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The quantum non-linear Schrödinger model with point-like defect

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Abstract

We establish a family of point–like impurities which preserve the quantum integrability of the non–linear Schrödinger model in 1+1 space–time dimensions. We briefly describe the construction of the exact second quantized solution of this model in terms of an appropriate reflection–transmission algebra. The basic physical properties of the solution, including the space–time symmetry of the bulk scattering matrix, are also discussed.

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1 Preliminaries

We present in this Letter the exact solution of the quantum non-linear Schrödinger (NLS) model with point-like impurity in 1+1 space-time dimensions. We focus mainly on the physical properties of the solution, referring for the mathematical details and proofs to [1]. Being the first exactly solvable example with non–trivial bulk scattering matrix, the NLS model provides valuable information about the interplay between point–like impurities, integrability and symmetries.

Assuming that the impurity is localized at $x = 0$, the model we are concerned with is defined by the equation of motion

$$(i\partial_t + \partial_x^2)\Phi(t,x) - 2g|\Phi(t,x)|^2\Phi(t,x) = 0\, , \qquad x \neq 0\, , \tag{1.1}$$

and the impurity boundary conditions

$$\begin{pmatrix} \Phi(t,+0) \\ \partial_x\Phi(t,+0) \end{pmatrix} = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Phi(t,-0) \\ \partial_x\Phi(t,-0) \end{pmatrix}\, , \tag{1.2}$$

where

$$\{a, ..., d \in \mathbb{R},\ \alpha \in \mathbb{C}\ :\ ad - bc = 1,\ \overline{\alpha}\alpha = 1\}\, . \tag{1.3}$$

Eq. (1.2) captures the interaction of the field Φ with the impurity [2, 3] and deserves some explanation. The parameters (1.3) label the self–adjoint extensions of the operator $-\partial_x^2$, defined on the space $C_0^\infty(\mathbb{R} \setminus \{0\})$ of smooth functions with compact support separated from the origin $x = 0$. This operator is not self–adjoint, but its closure admits self–adjoint extensions, which are parametrized [4] in terms of (1.3). In order to avoid the presence of bound states, we take below $g > 0$ and restrict further the parameters (1.3) according to:

$$\begin{cases} a + d + \sqrt{(a-d)^2 + 4} \leq 0\, , & b < 0\, , \\ c(a+d)^{-1} \geq 0\, , & b = 0\, , \\ a + d - \sqrt{(a-d)^2 + 4} \geq 0\, , & b > 0\, . \end{cases} \tag{1.4}$$

The operator $-\partial_x^2$ has no bound states in the domain (1.4). A complete orthonormal system of scattering states is given by

$$\psi_k^+(x) = \theta(-x)T_-^+(k)\mathrm{e}^{ikx} + \theta(x)\left[\mathrm{e}^{ikx} + R_+^+(-k)\mathrm{e}^{-ikx}\right]\, , \quad k < 0\, , \tag{1.5}$$

$$\psi_k^-(x) = \theta(x)T_+^-(k)\mathrm{e}^{ikx} + \theta(-x)\left[\mathrm{e}^{ikx} + R_-^-(-k)\mathrm{e}^{-ikx}\right]\, , \quad k > 0\, , \tag{1.6}$$

where θ denotes the standard Heaviside function and

$$R_+^+(k) = \frac{bk^2 + i(a-d)k + c}{bk^2 + i(a+d)k - c}\,, \qquad T_+^-(k) = \frac{2i\alpha k}{bk^2 + i(a+d)k - c}\,, \tag{1.7}$$

$$R_-^-(k) = \frac{bk^2 + i(a-d)k + c}{bk^2 - i(a+d)k - c}\,, \qquad T_-^+(k) = \frac{-2i\overline{\alpha}k}{bk^2 - i(a+d)k - c}\,, \tag{1.8}$$

are the *reflection* and *transmission coefficients* from the impurity. It is easily verified that the *reflection* and *transmission matrices*, defined by

$$\mathcal{R}(k) = \begin{pmatrix} R_+^+(k) & 0 \\ 0 & R_-^-(k) \end{pmatrix}, \qquad \mathcal{T}(k) = \begin{pmatrix} 0 & T_+^-(k) \\ T_-^+(k) & 0 \end{pmatrix}, \tag{1.9}$$

satisfy hermitian analyticity

$$\mathcal{R}(k)^\dagger = \mathcal{R}(-k)\,, \qquad \mathcal{T}(k)^\dagger = \mathcal{T}(k)\,, \tag{1.10}$$

and unitarity

$$\mathcal{T}(k)\mathcal{T}(k) + \mathcal{R}(k)\mathcal{R}(-k) = \mathbb{I}\,, \tag{1.11}$$

$$\mathcal{T}(k)\mathcal{R}(k) + \mathcal{R}(k)\mathcal{T}(-k) = 0\,. \tag{1.12}$$

