [12, 13] where the concept of generalized form factors was introduced. In that article consistency equations were formulated which are expected to be satisfied by these objects. Thereafter this approach was developed further and studied in the context of several explicit models by Smirnov [14] who proposed the form factor equations (see below) as extensions of similar formulae in the original article [12]. Further publications on form factors and in particular on sine-Gordon and sinh-Gordon form factors are [15]–[26]. Smirnow's approach in [15] is similar to that one used in the present article (see section 6).

Let $\mathcal{O}(x)$ be a local operator. The generalized form factors $\mathcal{O}_n(\theta_1, \dots, \theta_n)$ are defined by the vacuum – n-lowest breather matrix elements

$$\langle\, 0\,|\,\mathcal{O}(x)\,|\,p_1, \dots, p_n\,\rangle^{in} = e^{-ix(p_1+\cdots+p_n)}\,\mathcal{O}_n(\theta_1, \dots, \theta_n)\,, \quad \text{for}\;\; \theta_1 > \cdots > \theta_n$$

where the θ_i are the rapidities of the particles $p_i^\mu = m(\cosh\theta_i, \sinh\theta_i)$. In the other sectors of the variables the functions $\mathcal{O}_n(\theta_1, \dots, \theta_n)$ are given by analytic continuation with respect to the θ_i. General matrix elements are obtained from $\mathcal{O}_n(\underline{\theta})$ by crossing which means in particular analytic continuation $\theta_i \to \theta_i \pm i\pi$.

[1] This S-matrix element has been discussed before in [8, 9].

[2] Similar results were obtained by Zamolodchikov [11].

In [12] one of the present authors (M.K.) and Weisz showed that for the case of a diagonal S-matrix the n-particle form factor may be written as

$$\mathcal{O}_n(\underline{\theta}) = K_n^{\mathcal{O}}(\underline{\theta}) \prod_{1\leq i<j\leq n} F(\theta_{ij}) \tag{1}$$

where $\theta_{ij} = |\theta_i - \theta_j|$ and $F(\theta)$ is the two-particle form factor (see section 2). The K-function is an even $2\pi i$ periodic meromorphic function. In [2] we presented a general formula for soliton anti-soliton form factors in terms of an integral representation. Using the bound state fusion method we derived the general soliton-breather and pure breather form factor formula which we will investigate in this article in more detail. In particular for the case of lowest breathers the K-function turns out to be of the form[3]

$$K_n^{\mathcal{O}}(\underline{\theta}) = \sum_{l_1=0}^{1} \cdots \sum_{l_n=0}^{1} (-1)^{l_1+\cdots+l_n} \prod_{1\leq i<j\leq n} \left(1 + (l_i - l_j)\frac{i\sin\pi\nu}{\sinh\theta_{ij}}\right) p_n^{\mathcal{O}}(\underline{\theta},\underline{l}) \,. \tag{2}$$

The breather p-function $p_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n,l_1,\ldots,l_n)$ encodes the dependence on the operator $\mathcal{O}(x)$. It is obtained from the solitonic p-function $p_{sol,2n}^{\mathcal{O}}(\tilde\theta_1,\ldots,\tilde\theta_{2n};z_1,\ldots,z_n)$ (see [2]) by setting $\tilde\theta_{2i-1} = \theta_i + \frac{1}{2}iu^{(1)}$, $\tilde\theta_{2i} = \theta_i - \frac{1}{2}iu^{(1)}$ and $z_j = \theta_j - \frac{i\pi}{2}(1-(-1)^{l_j}\nu)$ with the fusion angle $u^{(1)} = \pi(1-\nu)$. In [2] we proposed the solitonic p-functions for several local operator. In this way we obtain all breather p-functions which are some sort of decedent of solitonic p-functions i.e. just we use the solitonic ones in the bound state points. In the present article we assume a little different point of view. We will obtain a wider class of p-functions corresponding to local operators with respect to the breather field, including also operators which are non-local with respect to solitonic field. The different point of view is the following:

As already mentioned, it has been shown in [12] that a form factor of n fundamental bosons (lowest breathers) is of the form (1) where the K-function $K_n^{\mathcal{O}}(\underline{\theta})$ is meromorphic, symmetric and periodic (under $\theta_i \to \theta_i + 2\pi i$) and it has to satisfy some further the conditions (see section 2). We consider eq. (2) as an Ansatz for the K-function which translates these conditions of the K-functions to simple conditions of the p-functions. In section 4 we present solutions of these conditions for the p-functions. In this article we propose the p-functions for several local operators. In particular we consider the infinite set of local currents $J_L^{\pm}(x)$, $(L = \pm1,\pm3,\ldots)$ which belong to the infinite set of conservation laws which are typical for integrable quantum field theories. For this example the correspondences of the operators, K-functions and the p-functions are (up to normalization constants)

$$J_L^{\pm}(x) \leftrightarrow K_n^{(L,\pm)}(\underline{\theta}) \leftrightarrow p_n^{(L,\pm)}(\underline{\theta},\underline{l}) \propto \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{i=1}^{n} e^{L\left(\theta_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)}.$$

Here the breather p-function is obtained from the corresponding solitonic one.

In contrast to this case the breather p-function for exponentials of the field $:e^{i\gamma\varphi}:(x)$ for generic real γ is not related to a solitonic p-function of any local operator (which

[3]For the sinh-Gordon model and the case of the exponential field an analogous representation was obtained by Brazhnikov and Lukyanov [21] by different methods.

means that $:e^{i\gamma\varphi}:(x)$ is not local with respect to the soliton fields). The correspondences are[4]

$$:e^{i\gamma\varphi}:(x) \leftrightarrow K_n^{(q)}(\underline{\theta}) \leftrightarrow p_n^{(q)}(\underline{l}) = N_n^{(q)} \prod_{i=1}^{n} q^{(-1)^{l_i}}$$

where $q = q(\gamma)$ (see section 4). Here and in the following $:\cdots:$ denotes normal ordering with respect to the physical vacuum which means in particular for the vacuum expectation value $\langle\, 0 \,|:\varphi^N:(x)\,|\, 0\,\rangle = 0$ and therefore $\langle\, 0\,|:\exp i\gamma\varphi:(x)|\, 0\,\rangle = 1$. In section 4 we present arguments to support these correspondences and also determine the normalization constants $N_n^{\mathcal{O}}$.

As applications of these results we investigate quantum operator equations. In particular we provide in particular exact expressions for all matrix elements of all powers of the fundamental bose field $\varphi(t,x)$ and its exponential $:\exp i\gamma\varphi:(t,x)$ for arbitrary γ. We find that the operator $\Box^{-1}:\sin\beta\varphi(x):$ is local. Moreover the quantum sine-Gordon field equation[5]

$$\Box\varphi(x) + m^2 \frac{\pi\nu}{\sin\pi\nu}\frac{1}{\beta} :\sin\beta\varphi:(x) = 0$$

is fulfilled for all matrix elements. The factor $\frac{\pi\nu}{\sin\pi\nu}$ modifies the classical equation and has to be considered as a quantum correction of the breather mass m as to be compared with the "bare" mass $\sqrt{\alpha}$. Further we find that the trace of the energy momentum tensor $T^{\mu\nu}$ satisfies

$$T^{\mu}_{\ \mu}(x) = -2\frac{\alpha}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)(:\cos\beta\varphi:(x) - 1)\,.$$

Again this operator equation is modified by a quantum correction $(1 - \beta^2/8\pi)$ compared to the classical one.

Also we show that the higher local currents $J^{\mu}_{M}(t,x)$ satisfy $\partial_\mu J^{\mu}_{M}(t,x) = 0$ and calculate all matrix elements of all higher conserved $Q_L = \int dx J^0_L(t,x)$

$$Q_L|\, p_1,\ldots,p_n\rangle^{in} = \sum_{i=1}^{n} e^{L\theta_i}|\, p_1,\ldots,p_n\rangle^{in}. \tag{3}$$

In particular for $L = \pm 1$ the currents yield the energy momentum tensor $T^{\mu\nu} = T^{\nu\mu}$ with $\partial_\mu T^{\mu\nu} = 0$.

The article is organized as follows: In section 2 we recall some formulae of [1, 2] in particular those for breather form factors which we need in the following. The properties of the form factors are translated to conditions for the 'K-functions' and finally to simple ones of the 'p-functions'. In section 3 we investigate the asymptotic behavior of bosonic form factors. In section 4 we discuss several explicit examples of local operators as general exponentials of the fundamental bose field, powers of the field, all higher conserved currents, and the energy momentum tensor. Using induction and Liouville's theorem we

[4] See footnote 3.

[5] In the framework of constructive quantum field theory quantum field equations where considered in [27, 28]. For the sine-Gordon model quantum field equations where dicussed by Smirnov in [15] (see also [20]).

prove some identities which means that the same operators may be represented in term of different p-functions. These results are used in section 5 to prove operator field equation as the quantum sine-Gordon field equation. In section 6 we present two other types representation of sine-Gordon breather form factors: a determinant formula and an integral representation. A proof is delegated to the appendix.

2 Breather form factors

Using the bound state fusion method we derived in [2] from a general formula for soliton anti-soliton form factors the pure breather form factor formula which in particular for the case of lowest breathers writes as (1) where $F(\theta)$ is the two particle form factor function. It satisfies Watson's equations

$$F(\theta) = F(-\theta)S(\theta) = F(2\pi i - \theta)$$

with the S-matrix given above. Explicitly it is given by the integral representation [12]

$$F(\theta) = N \exp \int_0^\infty \frac{dt}{t} \frac{\left(\cosh \frac{1}{2}t - \cosh(\frac{1}{2}+\nu)t\right)\left(1 - \cosh t(1-\frac{\theta}{i\pi})\right)}{\cosh \frac{1}{2}t \sinh t} \tag{4}$$

normalized such that $F(\infty) = 1$ which means $1/N = 1/F(i\pi) = Z^\varphi \frac{\beta^2}{8\pi\nu} \frac{\sin \pi\nu}{\pi\nu}$ (see [2] and eq. (18)).

