Universal boundary reflection amplitudes

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ABSTRACT: For all affine Toda field theories we propose a new type of generic boundary bootstrap equations, which can be viewed as a very specific combination of elementary boundary bootstrap equations. These equations allow to construct generic solutions for the boundary reflection amplitudes, which are valid for theories related to all simple Lie algebras, that is simply laced and non-simply laced. We provide a detailed study of these solutions for concrete Lie algebras in various representations.

1. Introduction

Similarly as in most other areas of physics, the majority of investigations on integrable quantum field theories consists of the study of specific examples, that is particular models. Certain general ideas and concepts can be studied very well in this manner. However, ultimately one would like to have formulations which go beyond particular examples as they will unravel better which features are model dependent and which ones are of a generic nature.

In the case of affine Toda field theory (ATFT) [1, 2] such type of formulation exists for the scattering matrices in (1+1) space-time dimensions [3, 4], where the space is a line extended infinitely in both directions. The formulae found are of generic validity independent of the particular algebra underlying the theory. The understanding is not this well developed when the theory is considered in half-space (or finite), i.e. when the line is restricted by a boundary in one direction (or possibly both). For such theories the Yang-Baxter equations [5, 6] with reflecting boundaries have been investigated first in [7, 8]. Recently some universal algebraic solutions for the Yang-Baxter equations for lattice models have been constructed [9]. For a full fletched quantum field theory one needs further properties of these solutions, such as unitarity, crossing invariance and the bootstrap equations, which were formulated in [10]. The solutions for the latter system of equations for some affine Toda field theories were first found in [10, 11]. Later on, several other types of solutions for these theories have been proposed and they have been investigated with respect to various aspects [12, 13, 14, 15, 16, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. In particular the sinh-Gordon model has attracted a considerable amount of attention [39, 40, 41, 42, 43, 44, 45]. Despite all this activities, up to now closed formulae similar to the ones mentioned for the bulk theories have not been provided for the corresponding scattering amplitudes when boundaries are included. Furthermore, for some algebras no solutions at all have been found yet, even on a case-by-case level. One of the purposes of this paper is to fill in the missing gaps, but the central aim is to supply universal, in the sense of being valid for all simple Lie algebras and all particle types, formulae for the boundary scattering amplitudes in affine Toda field theories.

Our manuscript is organized as follows: In section 2 we recapitulate the key ideas of the scattering theory with reflecting boundaries and emphasize the possibility of using certain ambiguity transformations to construct new solutions for the boundary reflection amplitudes. Section 3 contains our main result. We discuss here the solutions of the combined bootstrap equations. We first recall the analogue procedure for the bulk theory and thereafter adapt it to the situation with reflecting boundaries. We provide generic solutions for ATFT's in form of integral representations as well as the equivalent products of hyperbolic functions. In section 4 we provide the explicit evaluation of our generic expression for the reflection amplitudes for some ATFT's related to some concrete Lie algebras. In section 5 we demonstrate in detail how our solution can be used as a "seed" for the construction of other types of solutions, in particular we show how one may obtain from our solution, which respects the strong-weak duality in the coupling constant, a distinct solution in which this symmetry is broken. We provide a brief argument on how within the bootstrap context free parameters enter into theories related to non-simply laced algebras as well as the sinh-Gordon model. We state our conclusions in section 6. In an appendix we provide the details for the evaluation of the inverse q-deformed Cartan matrix and the kernel entering the integral representation of the reflection amplitudes.

2. Scattering theory with reflecting boundaries

We briefly recall some well known results in order to fix our notation and to state the problem. Exploiting the fact that the scattering of integrable theories in 1+1 dimensions is factorized, one may formulate the theory with the help of particle creation (annihilation) operators for the particle of type i moving with rapidity θ, say $Z_i(\theta)$, and a boundary in the state α, referred to as Z_α. Throughout this paper we denote particle types and boundary degrees of freedom by Latin and Greek letters, respectively. The operators are assumed to obey certain exchange relations, the so-called (extended) Zamolodchikov algebra,

$$Z_i(\theta_1)\, Z_j(\theta_2) = S_{ij}(\theta_{12}) Z_j(\theta_2)\, Z_i(\theta_1)\,, \tag{2.1}$$

$$Z_i(\theta)\, Z_\alpha = R_{i\alpha}(\theta)\, Z_i(-\theta)\, Z_\alpha\ . \tag{2.2}$$

We restrict our attention here to diagonal theories, i.e. absence of backscattering, and do not distinguish whether we have left or right half-spaces, i.e. if the particle hits the boundary from the left or right. This means we assume parity invariance. We abbreviate as usual $\theta_{12} = \theta_1 - \theta_2$. The equation (2.2) expresses the fact that the particle i is reflected off the boundary by picking up a boundary reflection amplitude R, is changing its sign of the momentum and of course that the particle always has to stay on one particular side of the boundary. The amplitudes obey the crossing and unitarity equations [46, 47, 48, 49]

$$S_{ij}(\theta)S_{ji}(-\theta) = 1, \qquad S_{i\bar{j}}(\theta) = S_{ji}(i\pi - \theta), \tag{2.3}$$

$$R_{i\alpha}(\theta)R_{i\alpha}(-\theta) = 1, \qquad R_{i\alpha}(\theta)R_{\bar{\imath}\alpha}(\theta + i\pi) = S_{ii}(2\theta) \; . \tag{2.4}$$

Most restrictive and specific to the particular theory under investigation are the bootstrap equations [50, 51, 52, 10]

$$S_{lk}(\theta) = S_{li}(\theta + i\eta_{ik}^j)S_{lj}(\theta - i\eta_{jk}^i), \tag{2.5}$$

$$R_{k\alpha}(\theta) = R_{i\alpha}(\theta + i\eta_{ik}^j)R_{j\alpha}(\theta - i\eta_{jk}^i)S_{ij}(2\theta + i\eta_{ik}^j + i\eta_{jk}^i), \tag{2.6}$$

where the $\eta_{ik}^j \in \mathbb{R}$ are fusing angles which encode the possibility that the process $i + j \to k$ takes place, i.e. particle k can be formed as a bound state in the scattering process between the particles i and j. The amplitude $R_{i\alpha}(\theta)$ might have single order poles and residues satisfying $-i \operatorname{Res} R(\theta) > 0$, at say $\theta = \eta_{i\alpha}^\beta$ which are interpreted as $i + \alpha \to \beta$, that is the particle i can cause the boundary to change from the state α into the state β. This process is encoded in a second type of boundary bootstrap equations [49]

$$R_{j\beta}(\theta) = R_{j\alpha}(\theta)S_{ij}(\theta + i\eta_{i\alpha}^\beta)S_{ij}(\theta - i\eta_{i\alpha}^\beta) \; . \tag{2.7}$$

As in the bulk theory the solutions to these equations are not unique and there are various ambiguities which can be used to construct from a known solution $R_{i\alpha}(\theta)$ of the equations (2.4), (2.6) and (2.7) a new solution $R'_{i\alpha}(\theta)$

$$R_{i\alpha}(\theta, B) \to R'_{i\alpha}(\theta, B) = R_{\bar{\imath}\alpha}(\theta + i\pi, B), \tag{2.8}$$

$$R_{i\alpha}(\theta, B) \to R'_{i\alpha}(\theta, B) = R_{i\alpha}(\theta, B) \prod\nolimits_j S_{ij}(\theta, B), \tag{2.9}$$

$$R_{i\alpha}(\theta, B) \to R'_{i\alpha}(\theta, B, B') = R_{i\alpha}(\theta, B) \prod\nolimits_{j=1}^{\ell} S_{ij}(\theta, B'), \tag{2.10}$$

$$R_{i\alpha}(\theta, B) \to R'_{i\alpha}(\theta, B') = R_{i\alpha}(\theta, B') \qquad \text{if} \quad S_{ij}(\theta, B) = S_{ij}(\theta, B') \; . \tag{2.11}$$

It is clear that (2.8) always holds [11] due to the fact that $S_{ij} = S_{\bar{\imath}\bar{j}}$. The validity of (2.9) was noted in [13] for some values of j and in general the new $R'_{i\alpha}(\theta)$ can be related to a boundary in a different state, such as for instance $R_{i\beta}(\theta)$ [13]. The possibility to construct a new solution in the form (2.10) was pointed out in [12], where ℓ denotes here the total amount of different particle types in the theory. We have also stated explicitly some dependence on the effective coupling B or B', which will be most important for what follows. The relevance of this is that we may change by means of (2.10) from a solution which respects a certain symmetry in the coupling constant, such as the strong-weak duality, to one in which this symmetry is broken. The relation (2.11) expresses the fact that once the bulk theory respects a certain symmetry we may construct a new solution for the boundary reflection amplitude in which this symmetry might be broken by replacing the coupling according to the bulk symmetry.

Let us briefly comment on the status of explicit solutions to the boundary reflection amplitude consistency equations (2.4), (2.6) and (2.7). For the particular example of affine Toda field theory related to simply laced algebras solutions to these equations were already constructed in [11]. Later on various other types of solutions have been proposed and investigated with respect to various aspects [12, 13, 14, 15, 16, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. As we shall demonstrate, essentially all these solutions can be related to each other or further solutions by means of (2.8)-(2.11). With regard to the above stated problem of finding closed solutions, not much progress has been made in the last ten years. Closed solutions which respect the bulk duality symmetry $B \to 2 - B$ for the A and D series were already found in [11]. Therefore, these type of solutions reduce in the strong as well as in the weak coupling limit to the same limit, such that if one would like to construct a solution which relates two different types of boundary conditions in these extremes, as proposed in [14], one has to break the duality symmetry. In [34] Fateev proposed a conjecture of such type for all simply laced algebras in form of an integral representation which generalizes a solution for the A series of [14, 27], the latter being simply related to the original one in [11] by the ambiguity transformations (2.8)-(2.11). However, apart from $D_n^{(1)}$, the conjecture of [34] provides in general only a solution of the crossing-unitarity relations (2.4). A solution for the boundary bootstrap equation (2.6) is only proposed in some cases for some particular amplitudes. A conjecture of a similar nature for some ATFT's related to some non-simply laced algebras ($B_n^{(1)}, C_n^{(1)}, A_{2n}^{(2)}$) was formulated in [36]. Here we aim to fill in the missing gaps, that is provide solutions for the amplitudes and algebras not treated so far. Moreover rather than just stating the solution as a conjecture, we propose a systematic and unified derivation for all Lie algebras, which was absent so far.

3. Solutions of the combined bootstrap equations

3.1 Bulk theory

We recall now the key idea of how a universal expression for the scattering matrix can be constructed in the bulk theory and adapt the procedure thereafter to the situation with reflecting boundaries. As already mentioned, the central equations for the construction of the scattering matrices when backscattering is absent are the bootstrap equations (2.5). These equations express a consequence of integrability, namely that when two particles (i and j) fuse to a third (k), it is equivalent to scatter with an additional particle (l) either with the two particles before the fusing takes place or with the resulting particle after the fusing process has happened. In principle, all these "basic" bootstrap equations (2.5), together with the constraints of crossing and unitarity (2.3), are sufficient to construct solutions for the scattering amplitudes. Proceeding this way is in general a quite laborious task when carried out for each algebra individually. However, in [3] it was noted that for affine Toda field theories there is one very special set of equations which may be obtained by substituting the previously mentioned "basic" bootstrap equations (2.5) into each other in a very particular way and which were therefore referred to as "combined bootstrap equations"

$$S_{ij}\left(\theta+\eta_i\right)S_{ij}\left(\theta-\eta_i\right)=\prod_{k=1}^{\ell}\prod_{n=1}^{I_{ik}}S_{jk}\left(\theta+\theta_{ik}^n\right)\ . \tag{3.1}$$

In order to keep the writing compact, the following abbreviations will be useful

$$\eta_i := \theta_h + t_i\theta_H, \qquad \theta_{ij}^n := (2n-1-I_{ij})\theta_H, \tag{3.2}$$

$$\theta_h := \frac{i\pi(2-B)}{2h} = i\pi\vartheta_h, \qquad \theta_H := \frac{i\pi B}{2H} = i\pi\vartheta_H. \tag{3.3}$$

The affine Toda field theory coupling constant β is encoded here into the effective coupling

$$B=\frac{2H\beta^2}{H\beta^2+4\pi h}\ . \tag{3.4}$$

We recall that ATFT's have to be considered in terms of some dual pairs of Lie algebras, where the classical Lagrangian related to one or the other algebra is obtained either in the weak or strong coupling limit, where h and H denote the respective (generalized) Coxeter numbers. The integers t_i symmetrize the incidence matrix I, i.e. $I_{ij}t_j = I_{ji}t_i$ and are either $t_i = 1$ or equal to the ratio of the length of long and short roots $t_i = \alpha_i^2/\alpha_s^2$, with α_s being a short root. For more details on the notation and the physics of these models see [3] and references therein.

The remarkable fact about equation (3.1) is that it contains the information about the entire bulk scattering theory. Just by solving these equations [3] one may derive universal expressions for the scattering amplitudes for *all* particle types i,j and *all* simple Lie algebras. In form of an integral representation the solutions acquire a particularly compact and neat form

$$S_{ij}(\theta,B)=\exp\int_0^\infty\frac{dt}{t}\ \Phi_{ij}(t)\sinh\left(\frac{\theta t}{i\pi}\right)\ , \tag{3.5}$$

with

$$\Phi_{ij}(t) = 8\sinh(\vartheta_h t)\sinh(t_i\vartheta_H t)K_{ij}^{-1}(t)\ , \tag{3.6}$$

$$K_{ij}(t) = 2\cosh(\vartheta_h t+t_i\vartheta_H t)\delta_{ij}-[I_{ij}]_{\bar q(t)} = (q\bar q^{t_i}+q^{-1}\bar q^{-t_i})\delta_{ij}-[I_{ij}]_{\bar q}, \tag{3.7}$$

where we used the standard notation $[n]_q = (q^n-q^{-n})/(q^1-q^{-1})$ for q-deformed integers. The deformation parameters are related to the coupling constant and are $q(t)=\exp(t\vartheta_h)$ and $\bar q(t)=\exp(t\vartheta_H)$. In fact the only relevant cases here for the deformed incidence matrix are $[0]_{\bar q(t)}=0$, $[1]_{\bar q(t)}=1$, $[2]_{\bar q(t)}=2\cosh(\vartheta_H t)$ and $[3]_{\bar q(t)}=1+2\cosh(2\vartheta_H t)$.

In [3] the combined bootstrap equations (3.1) were derived by translating an identity in the root space of the underlying simple Lie algebras into an expression for the scattering matrices. We present here a much simpler heuristic argument on how to obtain (3.1) which is suitable for a generalization to the situation with reflecting boundaries. For this purpose we can formally assume the following operator product identity

$$Z_i\left(\theta+\eta_i\right) Z_i\left(\theta-\eta_i\right)=\prod_{k=1}^{\ell} \prod_{n=1}^{I_{ik}} Z_k\left(\theta+\theta_{ik}^n\right) \ . \tag{3.8}$$

It is then clear that the combined bootstrap equations (3.1) follow immediately when we act on both sides of (3.8) with $Z_j\left(\theta'\right)$ from the right (left) and move it to the left (right) subject to the exchange relations (2.1). As such, this is a rather evident statement, but the relation (3.8) will lead to less obvious results when reflecting boundaries are included. Here we employ (3.8) only as a very useful computational tool, but it would be very interesting to have a deeper physical understanding of this identity as well as of the combined bootstrap equation (3.1). Note that for each concrete algebra we can disentangle precisely in which way (3.1) can be manufactured from the "basic" bootstrap equations (2.5), but at present we are not able to provide a general construction scheme which achieves this in a case independent manner.