Let us observe in passing that the reflection $x \mapsto -x$ leaves invariant eq. (1.1), but not always (1.2). The parity preserving impurities are selected by

$$a = d\,, \qquad \alpha = \overline{\alpha}\,. \tag{1.13}$$

We conclude this section by pointing out that the impurity boundary conditions (1.2) can be implemented, coupling the field Φ to an external potential with support in $x = 0$. The set

$$\{a = d = 1,\, b = 0,\, c = 2\eta;\, \alpha = 1\} \tag{1.14}$$

for instance, corresponds to the potential

$$V(x) = 2\eta\delta(x)\,, \tag{1.15}$$

known as δ–impurity. A general potential, which incorporates all four real parameters (1.3), has been suggested recently in [5].

2 The solution

When considered on the whole line $\mathbb{R}$, eq. (1.1) defines one of the most extensively studied integrable systems, which has been solved [6]–[10] by means of the inverse scattering transform [11]. We will show below that this method can be extended to eqs. (1.1,1.2) as well. For this purpose we will generalize to the case with impurity the Rosales [12, 13] series expansion of the solution in terms of the scattering data. A similar generalization has already been used for solving [14, 15] the boundary value problem associated with (1.1) on the the half–line $\mathbb{R}_+$.

It is instructive to display first the classical solution of eqs. (1.1,1.2). We introduce the fields $\Phi_\pm$ defined by

$$\Phi(t,x) = \begin{cases} \Phi_-(t,x)\,, & x<0\,, \\ \Phi_+(t,x)\,, & x>0\,, \end{cases} \tag{2.1}$$

and inspired by [12, 13] consider the series representation

$$\Phi_\pm(t,x) = \sum_{n=0}^{\infty} (-g)^n\, \Phi_\pm^{(n)}(t,x)\,, \tag{2.2}$$

with

$$\Phi_\pm^{(n)}(t,x) = \int \prod_{\substack{i=1\\ j=0}}^{n} \frac{dp_i}{2\pi}\frac{dq_j}{2\pi}\, \overline{\lambda}_\pm(p_1)\ldots\overline{\lambda}_\pm(p_n)\lambda_\pm(q_n)\ldots\lambda_\pm(q_0)\frac{e^{i\sum\limits_{j=0}^{n}(q_jx-q_j^2t)-i\sum\limits_{i=1}^{n}(p_ix-p_i^2t)}}{\prod\limits_{i=1}^{n}(p_i-q_{i-1})(p_i-q_i)}\,, \tag{2.3}$$

where the bar denotes complex conjugation and $\lambda_\pm$ define two solutions

$$\Phi_\pm^{(0)}(t,x) = \int \frac{dq}{2\pi}\lambda_\pm(q)\mathrm{e}^{i(qx-q^2t)}\,, \tag{2.4}$$

of the free Schrödinger equation. For sufficiently smooth $\lambda_\pm$ the series (2.2) converges and Rosales argument guarantees that Φ is a solution of (1.1). In order to satisfy the boundary condition (1.2), we take $\lambda_\pm$ of the form

$$\begin{pmatrix} \lambda_+(k) \\ \lambda_-(k) \end{pmatrix} = \begin{pmatrix} 1 & T_+^-(k) \\ T_+^-(k) & 1 \end{pmatrix}\begin{pmatrix} \mu_+(k) \\ \mu_-(k) \end{pmatrix} + \begin{pmatrix} R_+^+(k) & 0 \\ 0 & R_-^-(k) \end{pmatrix}\begin{pmatrix} \mu_+(-k) \\ \mu_-(-k) \end{pmatrix}, \tag{2.5}$$

where $\mu_\pm$ are smooth functions with certain decay and analyticity properties. Then the conditions (1.4) guarantee the smoothness of $\lambda_\pm$ and using unitarity (1.12), one easily verifies that $\lambda_\pm$ satisfy

$$\lambda_+(k) = T_+^-(k)\ \lambda_-(k) + R_+^+(k)\ \lambda_+(-k)\,, \tag{2.6}$$

$$\lambda_-(k) = T_-^+(k)\ \lambda_+(k) + R_-^-(k)\ \lambda_-(-k)\,. \tag{2.7}$$

Following [1], one can prove now that the boundary condition (1.2) holds order by order in g. The order $n = 0$ is a direct consequence of (2.6,2.7). For checking the higher orders it is convenient to introduce the new variables $\beta_\pm$ defined by

$$\begin{pmatrix} \beta_+(k) \\ \beta_-(k) \end{pmatrix} = \begin{pmatrix} \lambda_+(k) \\ \lambda_-(k) \end{pmatrix} + \begin{pmatrix} 0 & \alpha(d+ibk) \\ -\overline{\alpha}(d+ibk) & 0 \end{pmatrix} \begin{pmatrix} \lambda_+(-k) \\ \lambda_-(-k) \end{pmatrix}\,,$$

and use that

$$\beta_+(k) = -\frac{bk^2 - i(a+d)k - c}{bk^2 + i(a+d)k - c}\,\beta_+(-k)\ , \qquad \beta_-(k) = -\beta_-(-k)\,.$$

The freedom remaining in the choice of $\mu_\pm$ is fixed by the initial conditions. We will discuss this point at the quantum level, where the initial conditions are captured by the canonical commutation relations (see (2.8,2.9) below).

