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Form factors in the SS model and its RSOS restrictions

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Abstract

New integral representations for form factors in the two parametric SS model are proposed. Some form factors in the parafermionic sine-Gordon model and in an integrable perturbation of $SU(2)$ coset conformal field theories are straightforwardly obtained by different quantum group restrictions. Numerical checks on the value of the central charge are performed.

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1 Introduction

Form factors with n particles of an operator are the matrix elements of this operator between the vacuum and a n particles state. They are non pertubative objects that can be constructed, up to a normalization, as solution of "bootstrap" equations [1–3], once the exact scattering matrix is known. Besides being solutions of a nice mathematical problem, the form factors are useful tools to determine the long distance expansion of correlation functions by inserting a complete set of asymptotic states. It is often enough to approximate (with a good accuracy) the correlation functions of local operators with the contribution of the form factor with the smallest number of particles, due to the fast convergence of the spectral series [2].

In this paper, we construct form factors in a two parametric family of massive integrable quantum field theories known as the SS model, whose action can be found in [4], and which can be written in terms of three boson fields $\varphi_i,\ i = 1, 2, 3$ with an exponential interaction. We will

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[2] It is usually believed that the spectral series converges for local operators, but this has not been proven so far.

restrict ourselves to the unitary regime of this theory [4], where the model has a $U(1)\times U(1)$ symmetry described by two conserved topological charges:

$$Q_{\pm}=\frac{1}{2}(Q_1\pm Q_2),\quad Q_i=\int dx^1 j_i^0,\quad i=1,2\,,$$

where $j_i^{0,1}$ are the components of the $U(1)$ current. Both classically and in the quantum case, the charges (eigenvalues of the charges) satisfy the conditions:

$$Q_1,Q_2\in\mathbb{Z},\quad Q_1+Q_2\in 2\mathbb{Z}.$$

The $U(1)\times U(1)$ symmetry can be extended up to the symmetry generated by two quantum affine algebras $U_{q_1}(\hat{sl_2})\otimes U_{q_2}(\hat{sl_2})$. The name of the SS model is unfortunately justified by the expression of its two-particles S-matrix -first considered in [5]- in terms of two sine-Gordon S^{SG}-matrices with different coupling constants β_1,β_2:

$$S(\theta_{12})\equiv -S^{SG}_{p_1}(\theta_{12})\otimes S^{SG}_{p_2}(\theta_{12}),$$

where $p_1=\frac{\beta_1^2}{8\pi-\beta_1^2}$, $p_2=\frac{\beta_2^2}{8\pi-\beta_2^2}$ [6], and $\theta_{12}=\theta_1-\theta_2$ is the rapidity difference. The unitary regime is characterized by the following conditions on the parameters: $p_1,p_2>0$ and $p_1+p_2\geq 2$. We will need to introduce the negative parameter p_3 such that $p_1+p_2+p_3=2$, and the parameters α_i, $i=1,2,3$, defined by $p_i=2\alpha_i^2$ [4].

The SS model includes, as particular cases, the N=2 supersymmetric sine-Gordon model, whose S-matrix has the form [7]:

$$S(\theta_{12})=-S^{SG}_{2}(\theta_{12})\otimes S^{SG}_{p_2}(\theta_{12}),$$

as well as the $O(4)$ non linear sigma model (Principal chiral field model [8,9]), which possesses $SU(2)\times SU(2)$ symmetry; its S-matrix reads [6]

$$S(\theta_{12})=-S^{SG}_{\infty}(\theta_{12})\otimes S^{SG}_{\infty}(\theta_{12}),$$

which is nothing but the tensor product of two S-matrices of the $SU(2)$ invariant Thirring model. It also contains the anisotropic chiral field [10], which has the $U(1)\times SU(2)$ symmetry; its S-matrix is

$$S(\theta_{12})=-S^{SG}_{p_1}(\theta_{12})\otimes S^{SG}_{\infty}(\theta_{12}).$$

The SS model also includes other known integrable QFTs, like the $O(3)$ non linear sigma model, the sausage model [11] and the cosine-cosine model [12]. Let us note in passing that the ground state energy of the model in finite volume was determined in [4] by TBA method, and the finite size effects for the SS model were recently studied in [13].

In order to construct the form factors in this theory, we give in the first section a summary of the method used in [14,15] to construct form factors in the sine-Gordon model, and recall how to perform RSOS restriction [16] directly on sine-Gordon form factors [17]. In the second section, we propose a generic formula inspired from the SG model for form factors containing an even number of particles in the SS model; in particular we write down the form factors of the trace of the energy momentum tensor, first obtained in [5], and of some of the exponential fields (this problem was also recently considered in [18], with different methods). Two particles form factors of these operators can be expressed in terms of two particles form factors in the SG model. Then we do two different kinds of RSOS restrictions, namely [19] to the parafermionic sine-Gordon model (this model includes, amongst others, the N=1 and the restricted N=2 supersymmetric sine-Gordon models), and to the integrable perturbed coset CFT [20] $su(2)_{p_1-2}\otimes su(2)_{p_2-2}/su(2)_{p_1+p_2-4}$. A common limit of these QFT's is the Polyakov-Wiegmann model [8]. We propose expressions for form factors of the trace operator (they were first considered in [5]). In particular, we check our expressions for the two particles form factor of the trace of the energy momentum tensor by making a numerical estimation of the central charge and compare it with the exact result.