3.2 Theory with reflecting boundaries

Let us adapt the above arguments to the situation with reflecting boundaries. In that case we have besides the exchange relations (2.1) also the relations (2.2) at our disposal. We act now with each product of particle states on the left and right hand side in the identity (3.8) on the boundary state Z in such a way that each individual particle hits this boundary state. For simplicity we suppress here for the time being the explicit mentioning of the boundary degree of freedom ($Z_\alpha \rightarrow Z$) and assume that the boundaries remain in the same state during this process of subsequent bombardment with particles. Ensuring that all particles have contact with the boundary and considering thereafter the resulting state, amounts to saying that an asymptotic in-state is related to an out-state by a complete reversal of all signs in the momenta. Viewing then the asymptotic states obtained in this manner as equivalent, we derive a set of "combined boundary bootstrap equations"

$$\begin{aligned} R_i\left(\theta+\eta_i\right) R_i\left(\theta-\eta_i\right) S_{ii}(2\theta) &= \prod_{j=1}^{\ell} \prod_{n=1}^{I_{ij}} R_j\left(\theta+\theta_{ij}^n\right) \prod_{1\leq n<m\leq I_{ij}} S_{jj}(2\theta+\theta_{ij}^n+\theta_{ij}^m) \\ &\times \prod_{1\leq j<k\leq \ell} \prod_{n=1}^{I_{ij}} \prod_{m=1}^{I_{ik}} S_{jk}(2\theta+\theta_{ij}^n+\theta_{ik}^m) \ . \end{aligned} \tag{3.9}$$

The occurrence of the bulk scattering matrices in (3.9) is due to the fact that after a particle has hit the boundary a subsequent particle can only reach the boundary when it first scatters with the particle already returning back from the boundary, such that S always depends on the sum of the rapidities of the originally incoming particles. The product $\prod_{1\leq n<m\leq I_{ij}}$ involving particles of the same type only emerges for non-simply laced algebras. The equations (3.9) are central for our investigations and we can regard them as the analogues of (3.1). Therefore, we may expect that they contain all informations of the boundary reflection. Let us solve them similarly as in [10, 11, 3], that is we take the logarithm of (3.9) and subsequently use Fourier transforms. For this we define first

$$\ln R_j(\theta)=\frac{1}{2\pi}\int dt\ e^{it\theta} r_j(t) \qquad \text{and} \qquad \ln S_{kj}(2\theta)=\frac{1}{2\pi}\int dt\ e^{it\theta} s_{kj}(t) \tag{3.10}$$

such that from (3.9) follows

$$\sum_{j=1}^{\ell}\left[K_{ij}(t)r_j(t)-\sum_{1\leq n<m\leq I_{ij}} s_{jj}(t)e^{\frac{\theta_{ij}^n+\theta_{ij}^m}{2}}\right]=\sum_{1\leq j<k\leq \ell}\sum_{n=1}^{I_{ij}}\sum_{m=1}^{I_{ik}} s_{jk}(t)e^{\frac{\theta_{ij}^n+\theta_{ik}^m}{2}}-s_{ii}(t). \quad (3.11)$$

The important difference in comparison with the bulk theory is that this equation is non-homogeneous, in the sense that besides the quantity we want to determine, $r_j(t)$, it contains terms involving quantities we already know, namely $s_{ij}(t)$. We can use this to our advantage and solve this equation for $r_i(t)$, using the integral representation for the scattering matrix (3.5). Thus we obtain the main result of this paper, namely a closed expression for the boundary reflection matrix valid for affine Toda field theories related to all simple Lie algebras

$$\tilde{R}_j(\theta,B)=\exp\int_0^{\infty}\frac{dt}{t}\,\rho_j(t)\sinh\left(\frac{\theta t}{i\pi}\right), \quad (3.12)$$

with kernel

$$\rho_i(t)=\frac{1}{2}\sum_{j,k,p=1}^{\ell}\left[K^{-1}(t)\right]_{ij}\chi_j^{kp}(t)\Phi_{kp}(t/2)\,, \quad (3.13)$$

$$\chi_j^{kp}=(1-\delta_{pk})[I_{jk}]_{\bar{q}^{1/2}}[I_{jp}]_{\bar{q}^{1/2}}-2\delta_{jk}\delta_{jp}+2\sum_{n=1}^{I_{jk}-1}[n]_{\bar{q}}\delta_{kp}. \quad (3.14)$$

In the simply laced case the tensor χ reduces to

$$\chi_j^{kp}=I_{jk}I_{jp}-\delta_{pk}I_{jp}-2\delta_{jk}\delta_{jp}. \quad (3.15)$$

In the derivation we made use of parity invariance, that is we used $s_{ij}(t)=s_{ji}(t)$. To the particular solution we constructed from (3.9) we refer from now on always as $\tilde{R}_i(\theta,B)$ in order to distinguish it from other solutions which might be obtained by means of the ambiguities (2.8)-(2.11).

3.3 Integral representation versus blocks of hyperbolic functions

The integral representations (3.1) and (3.12) are very useful starting points for various applications such as the computations of form factors or the thermodynamic Bethe ansatz. However, one has to be cautious when one analytically continues them into the complex rapidity plane as one usually leaves the domain of convergence when one simply carries out shifts in θ. In addition, the singularity structure of the integral representation is not directly obvious. Therefore one would like to carry out the integrations which for the above type of integral always yield some finite products of hyperbolic functions. A further reason why we wish to carry out the integrals is that already many case-by-case solutions for the above theories exist in the literature, which we want to compare with.

When performing the integration, the scattering matrix of affine Toda field theory (3.5) may be represented in the form [3]

$$S_{ij}(\theta)=\prod_{x=1}^{h}\prod_{y=1}^{H}\{x,y\}_{\theta}^{2\mu_{ij}(x,y)}\ , \quad (3.16)$$

where

$$\{x,y\}_\theta := \frac{[x,y]_\theta}{[x,y]_{-\theta}} = \exp\int_0^\infty \frac{dt}{t\sinh t}\; f_{x,y}^{h,H}(t)\sinh\left(\frac{\theta t}{i\pi}\right), \tag{3.17}$$

with

$$[x,y]_\theta := \frac{\sinh\frac{1}{2}[\theta+(x-1)\theta_h+(y-1)\theta_H]\sinh\frac{1}{2}[\theta+(x+1)\theta_h+(y+1)\theta_H]}{\sinh\frac{1}{2}[\theta+(x-1)\theta_h+(y+1)\theta_H]\sinh\frac{1}{2}[\theta+(x+1)\theta_h+(y-1)\theta_H]}, \tag{3.18}$$

$$f_{x,y}^{h,H}(t) = 8\sinh(\vartheta_h t)\sinh(\vartheta_H t)\sinh(t - x\vartheta_h t - y\vartheta_H t)\ . \tag{3.19}$$

The powers $\mu_{ij}(x,y)$ are semi-integers, which can be computed in general from some inner products between roots and weights rotated by some q-deformed Coxeter element [4, 3][1]. Alternatively, one can determine them also from the generating function

$$M_{ij}(q,\bar q) = \sum_{x=1}^{2h}\sum_{y=1}^{2H}\mu_{ij}(x,y)q^x\bar q^y = \frac{1-q^{2h}\bar q^{2H}}{2}K_{ij}^{-1}(t)\,[t_j]_{\bar q}\ . \tag{3.20}$$

For this we have to view $K_{ij}^{-1}(t)$ in the q-deformed formulation (3.7) and expand the right hand side of (3.20) into a polynomial in q and $\bar q$. For simply laced theories one could use simpler functions as in that case the two dual algebras coincide, such that $h = H$ and $\{x,x\}_\theta =: \{x\}_\theta$. The advantage of the formulation (3.20) is that it allows for a unified treatment of all algebras.

We can proceed now similarly for the reflection amplitudes and seek to represent them in the form

$$\tilde R_i(\theta) = \prod_{x=1}^{2h}\prod_{y=1}^{2H} \|x,y\|_\theta^{2\bar\mu_i(x,y)}\ , \tag{3.21}$$

where

$$\|x,y\|_\theta := \frac{\langle x,y\rangle_\theta}{\langle x,y\rangle_{-\theta}} = \exp\int_0^\infty \frac{dt}{t\sinh t}\; \bar f_{x,y}^{h,H}(t)\sinh\left(\frac{\theta t}{i\pi}\right), \tag{3.22}$$

with

$$\langle x,y\rangle_\theta := \frac{\sinh\frac{1}{2}\left[\theta+\frac{x-1}{2}\theta_h+\frac{y-1}{2}\theta_H\right]\sinh\frac{1}{2}\left[\theta+\frac{x+1}{2}\theta_h+\frac{y+1}{2}\theta_H\right]}{\sinh\frac{1}{2}\left[\theta+\frac{x-1}{2}\theta_h+\frac{y+1}{2}\theta_H\right]\sinh\frac{1}{2}\left[\theta+\frac{x+1}{2}\theta_h+\frac{y-1}{2}\theta_H\right]}, \tag{3.23}$$

$$\bar f_{x,y}^{h,H}(t) = 8\sinh(\vartheta_h t/2)\sinh(\vartheta_H t/2)\sinh(t - x\vartheta_h t/2 - y\vartheta_H t/2)\ . \tag{3.24}$$

In this case we deduce the semi-integers $\bar\mu_i(x,y)$ from

$$\bar M_i(q,\bar q) = \sum_{x=1}^{2h}\sum_{y=1}^{2H}\bar\mu_i(x,y)q^{\frac{x}{2}}\bar q^{\frac{y}{2}} = \frac{1-q^{2h}\bar q^{2H}}{2}\left[K^{-1}(t)\right]_{ij}\chi_j^{kp}\left[K^{-1}(t/2)\right]_{kp}[t_p]_{\bar q^{1/2}}\ . \tag{3.25}$$

[1] As is known for more than ten years, in the special case of simply laced Lie algebras one can use the simpler formulation in terms of ordinary Coxeter elements [53, 54]. However, none of the formulations will be used here.

Once again for the simply laced cases this becomes easier $\|x,x\|_\theta =: \|x\|_\theta$, which equal the blocks $\mathcal{W}_{h-x}(\theta)$ used in [11]. For the non-simply laced cases we have in principle two possible algebras, whose Lie algebraic properties we can relate to. We make here the choice to express everything in terms of the non-twisted algebra. Clearly one can also formulate equivalently a generating function in terms of its dual as carried out for the bulk theory in [3], but as this does not yield new physical information, we shall be content here to do so for one algebra only.

In the following, we also abbreviate some products of the above blocks in a more compact form

$$\{x, y_n\}_\theta := \prod_{l=0}^{n-1} \{x, y+2l\}_\theta\,, \qquad \|x, y_n\|_\theta := \prod_{l=0}^{n-1} \|x, y+2l\|_\theta\,, \tag{3.26}$$

and

$$\left\{x_1, y_1^{\mu_1}; x_2, y_2^{\mu_2}; \cdots; x_n, y_n^{\mu_n}\right\}_\theta := \{x_1, y_1\}_\theta^{\mu_1} \{x_2, y_2\}_\theta^{\mu_2} \ldots \{x_n, y_n\}_\theta^{\mu_n}\,, \tag{3.27}$$

$$\left\|x_1, y_1^{\mu_1}; x_2, y_2^{\mu_2}; \cdots; x_n, y_n^{\mu_n}\right\|_\theta := \|x_1, y_1\|_\theta^{\mu_1} \|x_2, y_2\|_\theta^{\mu_2} \ldots \|x_n, y_n\|_\theta^{\mu_n}\ . \tag{3.28}$$

For completeness we also introduce here a more elementary block which will be useful for the comparison with results in the literature

$$(x)_\theta := \frac{\sinh(\theta + i\pi x/h)/2}{\sinh(\theta - i\pi x/h)/2} = -\exp\left(2\int_0^\infty \frac{dt}{t\sinh t}\ \sinh t(1-x/h)\sinh\frac{\theta t}{i\pi}\right). \tag{3.29}$$

We shall also use below the blocks

$$\widehat{\|x\|}_\theta\ : = \frac{\left(\frac{x-1}{2}\right)\left(\frac{x+1}{2} - h\right)}{\left(\frac{x-1+B}{2} - h\right)\left(\frac{x+1-B}{2}\right)}, \tag{3.30}$$

$$\overline{\|x\|}_\theta\ : = \frac{\left(\frac{h+x-1}{2}\right)\left(\frac{h-x+1}{2}\right)\left(\frac{h+x-1+B}{2}\right)\left(\frac{h-x+1-B}{2}\right)}{\left(\frac{h+x+1}{2}\right)\left(\frac{h-x-1}{2}\right)\left(\frac{h+x+1-B}{2}\right)\left(\frac{h-x-1+B}{2}\right)}, \tag{3.31}$$

which break the strong weak-duality.

By evaluating (3.25), we can determine case-by-case the powers in (3.21). For the simply laced case, it will turn out that our solutions coincide with the ones found by Kim [55] upon the use of the ambiguity (2.8)[2]. For the non-simply laced cases only two specific examples have been treated in [56]. On further solutions related to non-simply laced algebras we shall comment below.

4. $\tilde{R}_i(\theta, B)$ case-by-case

We shall now be more concrete and evaluate our generic solution $\tilde{R}_i(\theta, B)$ in more detail for some specified Lie algebras. We compare with some solutions previously found in the

[2]We are grateful to J.D. Kim for informing us that hep-th/9506031 v2 is published in [55] and that there is some discrepancy between the two versions.

literature. As our solutions are invariant under the strong-weak duality transformation we commence by comparing with those being of this type also. Apart from the $A_2^{(1)}$-case, we postpone the comparison with other types of solutions to section 5.

For the simply laced algebras the closed solution (3.12) admits an even simpler general block formulation

$$\tilde{R}_i(\theta+i\pi,B)=\prod_{x=1}^{h}\|2x-1\|_\theta^{\kappa_i}\left(\prod_{x\in\tilde{\mathcal{X}}_i}\|x\|_\theta\,\|x-2h\|_\theta\right)\ ,\tag{4.1}$$

where the integers κ_i are defined through the relation $\prod_{j=1}^{\ell}S_{ij}(\theta)=\prod_{x=1}^{h}\{x\}_\theta^{\kappa_i}$ and the sets $\tilde{\mathcal{X}}_i$ are specific to each algebra. At present we do not know how a general case independent formula which determines the sets $\tilde{\mathcal{X}}_i$.