We turn now to the quantum case, fixing first of all the basic structures which are involved in the second quantization of eqs. (1.1,1.2). They are:

- A Hilbert space $\mathcal{H}$ with positive definite scalar product $\langle \cdot\,, \cdot \rangle$, which describes the states of the system;

- An operator valued distribution $\Phi(t,x)$, defined on a dense domain $\mathcal{D} \subset \mathcal{H}$ and satisfying the equation of motion (1.1) and the impurity boundary condition (1.2) in mean value on $\mathcal{D}$, as well as the equal time canonical commutation relations

$$[\Phi(t,x)\,,\,\Phi(t,y)] = [\Phi^*(t,x)\,,\,\Phi^*(t,y)] = 0\,, \tag{2.8}$$

$$[\Phi(t,x)\,,\,\Phi^*(t,y)] = \delta(x-y)\,, \tag{2.9}$$

  where Φ^* is the Hermitian conjugate of Φ;

- A distinguished normalizable state $\Omega \in \mathcal{D}$ – the vacuum, which is cyclic with respect to the field Φ^*.

Our goal now is to describe the construction of the elements $\{\mathcal{H}, \mathcal{D}, \Omega, \Phi\}$ with the above properties. A convenient starting point is the well–known bulk scattering matrix

$$S(k_1 - k_2) = \frac{k_1 - k_2 - ig}{k_1 - k_2 + ig}\,, \qquad (2.10)$$

of the quantum NLS model without impurity. In terms of (2.10) we define the 4×4 matrix

$$\mathcal{S}_{\alpha_1\alpha_2}^{\beta_1\beta_2}(k_1, k_2) = S(\alpha_1 k_1 - \alpha_2 k_2)\delta_{\alpha_1}^{\beta_1}\,\delta_{\alpha_2}^{\beta_2}\,, \qquad \alpha_i,\, \beta_i = \pm\,, \qquad (2.11)$$

which will turn out to be the bulk scattering matrix with impurity. As a preliminary step in verifying this statement, one can show that $\mathcal{S}$ satisfies unitarity

$$\mathcal{S}_{12}(k_1, k_2)\,\mathcal{S}_{21}(k_2, k_1) = \mathbb{I} \otimes \mathbb{I}\,, \qquad (2.12)$$

hermitian analyticity

$$\mathcal{S}_{12}^{\dagger}(k_1, k_2) = \mathcal{S}_{21}(k_2, k_1)\,, \qquad (2.13)$$

the quantum Yang–Baxter equation

$$\mathcal{S}_{12}(k_1, k_2)\mathcal{S}_{13}(k_1, k_3)\mathcal{S}_{23}(k_2, k_3) = \mathcal{S}_{23}(k_2, k_3)\mathcal{S}_{13}(k_1, k_3)\mathcal{S}_{12}(k_1, k_2)\,, \qquad (2.14)$$

and the boundary Yang–Baxter equation

$$\begin{gathered}\mathcal{S}_{12}(k_1, k_2)\,\mathcal{R}_1(k_1)\,\mathcal{S}_{21}(k_2, -k_1)\,\mathcal{R}_2(k_2) = \\ \mathcal{R}_2(k_2)\,\mathcal{S}_{12}(k_1, -k_2)\,\mathcal{R}_1(k_1)\,\mathcal{S}_{21}(-k_2, -k_1)\,, \end{gathered} \qquad (2.15)$$

where $\mathcal{R}$ is the reflection matrix (1.9) and the conventional tensor notation has been used. It is worth stressing that the entries $\mathcal{S}_{++}^{++}$ and $\mathcal{S}_{--}^{--}$ depend on $k_1 - k_2$ and are therefore Galilean invariant. On the contrary, $\mathcal{S}_{+-}^{+-}$ and $\mathcal{S}_{-+}^{-+}$ being functions of $k_1 + k_2$ break this symmetry.