In general form factors of one kind of bosonic particles (i.e. with a diagonal S-matrix) satisfy the following properties [12, 14, 1].

Properties of the form factors The form factor function $\mathcal{O}_n(\underline{\theta})$ is meromorphic with respect to all variables $\theta_1, \ldots, \theta_n$. It satisfies Watson's equations

$$\mathcal{O}_n(\ldots, \theta_i, \theta_j, \ldots) = \mathcal{O}_n(\ldots, \theta_j, \theta_i, \ldots)\, S(\theta_{ij}). \tag{5}$$

The crossing relation means for the connected part (see e.g. [2]) of the matrix element

$$\langle\, p_1 \,|\, \mathcal{O}(0) \,|\, p_2, \ldots, p_n \,\rangle^{in}_{conn.} = \mathcal{O}_n(\theta_1 + i\pi, \theta_2, \ldots, \theta_n) = \mathcal{O}_n(\theta_2, \ldots, \theta_n, \theta_1 - i\pi)$$

which implies in particular

$$\mathcal{O}_n(\theta_1, \theta_2, \ldots, \theta_n,) = \mathcal{O}_n(\theta_2, \ldots, \theta_n, \theta_1 - 2\pi i) \tag{6}$$

The function $\mathcal{O}_n(\underline{\theta})$ has poles determined by one-particle states in each sub-channel. In particular it has the so-called annihilation poles at for example $\theta_{12} = i\pi$ such that the recursion formula[6] is satisfied

$$\operatorname*{Res}_{\theta_{12}=i\pi} \mathcal{O}_n(\theta_1, \ldots, \theta_n) = 2i\, \mathcal{O}_{n-2}(\theta_3, \ldots, \theta_n)\left(\mathbf{1} - S(\theta_{2n}) \ldots S(\theta_{23})\right). \tag{7}$$

Since we are dealing with relativistic quantum field theories Lorentz covariance in the form

$$\mathcal{O}_n(\theta_1, \ldots, \theta_n) = e^{-s\mu}\, \mathcal{O}_n(\theta_1 + \mu, \ldots, \theta_n + \mu) \tag{8}$$

holds if the local operator transforms as $\mathcal{O} \to e^{s\mu}\mathcal{O}$ where s is the "spin" of $\mathcal{O}$.

[6] This formula has been proposed in [14] as a generalization of formulae in [12] and it has been proven in [1] using LSZ assumptions.

Conditions for the K-functions Form factors of one kind of bosonic particles (which means that there is no backward scattering) may be expressed by eq. (1) in terms of the K-functions. Therefore properties of the form factors can be translated to

$$K_n^{\mathcal{O}}(\dots,\theta_i,\theta_j,\dots) = K_n^{\mathcal{O}}(\dots,\theta_j,\theta_i,\dots) \tag{9}$$

$$K_n^{\mathcal{O}}(\underline{\theta}) = K_n^{\mathcal{O}}(\theta_1 - 2\pi i,\theta_2,\dots,\theta_n) \tag{10}$$

$$\operatorname*{Res}_{\theta_{12}=i\pi} K_n^{\mathcal{O}}(\underline{\theta}) = \frac{2i}{F(i\pi)} \prod_{i=3}^{n} \frac{1}{F(\theta_{2i}+i\pi)F(\theta_{2i})} \left(1 - \prod_{i=3}^{n} S(\theta_{2i})\right) K_{n-2}^{\mathcal{O}}(\underline{\theta}'') \tag{11}$$

$$K_n^{\mathcal{O}}(\underline{\theta}) = e^{-s\mu}\, K_n^{\mathcal{O}}(\theta_1+\mu,\dots,\theta_n+\mu) \tag{12}$$

where $\underline{\theta} = \theta_1,\dots,\theta_n$ and $\underline{\theta}'' = \theta_3,\dots,\theta_n$.

Conditions for the p-functions Starting with a general integral representation for solitonic form factors and using the bound state fusion method we have shown in [2] that the lowest breather K-functions may be expressed by eq. (2) in terms of breather p-functions which follows from solitonic p-functions. As already mentioned in the introduction in the present article we make the *Ansatz* that the K-function is of the form (2) and we allow more general breather p-functions. The *Ansatz* (2) transforms the conditions for the K-function $K_n^{\mathcal{O}}(\underline{\theta})$ to simpler conditions for the p-function $p_n^{\mathcal{O}}(\underline{\theta},\underline{l})$:

The p-function $p_n^{\mathcal{O}}(\underline{\theta},\underline{l})$ is holomorphic with respect to all variables $\theta_1,\dots,\theta_n$, symmetric with respect to the exchange of the variables θ_i and $l_i$at the same time and periodic with period $2\pi i$.

$$\begin{aligned} p_n^{\mathcal{O}}(\dots,\theta_i,\theta_j,\dots,l_i,l_j,\dots) &= p_n^{\mathcal{O}}(\dots,\theta_j,\theta_i,\dots,l_j,l_i,\dots) && (13)\\ p_n^{\mathcal{O}}(\underline{\theta},\underline{l}) &= p_n^{\mathcal{O}}(\theta_1-2\pi i,\theta_2,\dots,\theta_n,\underline{l}) && (14) \end{aligned}$$

With the short notation $\underline{\theta}' = \theta_2,\dots,\theta_n$, $\underline{\theta}'' = \theta_3,\dots,\theta_n$ and $\underline{l}'' = l_3,\dots,l_n$ the recursion relation

$$p_n^{\mathcal{O}}(\theta_2+i\pi,\underline{\theta}',\underline{l}) = g(l_1,l_2)p_{n-2}^{\mathcal{O}}(\underline{\theta}'',\underline{l}'') + h(l_1,l_2) \tag{15}$$

holds where $g(0,1) = g(1,0) = 2/(F(i\pi)\sin\pi\nu)$ and $h(l_1,l_2)$ is independent of $\underline{l}''$. Lorentz covariance reads as

$$p_n^{\mathcal{O}}(\theta_1+\mu,\dots,\theta_n+\mu,\underline{l}) = e^{s\mu}\, p_n^{\mathcal{O}}(\theta_1,\dots,\theta_n,\underline{l}). \tag{16}$$

We now show that these conditions of the p-function are sufficient to guaranty the properties of the form factors.

Theorem 1 *If the p-function $p_n^{\mathcal{O}}(\underline{\theta},\underline{l})$ satisfies the conditions (13–16), the K-function $K_n^{\mathcal{O}}(\underline{\theta})$ satisfies the conditions (9–12) and therefore the form factor function $\mathcal{O}_n(\underline{\theta})$ satisfies the properties (5–8).*

Proof. Except of (11) all points are obvious. Taking the residue of (2) and inserting (11) we obtain (with $a = i \sin \pi\nu$)

$$
\begin{aligned}
\operatorname*{Res}_{\theta_{12}=i\pi} K_n(\underline{\theta}) &= -a \sum_{l_3=0}^{1} \cdots \sum_{l_r=0}^{1} (-1)^{l_3+\cdots+l_n} \prod_{3=i<j}^{n} \left(1 + \frac{l_i - l_j}{\sinh \theta_{ij}} a\right) \\
&\quad \times \sum_{l_1 \neq l_2} (-1)^{l_1+l_2} (l_1 - l_2) \prod_{i=3}^{n} \left(\left(1 + \frac{l_1 - l_i}{\sinh(\theta_{2i} + i\pi)} a\right) \left(1 + \frac{l_2 - l_i}{\sinh \theta_{2i}} a\right) \right) \\
&\qquad \times \left(g(l_1, l_2) p_{n-2}^{\mathcal{O}}(\underline{\theta}'', \underline{l}'') + h(l_1, l_2) \right) \\
&= \frac{2i}{F(i\pi)} K_{n-2}(\underline{\theta}'') \left(\prod_{i=3}^{n} \left(1 + \frac{a}{\sinh \theta_{2i}}\right) - \prod_{i=3}^{n} \left(1 - \frac{a}{\sinh \theta_{2i}}\right) \right) + h\text{-term}
\end{aligned}
$$

It has been used that

$$
\begin{aligned}
&\sum_{l_1 \neq l_2} (-1)^{l_1+l_2} (l_1 - l_2) \prod_{i=3}^{n} \left(\left(1 + \frac{l_1 - l_i}{\sinh(\theta_{2i} + i\pi)} a\right) \left(1 + \frac{l_2 - l_i}{\sinh \theta_{2i}} a\right) \right) g(l_1, l_2) \\
&\qquad = g(0,1) \prod_{i=3}^{n} \left(1 + \frac{a}{\sinh \theta_{2i}}\right) - g(1,0) \prod_{i=3}^{n} \left(1 - \frac{a}{\sinh \theta_{2i}}\right)
\end{aligned}
$$

independent of the l_i, $(i \geq 3)$ (similar for $g \to h$). The condition (11) follows finally from $g(0,1) = g(1,0) = 2/(F(i\pi) \sin \pi\nu)$ and

$$
F(\theta + i\pi) F(\theta) = 1 \Big/ \left(1 - \frac{i \sin \pi\nu}{\sinh \theta}\right)
$$

(which is easily obtained from the integral representation (4)), provided that the h-term vanishes. This is a consequence of the following lemma. Note that the h-term is proportional to a $K_{n-2}(\underline{\theta}'')$ given by the formula (2) with a p-function independent of the $\underline{l}''$. ■

Lemma 2 *If the 'p-function' in (2) does not depend on $l_1, \ldots, l_n$ then the corresponding K-function vanishes.*

Proof. The proof is simple by induction and Liouville's theorem: We easily obtain $K_1(\underline{\theta}) = K_2(\underline{\theta}) = 0$. As induction assumptions we take $K_{n-2}(\underline{\theta}'') = 0$. The function $K_n(\underline{\theta})$ is a meromorphic functions in terms of the $x_i = e^{\theta_i}$ with at most simple poles at $x_i = \pm x_j$ since $\sinh \theta_{ij} = (x_i + x_j)(x_i - x_j)/(2x_i x_j)$. The residues of the poles at $x_i = x_j$ vanish because of the symmetry. Further more the residues at $x_i = -x_j$ are proportional to $K_{n-2}(\underline{\theta}'')$ because similar as in the proof of theorem 1 we have

$$
\operatorname*{Res}_{\theta_{12}=i\pi} K_n(\underline{\theta}) = a K_{n-2}(\underline{\theta}'') \left(\prod_{i=3}^{n} \left(1 + \frac{a}{\sinh \theta_{2i}}\right) - \prod_{i=3}^{n} \left(1 - \frac{a}{\sinh \theta_{2i}}\right) \right)
$$