2 Form factors in the sine-Gordon model and RSOS restriction

In this section we recapitulate known results on form factors in the SG model in the repulsive regime. They will be useful in the next section for the construction of the form factors in the SS model.
The Sine-Gordon model alias the massive Thirring model is defined by the Lagrangians:

$$\begin{aligned}\mathcal{L}^{SG} &= \frac{1}{2}(\partial_\mu\varphi)^2+\frac{\alpha}{\beta^2}(\cos\beta\varphi-1), \\ \mathcal{L}^{MTM} &= \bar\psi(\mathrm{i}\gamma\partial-M)\psi-\frac{1}{2}g(\bar\psi\gamma_\mu\psi)^2,\end{aligned}\tag{1}$$

respectively. The fermi field ψ correspond to the soliton and antisoliton and the bose field φ to the lowest breather which is the lowest soliton antisoliton bound state. The relation between the coupling constants was found in [21] within the framework of perturbation theory:

$$p\equiv\frac{\beta^2}{8\pi-\beta^2}=\frac{\pi}{\pi+2g}.$$

The two soliton sine-Gordon S-matrix contains the following scattering amplitudes: the two-soliton amplitude $a_p(\theta)$, the forward and backward soliton anti-soliton amplitudes $b_p(\theta)$ and $c_p(\theta)$:

$$b_p(\theta)=\frac{\sinh\theta/p}{\sinh(i\pi-\theta)/p}a_p(\theta),\quad c_p(\theta)=\frac{\sinh i\pi/p}{\sinh(i\pi-\theta)/p}a_p(\theta),$$

$$a_p(\theta)=\exp\int_0^\infty\frac{dt}{t}\frac{\sinh\frac{1}{2}(1-p)t\sinh\frac{t\theta}{i\pi}}{\cosh\frac{t}{2}\sinh\frac{1}{2}pt}.$$

This S-matrix satisfies the Yang-Baxter equation as well as the unitarity condition:

$$S_p^{SG}(\theta)S_p^{SG}(-\theta)=1,$$

which can be rewritten for the amplitudes as:

$$a_p(\theta)a_p(-\theta)=1,\quad b_p(\theta)b_p(-\theta)+c_p(\theta)c_p(-\theta)=1.$$

The crossing symmetry condition reads for the amplitudes:

$$a_p(\mathrm{i}\pi-\theta)=b_p(\theta),\quad c_p(\mathrm{i}\pi-\theta)=c_p(\theta).$$

The form factors $f(\theta_1, \cdots, \theta_{2n})$ of a local operator[3] in the SG model are covector valued functions that satisfy a system of equations [3], which consist of a Riemann-Hilbert problem:

$$f(\theta_1, \cdots, \theta_i, \theta_{i+1}, \cdots, \theta_{2n}) S_p^{SG}(\theta_i - \theta_{i+1}) = f(\theta_1, \cdots, \theta_{i+1}, \theta_i, \cdots, \theta_{2n}),$$
$$f(\theta_1, \cdots, \theta_{2n-1}, \theta_{2n} + 2i\pi) = f(\theta_{2n}, \theta_1, \cdots, \theta_{2n-1}),$$

and a residue equation at $\theta_1 = \theta_{2n} + i\pi$:

$$\mathrm{res} f(\theta_1, \cdots, \theta_{2n}) = -2i\, f(\theta_2, \cdots, \theta_{2n-1}) \left(1 - \prod_{i=2}^{2n-1} S_p^{SG}(\theta_i - \theta_{2n}) \right) e_0, \quad e_0 = s_1 \otimes \bar{s}_{2n} + \bar{s}_1 \otimes s_{2n},$$

where s (or +) corresponds to the solitonic state (highest weight state), and $\bar{s}$ (or -) to the antisolitonic state. Form factors containing an arbitrary number of particles for the energy momentum tensor, the topological current and the semi-local operator $e^{\pm i \frac{\beta}{2} \varphi_{SG}}$ were first constructed in [3] by Smirnov; form factors of non local exponential fields $e^{ia\varphi_{SG}}$ were constructed in [22] by Lukyanov, using free field representation techniques that provide integral representations different from those of [3]. Below, we make the choice to present the posterior construction presented in [14, 15] by Babujian and Karowski *et. al.*
We first introduce the minimal form factor $f_p(\theta_{12})$ of the SG model: it satisfies the relation

$$f_p(\theta) = -f_p(-\theta) a_p(\theta) = f_p(2i\pi - \theta),$$

and reads explicitly

$$f_p(\theta) = -i \sinh\frac{\theta}{2} f_p^{min}(\theta) = -i \sinh\frac{\theta}{2} \exp \int_0^\infty \frac{dt}{t} \frac{\sinh\frac{1}{2}(1-p)t}{\sinh\frac{1}{2}pt \cosh\frac{1}{2}t} \frac{1 - \cosh t(1 - \frac{\theta}{i\pi})}{2\sinh t}.$$