The matrix $\mathcal{S}$ with the properties (2.12-2.15) identifies a reflection–transmission (RT) algebra $\mathcal{C}_\mathcal{S}$ [16, 17], which is the basic tool of our construction. The general concept of RT algebra has been designed for describing factorized scattering in integrable models with impurities. In what follows we will show that in the NLS model the algebra $\mathcal{C}_\mathcal{S}$ allows to reconstruct the off-shell quantum field Φ as well. $\mathcal{C}_\mathcal{S}$ is an associative algebra with identity $\mathbf{1}$, particle $\{a^{*\alpha}(k),\, a_\alpha(k)\}$ and impurity (defect) $\{r_\alpha^\beta(k),\, t_\alpha^\beta(k)\}$ generators obeying:

(i) bulk exchange relations

$$a_{\alpha_1}(k_1)\; a_{\alpha_2}(k_2) \;-\; \mathcal{S}_{\alpha_2\alpha_1}^{\beta_2\beta_1}(k_2, k_1)\; a_{\beta_2}(k_2)\, a_{\beta_1}(k_1) \;=\; 0\,, \qquad (2.16)$$

$$a^{*\alpha_1}(k_1)\, a^{*\alpha_2}(k_2) - a^{*\beta_2}(k_2)\, a^{*\beta_1}(k_1)\, \mathcal{S}_{\beta_2\beta_1}^{\alpha_2\alpha_1}(k_2,k_1) \;=\; 0\,, \tag{2.17}$$

$$\begin{aligned} a_{\alpha_1}(k_1)\, a^{*\alpha_2}(k_2) \;-\; a^{*\beta_2}(k_2)\, \mathcal{S}_{\alpha_1\beta_2}^{\beta_1\alpha_2}(k_1,k_2)\, a_{\beta_1}(k_1) \;=\; \\ 2\pi\, \delta(k_1-k_2)\, \left[\delta_{\alpha_1}^{\alpha_2}\, \mathbf{1} + t_{\alpha_1}^{\alpha_2}(k_1)\right] + 2\pi\, \delta(k_1+k_2)\, r_{\alpha_1}^{\alpha_2}(k_1)\,; \end{aligned} \tag{2.18}$$

(ii) defect exchange relations

$$\begin{aligned} \mathcal{S}_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}(k_1,k_2)\, r_{\gamma_1}^{\delta_1}(k_1)\, \mathcal{S}_{\gamma_2\delta_1}^{\delta_2\beta_1}(k_2,-k_1)\, r_{\delta_2}^{\beta_2}(k_2) = \\ r_{\alpha_2}^{\gamma_2}(k_2)\, \mathcal{S}_{\alpha_1\gamma_2}^{\delta_1\delta_2}(k_1,-k_2)\, r_{\delta_1}^{\gamma_1}(k_1)\, \mathcal{S}_{\delta_2\gamma_1}^{\beta_2\beta_1}(-k_2,-k_1)\,; \end{aligned} \tag{2.19}$$

$$\begin{aligned} \mathcal{S}_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}(k_1,k_2)\, t_{\gamma_1}^{\delta_1}(k_1)\, \mathcal{S}_{\gamma_2\delta_1}^{\delta_2\beta_1}(k_2,k_1)\, t_{\delta_2}^{\beta_2}(k_2) = \\ t_{\alpha_2}^{\gamma_2}(k_2)\, \mathcal{S}_{\alpha_1\gamma_2}^{\delta_1\delta_2}(k_1,k_2)\, t_{\delta_1}^{\gamma_1}(k_1)\, \mathcal{S}_{\delta_2\gamma_1}^{\beta_2\beta_1}(k_2,k_1)\,; \end{aligned} \tag{2.20}$$

$$\begin{aligned} \mathcal{S}_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}(k_1,k_2)\, t_{\gamma_1}^{\delta_1}(k_1)\, \mathcal{S}_{\gamma_2\delta_1}^{\delta_2\beta_1}(k_2,k_1)\, r_{\delta_2}^{\beta_2}(k_2) = \\ r_{\alpha_2}^{\gamma_2}(k_2)\, \mathcal{S}_{\alpha_1\gamma_2}^{\delta_1\delta_2}(k_1,-k_2)\, t_{\delta_1}^{\gamma_1}(k_1)\, \mathcal{S}_{\delta_2\gamma_1}^{\beta_2\beta_1}(-k_2,k_1)\,; \end{aligned} \tag{2.21}$$

(iii) mixed exchange relations

$$a_{\alpha_1}(k_1)\, r_{\alpha_2}^{\beta_2}(k_2) = \mathcal{S}_{\alpha_2\alpha_1}^{\gamma_2\gamma_1}(k_2,k_1)\, r_{\gamma_2}^{\delta_2}(k_2)\, \mathcal{S}_{\gamma_1\delta_2}^{\delta_1\beta_2}(k_1,-k_2)\, a_{\delta_1}(k_1)\,, \tag{2.22}$$

$$r_{\alpha_1}^{\beta_1}(k_1)\, a^{*\alpha_2}(k_2) = a^{*\delta_2}(k_2)\, \mathcal{S}_{\alpha_1\delta_2}^{\delta_1\gamma_2}(k_1,k_2)\, r_{\delta_1}^{\gamma_1}(k_1)\, \mathcal{S}_{\gamma_2\gamma_1}^{\alpha_2\beta_1}(k_2,-k_1)\ , \tag{2.23}$$