Therefore the function $K_n(\underline{\theta})$ is holomorphic everywhere. Furthermore for $x_1 \to \infty$ we have the asymptotic behavior

$$
\begin{aligned}
K_n(\underline{\theta}) &= \sum_{l_2=0}^{1} \cdots \sum_{l_n=0}^{1} (-1)^{l_2+\cdots+l_n} \prod_{2 \leq i < j \leq n} \left(1 + (l_i - l_j) \frac{i \sin \pi\nu}{\sinh \theta_{ij}}\right) \\
&\qquad \times \sum_{l_1=0}^{1} (-1)^{l_1} \prod_{j=2}^{n} \left(1 + (l_1 - l_j) \frac{i \sin \pi\nu}{\sinh \theta_{1j}}\right) \to 0 \quad (17)
\end{aligned}
$$

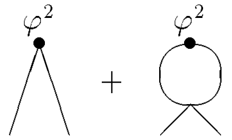
Therefore $K_n(\underline{\theta})$ vanishes identically by Liouville's theorem. ■

3 Asymptotic behavior of bosonic form factors

In this section we derive the asymptotic behavior of bosonic form factors by mean of general techniques of renormalized local quantum field theory. In particular we use perturbation theory in term of Feynman graphs. As the simplest example we investigate first the asymptotic behavior for $p_1 \to \infty$ or $\theta_{12} \to \infty$ of

$$\begin{aligned} \langle\, 0\,|\!:\!\varphi^2\!:\!|\,p_1, p_2\,\rangle^{in} &= 2\,\langle\, 0\,|\,\varphi(0)\,|\,p_1\rangle\,\langle\, 0\,|\,\varphi(0)\,|\,p_2\,\rangle + o(1) \\ &= 2Z^\varphi + o(1) \end{aligned}$$

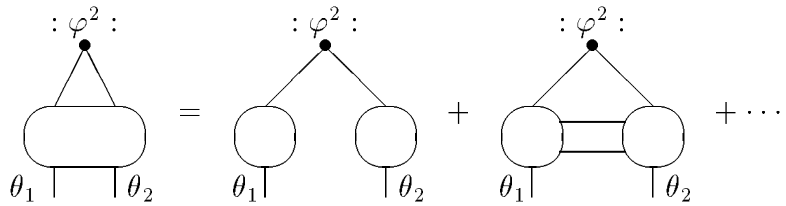
where $:\cdots:$ means normal ordering with respect to the physical vacuum. This may be seen in perturbation theory as follows: Feynman graph expansion in lowest order means

$$\langle\, 0\,|\!:\!\varphi^2\!:\!|\,p_1, p_2\,\rangle^{in} = 2\Bigg($$


$$\Bigg) + O(\beta^4)$$

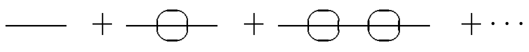
$$\begin{aligned} &= 2\left(1 + i\alpha\beta^2\tfrac{1}{2}\int \frac{d^2k}{(2\pi)^2}\frac{i}{k^2 - m_1^2}\frac{i}{(p-k)^2 - m_1^2}\right) + O(\beta^4) \\ &= 2 + \frac{\beta^2}{4\pi}\frac{i\pi - \theta_{12}}{\sinh\theta_{12}} + O(\beta^4) \end{aligned}$$

The second graph is of order $O(\ln p_1/p_1)$ for $p_1 \to \infty$. This is typical also for all orders in perturbation theory:



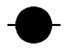
Weinberg's power counting theorem says that the second term and also all higher terms where more lines connect the two bubbles are again at least of order $O(\ln p_1/p_1)$ for $p_1 \to \infty$.

The wave function renormalization constant Z^φ is defined as usual by the two-point function of the (unrenormalized) field

$$\int \langle\, 0\,|T\varphi(x)\varphi(0)|\,0\,\rangle\, e^{ipx}\, d^2x =$$


$$= \frac{i}{p^2 - \alpha - \Pi(p^2)} = \frac{iZ^\varphi}{p^2 - m^2 - \Pi_{ren}(p^2)}$$

where $\Pi(p)$ is the self energy, which means that it is given by the sum of all amputated one-particle-irreducible graphs

$$-i\Pi(p^2) =$$


The physical breather mass m, the wave function renormalization constant Z^φ and the renormalized breather self energy are given by

$$\begin{aligned} m^2 &= \alpha + \Pi(m^2) \\ \frac{1}{Z^\varphi} &= 1 - \Pi'(m^2) \\ \Pi_{ren}(p^2) &= Z^\varphi \left(\Pi(p^2) - \Pi(m^2) - \left(p^2 - m^2\right) \Pi'(m^2) \right). \end{aligned}$$

Since the sine-Gordon model is a 'super renormalizable quantum field theory' both renormalization constants $\Pi(m^2)$ and $\Pi'(m^2)$ become finite after taking normal ordering in the interaction Lagrangian.. They can be calculated exactly. The wave function renormalization constant was obtained in [12]

$$Z^\varphi = (1+\nu) \frac{\frac{\pi}{2}\nu}{\sin \frac{\pi}{2}\nu} \exp\left(-\frac{1}{\pi} \int_0^{\pi\nu} \frac{t}{\sin t} dt \right) \tag{18}$$

and the relation of the unrenormalized and the physical mass is calculated in this article (see section 5)

$$\alpha = m^2 \frac{\pi\nu}{\sin \pi\nu}.$$

Both relations have been checked in perturbation theory.

Remark 1 *Usually in renormalized quantum field theory (in particular when Z is infinite) one would introduce the renormalized field with*

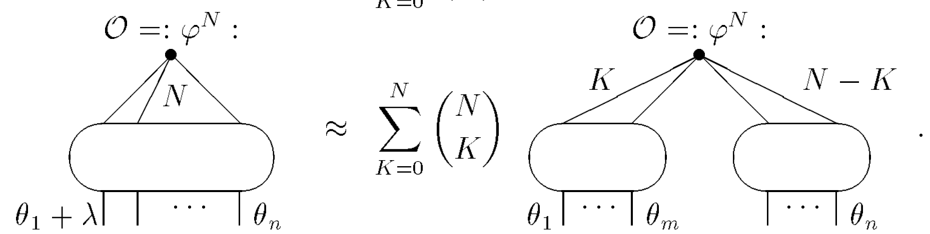
$$\langle\, 0 \,|\varphi_{ren}(0)|\, p\rangle = 1$$

by $\varphi(x) = \sqrt{Z^\varphi}\varphi_{ren}(x)$. However, since Coleman [4] the convention for the sine-Gordon model is to use the unrenormalized field $\varphi(x)$. Therefore we have the normalization

$$\langle\, 0 \,|\varphi(0)|\, p\rangle = \sqrt{Z^\varphi}.$$

As a generalization we now consider general n-particle form factors of an normal ordered arbitrary power of the field $\mathcal{O} =: \varphi^N :$ and let m of the momenta go to infinity. If $\underline{\theta}(\lambda) = (\theta_1 + \lambda, \ldots, \theta_m + \lambda, \theta_{m+1}, \ldots, \theta_n)$, $\underline{\theta}' = (\theta_1, \ldots, \theta_m)$ and $\underline{\theta}'' = (\theta_{m+1}, \ldots, \theta_n)$ Weinberg's power counting theorem for bosonic Feynman graphs says that for $\operatorname{Re}\lambda \to \infty$

$$\left[\varphi^N\right]_n (\underline{\theta}(\lambda)) \approx \sum_{K=0}^{N} \binom{N}{K} \left[\varphi^K\right]_m (\underline{\theta}') \left[\varphi^{N-K}\right]_{n-m} (\underline{\theta}'') \tag{19}$$



using the notation $\left[\varphi^N\right]_n (\underline{\theta}) = \langle\, 0 \,|: \varphi^N :(0)\,|\, p_1, \ldots, p_n \,\rangle^{in}$. Therefore for the special case of a local operator which is an exponential of the fundamental bose field $\mathcal{O} =: e^{i\gamma\varphi} :$ (for some γ) we have

$$\left[e^{i\gamma\varphi}\right]_n (\underline{\theta}(\lambda)) = \left[e^{i\gamma\varphi}\right]_m (\underline{\theta}') \left[e^{i\gamma\varphi}\right]_{n-m} (\underline{\theta}'') + O(e^{-\lambda})$$

in particular for $m = 1$ and $\operatorname{Re}\theta_1 \to \infty$

$$\left[e^{i\gamma\varphi}\right]_n (\theta_1, \theta_2, \ldots, \theta_n) = \left[e^{i\gamma\varphi}\right]_1 (\theta_1) \left[e^{i\gamma\varphi}\right]_{n-1} (\theta_2, \ldots, \theta_n) + O(e^{-\theta_1}). \tag{20}$$

4 Examples of operators

In this section we present some examples of p-functions which satisfy the conditions of section 2 in particular (13) – (16) and propose the correspondence of local operators, K-functions and p-functions due to eqs. (1) and (2) for these examples

$$\mathcal{O} \leftrightarrow K_n^{\mathcal{O}}(\underline{\theta}) \leftrightarrow p_n^{\mathcal{O}}(\underline{\theta}, \underline{l}) .$$

4.1 Classical local operators

The classical sine-Gordon Lagrangian is

$$\mathcal{L}^{SG} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\alpha}{\beta^2}(\cos\beta\varphi - 1)$$

We consider the following classical local operators:

1. $\exp(i\gamma\varphi(x))$ for arbitrary real γ,
2. $\varphi^N(x)$
3. higher conserved currents for $(L = 1, 3, 5. \dots)$

$$J_L^\rho = \begin{cases} J_L^+ = \partial^+\varphi\,(\partial^+)^L\,\varphi + O(\varphi^4) \\ J_L^- = \left((\partial^+)^{L-1}\,\varphi + O(\varphi^2)\right)\sin\varphi \end{cases} .$$

   A second set of conserved currents is obtained by replacing $\partial^+ \to \partial^-$. The higher charges are of the form

$$Q_L = \int dx\left(\partial^0\varphi\partial^{+L}\varphi + O(\varphi^4)\right), \quad L = 1, 3, 5, \dots$$

   and the charges for even L vanish.