4.1 $\mathbf{A}_\ell^{(1)}$-affine Toda field theory

4.1.1 $A_2^{(1)}$-affine Toda field theory

Let us exemplify the working of the above formulae with some easy example. As the sinh-Gordon model ($A_1^{(1)}$-ATFT) is very special [39, 40, 41, 42, 43, 44, 45] and exhibits a distinguished behaviour from all other ATFT's related to simply laced Lie algebras, we consider the next simple case, namely $A_2^{(1)}$-ATFT. This was already studied in [10, 11, 14] and especially detailed in [26]. The Coxeter number is $h=3$ in this case. The essential Lie algebraic input here is the inverse of the q-deformed Cartan matrix (3.7)

$$K^{-1}(t)=\frac{1}{1+2\cosh 2t/h}\begin{pmatrix}2\cosh t/h & 1\\ 1 & 2\cosh t/h\end{pmatrix}\ .\tag{4.2}$$

With this we compute from (3.13) and (3.15)

$$\rho_1(t)=\rho_2(t)=16\frac{\sinh[(B-2)t/12]\sinh(Bt/12)\cosh(t/6)}{1+2\cosh(2t/3)},\tag{4.3}$$

and (3.25) yields

$$\begin{aligned}\tilde{R}_1(\theta,B)&=\tilde{R}_2(\theta,B)=\tilde{R}_1(\theta,2-B)=\|7,7\|_\theta\,\|9,9\|_\theta\,, &(4.4)\\ &=-(-1)_\theta(-2)_\theta^2(1+B/2)_\theta(3-B/2)_\theta(B/2+2)_\theta(2-B/2)_\theta\ . &(4.5)\end{aligned}$$

We compare now with various solutions constructed before in the literature and demonstrate that they can all be related to our solution $\tilde{R}$ by means of the ambiguities (2.8)-(2.11). We can drop the subscripts and use $R_1=R_2=R$. In [26] the following solutions were studied in detail

$$\begin{aligned}R^{\mathrm{Neu}}(\theta,B)&=R^{\mathrm{Neu}}(\theta,B-2-2h)=R^{++}(\theta,2-B)=(-2)_\theta(-B/2)_\theta(2+B/2)_\theta, &(4.6)\\ R^{--}(\theta,B)&=R^{--}(\theta,B-2-2h)=-(-1)_\theta(B/2-1)_\theta(3-B/2)_\theta, &(4.7)\\ R^{++}(\theta,B)&=R^{++}(\theta,B-2-2h)=R^{\mathrm{Neu}}(\theta,2-B)=(-2)_\theta(B/2-1)_\theta(3-B/2)_\theta. &(4.8)\end{aligned}$$

The solution $R^{\rm Neu}(\theta,B)$ was already found in [10] and several arguments were provided in [26] to identify it with the Neumann boundary condition. In addition, $R^{++}(\theta,B)$ was related to the fixed boundary condition. For $R^{--}(\theta,B)$ doubts on a conclusive identification were raised. Using now the expressions for the scattering matrix [57]

$$S_{11}(\theta,B) = S_{22}(\theta,B) = (2)_\theta(B-2)_\theta(-B)_\theta, \tag{4.9}$$

$$S_{12}(\theta,B) = S_{21}(\theta,B) = -(1)_\theta(3+B)_\theta(-1-B)_\theta, \tag{4.10}$$

it is easy to see that our solution $\tilde{R}$ is relatable to the above ones

$$R^{\rm Neu}(\theta,B) = \tilde{R}(\theta,B)S_{11}(\theta,B/2)S_{12}(\theta,B/2), \tag{4.11}$$

$$R^{--}(\theta,B) = \tilde{R}(\theta+i\pi,B)/S_{11}(\theta,B/2)/S_{12}(\theta,B/2), \tag{4.12}$$

$$R^{++}(\theta,B) = \tilde{R}(\theta,B)S_{11}(\theta,1-B/2)S_{12}(\theta,1-B/2). \tag{4.13}$$

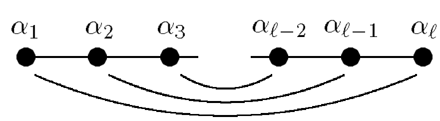
Thus we have changed by means of some ambiguities from a solution which respects the strong-weak duality transformation $B\to 2-B$ to one in which this symmetry is broken and replaced by the new symmetry $B\to B-2-2h$. The solution investigated in [55] is related to our solution by (2.9)

$$R^{\rm K}(\theta,B) = \tilde{R}(\theta+i\pi,B)\ . \tag{4.14}$$

For all amplitudes which were computed in [55] related to simply laced Lie algebras, the relation (4.14) always holds.

4.1.2 Generic $\mathbf{A}_\ell^{(1)}$-affine Toda field theory

We label the particles according to the Dynkin diagram:



The Coxeter number is $h=\ell+1$ in this case. We indicated also the automorphism which relates the particles of type j to their anti-particles $h-j$. From the formulae derived in the appendix A.1, we compute now the kernel of the integral representation (3.12) to

$$\rho_j^{\mathbf{A}_\ell}(t) = \frac{4\sinh(\frac{2-B}{4h})t\sinh\frac{Bt}{4h}\cosh\frac{t}{2h}\sinh(\frac{1-h}{2h})t\sinh\frac{jt}{h}\sinh(\frac{h-j}{h})t}{\sinh t\cosh\frac{t}{2}\sinh^2\frac{t}{h}}\ . \tag{4.15}$$

Solving the integral or more practical using the generating function (3.25), we transform this into the block representation (3.21) and find

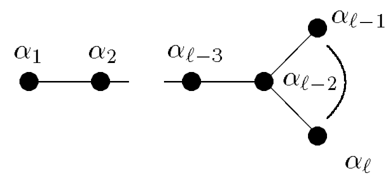
$$\tilde{R}_j(\theta+i\pi,B) = \tilde{R}_{h-j}(\theta+i\pi,B) = \prod_{p=1}^{j}\prod_{k=p}^{h-p}\|2k-1\|_\theta \quad \text{for } j\le h/2\ . \tag{4.16}$$

We used here the well-known relation between particles and anti-particles indicated above. For $j=1$ our solution coincides with the amplitude found in [55] shifted by $i\pi$ in the rapidity. More solutions were not reported in [55] for this algebra.

Computing $\prod_{j=1}^{\ell}S_{ij}(\theta)=\prod_{p=1}^{i}\prod_{k=p}^{h-p}\{k\}_\theta$, we note here the additional structure (4.1) with $\tilde{\mathcal{X}}_i=\emptyset$ for $1\le i\le\ell$.

4.2 $\mathbf{D}_\ell^{(1)}$-affine Toda field theory

We proceed now similarly and label the particles according to the Dynkin diagram



In the $\mathbf{D}_\ell$-case the Coxeter number is $h = 2(\ell - 1)$. As indicated most particles are self-conjugate apart from the two "spinors" at the end which are conjugate to each other. From the formulae derived in the appendix A.2,We compute now the kernel in (3.12) for $1 \leq j \leq \ell - 2$ to

$$\rho_j^{\mathbf{D}_\ell}(t) = \frac{16 \sinh(\frac{2-B}{4h})t \sinh\frac{Bt}{4h} \cosh\frac{(h-2)t}{4h} \sinh\frac{t}{4} \sinh\frac{(j-h)t}{2h} \sinh\frac{jt}{2h}}{\sinh t \sinh^2\frac{t}{2h}} \tag{4.17}$$

and for the spinors

$$\rho_\ell^{\mathbf{D}_\ell}(t) = \rho_{\ell-1}^{\mathbf{D}_\ell}(t) = \frac{8 \sinh(\frac{2-B}{4h})t \sinh\frac{Bt}{4h} \sinh\frac{(1-h)t}{2h} \sinh\frac{(h+1-2[\ell/2])t}{2h} \sinh\frac{t[\ell/2]}{h}}{\sinh t \sinh\frac{t}{h} \sinh\frac{t}{2h}}. \tag{4.18}$$

Solving the integral in (3.12) or using the generating function (3.25), we find the following compact and closed expressions for the reflection matrices in terms of hyperbolic functions

$$\tilde{R}_j(\theta + i\pi) = \left[\prod_{k=1}^{j} \|h - 2k + 1\|\right] \prod_{p=1}^{j} \prod_{k=p}^{h-p} \|2k - 1\| \qquad \text{for} \qquad j = 1, \ldots, \ell - 2, \tag{4.19}$$

$$\tilde{R}_\ell(\theta + i\pi) = \tilde{R}_{\ell-1}(\theta + i\pi) = \prod_{p=1}^{[\ell/2]} \prod_{k=2p-1}^{h-2p+1} \|2k - 1\| \,. \tag{4.20}$$

For $D_4^{(1)}$ our solution agrees with the one reported in [58] when shifted by $i\pi$ in the rapidity. This is one of the few examples for which a perturbative calculation has been carried out, using Neumann boundary conditions in this case. For higher ranks only a solution for $j = 1$ was also reported in [55], which once again coincides with ours subject to the relation (4.14).

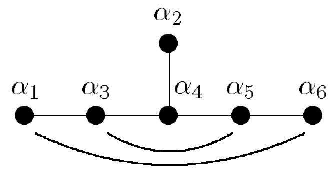
Computing now $\prod_{j=1}^{\ell} S_{ij}(\theta)_\theta$, we note that $\tilde{R}$ admits the alternative form (4.1) with

$$\tilde{\mathcal{X}}_i = \emptyset \qquad \text{for } i = 1, \ell - 1, \ell \tag{4.21}$$

$$\tilde{\mathcal{X}}_i = \bigcup_{1 \leq k < [(2i+1)/4]} \{h + 4k - 2i - 1\} \qquad \text{for } 2 \leq i \leq \ell - 2 \,. \tag{4.22}$$

4.3 $E_6^{(1)}$-affine Toda field theory

The labeling of the particle types is now according to the Dynkin diagram



The Coxeter number equals $h = 12$ in this case. We indicated the conjugation properties. From the formulae derived in the appendix A.3, we can obtain the integral representation (3.12) from which we deduce the block representation (3.21) directly or use the generating function (3.25). We find

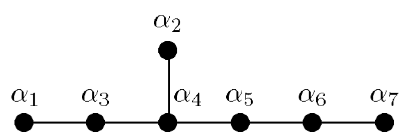
$$\tilde{R}_1(\theta + i\pi) = \tilde{R}_6(\theta + i\pi) = \left\|1; 3; 5; 7^2; 9^2; 11^2; 13^2; 15^2; 17; 19; 21\right\|_\theta , \tag{4.23}$$
$$\tilde{R}_3(\theta + i\pi) = \tilde{R}_5(\theta + i\pi) = \left\|1; 3^2; 5^3; 7^4; 9^4; 11^4; 13^4; 15^3; 17^2; 19^2; 21\right\|_\theta , \tag{4.24}$$
$$\tilde{R}_2(\theta + i\pi) = \left\|1; 3; 5^2; 7^3; 9^3; 11^3; 13^2; 15^3; 17^2; 19; 21\right\|_\theta , \tag{4.25}$$
$$\tilde{R}_4(\theta + i\pi) = \left\|1; 3^3; 5^5; 7^6; 9^6; 11^6; 13^5; 15^4; 17^3; 19^2; 21\right\|_\theta . \tag{4.26}$$

This solution coincides precisely with the amplitudes found in [55] shifted by $i\pi$ in the rapidity. We note here that the structure of the blocks in (4.23)-(4.26) can be encoded elegantly into the form (4.1) with

$$\tilde{\mathcal{X}}_1^{E_6} = \tilde{\mathcal{X}}_6^{E_6} = \emptyset, \quad \tilde{\mathcal{X}}_3^{E_6} = \tilde{\mathcal{X}}_5^{E_6} = \{7\}, \quad \tilde{\mathcal{X}}_2^{E_6} = \{11\}, \quad \tilde{\mathcal{X}}_4^{E_6} = \{5, 7, 9\}. \tag{4.27}$$

4.4 $E_7^{(1)}$-affine Toda field theory

The labeling of the particle types is now according to the Dynkin diagram

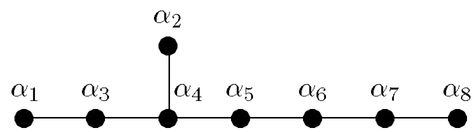


The Coxeter number equals $h = 18$ for E_7. All particles are self-conjugate. Using the formulae of appendix A.4, we can again either solve the integral (3.12) or use the generating function (3.25) and deduce the block representation (3.21). We find that these amplitudes coincide precisely with those reported in [55] (published version) when shifted by $i\pi$ in the rapidity. We note here once more, that they admit the additional structure (4.1) with

$$\tilde{\mathcal{X}}_1^{E_7} = \{17\}, \quad \tilde{\mathcal{X}}_2^{E_7} = \{11\}, \quad \tilde{\mathcal{X}}_3^{E_7} = \{7, 11, 15\}, \quad \tilde{\mathcal{X}}_4^{E_7} = \{5, 7, 9^2, 11, 13^2, 17\}, \tag{4.28}$$
$$\tilde{\mathcal{X}}_5^{E_7} = \{7, 9, 11, 15\}, \quad \tilde{\mathcal{X}}_6^{E_7} = \{9, 17\}, \quad \tilde{\mathcal{X}}_7^{E_7} = \emptyset. \tag{4.29}$$

4.5 $\mathbf{E}_8^{(1)}$-affine Toda field theory

In this case we label the particles according to the Dynkin diagram



The Coxeter number equals $h = 30$ for E_8. All particles are self-conjugate. Using the formulae of appendix A.5, we can solve the integral (3.12) or use the generating function (3.25) and deduce the block representation (3.21). Once more we find that these amplitudes coincide precisely with those reported in [55] (published version) when shifted by $i\pi$ in the rapidity. They admit the additional structure (4.1) with

$$\tilde{\mathcal{X}}_1^{E_8} = \{17, 29\}, \quad \tilde{\mathcal{X}}_2^{E_8} = \{11, 15, 19, 23\}, \quad \tilde{\mathcal{X}}_3^{E_8} = \{7, 11, 13, 15, 17, 19^2, 23, 27\}, \tag{4.30}$$

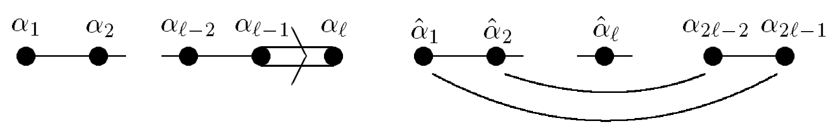
$$\tilde{\mathcal{X}}_4^{E_8} = \{5, 7, 9^2, 11^2, 13^3, 15^2, 17^3, 19^2, 21^3, 23, 25^2, 29\}, \tag{4.31}$$

$$\tilde{\mathcal{X}}_5^{E_8} = \{7, 9^2, 11^2, 13, 15^2, 17, 19^2, 21, 23^2, 27\}, \tag{4.32}$$

$$\tilde{\mathcal{X}}_6^{E_8} = \{9, 11, 13, 17, 19, 21, 25, 29\}, \quad \tilde{\mathcal{X}}_7^{E_8} = \{11, 19, 27\}, \quad \tilde{\mathcal{X}}_8^{E_8} = \{29\}\ . \tag{4.33}$$

4.6 $(\mathbf{B}_\ell^{(1)},\mathbf{A}_{2\ell-1}^{(2)})$-affine Toda field theory