$$a_{\alpha_1}(k_1)\, t_{\alpha_2}^{\beta_2}(k_2) = \mathcal{S}_{\alpha_2\alpha_1}^{\gamma_2\gamma_1}(k_2,k_1)\, t_{\gamma_2}^{\delta_2}(k_2)\, \mathcal{S}_{\gamma_1\delta_2}^{\delta_1\beta_2}(k_1,k_2)\, a_{\delta_1}(k_1)\,, \tag{2.24}$$

$$t_{\alpha_1}^{\beta_1}(k_1)\, a^{*\alpha_2}(k_2) = a^{*\delta_2}(k_2)\, \mathcal{S}_{\alpha_1\delta_2}^{\delta_1\gamma_2}(k_1,k_2)\, t_{\delta_1}^{\gamma_1}(k_1)\, \mathcal{S}_{\gamma_2\gamma_1}^{\alpha_2\beta_1}(k_2,k_1)\ , \tag{2.25}$$

(iv) unitarity

$$t_{\alpha_1}^{\beta}(k)t_{\beta}^{\alpha_2}(k) + r_{\alpha_1}^{\beta}(k)r_{\beta}^{\alpha_2}(-k) = \delta_{\alpha_1}^{\alpha_2}\,, \tag{2.26}$$

$$t_{\alpha_1}^{\beta}(k)r_{\beta}^{\alpha_2}(k) + r_{\alpha_1}^{\beta}(k)t_{\beta}^{\alpha_2}(-k) = 0\,. \tag{2.27}$$

As suggested by (1.9), we assume that $r(k)$ is a diagonal matrix while $t(k)$ is an anti-diagonal one. Then, due to the particular form of the $\mathcal{S}$–matrix, the defect relations (ii) are equivalent to

$$[r_{\alpha_1}^{\beta_1}(k_1), r_{\alpha_2}^{\beta_2}(k_2)] = 0\,, \quad [r_{\alpha_1}^{\beta_1}(k_1), t_{\alpha_2}^{\beta_2}(k_2)] = 0\,, \quad [t_{\alpha_1}^{\beta_1}(k_1), t_{\alpha_2}^{\beta_2}(k_2)] = 0\,.$$

The Fock representations $\mathcal{F}(\mathcal{C}_\mathcal{S})$ of $\mathcal{C}_\mathcal{S}$ have been classified and explicitly constructed in [17]. As usual, each Fock representation involves a cyclic (vacuum) state Ω obeying

$$a_{\pm}(k)\, \Omega = 0\,. \tag{2.28}$$

We recall also that each $\lambda_{\mathcal{R},\mathcal{T}} \in \mathcal{F}(\mathcal{C}_\mathcal{S})$ is uniquely defined by the doublet $\{\mathcal{R},\, \mathcal{T}\}$, satisfying eqs. (1.10–1.12,2.15). The quantum version of eqs. (2.6,2.7) is

$$\begin{aligned} a_\alpha(k) &= t_\alpha^\beta(k) a_\beta(k) + r_\alpha^\beta(k) a_\beta(-k)\,, &(2.29)\\ a^{*\alpha}(k) &= a^{*\beta}(k) t_\beta^\alpha(k) + a^{*\beta}(-k) r_\beta^\alpha(-k)\,, &(2.30)\end{aligned}$$

which hold in any $\lambda_{\mathcal{R},\mathcal{T}}$.

The attention in [16, 17] has been mainly focused on the subclass $\widetilde{\mathcal{F}}(\mathcal{C}_\mathcal{S}) \subset \mathcal{F}(\mathcal{C}_\mathcal{S})$ of representations, characterized by reflection matrices satisfying

$$\mathcal{S}_{12}(k_1,k_2)\mathcal{R}_2(k_1) = \mathcal{R}_2(k_1)\mathcal{S}_{12}(-k_1,k_2)\,, \tag{2.31}$$

which is stronger than the boundary Yang-Baxter equation (2.15). We stress in this respect that $\mathcal{S}$ and $\mathcal{R}$ in the impurity NLS model obey (2.15) but not (2.31), i.e. in our case $\lambda_{\mathcal{R},\mathcal{T}} \notin \widetilde{\mathcal{F}}(\mathcal{C}_\mathcal{S})$.