Its asymptotic behaviour when $\theta \to \pm\infty$ is given by $f_p(\theta) \sim \mathcal{C}_p\, e^{\pm\frac{1}{4}(\frac{1}{p}+1)(\theta - i\pi)}$, with the constant

$$\mathcal{C}_p = \frac{1}{2} \exp \frac{1}{2} \int_0^\infty \frac{dt}{t} \left(\frac{\sinh\frac{1}{2}(1-p)t}{\sinh\frac{1}{2}pt \cosh\frac{1}{2}t \sinh t} - \frac{1-p}{pt} \right). \tag{2}$$

It is proposed in [14] that form factors in SG can be generically written[4]:

$$f(\theta_1, \theta_2, \ldots, \theta_n) = N_n \prod_{i<j} f_p(\theta_{ij}) \int_{C_\theta} du_1 \ldots \int_{C_\theta} du_m\ h_p(\theta, u) p_n(\theta, u) \Psi^p(\theta, u), \tag{3}$$

where we introduced the scalar function (completely determined by the S-matrix)

$$h_p(\theta, u) = \prod_{i=1}^{2n} \prod_{j=1}^{m} \phi_p(\theta_i - u_j) \prod_{1 \le r < s \le m} \tau_p(u_r - u_s),$$

with

$$\phi_p(u) = \frac{1}{f_p(u) f_p(u + i\pi)}, \quad \tau_p(u) = \frac{1}{\phi_p(u)\phi_p(-u)}.$$

[3] We shall consider in the following form factors with an even number of particles only.
[4] This representation holds whether operators are local or not, topologically neutral or not.

$\Psi^p(\theta,u)$ is the Bethe ansatz state covector: we first define the monodromy matrix T_p as

$$\begin{pmatrix} A(\theta_1,\dots,\theta_n,u) & B(\theta_1,\dots,\theta_n,u) \\ C(\theta_1,\dots,\theta_n,u) & D(\theta_1,\dots,\theta_n,u) \end{pmatrix} \equiv T_p(\theta_1,\dots,\theta_n,u) = S_p^{SG}(\theta_1-u)\dots S_p^{SG}(\theta_n-u),$$

the definition of the Bethe ansatz covector is given by

$$\Psi^p(\theta,u) = \Omega_{1\dots n}\, C(\theta_1,\dots,\theta_n,u_1)\dots C(\theta_1,\dots,\theta_n,u_m),$$

where $\Omega_{1\dots n}$ is the pseudo vacuum consisting only of solitons

$$\Omega_{1\dots n} = s\otimes\dots\otimes s.$$

The number of integration variables m is related to the topological charge Q ($Q\in\mathbb{Z}$) of the operator considered and the number n of particles through the relation $Q=n-2m$. For example, for $n=2$ and $Q=0$,

$$\Psi^p(\theta_1,\theta_2,u) = \Psi^p_{+-}(\theta_1,\theta_2,u)+\Psi^p_{-+}(\theta_1,\theta_2,u) = b(\theta_1-u)c(\theta_2-u)s_1\otimes\bar{s}_2+c(\theta_1-u)a(\theta_2-u)\bar{s}_1\otimes s_2.$$

The function $p_n(\theta,u)$ is the only ingredient in formula (3) which depends on the operator considered. If the operator is chargeless, the form factors contain an even number of particles, and if in addition the operator is local, then the p-function satisfies the conditions[5]:

1. $p_{2n}(\theta,u)$ is a polynomial in $e^{\pm u_j}$, $(j=1,\dots,n)$ and $p_{2n}(\theta,u) = p_{2n}(\dots,\theta_i-2i\pi,\dots,u)$
2. $p_{2n}(\theta_1=\theta_{2n}+i\pi,\dots\theta_{2n};u_1\dots u_n=\theta_{2n}) = p_{2n-2}(\theta_2\dots\theta_{2n-1};u_1\dots u_{n-1})+\tilde{p}^1(\theta_2\dots\theta_{2n-1})$,
   $p_{2n}(\theta_1=\theta_{2n}+i\pi,\dots\theta_{2n};u_1\dots u_n=\theta_{2n}\pm i\pi) = p_{2n-2}(\theta_2\dots\theta_{2n-1};u_1\dots u_{n-1})+\tilde{p}^2_\pm(\theta_2\dots\theta_{2n-1})$,
   where $\tilde{p}^{1,2}(\theta_2\dots\theta_{2n-1})$ are independent of the integration variables.
3. $p_{2n}(\theta,u)$ is symmetric with respect to the θ's and the u's.
4. $p_{2n}(\theta+\ln\Lambda, u+\ln\Lambda) = \Lambda^{\mathrm{s}}\, p_{2n}(\theta,u)$ where s is the Lorentz spin of the operator.

Finally, the integration contours C_θ consist of several pieces for all integration variables u_j : a line from $-\infty$ to ∞ avoiding all poles such that $\mathrm{Im}\theta_i-\pi-\epsilon<\mathrm{Im}u_j<\mathrm{Im}\theta_i-\pi$, and clockwise oriented circles around the poles (of the $\phi(\theta_i-u_j)$) at $\theta_i=u_j$, $(j=1,\dots,m)$.