As not many examples for reflection amplitudes of ATFT's related to non-simply laced Lie algebras have been computed, we consider it useful to start with some specific example before turning to the generic case. In general we label the particle types according to the Dynkin diagram



4.6.1 $(B_2^{(1)}, A_3^{(2)})$-affine Toda field theory

In this case we have for the (generalized) Coxeter numbers $h = 4$, $H = 6$, the incidence matrix $I_{12} = 2$, $I_{21} = 1$ and the symmetrizers $t_1 = 2$ and $t_2 = 1$. This is already enough Lie algebraic information needed for the computation of the relevant matrices in equation (3.13). We obtain

$$K^{-1}(t) = \frac{1}{\cosh t/2} \begin{pmatrix} \cosh t(\vartheta_h + \vartheta_H) & \cosh t\vartheta_H \\ 1/2 & \cosh t(\vartheta_h + 2\vartheta_H) \end{pmatrix} . \tag{4.34}$$

With the help of this matrix we can evaluate the scattering amplitudes (3.5) and (3.12). Alternatively we may compute the representation in terms of blocks from this. The expression (3.16) yields

$$S_{11}(\theta) = \{1,1\}\{1,3\}\{3,3\}\{3,5\}_\theta\,, \quad S_{22}(\theta) = \{1,1\}\{3,5\}_\theta\,, \quad S_{12}(\theta) = \{2,2\}\{2,4\}_\theta \tag{4.35}$$

and (3.21)

$$\tilde{R}_1(\theta + i\pi) = \|1,1\|\,\|1,3\|\,\|3,3\|\,\|3,5\|\,\|3,7\|\,\|5,5\|_\theta \ , \tag{4.36}$$

$$\tilde{R}_2(\theta + i\pi) = \|1,1\|\,\|3,3\|\,\|3,5\|\,\|5,9\|_\theta \ . \tag{4.37}$$

The solutions (4.36), (4.37) correspond precisely to those found by J.D. Kim in [56] after re-defining the effective coupling as $B \to B/2$ and shifting θ by $i\pi$. These solutions are especially trustworthy as they have also been double checked against perturbation theory. As the non-simply laced cases are not yet covered very much in the literature, we consider it useful to perform some more analysis at least for this case. Let us study the bootstrap equation (2.7) which relates different boundary states to each other in more detail. Adopting here the same principle as in the bulk, see [3, 13] and references therein, namely that $-i \operatorname{Res} R(\theta = \eta) > 0$ in the entire range of the coupling constant we find here

$$-i \operatorname*{Res}_{\theta \to \eta_{2\alpha}^{\beta} = \theta_h + \theta_H} \tilde{R}_{i\alpha}(\theta + i\pi) > 0 \ . \tag{4.38}$$

Solving for this angle $\eta_{2\alpha}^{\beta}$ the bootstrap equation (2.7) yields

$$R_{i\alpha}(\theta) = S_{i1}(-\theta) R_{i\beta}(\theta) \ . \tag{4.39}$$

Considering now the new solution $R_{i\beta}(\theta)$, we observe that

$$-i \operatorname*{Res}_{\theta \to \eta_{2\beta}^{\alpha} = 3\theta_h + 5\theta_H} \tilde{R}_{i\beta}(\theta + i\pi) > 0 \ . \tag{4.40}$$

These are the only poles with the property to have positive definitive sign in the entire range of the coupling constant, such that we have just the two boundary states α and β. The corresponding energies are computed in the same way as in [49, 13]. Using that

$$m_1 = m \sinh(2\theta_h + 4\theta_H) \quad \text{and} \quad m_2 = m \sinh(\theta_h + \theta_H) \ , \tag{4.41}$$

with m being an overall mass scale, we find for the energies of the two boundary states

$$E_\alpha = E_\beta - m_2 \cosh(\theta_h + \theta_H) = E_\beta - m_1/2 \ , \tag{4.42}$$

such that it appears that Kim's solution is not the ground state. When performing the same analysis for our solution $\tilde{R}_i(\theta)$ we find that there is no simple order pole which respects (4.38), such that there is only one state in that case.

4.6.2 $(B_3^{(1)}, A_5^{(2)})$-affine Toda field theory

As the previous case can be related trivially to a solution which may be found already in the literature, let us present a case not dealt with so far. The (generalized) Coxeter numbers for $(B_3^{(1)}, A_5^{(2)})$ are $h = 6$ and $H = 10$. According to the corresponding Dynkin diagram of $B_3^{(1)}$, we have $t_1 = t_2 = 2$ and $t_3 = 1$. We evaluate

$$K^{-1} = \frac{1}{\det K} \begin{pmatrix} q^2\bar{q}^3 + q^{-2}\bar{q}^{-3} & q\bar{q} + q^{-1}\bar{q}^{-1} & \bar{q} + \bar{q}^{-1} \\ q\bar{q} + q^{-1}\bar{q}^{-1} & \bar{q} + \bar{q}^{-1} + q^2\bar{q}^3 + q^{-2}\bar{q}^{-3} & q\bar{q} + q^{-1}\bar{q}^{-1} + q\bar{q}^3 + q^{-1}\bar{q}^{-3} \\ 1 & q\bar{q}^2 + q^{-1}\bar{q}^{-2} & 1 + q^2\bar{q}^4 + q^{-2}\bar{q}^{-4} \end{pmatrix}, \tag{4.43}$$

with $\det K = q^3 \bar{q}^5 \ + q^{-3} \bar{q}^{-5}$. From this we obtain

$$\tilde{R}_1(\theta + i\pi) = \|1, 1_2; 3, 5_2; 5, 7_3; 7, 9; 9, 13\|_\theta \,, \tag{4.44}$$

$$\tilde{R}_2(\theta + i\pi) = \|1, 1_2; 3, 3_4; 3, 5_3; 5, 5_3; 7, 9_2; 7, 15; 9, 13\|_\theta \,, \tag{4.45}$$

$$\tilde{R}_3(\theta + i\pi) = \|1, 1; 3, 3_2; 5, 7_2; 5, 9; 7, 11_2; 9, 17\|_\theta \,. \tag{4.46}$$

We are not aware of any solution of this type occurring in the literature for this algebra.

4.6.3 Generic $(\mathbf{B}_\ell^{(1)}, \mathbf{A}_{2\ell-1}^{(2)})$-affine Toda field theory

In this case we have $h = 2\ell$ and $H = 2(2\ell - 1)$. The task of finding general block expressions for all reflection amplitudes, such as for instance in (4.16) for $A_\ell^{(1)}$ turns out to be quite involved in this case and therefore we present only closed expressions corresponding to some specific particles. For the first particle we find for $\ell \neq 2$

$$\tilde{R}_1^{(\mathbf{B}_\ell^{(1)}, \mathbf{A}_{2\ell-1}^{(2)})}(\theta + i\pi, B) = \|h-1, [H-3]_3\,; h+1, H-1\|_\theta \prod_{k=\ell+1}^{2(\ell-1)} \|2k+1, [4k-3]_2\|_\theta \times \prod_{k=0}^{\ell-2} \|2k+1, [4k+1]_2\|_\theta \,, \tag{4.47}$$

whereas for the second with $\ell \neq 2, 3$ we obtain

$$\begin{aligned}\tilde{R}_2^{(\mathbf{B}_\ell^{(1)}, \mathbf{A}_{2\ell-1}^{(2)})}(\theta + i\pi, B) = {} & \|1, 1_2; h-3, [H-7]_3\,; h-3, [H-5]_2\,; h-1, [H-5]_4\,; \\ & h-1, H-1; h+1, [H-1]_2\,; h+1, H+5; h+3, [H+3]_2\,; \\ & h+3, H+3; 2h-3, [2H-7]_2\|_\theta \prod_{k=0}^{\ell-2} \|2k+3, [4k+5]_2\|_\theta^2 \\ & \times \prod_{k=\ell+1}^{2(\ell-1)} \|2k+3, [4k+1]_2\|_\theta^2 \,. \end{aligned} \tag{4.48}$$

For the last two particles the amplitudes are

$$\begin{aligned}\tilde{R}_{\ell-1}^{(\mathbf{B}_\ell^{(1)}, \mathbf{A}_{2\ell-1}^{(2)})}(\theta + i\pi, B) = {} & \prod_{k=0}^{\ell-2} \|2k+1, [4k+1]_2\,; 2k+3, [4k+3]_3\,; 4k+5, 8k+5\|_\theta \\ & \times \prod_{k=1}^{\ell-2} \|2k+3, 4k+5; 4k+3, [8k+1]_2\,; 4k+3, 8k+7\|_\theta \\ & \times \prod_{n=0}^{\ell-4} \prod_{k=\ell-n}^{2(\ell-n-2)} \|2k+1, [4k-3]_4\|_\theta \,, \end{aligned} \tag{4.49}$$

$$\tilde{R}_\ell^{(\mathbf{B}_\ell^{(1)}, \mathbf{A}_{2\ell-1}^{(2)})}(\theta + i\pi, B) = \left[\prod_{k=0}^{\ell-1} \|4k+1, 8k+1\|_\theta \right] \prod_{n=0}^{\ell-2} \prod_{k=\ell-n}^{2(\ell-n-1)} \|2k-1, [4k-5]_2\|_\theta \,, \tag{4.50}$$

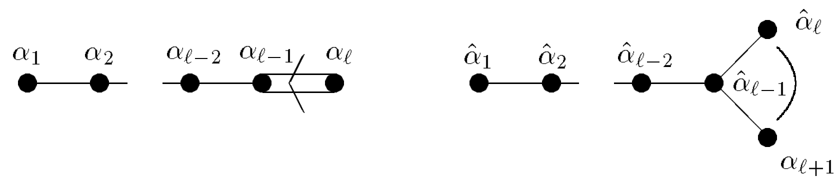
In particular, one can easily specialize these functions to reproduce the examples $\ell = 2, 3$ treated in the previous subsections. We also report here for $i = \ell$, the corresponding integral representation which is given in terms of the kernel,

$$
\begin{aligned}
&\sum_{j,k,p=1}^{\ell} \left[K^{(\mathbf{B}_\ell^{(1)},\mathbf{A}_{2\ell-1}^{(2)})}(t)\right]^{-1}_{\ell j} \chi_j^{kp} \left[K^{(\mathbf{B}_\ell^{(1)},\mathbf{A}_{2\ell-1}^{(2)})}(t/2)\right]^{-1}_{kp} [t_p]_{\bar{q}^{1/2}} \\
&= \frac{4\sinh\frac{th(\vartheta_h+2\vartheta_H)}{4}}{\sinh t \sinh\frac{t(\vartheta_h+2\vartheta_H)}{2}} \left(\frac{\cosh\frac{t\vartheta_H}{2}\sinh\frac{t(\vartheta_h+2\vartheta_H)(2-h)}{4}\sinh\frac{t(\vartheta_h(h-1)+2\vartheta_H(h-2))}{4}}{\sinh t(\vartheta_h+2\vartheta_H)} \right. \\
&\quad \left. + \cosh\frac{th(\vartheta_h+2\vartheta_H)}{4}\sinh\frac{t(2(1-H)\vartheta_H - H\vartheta_h)}{4} \right) . \qquad (4.51)
\end{aligned}
$$

As the expressions for the other amplitudes turn out to be rather lengthy we do not report them here, but it should be clear how to obtain them.

4.7 $(\mathbf{C}_\ell^{(1)},\mathbf{D}_{\ell+1}^{(2)})$-affine Toda field theory

We label the particle types according to the Dynkin diagram



The (generalized) Coxeter numbers are $h = 2\ell$, $H = 2\ell + 2$ in this case. Similarly as in the previous section, we present only closed formulae for some particles. For the first particle we find for $\ell \neq 2$

$$
\begin{aligned}
\tilde{R}_1^{(\mathbf{C}_\ell^{(1)},\mathbf{D}_{\ell+1}^{(2)})}(\theta + i\pi, B) &= \|h-1, [h-1]_2\, ; 2h-3, 2h+1\|_\theta \prod_{k=0}^{\ell-2} \|2k+1, 2k+1\|_\theta \\
&\quad \times \prod_{k=\ell+1}^{2(\ell-1)} \|2k-1, 2k+3\|_\theta \ , \qquad (4.52)
\end{aligned}
$$

whereas for the second we obtain

$$
\begin{aligned}
\tilde{R}_2^{(\mathbf{C}_\ell^{(1)},\mathbf{D}_{\ell+1}^{(2)})}(\theta + i\pi, B) &= \|1, 1; h-3, [h-3]_2\, ; h-3, h-3; h-1, [h-1]_3\, ; h+1, h+1; \\
&\quad h+1, h+5; 2h-3, 2h+1\|_\theta \prod_{k=\ell+2}^{2(\ell-1)} \|2k-1, 2k+3\|_\theta^2 \\
&\quad \times \prod_{k=0}^{\ell-4} \|2k+3, 2k+3\|_\theta^2 \qquad (4.53)
\end{aligned}
$$

with $\ell \neq 2, 3$. For the amplitudes related to the last two particles we find

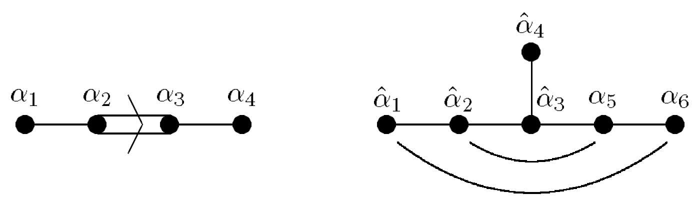
$$\tilde{R}_{\ell-1}^{(\mathbf{C}_\ell^{(1)},\mathbf{D}_{\ell+1}^{(2)})}(\theta + i\pi, B) = \prod_{k=0}^{\ell-2} \|2k+1, 2k+1; 2k+3, [2k+3]_2\,; 4k+5, 4k+9\|_\theta \times \prod_{n=0}^{\ell-3} \prod_{k=\ell+1-n}^{2(\ell-n-1)} \|2k-1, 2k-1; 2k-1, 2k+3\|_\theta\,, \tag{4.54}$$

$$\tilde{R}_{\ell}^{(\mathbf{C}_\ell^{(1)},\mathbf{D}_{\ell+1}^{(2)})}(\theta + i\pi, B) = \|1, 1_2\|_\theta \prod_{k=0}^{\ell-2} \|2k+3, [2k+3]_3\,; 2k+5, 2k+5\|_\theta \times \prod_{n=0}^{\ell-4} \prod_{k=\ell-n}^{2(\ell-n-2)} \|2k+3, 2k+3; 2k+3, 2k+7\|_\theta \times \prod_{k=1}^{\ell-2} \|2k+5, 2k+9; 4k+5, 4k+5\|_\theta\,. \tag{4.55}$$

Once again we do report the remaining amplitudes as their expressions turn out to rather lengthy.