The boundary Yang–Baxter equation (2.15) is actually the vacuum expectation value of the defect exchange relation (2.19) in the representation $\lambda_{\mathcal{R},\mathcal{T}}$. Taking the vacuum expectation value of the remaining relations (2.20,2.21), one obtains the transmission Yang–Baxter equation

$$\begin{aligned}&\mathcal{S}_{12}(k_1,k_2)\, \mathcal{T}_1(k_1)\, \mathcal{S}_{21}(k_2,k_1)\, \mathcal{T}_2(k_2) =\\ &\quad \mathcal{T}_2(k_2)\, \mathcal{S}_{12}(k_1,k_2)\, \mathcal{T}_1(k_1)\, \mathcal{S}_{21}(k_2,k_1)\,,\end{aligned} \tag{2.32}$$

and the mixed reflection–transmission Yang–Baxter equation

$$\begin{aligned}&\mathcal{S}_{12}(k_1,k_2)\, \mathcal{T}_1(k_1)\, \mathcal{S}_{21}(k_2,k_1)\, \mathcal{R}_2(k_2) =\\ &\mathcal{R}_2(k_2)\, \mathcal{S}_{12}(k_1,-k_2)\, \mathcal{T}_1(k_1)\, \mathcal{S}_{21}(-k_2,k_1)\,.\end{aligned} \tag{2.33}$$

The relations (2.32,2.33) have been discovered in [17], where it is shown that they are a general consequence of (1.10-1.12,2.15). The validity of (2.32,2.33) in our case can be checked directly, inserting (1.9,2.11).

At this stage we can define the basic structure $\{\mathcal{H}, \mathcal{D}, \Omega, \Phi\}$ in terms of $\lambda_{\mathcal{R},\mathcal{T}}$ as follows:

- $\mathcal{H}$, $\langle\cdot\,,\,\cdot\rangle$ and Ω are the Hilbert space, the scalar product and the vacuum state of $\lambda_{\mathcal{R},\mathcal{T}}$, where $\{\mathcal{R},\, \mathcal{T}\}$ and $\mathcal{S}$ are given by (1.7-1.9) and (2.11) respectively.

- The quantum fields $\Phi_\pm$, defined by (2.1), admit the series representation (2.2), where

$$\Phi_\pm^{(n)}(t,x) = \int \prod_{\substack{i=1\\ j=0}}^{n} \frac{dp_i}{2\pi}\frac{dq_j}{2\pi}\, a^{*\pm}(p_1)\ldots a^{*\pm}(p_n) a_\pm(q_n)\ldots a_\pm(q_0)\,\cdot$$
$$\frac{e^{i\sum\limits_{j=0}^{n}(q_j x - q_j^2 t) - i\sum\limits_{i=1}^{n}(p_i x - p_i^2 t)}}{\prod\limits_{i=1}^{n}(p_i - q_{i-1} \mp i\varepsilon)(p_i - q_i \mp i\varepsilon)}\,. \tag{2.34}$$

- The domain $\mathcal{D}$ is the finite particle subspace of $\lambda_{\mathcal{R},\mathcal{T}}$, which is well–known to be dense in $\mathcal{H}$.

The mere fact that our system interacts with an impurity shows up at the algebraic level, turning the Zamolodchikov–Faddeev (ZF) algebra from the impurity–free case [6]–[10] to an RT algebra (2.16–2.27). The details characterizing the impurity enter the construction at the level of representation by means of the reflection and transmission matrices (1.9). Notice also that the series (2.2) is actually a finite sum when Φ is acting on $\mathcal{D}$. The coupling constant g appears explicitly in (2.2) and implicitly in a_α and $a^{*\alpha}$ which depend on g through $\mathcal{S}$. The properties of the quantum field Φ, defined above, are summarized in the following

Proposition: $\Phi(t,x)$ is a well–defined operator–valued distribution satisfying the canonical commutation relations (2.8,2.9) on $\mathcal{D}$, as well as the equation of motion

$$(i\partial_t + \partial_x^2)\langle \varphi\,,\, \Phi(t,x)\psi\rangle = 2g\,\langle \varphi\,,\, :\Phi\Phi^*\Phi:(t,x)\psi\rangle\,, \qquad x\neq 0\,, \tag{2.35}$$

and the boundary conditions

$$\lim_{x\downarrow 0}\begin{pmatrix} \langle \varphi\,,\, \Phi(t,x)\psi\rangle \\ \partial_x\langle \varphi\,,\, \Phi(t,x)\psi\rangle \end{pmatrix} = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \lim_{x\uparrow 0}\begin{pmatrix} \langle \varphi\,,\, \Phi(t,x)\psi\rangle \\ \partial_x\langle \varphi\,,\, \Phi(t,x)\psi\rangle \end{pmatrix}\,, \tag{2.36}$$

$$\lim_{x\to\pm\infty}\langle \varphi\,,\, \Phi(t,x)\psi\rangle = 0\,, \tag{2.37}$$

for any φ, $\psi \in \mathcal{D}$.

For the proof of this statement we refer to [1], where the δ–impurity (see eq. (1.14)) is considered in detail. Following [15], the normal product $:\cdots:$ in (2.35) preserves the original order of the creators; the original order of two annihilators is preserved if both belong to the same Φ or Φ^* and inverted otherwise. Since Φ and the hermitian conjugate Φ^* are unbounded operators, the delicate points in proving the above proposition are essentially domain problems. They are solved taking into account that the reflection and transmission amplitudes R_+^+ and T_+^- (R_-^- and T_-^+) have no poles in the complex upper (lower) half–plane, which is a consequence of condition (1.4) ensuring the absence of impurity bound states.