Trace of the energy momentum tensor. The trace operator is a spinless and chargeless local operator. Its p-function is [15]:

$$p^{\Theta}_{SG}(\theta,u) = -\left(\sum_{i=1}^{2n} e^{-\theta_i}\sum_{j=1}^{n} e^{u_j} - \sum_{i=1}^{2n} e^{\theta_i}\sum_{j=1}^{n} e^{-u_j}\right). \tag{4}$$

The residue equation gives the following relation for the normalization N^Θ_{2n} (see [14]):

$$N^\Theta_{2n} = N^\Theta_{2n-2}\frac{\left(f_p^{min}(0)\right)^2}{4n\pi} \quad\to N^\Theta_{2n} = N^\Theta_2\frac{1}{n!}\left(\frac{\left(f_p^{min}(0)\right)^2}{4\pi}\right)^{n-1}.$$

[5] We consider here only the case where the operator is of bosonic type and the particles are of fermionic type. If both are fermionic, there is an extra statistic factor to be taken into account, see [15].

The two particles form factor can be computed explicitly:

$$f_{SG}^{\Theta}(\theta_{12}) = \frac{2\pi N_2^{\Theta}}{\mathcal{C}_p^4} f_p(\theta_{12}) \frac{\cosh\frac{\theta_{12}}{2}}{\sinh\frac{1}{2p}(i\pi - \theta_{12})} (s_1 \otimes \bar{s}_2 + \bar{s}_1 \otimes s_2),$$

in agreement with the result first obtained by diagonalization of the S^{SG}-matrix in [1]. The normalization for two particles is chosen to be $N_2^{\Theta} = \frac{1}{p} i M^2 \mathcal{C}_p^4$ (M being the mass of the soliton), in order to have:

$$f_{SG}^{\Theta}(\theta_1 + i\pi, \theta_1) = 2\pi M^2 (s_1 \otimes \bar{s}_2 + \bar{s}_1 \otimes s_2),$$

Exponential fields. The p-function of the spinless, chargeless, non local, exponential field $e^{ia\varphi_{SG}(x)}$ in the SG model is not written in [15], though it is a minor extension of the results contained in this paper. It reads

$$p_{SG}^{\frac{a}{\beta}}(\theta, u) = \frac{1}{e^{\frac{i\pi a}{\beta}}} \frac{\prod_{j=1}^{n} e^{\frac{2au_j}{\beta}}}{\prod_{i=1}^{2n} e^{\frac{a\theta_i}{\beta}}}. \tag{5}$$

The conditions 1. and 2. are modified into:

1. $p_{SG}^{\frac{a}{\beta}}(\theta, u) = e^{-\frac{2i\pi a}{\beta}} p_{SG}^{\frac{a}{\beta}}(\ldots, \theta_i - 2i\pi, \ldots, u)$,
2. $p_{SG}^{\frac{a}{\beta}}(\theta_1 = \theta_{2n} + i\pi, \ldots \theta_{2n}; u_1 \ldots u_n = \theta_{2n}) = e^{\frac{-2i\pi a}{\beta}} p_{SG}^{\frac{a}{\beta}}(\theta_2 \ldots \theta_{2n-1}; u_1 \ldots u_{n-1})$,
   $p_{SG}^{\frac{a}{\beta}}(\theta_1 = \theta_{2n} + i\pi, \ldots \theta_{2n}; u_1 \ldots u_n = \theta_{2n} + i\pi) = p_{SG}^{\frac{a}{\beta}}(\theta_2 \ldots \theta_{2n-1}; u_1 \ldots u_{n-1})$,
   $p_{SG}^{\frac{a}{\beta}}(\theta_1 = \theta_{2n} + i\pi, \ldots \theta_{2n}; u_1 \ldots u_n = \theta_{2n} - i\pi) = e^{\frac{-4i\pi a}{\beta}} p_{SG}^{\frac{a}{\beta}}(\theta_2 \ldots \theta_{2n-1}; u_1 \ldots u_{n-1})$.

The form factors of exponential fields were first constructed in [22] with a different representation. For $a = \frac{k}{2}\beta$, with $k \in \mathbb{Z}$, the form factors can be computed explicitly, and their expression can be found in [22]; In particular, the form factors of the semi-local operator $e^{\pm i\frac{\beta}{2}\varphi_{SG}(x)}$ were first obtained in [3]; their expression with two particles is[6]:

$$f_{SG}^{\pm\frac{1}{2}}(\theta_{12}) = f_p(\theta_{12}) \frac{i\pi}{\mathcal{C}_p^4 \sinh\frac{1}{p}(i\pi - \theta_{12})} \left(e^{\pm\frac{i\pi - \theta_{12}}{2p}} s_1 \otimes \bar{s}_2 + e^{\mp\frac{i\pi - \theta_{12}}{2p}} \bar{s}_1 \otimes s_2 \right).$$