4.8 $(\mathbf{F}_4^{(1)},\mathbf{E}_6^{(2)})$-affine Toda field theory

We label the particle types according to the Dynkin diagram



The (generalized) Coxeter numbers are $h = 12$ and $H = 18$ in this case, with $t_1 = t_2 = 2$ and $t_3 = t_4 = 1$. We compute

$$\tilde{R}_1(\theta + i\pi) = \|1, 1_2; 3, 5_2; 5, 7_3; 7, 9; 7, 9_3; 9, 11_3; 9, 15; 11, 13_4; 11, 17; 13, 17_2; 13, 23; 15, 21; 15, 21_3; 17, 23; 17, 27; 19, 25; 21, 29_2\|_\theta \tag{4.56}$$

$$\tilde{R}_2(\theta + i\pi) = \big\|1, 1_2; 3, 3_3; 3, 5_2; 5, 5_4; 5, 5_3; 5, 9; 7, 7_5; 7, 9_2; 7, 11^2; 9, 9_5; 9, 13_2; 9, 13^2; 11, 13_4; 11, 15_3; 11, 15; 11, 19; 13, 17_4; 13, 17^2; 13, 21; 15, 19_3; 15, 23; 15, 27; 17, 21; 17, 25; 17, 29; 19, 25_2; 19, 31; 21, 29; -19, -29\|_\theta \tag{4.57}$$

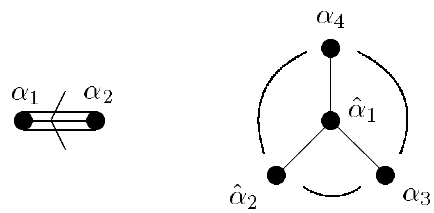
$$\tilde{R}_3(\theta + i\pi) = \big\|1, 1; 3, 3; 3, 3_2; 5, 5_3; 5, 7_2; 7, 7_4; 7, 11^2; 9, 11_4; 9, 13^2; 11, 15_3; 11, 15; 11, 15_2; 13, 17_3; 13, 21^2; 15, 19; 15, 21_2; 15, 23; 17, 23_2; 17, 29; 19, 27; 19, 31; 21, 33\big\|_\theta \tag{4.58}$$

$$\tilde{R}_4(\theta + i\pi) = \|1, 1; 3, 3; 5, 5_2; 7, 9_2; 7, 11; 9, 13_2; 9, 13; 11, 15_3; 13, 17; 13, 21; 15, 21_2; 15, 23; 17, 25_2; 19, 31; 21, 33\|_\theta\ . \tag{4.59}$$

We are not aware of any kind of solution known in the literature related to this algebra.

4.9 ($\mathbf{G}_2^{(1)}$,$\mathbf{D}_4^{(3)}$)-affine Toda field theory

We label the particle types according to the Dynkin diagram



The (generalized) Coxeter numbers are now $h = 12$ and $H = 18$. In this case we compute the integral representation

$$\rho_1^{\mathbf{G}_2^{(1)}}(t) = \frac{16 \sinh \frac{\vartheta_h t}{2} \sinh \frac{\vartheta_H t}{2} (\sinh \frac{t}{12} \sinh \frac{Bt}{16} - \cosh \frac{(B+4)t}{48} \cosh \frac{(B+4)t}{24})}{\frac{1}{2} - \cosh \frac{t}{3} + \cosh \frac{t}{2}}, \tag{4.60}$$

$$\rho_2^{\mathbf{G}_2^{(1)}}(t) = \frac{16 \sinh \frac{\vartheta_h t}{2} \sinh \frac{3\vartheta_H t}{2} \left[(2 \cosh \frac{t}{6} - 1) \sinh \frac{\vartheta_h t}{2} - \frac{1}{2} \cosh \frac{(B-4)t}{48} \right]}{\frac{1}{2} - \cosh \frac{t}{3} + \cosh \frac{t}{2}}. \tag{4.61}$$

When computing the block representation (3.21) we find complete agreement with the solution found in [56] shifted by $i\pi$ in the rapidities up to some obvious typos. We therefore do not need to report it here. The solutions differ from the ones reported in [12].

5. Breaking of the strong-weak duality

The above solutions are very general and can be related easily by means of the ambiguities (2.8)-(2.11) to all other solutions which are reported in the literature so far. Let us consider one particular ambiguity in more detail

$$\hat{R}_i(\theta, B) = \tilde{R}_i(\theta, B) \prod_{j=1}^{\ell} S_{ij}(\theta, 1 - B/2) \; . \tag{5.1}$$

At first sight there seems to be nothing special about $\hat{R}_i(\theta, B)$. Nonetheless, certain evident features can be seen from (5.1). Our solution $\tilde{R}_i(\theta, B)$ for the reflection amplitude shares with the bulk scattering amplitude $S_{ij}(\theta, B)$ the property of being invariant under the strong-weak duality transformation $B \to 2 - B$. Since $S_{ij}(\theta, 1 - B/2) \neq S_{ij}(\theta, B/2)$ it is clear from (5.1) that $\hat{R}_i(\theta, B)$ is not invariant under the strong-weak transformation. As was argued in [14], it is desirable to construct such solutions for the reflection amplitudes, because unlike $\tilde{R}_i(\theta, B)$ which tends to 1 in the weak and strong classical limit, i.e. $B \to 0, 2$, we have now simply

$$\hat{R}_i(\theta, B = 0) = \prod_{j=1}^{\ell} S_{ij}(\theta, B = 1), \tag{5.2}$$

$$\hat{R}_i(\theta, B = 2) = 1 \; . \tag{5.3}$$

This means, whilst $\tilde{R}_i(\theta, B)$ reduces in the classical limit to a theory with Neumann (free) boundary condition, the amplitude $\hat{R}_i(\theta, B)$ tends to a theory with fixed boundary conditions for $B \to 0$, but for $B \to 2$ to a theory with Neumann boundary condition. Hence the formulation (5.1) constitutes a simple mechanism of breaking consistently the duality and changing from one type of boundary conditions to another. This picture of obtaining two different classical Lagrangians is familiar for the bulk theories of ATFT related to non-simply laced Lie algebras and was put forward for theories with boundaries in [14] based on observations of the classical theory.

Let us evaluate the solution (5.1) in detail. From the above data and in particular the formulae provided in the appendix, we compute for the simply laced algebras an integral representation for $\hat{R}$ analogue to (3.12), where the corresponding kernel is

$$\hat{\rho}_i(t) = 4 \frac{\sinh \frac{t(2-B)}{4h}}{\cosh \frac{t}{2}} \left[\sinh \frac{t}{2}(1 + \frac{B}{2h}) \left[K^{-1}(t/2)\right]_{ii} - 2 \cosh \frac{Bt}{4h} \sum_{x \in \hat{\mathcal{X}}_i} \sinh \frac{xt}{2h} \right] . \qquad (5.4)$$

The $\hat{\mathcal{X}}_i$ are sets specific to the algebras and particle types. We find (see the appendix for some details on this calculations) that

$$\begin{aligned}
\hat{\mathcal{X}}_i^{\mathbf{A}_\ell} &= \emptyset \qquad \text{for } 1 \leq i \leq \ell && (5.5)\\
\hat{\mathcal{X}}_i^{\mathbf{D}_\ell} &= \emptyset \qquad \text{for } i = 1, \ell-1, \ell && (5.6)\\
\hat{\mathcal{X}}_i^{\mathbf{D}_\ell} &= \bigcup_{1 \leq k < [(2i+1)/4]} \{2i+1-4k\} \qquad \text{for } 2 \leq i \leq \ell-2 && (5.7)\\
\hat{\mathcal{X}}_1^{E_6} &= \hat{\mathcal{X}}_6^{E_6} = \emptyset, \quad \hat{\mathcal{X}}_3^{E_6} = \hat{\mathcal{X}}_5^{E_6} = \{5\}, \quad \hat{\mathcal{X}}_2^{E_6} = \{1\}, \quad \hat{\mathcal{X}}_4^{E_6} = \{3,5,7\}, && (5.8)\\
\hat{\mathcal{X}}_1^{E_7} &= \{1\}, \quad \hat{\mathcal{X}}_2^{E_7} = \{7\}, \quad \hat{\mathcal{X}}_3^{E_7} = \{3,7,11\}, \quad \mathcal{X}_4^{E_7} = \{1,5^2,7,9^2,11,13\}, && (5.9)\\
\hat{\mathcal{X}}_5^{E_7} &= \{3,7,9,11\}, \quad \hat{\mathcal{X}}_6^{E_7} = \{1,9\}, \quad \hat{\mathcal{X}}_7^{E_7} = \emptyset, && (5.10)\\
\hat{\mathcal{X}}_1^{E_8} &= \{1,13\}, \quad \hat{\mathcal{X}}_2^{E_8} = \{7,11,15,19\}, \quad \hat{\mathcal{X}}_3^{E_8} = \{3,7,11^2,13,15,17,19,23\}, && (5.11)\\
\hat{\mathcal{X}}_4^{E_8} &= \{1,5^2,7,9^3,11^2,13^3,15^2,17^3,19^2,21^2,23,25\}, && (5.12)\\
\hat{\mathcal{X}}_5^{E_8} &= \{3,7^2,9,11^2,13,15^2,17,19^2,21^2,23\}, && (5.13)\\
\hat{\mathcal{X}}_6^{E_8} &= \{1,5,9,11,13,17,19,21\}, \quad \hat{\mathcal{X}}_7^{E_8} = \{3,11,19\}, \quad \hat{\mathcal{X}}_8^{E_8} = \{1\} . && (5.14)
\end{aligned}$$

Up to some minor typo, the expression (5.4) corresponds for $\mathbf{A}_\ell$ to the formula proposed by Fateev in [34], which was obtained by changing from the block form (3.21) provided in [14] to an integral representation. For certain amplitudes, namely when the Kac label $n_i = \psi \cdot \lambda_i = 1$, with ψ being the highest root and λ_i the fundamental weight, a conjecture was put forward in [34], which corresponds precisely to our expression (5.4) when $\hat{\mathcal{X}}_i$ is the empty set $\emptyset$. At present the condition for the Kac labels is only an observation and has no deeper physical or mathematical meaning, but probably when one computes the quantities in terms of inner products of simple roots and weights, analogue to computations in [3] for the bulk S-matrix, one can provide a reasoning for it. Note that the two sets $\hat{\mathcal{X}}_i$ and $\tilde{\mathcal{X}}_i$ can be obtained from each other when replacing each element $x \in \tilde{\mathcal{X}}_i$ by $(h-x) \in \hat{\mathcal{X}}_i$.

We may carry out the sum

$$\sum_{x \in \hat{\mathcal{X}}_i^{\mathbf{D}_\ell}} \sinh \frac{tx}{2h} = \left[\sinh \frac{it}{2h} \sinh \frac{(i-1)t}{2h} \right] \sinh^{-1} \frac{t}{h} \tag{5.15}$$

for $2 \leq i \leq \ell - 2$ and obtain the only amplitudes which were provided in [34] not satisfying the condition $n_i = 1$. In this case we find agreement with our solution up to a minor typo.

Alternatively, we can turn (5.4) into a block form formulation

$$\hat{R}_i(\theta, B = 0) = \prod_{x=1}^{h} \widehat{\|x\|}_\theta^{2\mu_{ii}} \prod_{x \in \hat{\mathcal{X}}_i} \overline{\|x\|}_\theta, \tag{5.16}$$

where the powers μ_{ii} relate to the bulk scattering matrix as defined in (3.16) and the blocks $\widehat{\|x\|}$, $\overline{\|x\|}_\theta$ were introduced in (3.30) and (3.31). Note that $\widehat{\|x\|}_\theta \widehat{\|x\|}_{\theta+i\pi} = \{x\}_{2\theta}$ and $\overline{\|x\|}_\theta \overline{\|x\|}_{\theta+i\pi} = 1$, such that we see that the crossing relation (2.4) block-wise trivially satisfied when $\hat{R}_k = \hat{R}_{\bar{k}}$.

In principle the formula (5.1) also holds for the non-simply laced case and a similar reasoning as for the simply laced cases can be carried out. However, we expect now also the occurrence of some free parameters according to the arguments of [14, 16, 59]. This means some modifications are needed here. Even for special choices of the parameters the conjecture put forward in [36] does not seem to agree with (5.1). Let us briefly comment on the mechanisms, which leads to free parameters within the bootstrap approach. We commence with the easiest model which exhibits such features, that is the sinh-Gordon model ($A_1^{(1)}$-ATFT). Our solution for the reflection amplitude for the one particle in the model reads in this case

$$\tilde{R}(\theta, B) = (1)_\theta (-B/2)_\theta (B/2 - 1)_\theta \ . \tag{5.17}$$

The S-matrix is well known to be [60, 61, 50]

$$S(\theta, B) = -(-B)_\theta (B - 2)_\theta \ . \tag{5.18}$$

We can relate our solution easily to an expression analyzed relatively recently by Chenghlou and Corrigan [43] against perturbation theory. In their notation we find

$$R(\theta, B) = \tilde{R}(\theta, B) \frac{S(\theta, 1-E) S(\theta, 1-F)}{S(\theta, B/2) S(\theta, 1 + B/2)} \ , \tag{5.19}$$

where E and F are free parameters. As there is no bootstrap in the sinh-Gordon model, it is clear that every solution for R multiplied by $S(\theta, B')$ constitutes also a perfectly consistent solution from the bootstrap point of view. If then in addition the effective coupling is taken to be in the range $0 \leq B' \leq 2$ there will be no additional poles introduced by this multiplier, such that the bootstrap equation (2.7) is not coming into play. An important consequence is that the energy of the corresponding boundary state of this solution will be the same for all values of the free parameter B' in the stated regime. The factors $S(\theta, 1 - E)$ and $S(\theta, 1 - F)$ are precisely of this type. This argument is not yet sufficient to explain why there are precisely two free parameters (as in principle it would allow the introduction of an arbitrary number), but it explains when they might arise. Similarly, we obtain a solution which was found in [62] for the sinh-Gordon model with dynamical boundary conditions[3]. The solution found in there relates to ours as $R(\theta, B) = \tilde{R}(\theta, B)/S(\theta, 1)$.

Let us look at a more complicated model which involves a non-trivial bootstrap and for which we also expect this phenomenon: $(B_2^{(1)}, A_3^{(2)})$-ATFT. In that case we can define the new amplitudes

$$R_1(\theta, B, B') \to R_1(\theta, B) S_{11}(\theta, B') \quad \text{and} \quad R_2(\theta, B, B') \to R_2(\theta, B) S_{12}(\theta, B') \tag{5.20}$$

where the parameter $0 \leq B' \leq 2$ is kept free. Clearly there is no problem with crossing, unitarity (2.4) and by construction also the boundary bootstrap equation (2.6) is satisfied. As the amplitudes S_{11} and S_{12} introduce no new poles whose residues satisfy (4.38), we have similarly as for sinh-Gordon a new solution whose energies of the bound states are the same as for the original solution for all possible values of the free parameter B'. In comparison we can look at the $A_2^{(1)}$-ATFT, where such freedom does not exist. In that theory the process $1+1$ and $2+2$ lead to new bound states, such that we can not multiply with the corresponding S-matrices without changing the energies of the boundary states.

We have indicated here briefly how free parameters may emerge naturally in the bootstrap approach. A more detailed analysis of this argument we shall present elsewhere [63].