For $\alpha = a = d = 1$ and $b = c = 0$ one expects to recover from (2.34) the solution of the NLS equation without impurity. We will show now that this is indeed the case. First of all we observe that in this limit

$$\mathcal{R}(k) = 0\,, \qquad \mathcal{T}(k) = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)\,. \tag{2.38}$$

Because of (2.6,2.7), in the classical case one finds $\lambda_-(k) = \lambda_+(k)$ and Φ defined by (2.1–2.3) precisely reproduces the classical solution without impurity. The quantum case is slightly more involved. The data (2.38) fix a Fock representation of $\mathcal{C}_\mathcal{S}$ in which

$$r(k) = 0\,, \qquad t(k) = \left(\begin{array}{cc} 0 & t_+^- \\ t_-^+ & 0 \end{array}\right)\,. \tag{2.39}$$

From eq. (2.26) one deduces that

$$t_+^-\, t_-^+ = t_-^+\, t_+^- = \mathbf{1}\,, \tag{2.40}$$

where $\mathbf{1}$ is the identity operator in $\mathcal{H}$. We stress however that t_+^- and t_-^+ are *not* proportional to $\mathbf{1}$, since they do not commute with $a_\pm(k)$ (see eq. (2.24)). In agreement with this fact and consistently with the exchange relations (2.16–2.18) and the form of the bulk scattering matrix, one has $a_-(k) \neq a_+(k)$. Therefore the argument used at the classical level does not apply and one must proceed in the quantum case differently. We observe in this respect that inserting (2.39) in (2.18), one concludes that the polynomials of the operators $\{a^{*+}(k),\, a_+(k),\, \mathbf{1}\}$ close a ZF algebra $\mathcal{A}_+$ with exchange factor $\mathcal{S}_{++}^{++}(k_1,k_2) = S(k_1 - k_2)$. Applied on the vacuum Ω, the elements of $\mathcal{A}_+$ generate a subspace $\mathcal{H}_+ \subset \mathcal{H}$. By construction the quantum field Φ_+ leaves invariant $\mathcal{D}_+ = \mathcal{D} \cap \mathcal{H}_+$ and its restriction $\Phi_+|_{\mathcal{D}_+}$ on $\mathcal{D}_+$ solves [10] the impurity-free NLS equation. Analogously, the algebra $\mathcal{A}_-$ generated by $\{a^{*-}(k),\, a_-(k),\, \mathbf{1}\}$ is a ZF algebra with exchange factor $\mathcal{S}_{--}^{--}(k_1,k_2) = S(-k_1 + k_2)$. The counterpart $\mathcal{H}_-$ of $\mathcal{H}_+$ defines the domain $\mathcal{D}_- = \mathcal{D} \cap \mathcal{H}_-$, which is invariant under Φ_-. The restriction $\Phi_-|_{\mathcal{D}_-}$ is also a solution of the NLS equation without impurity. Being related by a parity transformation $x \mapsto -x$, which is a symmetry in this case, $\Phi_+|_{\mathcal{D}_+}$ and $\Phi_-|_{\mathcal{D}_-}$ are unitary equivalent. Finally, the fact that in momentum space parity is implemented by $k \mapsto -k$, explains the relation

$$\mathcal{S}_{++}^{++}(k_1,k_2) = \mathcal{S}_{--}^{--}(-k_1,-k_2)\, . \tag{2.41}$$

Turning back to the general impurity case, one can directly verify by means of (2.34) that the Hamiltonian H, which generates the time evolution

$$\Phi(t,x) = \mathrm{e}^{itH}\, \Phi(0,x)\, \mathrm{e}^{-itH}\, , \tag{2.42}$$

has the familiar quadratic form

$$H = \int \frac{dk}{2\pi} k^2 a^{*\alpha}(k) a_\alpha(k)\, . \tag{2.43}$$

H is actually the second term of a whole sequence [21, 22]

$$H_n = \int \frac{dk}{2\pi} k^{2n} a^{*\alpha}(k) a_\alpha(k)\, , \qquad n = 0, 1, 2, ... \tag{2.44}$$

of integrals of motion in involution. In this sense the impurity system under consideration is integrable. The simple form of H_n is among the advantages of the RT algebra approach.