An important remark to be made is that there exists an alternative p^{Θ}-function to eq. (4) for the trace of the energy momentum tensor. Indeed, if one remembers that in the SG model the trace Θ is identified as the term in the action $\cos\beta\varphi_{SG}$, up to inessential coefficients, then one can rewrite its p-function in a suggestive form as a sum of p-functions for exponential fields with $a = \beta$ and $a = -\beta$, *i.e.*:

$$p_{SG}^{\Theta}(\theta, u) = p_{SG}^{1}(\theta, u) + p_{SG}^{-1}(\theta, u). \tag{6}$$

Indeed, using the expression

$$f_{SG}^{\pm 1}(\theta_{12}) = f_p(\theta_{12}) \frac{2\pi \operatorname{cotan}\frac{\pi p}{2}}{\mathcal{C}_p^4} \frac{\cosh\frac{\theta_{12}}{2}}{\sinh\frac{1}{p}(i\pi - \theta_{12})} \left(e^{\pm\frac{i\pi - \theta_{12}}{2p}} s_1 \otimes \bar{s}_2 + e^{\mp\frac{i\pi - \theta_{12}}{2p}} \bar{s}_1 \otimes s_2 \right),$$

[6] We will not need the normalization factor of the exponential fields in the SG model. They can be found in [23].

we find

$$f^{\Theta}_{SG}(\theta_{12}) = \frac{2\pi \tilde{N}^{\Theta}_{2} \mathrm{cotan}\frac{\pi p}{2}}{\mathcal{C}^{4}_{p}} f_p(\theta_{12}) \frac{\cosh\frac{\theta_{12}}{2}}{\sinh\frac{1}{2p}(i\pi - \theta_{12})} (s_1 \otimes \bar{s}_2 + \bar{s}_1 \otimes s_2),$$

where the new normalization constant $\tilde{N}^{\Theta}_{2}$ is given this time by the formula $\tilde{N}^{\Theta}_{2} = \frac{iM^2\mathcal{C}^{4}_{p}}{p} \tan\frac{\pi p}{2}$. Let us note that the vacuum expectation value of the trace was obtained in [24] thanks to the thermodynamic Bethe ansatz, and is equal to $< \Theta >= -\pi M^2 \tan\frac{\pi p}{2}$; the following relation then holds:

$$\tilde{N}^{\Theta}_{2} = - < \Theta > \frac{\mathcal{C}^{4}_{p}}{p\,\pi}.$$

RSOS restriction [16]. The RSOS restriction describes the $\Phi_{1,3}$-perturbations of minimal models of CFT [25] for rational values of p. When p is an integer, we deal with $\Phi_{1,3}$-perturbations of minimal models M_p with central charge $c = 1 - \frac{6}{p(p+1)}$. In particular, we remind that $S^{RSOS}_{3} = -1$ is the Ising S-matrix. Form factors in the model M_p can be directly obtained from those of the SG model, as explained in [17]. For the trace operator, the RSOS procedure consists of 'taking the half' of the p-function[7] (4), such that its p-function reads:

$$p^{\Theta}_{RSOS}(\theta, u) = -\sum_{i=1}^{2n} e^{-\theta_i} \sum_{m=1}^{n} e^{u_m},$$

then we should modify the Bethe ansatz state:

$$\tilde{\Psi}^{p}_{\epsilon_1\epsilon_2\ldots\epsilon_{2n}} \equiv e^{\frac{1}{2p}\sum_i \epsilon_i\theta_i} \Psi^{p}_{\epsilon_1\epsilon_2\ldots\epsilon_{2n}}, \quad \epsilon_i = \pm, \ \sum_{i=1}^{2n} \epsilon_i = 0. \tag{7}$$

The two particles form factor of the trace operator reads explicitly:

$$f^{\Theta}_{RSOS}(\theta_{12}) = \frac{2\pi N^{\Theta}_{2}}{\mathcal{C}^{4}_{p}} f_p(\theta_{12}) \frac{\cosh\frac{\theta_{12}}{2}}{\sinh\frac{1}{p}(i\pi - \theta_{12})} (e^{\frac{i\pi}{2p}} s_1 \otimes \bar{s}_2 + e^{-\frac{i\pi}{2p}} \bar{s}_1 \otimes s_2) \tag{8}$$

and the normalization for two particles is chosen to be $N^{\Theta}_{2} = \frac{2}{p} iM^2\mathcal{C}^{4}_{p}$, in order to have:

$$f^{\Theta}_{RSOS}(\theta_1 + i\pi, \theta_1) = 2\pi M^2 (e^{\frac{i\pi}{2p}}\ s_1 \otimes \bar{s}_2 + e^{-\frac{i\pi}{2p}}\ \bar{s}_1 \otimes s_2).$$

The knowledge of the form factors of the trace of the stress energy tensor allows to estimate the variation of the central charge by means of the so-called "c-theorem" sum rule [26, 27]:

$$c_{UV} = \frac{3}{2}\int_0^{\infty} dr\ r^3 < \Theta(r)\Theta(0) > . \tag{9}$$

Since in the massive case any correlation function can be represented by its spectral expansion

$$\begin{aligned} &< O(x)O(0) >= \\ &\sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{+\infty} \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n} |F_n(\theta_1, \ldots, \theta_n)|^2 e^{-Mr\sum_{j=1}^{n}\cosh\theta_j}, \end{aligned} \tag{10}$$

[7] The rationale behind this is explained in [17].

p	$c^{(2)}_{\text{num}}$	c_{exact}
3	0.5	0.5
3.12	0.5331	0.533234
4	0.6988	0.7
5	0.7972	0.8
10	0.9373	0.9454...
20	0.9744	0.9857...
100	0.9864	0.9994...