4. $T^{\mu\nu}(x) = \partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\mathcal{L}^{SG}$ the energy momentum tensor or in terms of light cone coordinates ($\partial^\pm = \partial^0 \pm \partial^1$ etc.)

$$\begin{aligned} T^{\pm\pm} &= T^{00} \pm T^{01} \pm T^{10} + T^{11} = \partial^+\varphi\partial^+\varphi = \partial^-\varphi\partial^-\varphi \\ T^{+-} &= T^{00} - T^{01} + T^{10} - T^{11} = -2\frac{\alpha}{\beta^2}(\cos\beta\varphi - 1) = T^{-+} \end{aligned}$$

5. $\exp(i\beta\varphi(x))$ for the particular value $\gamma = \beta$

4.2 The normalization of form factors

The normalization constants are obtained in the various cases by the following observations:

a) The normalization a field annihilating a one-particle state is given by the vacuum one-particle matrix element, in particular for the fundamental bose field one has

$$\langle\, 0 \,|\, \varphi(0) \,|\, p \,\rangle = \sqrt{Z^\varphi}$$

   where Z^φ is the finite wave function renormalization constant (18) calculated in [12].

b) If an observable like a charge $Q = \int dx\, \mathcal{O}(x)$ belongs to a local operator we use the relation

$$\langle\, p' \,|\, Q \,|\, p \,\rangle = q \langle\, p' \mid p \,\rangle .$$

This will be applied for example to the higher conserved charges.

c) We use Weinberg's power counting theorem for bosonic Feynman graphs. As discussed in section 3 this yields in particular the asymptotic behavior for the exponentials of the boson field $\mathcal{O} = :e^{i\gamma\varphi}:$

$$\mathcal{O}_n(\theta_1,\theta_2,\dots) = \mathcal{O}_1(\theta_1)\, \mathcal{O}_{n-1}(\theta_2,\dots) + O(e^{-\operatorname{Re}\theta_1})$$

as $\operatorname{Re}\theta_1 \to \infty$ in any order of perturbation theory. This behavior is also assumed to hold for the exact form factors. Applying this formula iteratively we obtain from (2) relations[7] for the normalization constants of the operators $:e^{i\gamma\varphi}:$

d) The recursion relation (15) relates N_{n+2} and N_n. For all p-functions discussed below this means

$$N_{n+2} = N_n \frac{2}{\sin \pi\nu F(i\pi)} \quad (n \geq 1). \tag{21}$$

4.3 Local operators and their p-functions

For all cases discussed in the following the conditions (13-16) are again obvious except that of the recursion relation (15) which will be discussed in detail. For later convenience we also list for some cases the explicit expressions of $K_n(\underline{\theta})$ for $n = 1, 2$ and the asymptotic behavior of $K_n(\underline{\theta})$ for $\operatorname{Re}\theta_1 \to \infty$ which is easily obtained analogously to the calculation (17) in the proof of lemma 2. For convenience we will use the notation $K_n = N_n \tilde{K}_n$ in the following.

4.3.1 Exponentials of the breather field

We propose the correspondence

$$e^{i\gamma\varphi} \leftrightarrow N_n^{(q)} \tilde{K}_n^{(q)}(\underline{\theta}) \leftrightarrow p_n^{(q)}(\underline{l}) = N_n^{(q)} \prod_{i=1}^{n} q^{(-1)^{l_i}} \tag{22}$$

with $q = q(\gamma)$ (and $q(0) = 1$) to be determined below. One easily calculates the low particle K-functions

$$\begin{aligned} \tilde{K}_1^{(q)}(\theta) &= (q - 1/q) \\ \tilde{K}_2^{(q)}(\underline{\theta}) &= (q - 1/q)^2 \end{aligned} \tag{23}$$

and the asymptotic behavior

$$\tilde{K}_n^{(q)}(\underline{\theta}) \approx \tilde{K}_1^{(q)}(\theta_1) \tilde{K}_{n-1}^{(q)}(\underline{\theta}') \,.$$

[7] This type of arguments has been also used in [12, 18, 19, 20].

The last formula is obtained similar as in the proof of lemma 2. The proposal that the p-function $p_n^{(q)}(\underline{l})$ correspond to an exponential of a bosonic field is supported by the following observation. The asymptotic behavior is consistent with that of the form factors of exponentials of bosonic fields (20) as discussed in section 3 which reads in terms of the K-functions $K_n^{(q)}(\underline{\theta}) \approx K_1^{(q)}(\theta_1)K_{n-1}^{(q)}(\underline{\theta}')$ (since $F(\infty) = 1$) provided the normalization constants satisfy

$$N_n^{(q)} = N_1^{(q)} N_{n-1}^{(q)} \quad \Rightarrow \quad N_n^{(q)} = \left(N_1^{(q)}\right)^n .$$

This is what we discussed above under point c) to determine the normalization constants; point d) means the following: The recursion condition (15) is satisfied since in this case we have $g(l_1, l_2) = q^{(-1)^{l_1}+(-1)^{l_2}} N_n^{(q)}/N_{n-2}^{(q)}$ which is symmetric and $h(l_1, l_2) = 0$. The condition (15) $g(0,1) = g(1,0) = 2/(F(i\pi)\sin\pi\nu)$ implies the recursion relation for the normalization constants (21) which finally yields

$$\begin{aligned} N_1^{(q)} &= \sqrt{\frac{2}{F(i\pi)\sin\pi\nu}} = \sqrt{Z^\varphi}\frac{\beta}{2\pi\nu} \\ N_n^{(q)} &= \left(\sqrt{Z^\varphi}\frac{\beta}{2\pi\nu}\right)^n \end{aligned}$$

where Z^φ is the breather wave function renormalization constant (18). The relation of $F(i\pi)$ with Z^φ is obtained by elementary manipulations of the integral representation (4) and (18). It has been also used that normal ordering implies $N_0^{(q)} = 1$.

4.3.2 Powers of a bose field

Motivated by the expansion of (22) with respect to q we propose the correspondence

$$\varphi^N \leftrightarrow N_n^{(N)} \tilde{K}_n^{(N)}(\underline{\theta}) \leftrightarrow p_n^{(N)}(\underline{l}) = N_n^{(N)} \left(\sum_{i=1}^n (-1)^{l_i}\right)^N$$

Again one easily calculates (with $\tilde{K}_n^{(N)} = K_n^{(N)}/N_n^{(N)}$) the low particle K-functions

$$\begin{aligned} &\tilde{K}_1^{(N)}(\theta) = 2 \\ &\tilde{K}_2^{(N)}(\underline{\theta}) = 2^{N+1} \\ &\tilde{K}_3^{(N)}(\underline{\theta}) = 2(3^N - 3) - \sin^2\pi\nu \prod_{i<j} \frac{1}{\cosh\frac{1}{2}\theta_{ij}} \end{aligned}$$

and the asymptotic behavior

$$\tilde{K}_n^{(N)}(\underline{\theta}) \approx \sum_{K=1}^N \binom{N}{K} \tilde{K}_1^{(K)}(\theta_1)\tilde{K}_{n-1}^{(N-K)}(\underline{\theta}')$$

where $K_n^{(N)}$ is only nonvanishing for $N - n =$ even. Expanding $e^{i\gamma\varphi} \leftrightarrow p_n^{(q)}(\underline{l})$ in powers of γ one obtains

$$N_n^{(N)} = N_n^{(q)} \left(\frac{dq}{qid\gamma}\right)^N .$$

where $\frac{dq}{qid\gamma}$ must not dependent on γ. Using for the special values $n = N = 1$ the normalization condition a)

$$\langle\, 0\,|\,\varphi(0)\,|\,p\,\rangle = \sqrt{Z^\varphi} = K_1^{(1)}(\theta) = 2N_1^{(1)} = 2\sqrt{Z^\varphi}\frac{\beta}{2\pi\nu}\frac{dq}{qid\gamma}$$

we conclude the dependence of the number q as a function of γ

$$\frac{dq}{qid\gamma} = \frac{\pi\nu}{\beta} \;\Rightarrow\; q = \exp\left(i\frac{\pi\nu}{\beta}\gamma\right)$$

and finally the normalization constants

$$N_n^{(N)} = \left(\frac{1}{2}\sqrt{Z^\varphi}\right)^n \left(\frac{\pi\nu}{\beta}\right)^{N-n}. \tag{24}$$

Note that this normalization is also consistent with the asymptotic behavior (19) which follows from Weinberg's power counting argument. We compare these general results with know special ones [12]. In particular for $n = N = 2$ we have

$$\begin{aligned}\langle\, 0\,|\!:\!\varphi^2\!:\!(0)\,|\,p_1,p_2\,\rangle^{in} &= K_2^{(2)}(\theta_{12})F(\theta_{12}) = 8N_2^{(2)}F(\theta_{12}) \\ &= 2Z^\varphi F(\theta_{12})\end{aligned}$$

which agrees with formulae (4.4–4.6) of [12]. Further for $n = 3$ and $N = 1$ we have

$$\begin{aligned}\langle\, 0\,|\,\varphi(0)\,|\,p_1,p_2\,,p_3\,\rangle^{in} &= K_3^{(1)}(\underline{\theta})\prod_{i<j}F(\theta_{ij}) \\ &= -\,(Z^\varphi)^{3/2}\,\frac{1}{8}\left(\beta\frac{\sin\pi\nu}{\pi\nu}\right)^2\prod_{i<j}\frac{F(\theta_{ij})}{\cosh\frac{1}{2}\theta_{ij}}\end{aligned}$$

which again agrees with formulae (4.9–4.12) of [12].