Employing (2.2,2.34), one can construct all correlation functions of Φ and Φ^*. The structure of (2.34) implies that for the $2n$–point function one needs at most the $(n-1)$–th order contribution in (2.2). In fact, one has for example

$$\langle \Omega, \Phi(t_1,x_1)\Phi^*(t_2,x_2)\Omega\rangle = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \mathrm{e}^{-ik^2 t_{12}} \cdot$$
$$\Big\{\theta(x_1)\theta(x_2)\left[\mathrm{e}^{ikx_{12}} + R_+^+(k)\mathrm{e}^{ik\widetilde{x}_{12}}\right] + \theta(-x_1)\theta(-x_2)\left[\mathrm{e}^{ikx_{12}} + R_-^-(k)\mathrm{e}^{ik\widetilde{x}_{12}}\right] +$$
$$\theta(x_1)\theta(-x_2)T_+^-(k)\mathrm{e}^{ikx_{12}} + \theta(-x_1)\theta(x_2)T_-^+(k)\mathrm{e}^{ikx_{12}}\Big\}\, , \tag{2.45}$$

where $t_{12} = t_1 - t_2$, $x_{12} = x_1 - x_2$ and $\widetilde{x}_{12} = x_1 + x_2$. Analogous, but more involved integral representations hold for the $2n$-point functions with $n > 1$.

In [1] it is also shown that Φ and Φ^* admit asymptotic limits in a suitably adapted to the impurity case Haag–Ruelle scattering theory. The net result is that the asymptotic states are obtained applying the creation operators $a^{*\pm}$ to the vacuum Ω: in $\mathbb{R}_+$ and $\mathbb{R}_-$ the asymptotic incoming particles are generated by $\{a^{*+}(k)\, :\, k < 0\}$ and $\{a^{*-}(k)\, :\, k > 0\}$ respectively, while the outgoing particles are created by

$\{a^{*+}(k) \ : \ k > 0\}$ and $\{a^{*-}(k) \ : \ k < 0\}$. The scattering amplitudes are thus derived in a purely algebraic way, using the exchange relation (2.18) and the fact that according to (2.28) $a_\pm$ annihilate Ω. As expected, the total scattering operator $\mathbf{S}$ factorizes, the factors being the bulk scattering matrix $\mathcal{S}$ (2.11) and the reflection and transition matrices $\mathcal{R}$ and $\mathcal{T}$ (1.9).

Summarising, we have established a family (1.2-1.4) of point–like impurities interacting with the NLS field, which preserve quantum integrability. These systems can be investigated by the inverse scattering method. We have shown in this respect that the RT algebra $\mathcal{C}_\mathcal{S}$ and its Fock representation $\lambda_{\mathcal{R},\mathcal{T}}$ allow to construct not only the scattering operator but also the off–shell quantum field $\Phi(t,x)$.

3 Discussion

A debated and physically relevant question in the theory of integrable systems with impurities concerns the space–time symmetry of the *bulk* scattering matrix $\mathcal{S}$. It is well–known that impurities break down Galilean (Lorentz) invariance in the *total* scattering matrix $\mathbf{S}$. However, since $\mathcal{S}$ describes the scattering away from the impurity, one might be tempted to assume [18]–[20] that $\mathcal{S}$ preserves these symmetries and that the breaking in $\mathbf{S}$ is generated exclusively by the reflection and transmission coefficients $\mathcal{R}$ and $\mathcal{T}$. Unfortunately however, the conditions of factorized scattering then imply [18, 20] that $\mathcal{S}$ is constant. Being too restrictive, this property limits very much the interest in such systems. In order to avoid the problem, a consistent factorized scattering theory of a unitary scattering operator has been developed in [16, 17] in terms of RT algebras, without necessarily assuming that $\mathcal{S}$ is Galilean (Lorentz) invariant. The impurity NLS model considered above, is the first concrete application of this framework with non–trivial bulk scattering. The lesson from it is quite instructive. Focusing on (2.11), we see that Galilean invariance is indeed broken by the entries of $\mathcal{S}$, which describe the scattering of two incoming particles localized for $t \to -\infty$ on $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively. In fact, these entries depend on $k_1 + k_2$ and not on $k_1 - k_2$. The intuitive reason behind this breaking is that before such particles scatter, one of them must necessarily cross the impurity. The non–trivial transmission is therefore the origin of the symmetry breaking in $\mathcal{S}$. This conclusion agrees with the observation that in systems which allow only reflection (e.g. models on the half–line), one can have both Galilean (Lorentz) invariant and non–constant bulk scattering matrices.

For simplicity we focused in this paper on linear impurity boundary conditions. One can expect however that there exist non-liner boundary conditions of the type proposed in [23, 24] for the Toda model, which also preserve the integrability of the NLS equation.

Another aspect which deserves further investigation is the issue of internal symmetries in the presence of impurities. This question has been partially addressed in [21, 22], where the role of the reflection and transmission elements of the RT algebra as symmetry generators has been established. It will be interesting in this respect to extend the analysis [25] of the $SU(N)$–NLS model on the half–line to the impurity case.

Let us conclude by observing that the concept of RT algebra indeed represents a powerful tool for solving the NLS model with impurities. We strongly believe that this algebraic framework is actually universal and applies to the quantization of other systems as well.

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