6. Conclusion

In this manuscript we have provided a closed generic solution $\tilde{R}(\theta)$ for the boundary bootstrap equations valid for affine Toda field theories related to all simple Lie algebras, simply laced as well as non-simply laced. We have worked out this formula in detail for specific Lie algebras in form of an integral representation as well as in form of blocks of hyperbolic functions. Our solution $\tilde{R}(\theta)$ can be used as a seed to construct (all) other solutions related to various types of boundary conditions.

The non-uniqueness of the solution is related to the fact that one can make use of the transformations (2.8)-(2.11) and always produce new types of solutions. The natural question which arises is: Which of these solutions are meaningful? In the bulk theories one finds that essentially all solutions to the bootstrap equations subjected to minimal analyticity lead to meaningful quantum field theories. Very often there is no classical counterpart in form of a Lagrangian known to these solutions. Even though conceptually not needed, as an organizing principle classical Lagrangians are very useful. In the case of boundary theories it is the different types of boundary conditions which label the solutions (theories). In a sequence of papers the Durham/York-group [14, 16, 59] has investigated which type of classical boundary terms can be used to perturb an affine Toda field theory such that the integrability is preserved. The findings were that the theory has to be of the

[3]We are grateful to P.Baseihal for bringing [62] to our attention.

form

$$\mathcal{L} = \Theta(-x)\mathcal{L}_{ATFT} - \delta(x)\frac{m}{\beta^2}\sum_{i=0}^{\ell}\kappa_i\sqrt{n_i}e^{\beta\alpha_i\cdot\varphi/2} \tag{6.1}$$

where the n_i are the usual Kac labels occurring in $\mathcal{L}_{ATFT}$, defined through the expansion of the highest root $\psi = -\alpha_0 = \sum_{i=1}^{\ell} n_i\alpha_i$ in terms of simple roots α_i. For theories related to simply laced algebras (except sinh-Gordon $\equiv A_1^{(1)}$ where the two parameters are free) the constants κ_i can be either all zero $\kappa_i = 0$ for $\forall i$ (Neumann boundary condition) or $|\kappa_i| = 1$ for $\forall i$. For the non-simply laced case the κ_i are fixed depending on the algebra and there are up to two free parameters κ_i either exclusively related to the short or long roots (see appendix D in [16] for details).

How can our solution (3.12) be related to the different choices of the boundary in (6.1)? Let us consider the slightly generalized expression (5.1) for the $\mathbf{A}_\ell$-ATFT

$$\hat{R}_j^{\pm}(\theta, B) = \tilde{R}_j(\theta, B)^{\pm}\prod_{k=1}^{\ell} S_{jk}(\theta, 1 - B/2)^{\pm 1} \quad . \tag{6.2}$$

Computing now the classical limit, we find

$$\begin{aligned}
\lim_{B\to 0}\hat{R}_j^{\pm}(\theta, B) &= \tilde{R}_i(\theta, B = 0)^{\pm}\prod_{k=1}^{\ell} S_{jk}(\theta, 1)^{\pm 1} &(6.3)\\
&= \exp\left(\pm 8\int_0^{\infty}\frac{dt}{t}\sinh^2\frac{t}{2h}\sum\nolimits_{k=1}^{\ell} K_{jk}^{-1}(t)\sinh\frac{\theta t}{i\pi}\right) &(6.4)\\
&= \exp\left(\pm 4\int_0^{\infty}\frac{dt}{t}\sinh\frac{t}{2h}\tanh\frac{t}{2}K_{jj}^{-1}(t/2)\sinh\frac{\theta t}{i\pi}\right) &(6.5)\\
&= -(j)_\theta^{\pm}(h-j)_\theta^{\pm} &(6.6)\\
&= \frac{i\sinh\theta \mp 1/2mm_j}{i\sinh\theta \pm 1/2mm_j}, &(6.7)
\end{aligned}$$

where we used $m_j = 2m\sin(j\pi/h)$, with m being once more an overall mass scale. The expression (6.7) is what is predicted in this limit [14, 16, 59]. It is then clear that combinations of $\hat{R}_j^{\pm}(\theta, B)$ for different j can be used to construct all possible fixed boundary solutions, i.e. $\hat{R}_1^{+}(\theta, B), \hat{R}_2^{-}(\theta, B), \hat{R}_3^{-}(\theta, B), \ldots \to \{+, -, -, \ldots\}$. Similar limits can be carried out for the other Lie algebras. For non-simply laced algebra and the sinh-Gordon model, we gave a short argument which leads to the occurrence of free parameter within the bootstrap approach. As many solution give the same classical limit, it is clear that even in the simply laced cases the classical limit[4] is not enough to pin down the solutions and relate them one-to-one to one particular boundary condition. More information can be obtained from perturbative computations, as at order β^2 already many solutions start to differ from each other, although even at that order some distinct solutions still coincide. Unfortunately, there are not many computations of this kind existing in the literature to compare with.

[4]We do not see how this is compatible with the statement expressed in [27], where the opposite is claimed, namely that different boundary conditions share the same quantum reflection amplitude.

A further way to minimize the amount of solutions which can be generated from our generic solution $\tilde{R}$ and the ambiguities (2.8)-(2.11) is of course to close also the second type of bootstrap equation (2.7) [13, 27, 30] (see also section 4.6.1 for an example related to non-simply laced Lie algebras) and eliminate those solutions which do not allow for such a closure. A systematic study of this kind has not been carried out so far and we share the pessimistic viewpoint expressed in [27] concerning such an undertaking. Whereas it appears possible to show that some solutions do indeed close, it seems difficult to develop a systematic scheme which selects solution which do not close. Possibly when developing a formulation in terms of Coxeter geometry similarly as in the bulk [3], this can be understood better.

More in the spirit of exactly solvable models are considerations carried out in [64, 65], where the scaling functions (free energies) have been computed in two alternative ways. On one hand one can compute it by means of the thermodynamic Bethe ansatz and on the other by a semi-classical perturbation around the conformal field theory. Since in the former the boundary reflection amplitude enters as an input and in the latter the explicit boundary conditions one may compare the outcome and therefore indirectly relate solutions of the boundary bootstrap equations and classical boundary conditions. We leave this analysis for future investigations [66].

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A. Appendix: The inverse of K and the evaluation of ρ

The central object which enters into the computation of the bulk scattering amplitude (3.5) as well as into the reflection amplitude (3.12) is the inverse of the q-deformed Cartan matrix (3.7). We demonstrate that the entries of this matrix can be written in a closed form and furthermore that the sums over rows or columns can be carried out explicitly. Also this closed form can be used when they enter the expression for the kernel in (3.13) and (5.4).

Let us comment on the evaluation for this object in this appendix. We only present the formulae for the simply laced algebras **g** and first determine the determinant of $K(t)$. For simply laced algebras we know the eigenvalues of the incidence matrix I of **g** to be $I_{ij} y_j(n) = 2\cos(\pi s_n/h) y_i(n)$, where the s_n are the exponents of **g**. Therefore we conclude directly

$$K_{ij}^{\mathbf{g}}(t) y_j(n) = 4\cosh\left[(t+i\pi s_n)/2h\right]\cosh\left[(t-i\pi s_n)/2h\right] y_i(n) = \lambda_n^{\mathbf{g}} y_i(n) \ . \tag{A.1}$$

Appealing to the well-known relation between the eigenvalues of a matrix and its determinant we obtain

$$\det K^{\mathbf{g}}(t) = \prod_{n=1}^{\ell} \lambda_n^{\mathbf{g}} = \prod_{n=1}^{\ell} 4\cosh\left[(t+i\pi s_n)/2h\right]\cosh\left[(t-i\pi s_n)/2h\right] \ . \tag{A.2}$$

Having in mind to compute the inverse of $K(t)$ we also need to determine its cofactors. It turns out that the sub-matrix resulting from the elimination of the i-th row and the j-th column always decomposes into some matrices which can be identified as a deformed Cartan matrix of some new algebras $\tilde{\mathbf{g}}_i$ and $\tilde{\mathbf{g}}_j$

$$K^{\mathrm{g}/\tilde{\mathrm{g}}}(t) \to K^{\mathrm{g}/\tilde{\mathrm{g}}_i}(t) \oplus K^{\mathrm{g}/\tilde{\mathrm{g}}_j}(t) \ . \tag{A.3}$$

Here we introduced the matrices

$$K^{\mathrm{g}/\tilde{\mathrm{g}}}(t) := 2\cosh(t/h) - I^{\tilde{\mathrm{g}}} \ . \tag{A.4}$$

Hence $K^{\mathrm{g}/\tilde{\mathrm{g}}}(t)$ differs from $K^{\tilde{\mathrm{g}}}(t)$ in the sense that the Coxeter number h appearing in its diagonal belongs to $\mathbf{g}$ rather than $\tilde{\mathbf{g}}$. The same argument which lead to (A.1) then gives the eigenvalues of $K^{\mathrm{g}/\tilde{\mathrm{g}}}(t)$

$$\lambda_n^{\mathrm{g}/\tilde{\mathrm{g}}} = 4\cosh(t/2h + i\pi\tilde{s}_n/2\tilde{h})\cosh(t/2h + i\pi\tilde{s}_n/2\tilde{h}) \ . \tag{A.5}$$

Therefore we obtain the inverse of the (doubly) q-deformed Cartan matrix

$$\left[K^{\mathrm{g}}(t)^{-1}\right]_{ij} = \frac{\det K^{\mathrm{g}/\tilde{\mathrm{g}}_i}(t)\det K^{\mathrm{g}/\tilde{\mathrm{g}}_j}(t)}{\det K^{\mathrm{g}}(t)} = \frac{\prod_{n=1}^{\ell_i}\lambda_n^{\mathrm{g}/\tilde{\mathrm{g}}_i}\prod_{n=1}^{\ell_j}\lambda_n^{\mathrm{g}/\tilde{\mathrm{g}}_j}}{\prod_{n=1}^{\ell}\lambda_n^{\mathrm{g}}} \ . \tag{A.6}$$

What remains to be specified is the precise decomposition (A.3). We shall demonstrate this in detail. For this we need to specialize the formula (A.6) for some concrete algebras. As (A.6) consists of products it is not very suitable in that form and we therefore also present some alternative method which turns the products into sums. We also need to compute the sum over some rows or columns of $K(t)^{-1}$ and then we evaluate the sums in (3.13).

A.1 $\mathbf{A}_\ell$

Taking $\mathbf{g}$ to be $\mathbf{A}_\ell$ in (A.6) it is easy to convince oneself that

$$\left[K^{\mathbf{A}_\ell}(t)\right]_{ij}^{-1} = \left[K^{\mathbf{A}_\ell}(t)\right]_{ji}^{-1} = \frac{\det K^{\mathbf{A}_\ell/\mathbf{A}_{i-1}}\det K^{\mathbf{A}_\ell/\mathbf{A}_{n-j}}}{\det K^{\mathbf{A}_\ell}} \qquad \text{for} \quad i \le j \tag{A.7}$$

$$= \frac{\prod_{n=1}^{i-1}\lambda_n^{\mathbf{A}_\ell/\mathbf{A}_{i-1}}\prod_{n=1}^{\ell-j}\lambda_n^{\mathbf{A}_\ell/\mathbf{A}_{\ell-j}}}{\prod_{n=1}^{\ell}\lambda_n^{\mathbf{A}_\ell}} \ . \tag{A.8}$$

Having in mind to sum over some rows and columns of $K(t)^{-1}$, we present a different method to compute (A.7). For this we develop the determinant of $K^{\mathbf{A}_\ell/\mathbf{A}_n}$ with respect to the first row or column

$$\det K^{\mathbf{A}_\ell/\mathbf{A}_n} = \det K^{\mathbf{A}_\ell/\mathbf{A}_1}\det K^{\mathbf{A}_\ell/\mathbf{A}_{n-1}} - \det K^{\mathbf{A}_\ell/\mathbf{A}_{n-2}} \ . \tag{A.9}$$

Understanding that $\det K^{\mathbf{A}_\ell/\mathbf{A}_0} = 1$ and $\det K^{\mathbf{A}_\ell/\mathbf{A}_n} = 0$ for $n < 0$, we can view (A.9) as a recursive equation for $\det K^{\mathbf{A}_\ell/\mathbf{A}_n}$ in terms of $\det K^{\mathbf{A}_\ell/\mathbf{A}_1} = K^{\mathbf{A}_\ell/\mathbf{A}_1}$, which we can leave completely arbitrary at this point. We note that the equation (A.9) is the recursive equation for the Chebychev polynomials of the second kind $U_n(x)$, such that

$$\det K^{\mathbf{A}_\ell/\mathbf{A}_n} = \sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}\left(K^{\mathbf{A}_\ell/\mathbf{A}_1}\right)^{n-2k} = U_n\left(K^{\mathbf{A}_\ell/\mathbf{A}_1}/2\right) \ , \tag{A.10}$$

where $[x]$ denotes the integer part of x. We also need below

$$\sum_{n=0}^{p} \det K^{\mathbf{A}_\ell/\mathbf{A}_n} = \sum_{n=0}^{p} \sum_{k=0}^{[n/2]} (-1)^k \binom{p-n+k}{k} \left(K^{\mathbf{A}_\ell/\mathbf{A}_1}\right)^{p-n} . \tag{A.11}$$

Let us now fix $K^{\mathbf{A}_\ell/\mathbf{A}_1} = q + q^{-1}$. Then we obtain from (A.10)

$$\det K^{\mathbf{A}_\ell/\mathbf{A}_n} = U_n \left[(q+q^{-1})/2\right] = [1+n]_q \, , \tag{A.12}$$

such that (A.7) yields

$$\left[K^{\mathbf{A}_\ell}(t)\right]^{-1}_{ij} = \left[K^{\mathbf{A}_\ell}(t)\right]^{-1}_{ji} = \frac{[i]_q \, [h-j]_q}{[h]_q} \qquad \text{for} \quad i \leq j \; . \tag{A.13}$$

The same specialization reduces (A.11) to

$$\sum_{n=0}^{p-1} \det K^{\mathbf{A}_\ell/\mathbf{A}_n} = [(p+1)/2]_q \, [p]_{q^{1/2}} \; . \tag{A.14}$$

We can now also carry out the sum over the i-th row or column. From (A.7), (A.12), (A.13) and (A.14) follows

$$\begin{aligned}
\sum_{j=1}^{\ell} [K^{\mathbf{A}_\ell}(t)]^{-1}_{ij} &= \frac{1}{\det K^{\mathbf{A}_\ell}} \left[\det K^{\mathbf{A}_\ell/\mathbf{A}_{i-1}} \sum_{j=0}^{\ell-i} \det K^{\mathbf{A}_\ell/\mathbf{A}_j} + \det K^{\mathbf{A}_\ell/\mathbf{A}_{\ell-i}} \sum_{j=0}^{i-2} \det K^{\mathbf{A}_\ell/\mathbf{A}_j} \right] \\
&= \frac{1}{[h]_q} \left([i]_q \, [(h+1-i)/2]_q \, [h-i]_{q^{1/2}} + [h-i]_q \, [i/2]_q \, [i-1]_{q^{1/2}} \right) , \\
&= \frac{1}{2\cosh t/2} \, [h-i]_{q^{1/2}} \, [i]_{q^{1/2}} \; , \\
&= \frac{\tanh(t/2)}{2\sinh(t/2h)} (K^{\mathbf{A}_\ell}(t/2))^{-1}_{ii} \; .
\end{aligned} \tag{A.15}$$

To be able to compute (3.13) in more detail we derive from the above relations

$$[K^{\mathbf{A}_\ell}(t)]^{-1}_{ij} \, \chi^{kp}_j(t) [K^{\mathbf{A}_\ell}(t/2)]^{-1}_{kp} = \frac{\cosh(t/2h)\sinh[(1-h)t/2h]}{2\cosh t/2 \sinh(t/h)} (K^{\mathbf{A}_\ell}(t/2))^{-1}_{ii} \; . \tag{A.16}$$

It is the non obvious feature that the sum in (A.15) as well as the expressions in (A.15) are both proportional to $(K(t/2))^{-1}_{ii}$ which allows for the computation of (5.4). For the other algebras there are additional terms appearing as indicated in (5.5)-(5.14). We proceed similarly for them.