4.3.3 Higher currents

We propose the correspondence

$$J_L^\pm \leftrightarrow p_n^{(L,\pm)}(\underline{\theta},\underline{l}) = \pm N_n^{(J_L)}\sum_{i=1}^n e^{\pm\theta_i}\sum_{i=1}^n e^{L\left(\theta_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)}$$

for $n =$ even and zero for $n =$ odd ($L = \pm1, \pm3, \dots$). Again one easily calculates the 2 particle K-function

$$\tilde{K}_2^{(L,\pm)}(\underline{\theta}) = -2\,(-i)^L\sin\tfrac{1}{2}L\pi\nu\frac{\sin\pi\nu}{\sinh\theta_{12}}\left(e^{\pm\theta_1}+e^{\pm\theta_2}\right)\left(e^{L\theta_1}-e^{L\theta_2}\right) \tag{25}$$

The recursion condition (15) is satisfied since $g(l_1,l_2) = N_n^{(J_L)}/N_{n-2}^{(J_L)}$ is symmetric and $h(l_1,l_2) = \sum_{i=3}^n e^{\pm\theta_i}\sum_{i=1}^2 e^{L\left(\theta_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)}$ is independent of l_i, ($i > 2$). Again we have the recursion relation for the normalization constants (21). The two particle normalization we calculated by means of b) with the charges

$$\begin{aligned}\langle\, p' \,| Q_L \,|\, p\rangle &= \int_{-\infty}^{\infty} dx \,\langle\, p' \,|\tfrac{1}{2}\left(J_L^+(x) + J_L^-(x)\right)\,|\, p\rangle \\ &= \int_{-\infty}^{\infty} dx\, e^{i(p-p')x}\, \tfrac{1}{2}\left(K_2^{(L,+)} + K_2^{(L,-)}\right)(\theta'+i\pi,\theta)F(\theta'+i\pi-\theta) \\ &= 2\pi\delta(p-p')\tfrac{1}{2}\left(K_2^{(L,+)} + K_2^{(L,-)}\right)(\theta+i\pi,\theta)F(i\pi) \\ &= \langle\, p' \,|\, p\rangle\, e^{L\theta} \quad \text{if } L \text{ odd.}\end{aligned}$$

Using (25) we obtain

$$\begin{aligned}&\tfrac{1}{2}\left(K_2^{(L,+)} + K_2^{(L,-)}\right)(\theta+i\pi,\theta) \\ &= -N_n^{(J_L)} 2\,(-i)^L \sin\tfrac{1}{2}L\pi\nu \sin\pi\nu\cosh\theta e^{L\theta}\left(e^{Li\pi}-1\right) \\ &= 2m\cosh\theta\, e^{L\theta}/F(i\pi)\end{aligned}$$

for L odd. For even L the charges vanish as in the classical case. With the relation of the normalization constants (21) we finally obtain

$$N_n^{(J_L)} = \frac{m\, i^L}{4\sin\frac{1}{2}L\pi\nu}\left(\sqrt{Z^\varphi}\frac{\beta}{2\pi\nu}\right)^n .$$

Next we derive all eigenvalues of the higher charges (3). We show that for $n'+n>2$ the connected part of the matrix element ${}^{out}\langle\, p_1',\ldots,p_{n'}' \,|Q_L|\, p_1,\ldots,p_n\rangle^{in}$ vanishes. The analytic continuation $\mathcal{O}_{n'+n}(\underline{\theta}'+i\pi,\underline{\theta})$ yields this connected part. From the correspondence of operators and p-functions

$$\begin{aligned}Q_L = \int dx J_L^0(x) \leftrightarrow 2\pi\delta(P'-P)N_n^{(J_L)} m\left(-\sum_{i=1}^{n'}\sinh\theta_i' + \sum_{i=1}^{n}\sinh\theta_i\right) \\ \times\left(\sum_{i=1}^{n'} e^{L\left(\theta_i'+i\pi-\frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)} + \sum_{i=1}^{n} e^{L\left(\theta_i-\frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)}\right)\end{aligned}$$

the claim follows since for $n'+n>2$ there are no poles which may cancel the zero at $P'=P$ where $P^{(\prime)} = \sum p_i^{(\prime)}$. Therefore contributions to matrix element come from disconnected parts which contain (analytic continued) two-particle form factors:

$$\begin{aligned}&{}^{out}\langle\, p_1',\ldots,p_{n'}' \,|Q_L\,|\, p_1,\ldots,p_n\rangle^{in} \\ &= \sum_{i,j} {}^{out}\langle\, p_1',\ldots,\hat{p}_i',\ldots,p_{n'}' \,|\, p_1,\ldots,\hat{p}_j,\ldots,p_n\rangle^{in}\,\langle\, p_i' \,|Q_L\,|\, p_j\rangle \\ &= {}^{out}\langle\, p_1',\ldots,p_{n'}' \,|\, p_1,\ldots,p_n\rangle^{in} \sum_{i=1}^{n} e^{\theta_i L}\end{aligned}$$

where $\hat{p}_j$ means that this particles is missing in the state.

From the higher currents for $L=\pm 1$ we get the light cone components of the energy momentum tensor $T^{\rho\sigma} \propto J_\sigma^\rho$ with $\rho,\sigma=\pm$ (see also [20]).

4.3.4 The energy momentum tensor

We propose the correspondence

$$T^{\rho\sigma} \leftrightarrow p_n^{\rho\sigma}(\underline{\theta},\underline{l}) = \rho N_n^{(T)} \sum_{i=1}^{n} e^{\rho\theta_i} \sum_{i=1}^{n} e^{\sigma\left(\theta_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)} \tag{26}$$

for n even and $p_n^{\rho\sigma} = 0$ for n odd. The normalization is again determined by c) namely

$$\langle\, p' \,|\, P^\nu \,|\, p \rangle = \langle\, p' \,|\, \int dx T^{0\nu}(x) \,|\, p \rangle = \langle\, p' \,|\, p \rangle \, p^\nu$$

which gives similar as above

$$N_n^{(T)} = \frac{i\, m^2}{4 \sin \frac{1}{2}\pi\nu} \left(\sqrt{Z^\varphi} \frac{\beta}{2\pi\nu} \right)^n . \tag{27}$$

Note that the energy momentum tensor does not look symmetric. However, it is due to an identity proven in the next section (see eq. (30)). The conservation law follows as above and the eigenvalue equation of the energy momentum operator is

$$P^\nu \,|\, p_1, \ldots, p_n \rangle^{in} = \sum_{i=1}^{n} p_i^\nu \,|\, p_1, \ldots, p_n \rangle^{in}.$$

4.3.5 Special exponentials of the breather field

For the special cases of the exponential of the field $\gamma = \pm\beta$ we propose the correspondence alternative to (22)

$$e^{\pm i\beta\varphi} \leftrightarrow N_n^{\pm} \tilde{K}_n^{\pm}(\underline{\theta}) \leftrightarrow p_n^{\pm}(\underline{\theta},\underline{l}) = N_n^{\pm} \sum_{i=1}^{n} e^{\mp\theta_i} \sum_{i=1}^{n} e^{\pm\left(\theta_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)}. \tag{28}$$

Again one easily calculates the low particle K-functions

$$\begin{aligned} \tilde{K}_1^{\pm}(\theta) &= 2 \sin \tfrac{1}{2}\pi\nu \\ \tilde{K}_2^{\pm}(\underline{\theta}) &= \pm 4i \sin \tfrac{1}{2}\pi\nu \sin \pi\nu \end{aligned}$$

and the asymptotic behavior

$$\tilde{K}_n^{\pm}(\underline{\theta}) \approx \pm 2i \sin \pi\nu \, \tilde{K}_{n-1}^{\pm}(\underline{\theta}')\,.$$

The proof of the last formula is delegated to the appendix. The normalization constants are calculated analogously to the case of the general exponential

$$N_n^{\pm} = \pm i \frac{\sin \pi\nu}{\sin \frac{1}{2}\pi\nu} \left(\sqrt{Z^\varphi} \frac{\beta}{2\pi\nu} \right)^n . \tag{29}$$

4.4 Identities

It turns out that the correspondence of local operators and p-functions is not unique. In this subsection we prove some identities which we will need in the following section to prove operator equations. To have a consistent interpretation of $K_n^{(q)}(\underline{\theta})$ with $q = \exp(i\pi\nu\gamma/\beta)$ as the K-function of $\exp(i\gamma\varphi(x))$ it is necessary that $K_n^{(q)}(\underline{\theta})$ is even/odd for $n =$ even/odd under the exchange $q \leftrightarrow 1/q$. For $\gamma = \pm\beta$ the K-function of the general exponential should assume the K-functions of the special ones. These fact are expressed by the following lemma.