Table 1: Minimal models M_p

the computation of c turns out to be a non trivial check for the form factor $f^{\Theta}_{RSOS}(\theta_{12})$ in equation (8). In Table 1 above are presented the numerical results with two particles contribution for the central charge in the M_p model versus the theoretical result. As one may observe with the case $p = 3.12$, the parameter p can be taken continuous, as the observables depend continuously on it [8].
More generally [17], one can obtain the form factors of the primaries $\Phi_{1,k}$ using the identification $\Phi_{1,k} \sim e^{i\frac{(k-1)}{2}\beta\varphi_{SG}(x)}$, and then 'twisting' the Bethe ansatz state like in (7). In the two particle case, the form factors can be computed explicitly.

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This program (for $p_1, p_2 \geq 1$) was initiated for this model first by Smirnov [5] for form factors of the energy momentum tensor and the $U(1) \times U(1)$ current, then continued recently by Fateev and Lashkevich [18], who proposed expressions for the form factors of the exponential fields using the method of [22].
The form-factors $F(\theta_1, \cdots, \theta_{2n})$ of a local operator in the SS model satisfy the system of equations [5]:

$$\begin{aligned} &F(\theta_1, \cdots, \theta_i, \theta_{i+1}, \cdots, \theta_{2n}) S(\theta_i - \theta_{i+1}) = F(\theta_1, \cdots, \theta_{i+1}, \theta_i, \cdots, \theta_{2n}), \\ &F(\theta_1, \cdots, \theta_{2n-1}, \theta_{2n} + 2i\pi) = -F(\theta_{2n}, \theta_1, \cdots, \theta_{2n-1}), \end{aligned} \tag{11}$$

and the residue equation at $\theta_1 = \theta_{2n} + i\pi$:

$$\text{res} F(\theta_1, \cdots, \theta_{2n}) = -2i\, F(\theta_2, \cdots, \theta_{2n-1}) \left(1 - \prod_{i=2}^{2n-1} S_{p_1}(\theta_i - \theta_{2n}) S_{p_2}(\theta_i - \theta_{2n}) \right) e_0, \tag{12}$$

$e_0 = (s_1 \otimes \bar{s}_2 + \bar{s}_1 \otimes s_2) \otimes (s_1 \otimes \bar{s}_2 + \bar{s}_1 \otimes s_2)$.
The minimal form factor satisfies: $f_{ss}(\theta) = -f_{ss}(-\theta) a_{p_1}(\theta) a_{p_2}(\theta) = f_{ss}(2i\pi - \theta)$, and reads [5]:

$$f_{ss}(\theta_{12}) \equiv \frac{\cos \frac{\theta_{12}}{2i}}{\sin \frac{\theta_{12}}{2i}} f_{p_1}(\theta_{12}) f_{p_2}(\theta_{12}). \tag{13}$$

[8] We refer the reader to the Fig.5 of [28] for similar numerical tests on the central charge (the parameter p is taken continuous); in this article, the author constructed form factors with two kinks only, starting directly with the definition of the RSOS matrix. It is trivial to see that our correlation function $< \Theta\Theta >$ with two particles coincides with the one of [28].

It has no poles and no zeros in the physical strip $0 < \mathrm{Im}\,\theta_{12} < \pi$ and at most a simple zero at $\theta_{12} = 0$.
We make the following ansatz for the form factors in the SS model[9]:

$$F(\theta_1, \ldots, \theta_n) = \\ \mathcal{N}_n \prod_{i<j} f_{ss}(\theta_{ij}) \int_{C_\theta} du_1 \ldots \int_{C_\theta} du_s \; h_{p_1}(\theta, u) \Psi^{p_1}(\theta, u) \int_{C_\theta} dv_1 \ldots \int_{C_\theta} dv_t \; h_{p_2}(\theta, v) \Psi^{p_2}(\theta, v) \\ \times \mathcal{M}_n(\theta, u, v) \; p_n(\theta, u, v). \tag{14}$$

A few comments about our ansatz are in order:

- in the equation (12), the residue at the pole located at $\theta_1 = \theta_{2n} + i\pi$ is simple. As explained in [5], the introduction of the cosine term in the minimal form factor (13) gives a zero at $\theta_1 = \theta_{2n} + i\pi$, and consequently we should look for the residue at a second order pole of the integral representation proposed in equation (3).
- the properties for the p-function of a local operator are similar to those of the SG model presented in the previous section, though with a small modification at $\theta_1 = \theta_{2n} + i\pi$:

$$p_{2n}(\theta_1 \ldots \theta_{2n}, u_1 \ldots u_n, v_1 \ldots v_n) = -p_{2n-2}(\theta_2 \ldots \theta_{2n-1}, u_1 \ldots u_{n-1}, v_1 \ldots v_{n-1}) \\ + \tilde{p}^1(\theta_2 \ldots \theta_{2n-1}) \quad \text{at } u_n = v_n = \theta_{2n}, \\ p_{2n}(\theta_1 \ldots \theta_{2n}, u_1 \ldots u_n, v_1 \ldots v_n) = p_{2n-2}(\theta_2 \ldots \theta_{2n-1}, u_1 \ldots u_{n-1}, v_1 \ldots v_{n-1}) \\ + \tilde{p}^2_{\pm,\pm}(\theta_2 \ldots \theta_{2n-1}) \quad \text{at } u_n = \theta_{2n} \pm i\pi, v_n = \theta_{2n} \pm i\pi.$$

- clearly [5], the simple tensor product of the form factors of the SG model is not a solution as it spoils the residue equation. At $\theta_1 = \theta_{2n} + i\pi$, each of the $2n$ integration contours get pinched at $u_n = \theta_{2n}, \theta_{2n} \pm i\pi$ and $v_n = \theta_{2n}, \theta_{2n} \pm i\pi$ (the choice for the integration variables u_n and v_n is arbitrary because of symmetry). For this reason we introduced the function $\mathcal{M}_{2n}(\theta, u, v)$ which must have the properties at $\theta_1 = \theta_{2n} + i\pi$:
  - $\mathcal{M}_{2n}(\theta_1 \ldots \theta_{2n}, u_1 \ldots u_n, v_1 \ldots v_n) = \mathcal{M}_{2n-2}(\theta_2 \ldots \theta_{2n-1}, u_1 \ldots u_{n-1}, v_1 \ldots v_{n-1})$ when $u_n = v_n = \theta_{2n}$ or $u_n = \theta_{2n} \pm i\pi$ and $v_n = \theta_{2n} \pm i\pi$.
  - $\mathcal{M}_{2n}(\theta_1 \ldots \theta_{2n}, u_1 \ldots u_n, v_1 \ldots v_n) = 0$ when $u_n = \theta_{2n}$ and $v_n = \theta_{2n} \pm i\pi$, or when $v_n = \theta_{2n}$ and $u_n = \theta_{2n} \pm i\pi$.

We introduce the set $S = (1, \ldots, 2n)$ as well as $T \subset S$ and $\bar{T} \equiv S \backslash T$. A new result of this article is the following expression[10]:

$$\mathcal{M}_{2n}(\theta, u, v) = \frac{4}{\sum_{i=1}^{2n} e^{-\theta_i} \sum_{i=1}^{2n} e^{\theta_i}} \sum_{\substack{T \subset S, \\ \# T = n-1}} \frac{\prod_{\substack{k,l \in \bar{T} \\ k<l}} \cos^2 \frac{\theta_{kl}}{2i}}{\prod_{\substack{i \in T, \\ k \in \bar{T}}} \sin^2 \frac{\theta_{ki}}{2i}} \\ \times \frac{\prod_{\substack{i \in T, \\ r=1,\ldots,n}} \cos \frac{\theta_i - u_r}{2i}}{\prod_{1 \le r < s \le n} \cos \frac{u_r - u_s}{2i}} \frac{\prod_{\substack{i \in T, \\ r=1,\ldots,n}} \cos \frac{\theta_i - v_r}{2i}}{\prod_{1 \le r < s \le n} \cos \frac{v_r - v_s}{2i}}. \tag{15}$$

[9] The numbers s and t of integration variables u and v are related to the topological charges Q_1 and Q_2 as well as to the number of particles n through the formula $Q_1 = n - 2s$ and $Q_2 = n - 2t$.

[10] The notation '#' stands for 'number of elements'.

For two particles, this function is equal to $\mathcal{M}_2(\theta_1,\theta_2,u,v)=1$.
The function above is not the only one to have the required properties, but we conjecture that this is the one we need.

Trace of the stress energy tensor. We propose the following p-function:

$$p_{2n}^{\Theta}(\theta,u,v)=p_{SG}^{\Theta}(\theta,u)\otimes\left(p_{SG}^{\frac{1}{2}}(\theta,v)+p_{SG}^{-\frac{1}{2}}(\theta,v)\right)+\left(p_{SG}^{\frac{1}{2}}(\theta,u)+p_{SG}^{-\frac{1}{2}}(\theta,u)\right)\otimes p_{SG}^{\Theta}(\theta,v),$$

with the recursion relation for the normalization constant:

$$\mathcal{N}_{2n}^{\Theta}=-i\mathcal{N}_{2n-2}^{\Theta}\frac{\left(f_{p_1}^{min}(0)f_{p_2}^{min}(0)\right)^2}{8\pi n^2}.$$