A.2 $\mathbf{D}_\ell$

As in the previous section we find also a recursive relation for this case by expanding the determinant with respect to the first (last) row or column

$$\begin{aligned}
\det K^{\mathbf{D}_\ell/\mathbf{D}_n} &= K^{\mathbf{D}_\ell/\mathbf{A}_1} \det K^{\mathbf{D}_\ell/\mathbf{D}_{n-1}} - \det K^{\mathbf{D}_\ell/\mathbf{D}_{n-2}} , \\
&= K^{\mathbf{D}_\ell/\mathbf{A}_1} \left[\det K^{\mathbf{D}_\ell/\mathbf{A}_{n-1}} - \det K^{\mathbf{D}_\ell/\mathbf{A}_{n-3}} \right] \; .
\end{aligned} \tag{A.17}$$

From the second equality we observe that we can employ the results of the previous section and express the determinant in terms of sub-determinants related to A_n algebras. In this way we compute the following expansion in terms of $K^{\mathbf{D}_\ell/\mathbf{A}_1}$,

$$\det K^{\mathbf{D}_\ell/\mathbf{D}_n} = \sum_{k=0}^{[n/2]} (-1)^k \frac{n-1}{n-k-1} \begin{pmatrix} n-k-1 \\ k \end{pmatrix} \left(K^{\mathbf{D}_\ell/\mathbf{A}_1} \right)^{n-2k} . \tag{A.18}$$

When we fix $K^{\mathbf{D}_\ell/\mathbf{A}_1} = q + q^{-1} = 2\cosh t/h$, this equation becomes

$$\det K^{\mathbf{D}_\ell/\mathbf{D}_n} = [2]_q \left([\ell - n]_q - [\ell - n - 2]_q \right) = 4 \cosh \frac{t(\ell - n - 1)}{h} \cosh \frac{t}{h}. \tag{A.19}$$

The object which enters the general formulae for the reflection amplitudes is the inverse of the q-deformed Cartan matrix. With the help of the previous equalities, we can compute the cofactors and find

$$\left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{ij} = \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{ji} = \frac{\sinh it/h \cosh(\ell - j - 1)t/h}{\sinh t/h \cosh t/2} \quad \text{for } 1 \le i \le j \le \ell - 2, \tag{A.20}$$

$$\left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{ip} = \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{pi} = \frac{\sinh it/h}{2 \sinh t/h \cosh t/2} \qquad \text{for} \quad p = \ell, \ell - 1, \tag{A.21}$$

together with

$$\left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{\ell\ell-1} = \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{\ell-1\ell} = \frac{\sinh(\ell-2)t/h}{2 \sinh 2t/h \cosh t/2}, \tag{A.22}$$

$$\left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{\ell\ell} = \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{\ell-1\ell-1} = \frac{\sinh \ell t/h}{2 \sinh 2t/h \cosh t/2}. \tag{A.23}$$

Taking now the sums over a row or a column gives

$$\sum_{j=1}^{\ell} \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{ij} = \frac{\sinh \frac{it}{h}}{2 \cosh \frac{t}{2} \sinh \frac{t}{h}} + \frac{\sinh \frac{it}{2h} \sinh \frac{(h-i)t}{2h}}{\tanh \frac{t}{2h} \cosh \frac{t}{2} \sinh \frac{t}{h}} \quad \text{for } 1 \le i \le j \le \ell - 2, \tag{A.24}$$

and

$$\sum_{j=1}^{\ell} \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{j\ell} = \sum_{j=1}^{\ell} \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{j\ell-1} = \frac{\sinh \frac{\ell t}{2h} \sinh \frac{t}{4}}{2 \cosh \frac{t}{2} \sinh \frac{t}{h} \sinh \frac{t}{2h}}. \tag{A.25}$$

Combining now (A.20)-(A.21) and (A.24)-(A.25), we obtain

$$\frac{\sum_{j=1}^{\ell} \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{ij}}{[K^{\mathbf{D}_\ell}(t/2)]^{-1}_{ii}} = \frac{\tanh t/2}{2 \sinh t/2h}, \qquad \text{for} \quad i = 1, \ell - 1, \ell, \tag{A.26}$$

and

$$\frac{\sum_{j=1}^{\ell} \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{ij}}{[K^{\mathbf{D}_\ell}(t/2)]^{-1}_{ii}} = \frac{\left(\cosh \frac{it}{2h} \sinh \frac{t}{2h} + \sinh \frac{(h-i)t}{2h} \cosh \frac{t}{2h} \right) \cosh \frac{t}{4}}{\cosh \frac{t}{2} \sinh \frac{t}{h} \cosh \frac{(h-2i)t}{4h}}. \tag{A.27}$$

for the remaining values $2 \le i \le \ell - 2$. Finally, we may compute the quantity, which enters directly our expressions for $R_i(\theta)$. We find the following closed formulae for $1 \le i \le \ell - 2$

$$\sum_{j,k,p=1}^{\ell} \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{ij} \chi_j^{kp} \left[K^{\mathbf{D}_\ell}(t/2)\right]^{-1}_{kp} = \frac{4 \cosh \frac{(h-2)t}{4h} \sinh \frac{t}{4} \sinh \frac{(i-h)t}{2h} \sinh \frac{it}{2h}}{\sinh t \sinh^2 \frac{t}{2h}}, \tag{A.28}$$

and

$$\sum_{j,k,p=1}^{\ell} \left[K^{\mathbf{D}_\ell}(t)\right]^{-1}_{ij} \chi_j^{kp} \left[K^{\mathbf{D}_\ell}(t/2)\right]^{-1}_{kp} = \frac{2\sinh\frac{(1-h)t}{2h}\sinh\frac{(h+1-2[\ell/2])t}{2h}\sinh\frac{t[\ell/2]}{h}}{\sinh t \sinh\frac{t}{h}\sinh\frac{t}{2h}}, \qquad \text{(A.29)}$$

for $i = \ell-1, \ell$. Having these formulae at hand we can easily obtain the kernels (4.17) and (4.18). In this case we found that the proportionality to $(K(t/2))^{-1}_{ii}$, observed in the previous section, no longer holds and therefore the formulae are more lengthy when $i \neq 1, \ell-1, \ell$

A.3 $\mathbf{E}_6$

Developing the determinant gives again some recursive equation

$$\begin{aligned}\det K^{\mathbf{E}_6} &= \left[K^{\mathbf{E}_6/\mathbf{A}_1}\right]^2 \left[\det K^{\mathbf{E}_6/\mathbf{A}_4} - \det K^{\mathbf{E}_6/\mathbf{A}_2}\right] - \det K^{\mathbf{E}_6/\mathbf{A}_4} \\ &= K^{\mathbf{E}_6/\mathbf{A}_1}\det K^{\mathbf{E}_6/\mathbf{D}_5} - \det K^{\mathbf{E}_6/\mathbf{D}_4}. \end{aligned} \qquad \text{(A.30)}$$

We note that we can use once more the results of the previous sections. Specifying $K^{\mathbf{E}_6/\mathbf{A}_1} = 2\cosh t/12$ we compute the cofactors and obtain the inverse of the q-deformed Cartan matrix for the simply-laced algebra E_6

$$\left[K^{\mathbf{E}_6}(t)\right]^{-1} = \frac{1}{\mathbf{E}_6}\begin{pmatrix} \mathbf{D}_5 & \mathbf{A}_2 & \mathbf{A}_4 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_1^2 & \mathbf{A}_1 \\ \mathbf{A}_2 & \mathbf{A}_5 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_2^2 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_2 \\ \mathbf{A}_4 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_1\mathbf{A}_4 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1^3 & \mathbf{A}_1^2 \\ \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_2^2 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_2^2\mathbf{A}_1 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1\mathbf{A}_2 \\ \mathbf{A}_1^2 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_1^3 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1\mathbf{A}_4 & \mathbf{A}_4 \\ \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_1^2 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_4 & \mathbf{D}_5 \end{pmatrix} \qquad \text{(A.31)}$$

where we understand the entries of this matrix as $g \equiv \det K^{\mathbf{E}_6/\mathbf{g}}$. Taking now the sum of particular rows and column gives

$$\sum_{j=1}^{6}\left[K^{\mathbf{E}_6}(t)\right]^{-1}_{1j} = \sum_{j=1}^{6}\left[K^{\mathbf{E}_6}(t)\right]^{-1}_{6j} = \frac{2+2\sum_{k=1}^{3}\cosh\frac{kt}{12}}{2\cosh\frac{t}{3}-1}, \qquad \text{(A.32)}$$

$$\sum_{j=1}^{6}\left[K^{\mathbf{E}_6}(t)\right]^{-1}_{3j} = \sum_{j=1}^{6}\left[K^{\mathbf{E}_6}(t)\right]^{-1}_{5j} = \frac{3+2\sum_{k=1}^{3}(4-k)\cosh\frac{kt}{12}}{2\cosh\frac{t}{3}-1}, \qquad \text{(A.33)}$$

$$\sum_{j=1}^{6}\left[K^{\mathbf{E}_6}(t)\right]^{-1}_{2j} = \frac{3+2\cosh\frac{t}{12}+2\sum_{k=1}^{3}\cosh\frac{kt}{12}}{2\cosh\frac{t}{3}-1}, \qquad \text{(A.34)}$$

$$\sum_{j=1}^{6}\left[K^{\mathbf{E}_6}(t)\right]^{-1}_{4j} = \frac{5+8\cosh\frac{t}{12}+6\cosh\frac{t}{6}+2\cosh\frac{t}{4}}{2\cosh\frac{t}{3}-1}. \qquad \text{(A.35)}$$

From this we compute

$$\sum_{j,k,p=1}^{6} \left[K^{\mathbf{E}_6}(t)\right]^{-1}_{1j} \chi_j^{kp} \left[K^{\mathbf{E}_6}(\frac{t}{2})\right]^{-1}_{kp} = -\frac{1+2\sum_{k=1}^{5}\cosh\frac{kt}{12}}{\cosh\frac{t}{2}} \left[K^{\mathbf{E}_6}(\frac{t}{2})\right]^{-1}_{11},$$

$$\sum_{j,k,p=1}^{6} \left[K^{\mathbf{E}_6}(t)\right]^{-1}_{2j} \chi_j^{kp} \left[K^{\mathbf{E}_6}(\frac{t}{2})\right]^{-1}_{kp} = \frac{2-4\left[\sum\limits_{k=1}^{2}\cosh\frac{kt}{12}+\cosh\frac{t}{3}\right]}{\left(1-2\cosh\frac{t}{3}\right)\left(1-2\cosh\frac{t}{12}\right)} \left[K^{\mathbf{E}_6}(\frac{t}{2})\right]^{-1}_{22},$$

$$\sum_{j,k,p=1}^{6} \left[K^{\mathbf{E}_6}(t)\right]^{-1}_{3j} \chi_j^{kp} \left[K^{\mathbf{E}_6}(\frac{t}{2})\right]^{-1}_{kp} = \frac{4\sum_{k=2}^{3}\cosh\frac{kt}{12}}{1-2\cosh\frac{t}{3}} \left[K^{\mathbf{E}_6}(\frac{t}{2})\right]^{-1}_{33},$$

$$\sum_{j,k,p=1}^{6} \left[K^{\mathbf{E}_6}(t)\right]^{-1}_{4j} \chi_j^{kp} \left[K^{\mathbf{E}_6}(\frac{t}{2})\right]^{-1}_{kp} = \frac{2\left(1+2\cosh\frac{t}{4}\right)}{1-2\cosh\frac{t}{3}} \left[K^{\mathbf{E}_6}(\frac{t}{2})\right]^{-1}_{44}.$$

Taking the first subscript to be 5 or 6 equals the expressions for taking them to be 3 or 1, respectively. This is sufficient to compute the expressions for R.

A.4 $\mathbf{E}_7$

Developing the determinant gives now the recursive equations

$$\begin{aligned} \det K^{\mathbf{E}_7} &= \left[K^{\mathbf{E}_7/\mathbf{A}_1}\right]^2 \left[\det K^{\mathbf{E}_7/\mathbf{A}_5} - \det K^{\mathbf{E}_7/\mathbf{A}_3}\right] - \det K^{\mathbf{E}_7/\mathbf{A}_5} \\ &= K^{\mathbf{E}_7/\mathbf{A}_1} \det K^{\mathbf{E}_7/\mathbf{D}_6} - \det K^{\mathbf{E}_7/\mathbf{A}_5}, \end{aligned} \tag{A.36}$$

which can be evaluated again from the quantities already computed in the previous sections. Specializing $K^{\mathbf{E}_7/\mathbf{A}_1} = 2\cosh t/18$ we compute

$$\left[K^{\mathbf{E}_7}(t)\right]^{-1} = \frac{1}{\mathbf{E}_7} \begin{pmatrix} \mathbf{D}_6 & \mathbf{A}_3 & \mathbf{A}_5 & \mathbf{A}_1\mathbf{A}_3 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_1^2 & \mathbf{A}_1 \\ \mathbf{A}_3 & \mathbf{A}_6 & \mathbf{A}_1\mathbf{A}_3 & \mathbf{A}_2\mathbf{A}_3 & \mathbf{A}_2^2 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_2 \\ \mathbf{A}_5 & \mathbf{A}_1\mathbf{A}_3 & \mathbf{A}_1\mathbf{A}_5 & \mathbf{A}_1^2\mathbf{A}_3 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1^3 & \mathbf{A}_1^2 \\ \mathbf{A}_1\mathbf{A}_3 & \mathbf{A}_2\mathbf{A}_3 & \mathbf{A}_1^2\mathbf{A}_3 & \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 & \mathbf{A}_2^2\mathbf{A}_1 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1\mathbf{A}_2 \\ \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_2^2 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_2^2\mathbf{A}_1 & \mathbf{A}_2\mathbf{A}_4 & \mathbf{A}_1\mathbf{A}_4 & \mathbf{A}_4 \\ \mathbf{A}_1^2 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_1^3 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1\mathbf{A}_4 & \mathbf{A}_1\mathbf{D}_5 & \mathbf{D}_5 \\ \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_1^2 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_4 & \mathbf{D}_5 & \mathbf{E}_6 \end{pmatrix} \tag{A.37}$$