Lemma 3 *Let the K-functions*

$$K_n(\underline{\theta}) = \sum_{l_1=0}^{1} \cdots \sum_{l_r=0}^{1} (-1)^{l_1+\cdots+l_r} \prod_{1\le i<j\le n} \left(1 + (l_i - l_j)\frac{i\sin\pi\nu}{\sinh\theta_{ij}}\right) p_n(\underline{\theta}, \underline{l})$$

be given by the p-functions

$$\begin{aligned} K_n^{(q)}(\underline{\theta}) &\leftrightarrow p_n^{(q)}(\underline{l}) = N_n^{(q)} \prod_{i=1}^{n} q^{(-1)^{l_i}} \\ K_n^{\pm}(\underline{\theta}) &\leftrightarrow p_n^{\pm}(\underline{\theta}, \underline{l}) = N_n^{\pm} \sum_{i=1}^{n} e^{\mp\theta_i} \sum_{i=1}^{n} e^{\pm\left(\theta_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)} \\ K_n^{(1)}(\underline{\theta}) &\leftrightarrow p_n^{(1)}(\underline{l}) = N_n^{(1)} \sum_{i=1}^{n} (-1)^{l_i}. \end{aligned}$$

Then the following identities hold (again with $K_n(\underline{\theta}) = N_n\tilde{K}_n(\underline{\theta})$)

$$\begin{aligned} \tilde{K}_n^{(q)}(\underline{\theta}) &= -(-1)^n \tilde{K}_n^{(1/q)}(\underline{\theta}) \\ \tilde{K}_n^{+}(\underline{\theta}) &= -(-1)^n \tilde{K}_n^{-}(\underline{\theta}), \end{aligned} \tag{30}$$

in particular for $q = \exp(i\pi\nu)$

$$K_n^{+}(\underline{\theta}) = K_n^{(q)}(\underline{\theta}).$$

and further more

$$\tilde{K}_n^{(1)}(\underline{\theta}) = \frac{1}{2\sin\frac{1}{2}\pi\nu} \left(\sum_{i=1}^{n} e^{\theta_i} \sum_{i=1}^{n} e^{-\theta_i}\right)^{-1} \left(\tilde{K}_n^{+}(\underline{\theta}) + \tilde{K}_n^{-}(\underline{\theta})\right). \tag{31}$$

Proof. Again as in the proof of lemma 2 we apply induction and Liouville's theorem. We introduce the differences

$$\begin{aligned} f_n(\underline{\theta}) &= \tilde{K}_n^{(q)}(\underline{\theta}) + (-1)^n \tilde{K}_n^{(1/q)}(\underline{\theta}) \\ \text{or}\quad f_n(\underline{\theta}) &= \tilde{K}_n^{+}(\underline{\theta}) + (-1)^n \tilde{K}_n^{-}(\underline{\theta}) \\ \text{or}\quad f_n(\underline{\theta}) &= K_n^{+}(\underline{\theta}) - K_n^{(q=\exp(i\pi\nu))}(\underline{\theta}) \\ \text{or}\quad f_n(\underline{\theta}) &= \tilde{K}_n^{(1)}(\underline{\theta}) - \frac{1}{2\sin\frac{1}{2}\pi\nu} \left(\sum_{i=1}^{n} e^{\theta_i} \sum_{i=1}^{n} e^{-\theta_i}\right)^{-1} \left(\tilde{K}_n^{+}(\underline{\theta}) + \tilde{K}_n^{-}(\underline{\theta})\right). \end{aligned}$$

Then the results of the previous subsection imply in all cases $f_1(\theta) = f_2(\underline{\theta}) = 0$. As induction assumptions we take $f_{n-2}(\underline{\theta}'') = 0$. The functions $f_n(\underline{\theta})$ are a meromorphic functions in terms of the $x_i = e^{\theta_i}$ with at most simple poles at $x_i = \pm x_j$ since $\sinh\theta_{ij} = (x_i + x_j)(x_i - x_j)/(2x_i x_j)$. The residues of the poles at $x_i = x_j$ vanish because of the symmetry and again the residues at $x_i = -x_j$ are proportional to $f_{n-2}(\underline{\theta}'')$ because of the recursion relation (11). Furthermore for $x_i \to \infty$ again $f_n(\underline{\theta}) \to 0$. Therefore $f_n(\underline{\theta})$ vanishes identically by Liouville's theorem in all cases. For the last case it has been used that because of (30) for n even both $\tilde{K}$-terms cancel and that they are equal for odd n. Because due to (28) $\tilde{K}_n^+$ is proportional to $\sum_{i=1}^n e^{-\theta_i}$ and $\tilde{K}_n^-$ is proportional to $\sum_{i=1}^n e^{\theta_i}$ both are proportional to $\sum_{i=1}^n e^{-\theta_i} \sum_{i=1}^n e^{+\theta_i}$ which means that there are no extra poles at $\sum_{i=1}^n e^{\theta_i} = 0$ or $\sum_{i=1}^n e^{-\theta_i} = 0$. ■

5 Operator equations

The classical sine-Gordon model is given by the wave equation

$$\Box\varphi(t,x) + \frac{\alpha}{\beta}\sin\beta\varphi(t,x) = 0.$$

and if this fulfilled we have also the relation for the trace of energy-momentum tensor

$$T^{\mu}_{\ \mu}(t,x) = -2\frac{\alpha}{\beta^2}\left(\cos\beta\varphi(t,x) - 1\right).$$

In this section we construct the quantum version of this two classical equations. Here and in the following $:\cdots:$ denotes normal ordering with respect to the physical vacuum which means in particular for the vacuum expectation value $\langle\, 0\,|\ :\exp i\gamma\varphi:(t,x)\,|\,0\,\rangle = 1$. As consequences of the identities of subsection 4.4 we can prove quantum field equations.

Theorem 4 *The following operator equations are to be understood in term of all their matrix elements.*

1. *For the exceptional value $\gamma = \beta$ the operator $\Box^{-1} : \sin\gamma\varphi : (t,x)$ is local and the quantum sine-Gordon field equation holds*

$$\Box\varphi(t,x) + \frac{\alpha}{\beta} : \sin\beta\varphi : (t,x) = 0 \tag{32}$$

   if the "bare" mass $\sqrt{\alpha}$ is related to the renormalized one by[8]

$$\alpha = m^2\frac{\pi\nu}{\sin\pi\nu}. \tag{33}$$

   Here m is the physical mass of the fundamental boson.

2. *The energy momentum tensor is symmetric and its trace satisfies*

$$T^{\mu}_{\ \mu}(t,x) = -2\frac{\alpha}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)(: \cos\beta\varphi : (t,x) - 1). \tag{34}$$

[8]Before this formula was found in [29, 30] by different methods.

3. For all higher currents the conservation laws hold

$$\partial_\mu J_L^\mu(x) = 0 \quad (L = \pm 1, \pm 3, \dots).$$

Proof.

1. From (28) we have the correspondence of operators and K-functions

$$\Box^{-1} \sin \beta\varphi \leftrightarrow \frac{K_n^+(\underline{\theta}) - K_n^-(\underline{\theta})}{2i \sum_{i=1}^n e^{\theta_i} \sum_{i=1}^n e^{-\theta_i}}.$$

   As shown in the proof of lemma 3 there are no poles at $\sum_{i=1}^n e^{\theta_i} = 0$ or $\sum_{i=1}^n e^{-\theta_i} = 0$. Therefore $\Box^{-1}$: $\sin \beta\varphi$: is a local operator. Furthermore by eq. (31)

$$\sum_{i=1}^n e^{\theta_i} \sum_{i=1}^n e^{-\theta_i} K_n^{(1)}(\underline{\theta}) = \frac{\pi\nu}{\beta \sin \pi\nu} \frac{1}{2i} \left(K_n^+(\underline{\theta}) - K_n^-(\underline{\theta})\right)$$

   where the normalizations (24) and (29) have been used which means in particular

$$\frac{N_n^{(1)}}{N_n^+} \frac{i}{\sin \frac{1}{2}\pi\nu} = \frac{\pi\nu}{\beta \sin \pi\nu}.$$

   In terms of operators this is just the quantum sine-Gordon field equation. Comparing this with the classical equation we obtain the relation eq. (33) of the bare and the physical mass

2. Using (26) and (28) we have the correspondence of operators and K-functions for n even

$$\begin{aligned} T^{+-} &\leftrightarrow N_n^{(T)} \tilde{K}_n^- \\ T^{-+} &\leftrightarrow -N_n^{(T)} \tilde{K}_n^+ \\ T^\mu{}_\mu \leftrightarrow K_n^{(T)}(\underline{\theta}) &= -\frac{1}{2} N_n^{(T)} \left(\tilde{K}_n^+ - \tilde{K}_n^-\right) \\ \cos \beta\varphi - 1 &\leftrightarrow \frac{1}{2} \left(K_n^+(\underline{\theta}) + K_n^-(\underline{\theta})\right) \end{aligned}$$

   The symmetry $T^{+-} = T^{-+}$ is again a consequence of (30). Further more the identity of K-functions follows

$$K_n^{(T)}(\underline{\theta}) = -\frac{\alpha \left(1 - \frac{\beta^2}{8\pi}\right)}{\beta^2} \left(K_n^+(\underline{\theta}) + K_n^-(\underline{\theta})\right)$$

   where the normalizations (27) and (29) have been used which means

$$\frac{N_n^{(T)}}{N_n^+} = 2 \frac{\alpha}{\beta^2} \left(1 - \frac{\beta^2}{8\pi}\right)$$

3. The claim follows since we have the correspondence of operators and p-functions

$$\partial_\mu J_L^\mu \leftrightarrow P^+ p_n^{(L,-)}(\underline{\theta},\underline{l}) + P^- p_n^{(L,+)}(\underline{\theta},\underline{l}) = -N_n^{(J_L)} m$$
$$\times \left(\sum_{i=1}^n e^{\theta_i} \sum_{i=1}^n e^{-\theta_i} - \sum_{i=1}^n e^{-\theta_i} \sum_{i=1}^n e^{\theta_i} \right) \sum_{i=1}^n e^{L\left(\theta_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)} = 0\,.$$

■

The factor $\frac{\pi\nu}{\sin\pi\nu}$ modifies the classical equation and has to be considered as a quantum correction. For the sinh-Gordon model an analogous quantum field equation has been obtained in [20][9]. Note that in particular at the 'free fermion point' $\nu \to 1$ ($\beta^2 \to 4\pi$) this factor diverges, a phenomenon which is to be expected from short distance investigations [31]. For fixed bare mass square α and $\nu \to 2, 3, 4, \ldots$ the physical mass goes to zero. These values of the coupling are known to be specific: 1. the Bethe Ansatz vacuum in the language of the massive Thirring model shows phase transitions [32] and 2. the model at these points is related [33, 34, 35] to Baxters RSOS-models which correspond to minimal conformal models with central charge $c = 1 - 6/(\nu(\nu+1))$ (see also [20]).

The second formula (34) is consistent with renormalization group arguments [36, 37]. In particular this means that $\beta^2/4\pi$ is the anomalous dimension of $\cos\beta\varphi$. Again this operator equation is modified by a quantum correction $(1 - \beta^2/8\pi)$. Obviously for fixed bare mass square α and $\beta^2 \to 8\pi$ the model will be conformal invariant which is related to a Berezinski-Kosterlitz-Thouless phase transition [7, 38, 39]. The conservation law $\partial_\mu T^{\mu\nu} = 0$ for the energy momentum tensor holds, because it is obtained from the higher currents for $L = \pm 1$. All the results may be checked in perturbation theory by Feynman

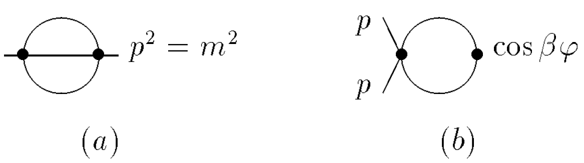


Figure 1: Feynman graphs

graph expansions. In particular in lowest order the relation between the bare and the renormalized mass (33) is given by Figure 1 (a). It had already been calculated in [12] and yields

$$m^2 = \alpha \left(1 - \frac{1}{6} \left(\frac{\beta^2}{8} \right)^2 + O(\beta^6) \right)$$

which agrees with the exact formula above. Similarly we check the quantum corrections of the trace of the energy momentum tensor (34) by calculating the Feynman graph of Figure 1 (b) with the result again taken from [12] as

$$\langle\, p \,|\!:\cos\beta\varphi - 1\!:\!|\, p \,\rangle = -\beta^2 \left(1 + \frac{\beta^2}{8\pi} \right) + O(\beta^6).$$

This again agrees with the exact formula above since the normalization given by eq. (3) implies $\langle\, p \,|\, T^\mu_{\ \mu} |\, p \,\rangle = 2m^2$.

[9]It should be obtained from (32) by the replacement $\beta \to ig$. However the relation between the bare and the renormalized mass in [20] differs from the analytic continuation of (33).

6 Other representations of form factors

6.1 Determinant representation of bosonic sine-Gordon form factors

The scaling Lee-Yang model is equivalent to the breather part of the sine-Gordon model for the coupling constant equal to $\nu = 1/3$ (in our notation). For this model Smirnov [15] derived a determinant formula for form factors (see also [17]). Generalizing this formula in [18, 19] form factors were proposed for the sinh-Gordon model in terms of determinants. The sinh-Gordon model form factors should be identified with sine-Gordon form factors by analytic continuation $\nu \to$ negative values. Using this one would propose for the sine-Gordon model an analogous determinant representation for the K-function of exponentials of the field $\mathcal{O} = :e^{ik\beta\varphi}:$

$$\tilde{K}_n^{(q)}(\underline{\theta}) = (q' - 1/q')^n \prod_{i,j=1}^{n} (x_i + x_j)^{-1} Det_n(\underline{x}, k)$$

where $\underline{x} = (x_1, \dots, x_n)$ with $x_i = e^{\theta_i}$ and $q' = \exp i\pi\nu$. The Determinant is

$$\begin{aligned} Det_n(\underline{x}, k) &= \det\left(\left((k+i-j)_{q'}\,\sigma_{2i-j-1}(\underline{x})\right)_{i,j=1}^{n}\right) \\ &= \begin{vmatrix} (k)_{q'}\,\sigma_0 & \cdots & (k-n+1)_{q'}\,\sigma_{-n+1} \\ \vdots & \ddots & \vdots \\ (k+n-1)_{q'}\,\sigma_{2n-2} & \cdots & (k)_{q'}\,\sigma_{n-1} \end{vmatrix}. \end{aligned}$$

where the symmetric polynomials $\sigma_l(\underline{x})$ are defined by

$$\prod_{l=1}^{n}(y + x_l) = \sum_{\lambda=0}^{n} y^{n-\lambda}\sigma_\lambda^{(n)}(\underline{x})\,.$$

This relation of $\tilde{K}_n^{(q)}(\underline{\theta})$ with $Det_n(\underline{x}, k)$ could be proven similar as in the proof of lemma 3, if the corresponding recursion relation (11) would have been proven for $Det_n(\underline{x}, k)$. This has been done [18] only for special values of the coupling constant $\nu = -1/2, -1/3$.

6.2 Integral representations of breather form factors

In [1] and [2] integral representations for solitonic form factors were proposed. These formulae are quite general and model independent, so analogous formulae should also hold for breather form factors. We propose for n lowest breather and $0 \le m \le n$

$$\mathcal{O}_n(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m\, h(\underline{\theta}, \underline{z})\, \check{p}^{\mathcal{O}}(\underline{\theta}, \underline{z}) \tag{35}$$

with the scalar function (c.f. [1])

$$h(\underline{\theta}, \underline{z}) = \prod_{1\le i<j\le n} F(\theta_{ij}) \prod_{i=1}^{n}\prod_{j=1}^{m} \tilde{\phi}(\theta_i - z_j) \prod_{1\le i<j\le m} \tau(z_i - z_j)$$

and

$$\tilde{\phi}(z) = \frac{S(z)}{F(z)\,F(z+i\pi)} = 1 + \frac{i\sin\pi\nu}{\sinh z} = \frac{\sinh z + i\sin\pi\nu}{\sinh z}.$$

$$\tau(z) = \frac{1}{\tilde{\phi}(z)\,\tilde{\phi}(-z)} = \frac{\sinh^2 z}{\sinh^2 z + \sin^2\pi\nu}.$$

The two breather form factor function $F(\theta)$ is again defined by eq. (4). For all integration variables z_j $(j = 1, \dots, m)$ the integration contours $\mathcal{C}_{\underline{\theta}}$ encloses clock wise oriented the points $z_j = \theta_i \, (i = 1, \dots, n)$. The above integral representation satisfies all form factor properties if suitable conditions for the new type of p-function[10] $\check{p}^{\mathcal{O}}(\theta, z)$ are assumed. Here we consider $\check{p} =$ constant. The form factors of the exponential of the field $\mathcal{O}(x) =:\exp i\gamma\varphi:(x)$ are given by linear combinations of expressions (35)

$$\mathcal{O}_n(\underline{\theta}) = \left(\sqrt{Z^\varphi}\frac{\beta}{2\pi\nu}\right)^n \prod_{1\le i<j\le n} F(\theta_{ij}) \sum_{m=0}^{n} q^{n-2m}(-1)^m I_{nm}(\underline{\theta}) \tag{36}$$

where again $q = \exp\left(i\frac{\pi\nu}{\beta}\gamma\right)$ and

$$\begin{aligned} I_{nm}(\underline{\theta}) &= \frac{1}{m!}\int_{\mathcal{C}_{\underline{\theta}}}\frac{dz_1}{R}\cdots\int_{\mathcal{C}_{\underline{\theta}}}\frac{dz_m}{R}\prod_{i=1}^{n}\prod_{j=1}^{m}\tilde{\phi}(\theta_i - z_j)\prod_{1\le i<j\le m}\tau(z_i - z_j) \\ &= \sum_{\substack{K\subseteq N \\ |K|=m}} \prod_{i\in N\setminus K}\prod_{k\in K}\tilde{\phi}(\theta_{ik}) \end{aligned}$$

with $R = \operatorname*{Res}_{\theta=0}\tilde{\phi}(\theta)$ and $N = \{1, \dots, n\}, K = \{k_1, \dots, k_m\}$. It is easy to verify that the asymptotic behavior (20) is satisfied. Also the low particle form factors agree with eqs. (23). This proves formula (36).

Another integral representation is directly obtained from the integral representations for solitonic form factors in [2]

$$\begin{aligned} \mathcal{O}(\underline{\theta}) = \tilde{N}_n^{\mathcal{O}} \prod_{i<j} \left(F(\theta_{ij})\tanh\tfrac{1}{2}\theta_{ij}\sinh\tfrac{1}{2}(\theta_{ij}+i\pi\nu)\sinh\tfrac{1}{2}(\theta_{ij}-i\pi\nu)\right) \\ \times \int_{\mathcal{D}_{\theta_1}} dz_1 \cdots \int_{\mathcal{D}_{\theta_n}} dz_r \prod_{i=1}^{n}\prod_{j=1}^{n}\chi(\theta_i - z_j)\prod_{i<j}\sinh z_{ij}\; p(\underline{\theta}, \underline{z}) \end{aligned} \tag{37}$$

where the contour $\mathcal{D}_{\theta_i}$ consists of two circles around the poles at $\theta_i - \frac{i\pi}{2}(1\pm\nu)$ and

$$\chi(\theta) = \frac{1}{\sinh\frac{1}{2}(\theta - \frac{i\pi}{2}(1-\nu))\sinh\frac{1}{2}(\theta - \frac{i\pi}{2}(1+\nu))}.$$

[10] A similar function was used by Cardy and Mussardo [16] in case of the scaling Ising model to represent the various operators.

As a matter of fact in [2] from this integral representation the representation (1) with the K-function (2) was derived using the identity

$$\sinh\tfrac{1}{2}(\theta_{ij}-i\pi\nu)\chi(\theta_i-z_j^{(l_j)})\chi(\theta_j-z_i^{(l_i)})\sinh(z_i^{(l_i)}-z_j^{(l_j)})$$
$$=\frac{2}{\tanh\frac{1}{2}\xi_{ij}\sinh\frac{1}{2}(\xi_{ij}+i\pi\nu)\sinh\frac{1}{2}(\xi_{ij}-i\pi\nu)}\left(1+(l_i-l_j)\frac{i\sin\pi\nu}{\sinh\xi_{ij}}\right)$$

for $l_i,l_j=0,1$ and $z_i^{(l_i)}=\xi_i-\frac{i\pi}{2}(1-(-1)^{l_i}\nu)$. Performing one integration in eq. (37) and using symmetry properties of the integrand one obtains an integral representations of the type as used by Smirnov in [15].