We abbreviated here $g \equiv \det K^{\mathbf{E}_7/\mathrm{g}}$. The sum of particular rows and column gives

$$\sum_{j=1}^{7} \left[K^{\mathbf{E}_7}(t)\right]^{-1}_{1j} = \frac{3+4\sum_{k=1}^{2}\cosh\frac{kt}{18}+2\sum_{k=3}^{5}\cosh\frac{kt}{18}}{2\cosh\frac{t}{3}-1}, \tag{A.38}$$

$$\sum_{j=1}^{7} \left[K^{\mathbf{E}_7}(t)\right]^{-1}_{2j} = \frac{\left[-1+2\cosh\frac{t}{18}\right]\left[1+2\sum_{k=1}^{2}\cosh\frac{kt}{18}\right]^2-\frac{1}{2}\cosh^{-1}\frac{t}{18}}{2\cosh\frac{t}{3}-1}, \tag{A.39}$$

$$\sum_{j=1}^{7} \left[K^{\mathbf{E}_7}(t)\right]^{-1}_{3j} = \frac{5\left[1+2\cosh\frac{t}{18}\right]+6\sum_{k=2}^{3}\cosh\frac{kt}{18}+2\left[2\cosh\frac{2t}{9}+\cosh\frac{5t}{18}\right]}{2\cosh\frac{t}{3}-1}, \tag{A.40}$$

$$\sum_{j=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{4j} = \sum_{j=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{3j} + \frac{3+6\cosh\frac{t}{9}+2\left[\sum_{k=3}^{4}\cosh\frac{kt}{18}+\cosh\frac{t}{18}\right]}{2\cosh\frac{t}{3}-1}, \tag{A.41}$$

$$\sum_{j=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{5j} = \sum_{j=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{3j} + \frac{2\sum_{k=2}^{3}\cosh\frac{kt}{18}+\frac{1}{2}\cosh^{-1}\frac{t}{18}}{2\cosh\frac{t}{3}-1}, \tag{A.42}$$

$$\sum_{j=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{6j} = \sum_{j=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{2j} + \frac{1-2\left[\cosh\frac{t}{18}+\cosh\frac{2t}{9}\right]+\frac{1}{2}\cosh^{-1}\frac{t}{18}}{2\cosh\frac{t}{3}-1}, \tag{A.43}$$

$$\sum_{j=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{7j} = \sum_{j=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{2j} + \frac{1+2\left[2\sum_{k=1}^{3}\cosh\frac{kt}{18}-\cosh\frac{t}{6}\right]}{1-2\cosh\frac{t}{3}}. \tag{A.44}$$

Therefore

$$\sum_{j,k,p=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{1j}\chi_j^{kp}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{kp} = \frac{4\left(\sum\limits_{k=2}^{3}\cosh\frac{kt}{18}+\cosh\frac{5t}{18}+\cosh\frac{7t}{18}\right)\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{11}}{\left(1-2\cosh\frac{t}{18}+2\cosh\frac{t}{9}\right)\left(1-2\cosh\frac{t}{3}\right)},$$

$$\sum_{j,k,p=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{2j}\chi_j^{kp}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{kp} = \frac{2-4\sum\limits_{k=2}^{5}\cosh\frac{kt}{18}+\cosh^{-1}\frac{t}{18}}{2\cosh\frac{t}{3}-1}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{22},$$

$$\sum_{j,k,p=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{3j}\chi_j^{kp}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{kp} = \frac{2\left(1+2\cosh\frac{2t}{9}+2\cosh\frac{5t}{18}\right)}{2\cosh\frac{t}{3}-1}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{33},$$

$$\sum_{j,k,p=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{4j}\chi_j^{kp}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{kp} = \frac{4\left(\cosh\frac{t}{18}+\cosh\frac{5t}{18}\right)-\cosh^{-1}\frac{t}{18}}{2\cosh\frac{t}{3}-1}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{44},$$

$$\sum_{j,k,p=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{5j}\chi_j^{kp}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{kp} = \frac{\left[4\left(1-\cosh\frac{t}{9}+\sum\limits_{k=4}^{5}\cosh\frac{kt}{18}\right)+\frac{1}{\cosh\frac{t}{18}}\right]\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{55}}{1-2\cosh\frac{t}{3}},$$

$$\sum_{j,k,p=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{6j}\chi_j^{kp}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{kp} = 4\frac{\sum\limits_{k=3}^{5}\cosh\frac{kt}{18}}{1-2\cosh\frac{t}{3}}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{66},$$

$$\sum_{j,k,p=1}^{7}\left[K^{\mathbf{E}_7}(t)\right]^{-1}_{7j}\chi_j^{kp}\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{kp} = \frac{\left[4\left(\sum\limits_{k=2}^{3}\cosh\frac{kt}{18}+\sum\limits_{k=6}^{7}\cosh\frac{kt}{18}\right)+\frac{1}{\cosh\frac{t}{18}}\right]\left[K^{\mathbf{E}_7}(\frac{t}{2})\right]^{-1}_{77}}{\left(1-2\cosh\frac{t}{9}\right)\left(2\cosh\frac{t}{3}-1\right)}$$

which suffices to compute R.

A.5 $\mathbf{E}_8$

Developing the determinant gives now the recursive equations

$$\begin{aligned}\det K^{\mathbf{E}_8} &= \left[K^{\mathbf{E}_8/\mathbf{A}_1}\right]^2\left[\det K^{\mathbf{E}_8/\mathbf{A}_6}-\det K^{\mathbf{E}_8/\mathbf{A}_4}\right]-\det K^{\mathbf{E}_8/\mathbf{A}_6}\\ &= K^{\mathbf{E}_8/\mathbf{A}_1}\det K^{\mathbf{E}_8/\mathbf{D}_7}-\det K^{\mathbf{E}_8/\mathbf{A}_6}.\end{aligned} \tag{A.45}$$

Again we can use the quantities already determined in the previous sections. Specializing $K^{\mathbf{E}_8/\mathbf{A}_1} = 2\cosh t/30$, we compute

$$\left[K^{\mathbf{E}_8}(t)\right]^{-1} = \frac{1}{\mathbf{E}_8}\begin{pmatrix} \mathbf{D}_7 & \mathbf{A}_4 & \mathbf{A}_6 & \mathbf{A}_1\mathbf{A}_4 & \mathbf{A}_1\mathbf{A}_3 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_1^2 & \mathbf{A}_1 \\ \mathbf{A}_4 & \mathbf{A}_7 & \mathbf{A}_1\mathbf{A}_4 & \mathbf{A}_2\mathbf{A}_4 & \mathbf{A}_2\mathbf{A}_3 & \mathbf{A}_2^2 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_2 \\ \mathbf{A}_6 & \mathbf{A}_1\mathbf{A}_4 & \mathbf{A}_1\mathbf{A}_6 & \mathbf{A}_1^2\mathbf{A}_4 & \mathbf{A}_1^2\mathbf{A}_3 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1^3 & \mathbf{A}_1^2 \\ \mathbf{A}_1\mathbf{A}_4 & \mathbf{A}_2\mathbf{A}_4 & \mathbf{A}_1^2\mathbf{A}_4 & \mathbf{A}_1\mathbf{A}_2\mathbf{A}_4 & \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 & \mathbf{A}_1\mathbf{A}_2^2 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1\mathbf{A}_2 \\ \mathbf{A}_1\mathbf{A}_3 & \mathbf{A}_2\mathbf{A}_3 & \mathbf{A}_1^2\mathbf{A}_3 & \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 & \mathbf{A}_3\mathbf{A}_4 & \mathbf{A}_2\mathbf{A}_4 & \mathbf{A}_1\mathbf{A}_4 & \mathbf{A}_4 \\ \mathbf{A}_1\mathbf{A}_3 & \mathbf{A}_2^2 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1\mathbf{A}_2^2 & \mathbf{A}_2\mathbf{A}_4 & \mathbf{D}_5\mathbf{A}_2 & \mathbf{D}_5\mathbf{A}_1 & \mathbf{D}_5 \\ \mathbf{A}_1^2 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_1^3 & \mathbf{A}_1^2\mathbf{A}_2 & \mathbf{A}_1\mathbf{A}_4 & \mathbf{D}_5\mathbf{A}_1 & \mathbf{A}_1\mathbf{E}_6 & \mathbf{E}_6 \\ \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_1^2 & \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_4 & \mathbf{D}_5 & \mathbf{E}_6 & \mathbf{E}_7 \end{pmatrix}. \tag{A.46}$$

We abbreviated here $g \equiv \det K^{\mathbf{E}_8/\mathrm{g}}$. The sum of particular rows and column gives

$$\sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{1j} = \frac{2\left(3\left[1+\sum_{k=3}^{4}\cosh\frac{kt}{30}\right]+5\sum_{k=1}^{2}\cosh\frac{kt}{30}+\sum_{k=5}^{7}\cosh\frac{kt}{30}+\cosh\frac{t}{6}\right)}{2\cosh\frac{t}{5}+2\cosh\frac{4t}{15}-2\cosh\frac{t}{15}-1},$$

$$\sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{2j} = \frac{2\left(4+\sum_{k=1}^{3}(9-k)\cosh\frac{kt}{30}+\sum_{k=4}^{5}(8-k)\cosh\frac{kt}{30}+\sum_{k=6}^{7}\cosh\frac{kt}{30}\right)}{2\cosh\frac{t}{5}+2\cosh\frac{4t}{15}-2\cosh\frac{t}{15}-1},$$

$$\sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{3j} = \sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{2j} + \frac{3\left[1+2\cosh\frac{t}{30}\right]+4\sum_{k=2}^{3}\cosh\frac{kt}{30}+2\sum_{k=4}^{6}\cosh\frac{kt}{30}}{2\cosh\frac{t}{5}+2\cosh\frac{4t}{15}-2\cosh\frac{t}{15}-1},$$

$$\sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{3j} = \sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{3j} + \frac{4\cosh^2\frac{t}{30}\left(1+2\sum_{k=1}^{4}\cosh\frac{kt}{30}+2\cosh\frac{t}{15}\right)}{2\cosh\frac{t}{5}+2\cosh\frac{4t}{15}-2\cosh\frac{t}{15}-1},$$

$$\sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{5j} = \sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{3j} + \frac{1+4\sum_{k=1}^{5}\cosh\frac{kt}{30}-2\cosh\frac{t}{6}}{2\cosh\frac{t}{5}+2\cosh\frac{4t}{15}-2\cosh\frac{t}{15}-1},$$

$$\sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{6j} = \sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{3j} + \frac{1+6\cosh\frac{t}{30}-2\left(\cosh\frac{t}{10}+\cosh\frac{2t}{15}\right)}{2\cosh\frac{t}{5}+2\cosh\frac{4t}{15}-2\cosh\frac{t}{15}-1},$$

$$\sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{7j} = \sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{3j} - \frac{2\left(3+\sum_{k=1}^{5}(6-k)\cosh\frac{kt}{30}+\cosh\frac{2t}{15}\right)}{2\cosh\frac{t}{5}+2\cosh\frac{4t}{15}-2\cosh\frac{t}{15}-1},$$

$$\sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{8j} = \sum_{j=1}^{8}\left[K^{\mathbf{E}_8}(t)\right]^{-1}_{7j} - \frac{4\cosh^2\frac{t}{30}\left(1+2\left[\cosh\frac{t}{30}+\sum_{k=3}^{4}\cosh\frac{kt}{30}\right]\right)}{2\cosh\frac{t}{5}+2\cosh\frac{4t}{15}-2\cosh\frac{t}{15}-1}.$$

Then we obtain

$$\sum_{j,k,p=1}^{8} \left[K^{\mathbf{E}_8}(t)\right]^{-1}_{1j} \chi_j^{kp} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{kp} = \frac{2\left(1+2\cosh\frac{t}{30}-2\sum_{k=3}^{9}\cosh\frac{kt}{30}\right)}{-1+2\cosh\frac{t}{3}} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{11},$$

$$\sum_{j,k,p=1}^{8} \left[K^{\mathbf{E}_8}(t)\right]^{-1}_{2j} \chi_j^{kp} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{kp} = \frac{8\cosh\frac{t}{30}\left(1+\cosh\frac{t}{30}\left[1-2\sum_{k=1}^{2}\cosh\frac{kt}{15}\right]\right)\left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{22}}{2\cosh\frac{t}{5}+2\cosh\frac{4t}{15}-2\cosh\frac{t}{15}-1},$$

$$\sum_{j,k,p=1}^{8} \left[K^{\mathbf{E}_8}(t)\right]^{-1}_{3j} \chi_j^{kp} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{kp} = \frac{4\left(\sum_{k=4}^{7}\cosh\frac{t}{6}-\cosh\frac{t}{30}\right)}{1+2\cosh\frac{t}{15}-2\cosh\frac{t}{5}-2\cosh\frac{4t}{15}} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{33},$$

$$\sum_{j,k,p=1}^{8} \left[K^{\mathbf{E}_8}(t)\right]^{-1}_{4j} \chi_j^{kp} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{kp} = \frac{4\left(\cosh\frac{t}{6}+\cosh\frac{7t}{30}\right)}{1+2\cosh\frac{t}{15}-2\cosh\frac{t}{5}-2\cosh\frac{4t}{15}} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{44},$$

$$\sum_{j,k,p=1}^{8} \left[K^{\mathbf{E}_8}(t)\right]^{-1}_{5j} \chi_j^{kp} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{kp} = \frac{2+4\left(\sum_{k=5}^{7}\cosh\frac{kt}{30}-\cosh\frac{t}{10}\right)}{1+2\cosh\frac{t}{15}-2\cosh\frac{t}{5}-2\cosh\frac{4t}{15}} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{55},$$

$$\sum_{j,k,p=1}^{8} \left[K^{\mathbf{E}_8}(t)\right]^{-1}_{6j} \chi_j^{kp} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{kp} = \frac{1+2\left(\sum_{k=5}^{9}\cosh\frac{kt}{30}+2\cosh\frac{7t}{30}\right)}{\left(1+2\cosh\frac{t}{15}-2\sum_{k=3}^{4}\cosh\frac{kt}{15}\right)\cosh\frac{t}{15}} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{66},$$

$$\sum_{j,k,p=1}^{8} \left[K^{\mathbf{E}_8}(t)\right]^{-1}_{7j} \chi_j^{kp} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{kp} = \frac{2+4\sum_{k=6}^{9}\cosh\frac{kt}{30}}{1-2\cosh\frac{t}{3}} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{77},$$

$$\sum_{j,k,p=1}^{8} \left[K^{\mathbf{E}_8}(t)\right]^{-1}_{8j} \chi_j^{kp} \left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{kp} = \frac{4\left(\cosh\frac{t}{30}+\sum_{k=6}^{10}\cosh\frac{kt}{30}\right)\left[K^{\mathbf{E}_8}(\frac{t}{2})\right]^{-1}_{88}}{\left(2\cosh\frac{t}{10}-1\right)\left(1+2\cosh\frac{t}{15}-2\cosh\frac{t}{5}-2\cosh\frac{4t}{15}\right)},$$

from which we can deduce directly R.

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