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Appendix: Asymptotic behavior

Lemma 5 *The K-functions defined by eq. (2) and the p-functions*

$$\begin{array}{ll} \text{a)} & \tilde p_n^{(q)}(\underline\theta)=\prod_{i=1}^n q^{(-1)^{l_i}} \\ \text{b)} & \tilde p_n^{(N)}(\underline\theta)=\left(\sum_{i=1}^n(-1)^{l_i}\right)^N \\ \text{c)} & \tilde p_n^{(\pm)}(\underline\theta)=\sum_{i=1}^n e^{\mp\theta_i}\sum_{i=1}^n e^{\pm z_i^{(l_i)}} \end{array}$$

satisfy for $\operatorname{Re}\theta_1\to\infty$ *the asymptotic behavior*

$$\begin{array}{ll} \text{a)} & \tilde K_n^{(q)}(\underline\theta)=\tilde K_1^{(q)}(\theta_1)\tilde K_{n-1}^{(q)}(\underline\theta')+O(e^{-\operatorname{Re}\theta_1}) \\ \text{b)} & \tilde K_n^{(N)}(\underline\theta)=\sum_{K=1}^{N-1}\binom{N}{K}\tilde K_1^{(K)}(\theta_1)\tilde K_{n-1}^{(N-K)}(\underline\theta')+O(e^{-\operatorname{Re}\theta_1}) \\ \text{c)} & \tilde K_n^{\pm}(\underline\theta)=\pm2i\sin\pi\nu\,\tilde K_{n-1}^{\pm}(\underline\theta')+O(e^{-\operatorname{Re}\theta_1}) \end{array}$$

with $\underline\theta'=(\theta_2,\dots,\theta_1)$. *In particular*

$$\tilde K_1^{(1)}(\theta)=const\quad\text{and}\quad\tilde K_n^{(1)}(\underline\theta)=O(e^{-\operatorname{Re}\theta_1})\quad\text{for } n>1.$$

Proof. The two first asymptotic relations are quit obvious. Note that $\tilde K_1^{(q)}=q-1/q$ and $\tilde K_1^{(1)}(\underline\theta)=2$.

a) For $\tilde K_n^{(q)}(\underline\theta)$ and $\operatorname{Re}\theta_1\to\infty$ we have

$$\tilde K_n^{(q)}(\underline\theta)=\sum_{l_1=0}^1(-1)^{l_1}q^{(-1)^{l_1}}\tilde K_{n-1}^{(q)}(\underline\theta')+O(e^{-\operatorname{Re}\theta_1})$$

b) For $\tilde{K}_n^{(N)}(\underline{\theta})$ we use

$$\left(\sum_{i=1}^{n}(-1)^{l_i}\right)^N = \sum_{K=0}^{N}\binom{N}{K}\left((-1)^{l_1}\right)^K\left(\sum_{i=2}^{n}(-1)^{l_i}\right)^{N-K}$$

which proves the claim as in the previous case.

c) For $\tilde{K}_n^{+}(\underline{\theta})$ and $\operatorname{Re}\theta_1 \to \infty$ we have

$$\begin{aligned}\sum_{i=1}^{n} e^{-\theta_i}\sum_{i=1}^{n} e^{z_i^{(l_i)}} &= \left(e^{-\theta_1}+\sum_{i=2}^{n} e^{-\theta_i}\right)\left(e^{\theta_1-\frac{i\pi}{2}(1-(-1)^{l_1}\nu)}+\sum_{i=2}^{n} e^{z_i^{(l_i)}}\right)\\ &= e^{-\frac{i\pi}{2}(1-(-1)^{l_1}\nu)}+\left(\sum_{i=2}^{n} e^{-\theta_i}\right)e^{\theta_1-\frac{i\pi}{2}(1-(-1)^{l_1}\nu)}\\ &\quad +\sum_{i=2}^{n} e^{-\theta_i}\sum_{i=2}^{n} e^{z_i^{(l_i)}}+O(e^{-\operatorname{Re}\theta_1}).\end{aligned}$$

We calculate the leading terms $O(1)$. The contribution of the first term consists of two types: one vanishes because of the lemma above and the other is of order $O(e^{-\operatorname{Re}\theta_1})$. The contribution of the third term vanishes after summation over l_1. The contribution of the second term is proportional

$$\begin{aligned}&\sum_{l_1=0}^{1}(-1)^{l_1}\left(1+\sum_{j=2}^{n}(l_1-l_j)\frac{i\sin\pi\nu}{\sinh\theta_{1j}}\right)e^{\theta_1-\frac{i\pi}{2}(1-(-1)^{l_1}\nu)}\\ &\qquad\approx -ie^{\theta_1}\left(e^{\frac{i\pi}{2}\nu}-e^{-\frac{i\pi}{2}\nu}\right)+2i\sin\pi\nu\sum_{j=2}^{n}\sum_{l_1=0}^{1}(-1)^{l_1}(l_1-l_j)e^{\theta_j-\frac{i\pi}{2}(1-(-1)^{l_1}\nu)}\end{aligned}$$

The first term again vanishes due to the lemma and the second one yields

$$\begin{aligned}&2i\sin\pi\nu\sum_{j=2}^{n}\sum_{l_1=0}^{1}(-1)^{l_1}(l_1-l_j)e^{\theta_j-\frac{i\pi}{2}(1-(-1)^{l_1}\nu)}\\ &= 2i\sin\pi\nu\sum_{j=2}^{n}\sum_{l_1=0}^{1}(-1)^{l_1}(l_1-l_j)e^{\theta_j-\frac{i\pi}{2}(1-(-1)^{1-l_j}\nu)}\\ &= -2i\sin\pi\nu\sum_{j=2}^{n}e^{\theta_j-\frac{i\pi}{2}(1+(-1)^{l_j}\nu)}\end{aligned}$$

Therefore we finally obtain the asymptotic behavior

$$\begin{aligned}\tilde{K}_n^{+}(\underline{\theta}) &= -2i\sin\pi\nu\sum_{l_2=0}^{1}\cdots\sum_{l_n=0}^{1}(-1)^{l_2+\cdots+l_n}\\ &\quad\times\prod_{2\le i<j\le n}\left(1+(l_i-l_j)\frac{i\sin\pi\nu}{\sinh\theta_{ij}}\right)\\ &\quad\times\sum_{i=2}^{n}e^{-\theta_i}\sum_{j=2}^{n}e^{\theta_j-\frac{i\pi}{2}(1+(-1)^{l_j}\nu)}+O(e^{-\operatorname{Re}\theta_1})\\ &= 2i\sin\pi\nu\,\tilde{K}_{n-1}^{+}(\underline{\theta}')+O(e^{-\operatorname{Re}\theta_1})\end{aligned}$$

with $\underline{\theta'} = (\theta_2, \ldots, \theta_n)$ It has been used that because of the lemma

$$\sum_{l_2=0}^{1} \cdots \sum_{l_n=0}^{1} (-1)^{l_2+\cdots+l_n} \prod_{2\le i<j\le n} \left(1 + (l_i - l_j)\frac{i\sin\pi\nu}{\sinh\theta_{ij}}\right)$$
$$\times \sum_{j=2}^{n} \left(e^{\theta_j - \frac{i\pi}{2}(1+(-1)^{l_j}\nu)} + e^{\theta_j - \frac{i\pi}{2}(1-(-1)^{l_j}\nu)}\right) = 0\,.$$

For $\tilde{K}_n^-(\underline{\theta})$ and $\operatorname{Re}\theta_1 \to \infty$ we have

$$\begin{aligned}
\sum_{i=1}^{n} e^{\theta_i} \sum_{i=1}^{n} e^{-z_i^{(l_i)}} &= \left(e^{\theta_1} + \sum_{i=2}^{n} e^{\theta_i}\right)\left(e^{-\theta_1 + \frac{i\pi}{2}(1-(-1)^{l_1}\nu)} + \sum_{i=2}^{n} e^{-z_i^{(l_i)}}\right) \\
&= e^{\frac{i\pi}{2}(1-(-1)^{l_1}\nu)} + e^{\theta_1}\sum_{i=2}^{n} e^{-\theta_j - \frac{i\pi}{2}(1-(-1)^{l_j}\nu)} \\
&\quad + \sum_{i=2}^{n} e^{\theta_i} \sum_{i=2}^{n} e^{-z_i^{(l_i)}} + O(e^{-\operatorname{Re}\theta_1}).
\end{aligned}$$

We again calculate the leading terms $O(1)$. Again the contribution of the first term consists of two types: one vanishes because of the lemma above and the other is of order $O(e^{-\operatorname{Re}\theta_1})$. The contribution of the third term vanishes after summation over l_1. The contribution of the second term is proportional

$$\begin{aligned}
&\sum_{l_1=0}^{1}(-1)^{l_1}\left(1 + \sum_{j=2}^{n}(l_1 - l_i)\frac{i\sin\pi\nu}{\sinh\theta_{1i}}\right) e^{\theta_1} \sum_{j=2}^{n} e^{-\theta_j + \frac{i\pi}{2}(1-(-1)^{l_j}\nu)} \\
&\approx 2i\sin\pi\nu \sum_{j=2}^{n} e^{\theta_i} \sum_{l_1=0}^{1}(-1)^{l_1}(l_1 - l_j)e^{-\theta_j + \frac{i\pi}{2}(1-(-1)^{l_j}\nu)} \\
&= -2i\sin\pi\nu \sum_{j=2}^{n} e^{\theta_i} \sum_{j=2}^{n} e^{-\theta_j + \frac{i\pi}{2}(1-(-1)^{l_j}\nu)}
\end{aligned}$$

Therefore we finally obtain the asymptotic behavior

$$\begin{aligned}
\tilde{K}_n^-(\underline{\theta}) &= -2i\sin\pi\nu \sum_{l_2=0}^{1}\cdots\sum_{l_n=0}^{1}(-1)^{l_2+\cdots+l_n} \\
&\quad \times \prod_{2\le i<j\le n}^{n}\left(1 + (l_i - l_j)\frac{i\sin\pi\nu}{\sinh\theta_{ij}}\right) \\
&\quad \times \sum_{i=2}^{n} e^{\theta_i} \sum_{j=2}^{n} e^{-\theta_j + \frac{i\pi}{2}(1+(-1)^{l_j}\nu)} + O(e^{-\operatorname{Re}\theta_1}) \\
&= -2i\sin\pi\nu\, \tilde{K}_{n-1}^-(\underline{\theta'}) + O(e^{-\operatorname{Re}\theta_1})\,.
\end{aligned}$$

■

Analogously one may discuss the behavior for $\operatorname{Re}\theta_1 \to -\infty$.

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