The p_{SG}^{Θ}-function is given indifferently by eq. (4) or eq. (6). In the two particles case, the form factor boils down to the sum of products of two form factors of the SG model, and reads explicitly:

$$F^{\Theta}(\theta_1,\theta_2)=\frac{2i\pi M^2}{p_1p_2}\cosh\frac{\theta_{12}}{2}\ f_{ss}(\theta_{12})\ \left(\frac{(s_1\otimes\bar{s}_2+\bar{s}_1\otimes s_2)\otimes(s_1\otimes\bar{s}_2+\bar{s}_1\otimes s_2)}{\sinh\frac{1}{2p_1}(i\pi-\theta_{12})\sinh\frac{1}{2p_2}(i\pi-\theta_{12})}\right). \qquad (16)$$

It is not difficult to check that this expression satisfies equations (11); this result agrees with [5]. The constant $\mathcal{N}_2^{\Theta}$ was chosen equal to $\mathcal{N}_2^{\Theta}=\frac{\left(\mathcal{C}_{p_1}\mathcal{C}_{p_2}\right)^4M^2}{p_1p_2\pi}$, such that the following relation holds: $F^{\Theta}(\theta_1+i\pi,\theta_1)=2\pi M^2\ e_0$, where M is the mass of the fundamental particles.
For $p_1=p_2=1$, $p_3=0$, we have two sine-Gordon models at the free fermion point; the expression of equation (16) becomes as expected:

$$F^{\Theta}(\theta_1,\theta_2)=-2\pi M^2\sinh\frac{\theta_{12}}{2}\ e_0.$$

A few numerical checks on the value of the central charge at the level of two particles contributions can be found in Table 2. Compared with the results of Table 1, one sees that the accuracy is rather poor. We have so far no compelling explanation for this phenomenon. It worth mentioning that in [29], it was already noticed that the two-particle approximation to the correlation function does not always give accurate results for the numerical estimation of the central charge, in contradiction with the common belief.

Exponential fields. We introduce a non locality factor $e^{2i\pi(\pm\frac{a_1}{\alpha_1}\pm\frac{a_2}{\alpha_2})}$ in front of the product of the S-matrices in the residue equation. We propose the p-function:

$$\begin{aligned}p_{2n}(\theta,u,v)=&\ p_{SG}^{\frac{a_1}{\alpha_1}}(\theta,u)\otimes p_{SG}^{\frac{a_2}{\alpha_2}+\frac{1}{2}}(\theta,v)+p_{SG}^{\frac{a_1}{\alpha_1}}(\theta,u)\otimes p_{SG}^{\frac{a_2}{\alpha_2}-\frac{1}{2}}(\theta,v)\\&+p_{SG}^{\frac{a_1}{\alpha_1}+\frac{1}{2}}(\theta,u)\otimes p_{SG}^{\frac{a_2}{\alpha_2}}(\theta,v)+p_{SG}^{\frac{a_1}{\alpha_1}-\frac{1}{2}}(\theta,u)\otimes p_{SG}^{\frac{a_2}{\alpha_2}}(\theta,v).\end{aligned} \qquad (17)$$

One notices that the following sum of two particle form factors:

$$F^{\frac{\alpha_1}{2},\frac{\alpha_2}{2}}(\theta_1,\theta_2)+F^{-\frac{\alpha_1}{2},-\frac{\alpha_2}{2}}(\theta_1,\theta_2)+F^{-\frac{\alpha_1}{2},\frac{\alpha_2}{2}}(\theta_1,\theta_2)+F^{\frac{\alpha_1}{2},-\frac{\alpha_2}{2}}(\theta_1,\theta_2)$$

	p_1	$c_{\rm num}^{(2)}$	$c_{\rm exact}$
$p_2=1$	1	2	2
	2	1.9848	3
$p_2=2$	2	2.1586	3
	3	2.196	
	5.2	2.213	
	15	2.217	
	150	2.217	
$p_2=3$	3	2.255	3
	10	2.29	
$p_2=5.2$	5.3	2.3	3
$p_2=10$	10	2.35	3
$p_2=15$	15	2.36	3

Table 2: SS model

has for p-function[11]

$$(p_{SG}^{1}+p_{SG}^{-1})\otimes(p_{SG}^{\frac{1}{2}}+p_{SG}^{-\frac{1}{2}})+(p_{SG}^{\frac{1}{2}}+p_{SG}^{-\frac{1}{2}})\otimes(p_{SG}^{1}+p_{SG}^{-1}).$$

Then, using the expression (6), we find

$$F^{\Theta}(\theta_1,\theta_2)= \tag{18}$$
$$\frac{2i\pi^2\tilde{N}_2^{\Theta}}{(\mathcal{C}_{p_1}\mathcal{C}_{p_2})^4}\left(\mathrm{cotan}\frac{\pi p_1}{2}+\mathrm{cotan}\frac{\pi p_2}{2}\right)\cosh\frac{\theta_{12}}{2}\, f_{ss}(\theta_{12})\,\left(\frac{(s_1\otimes\bar{s}_2+\bar{s}_1\otimes s_2)\otimes(s_1\otimes\bar{s}_2+\bar{s}_1\otimes s_2)}{\sinh\frac{1}{2p_1}(i\pi-\theta_{12})\sinh\frac{1}{2p_2}(i\pi-\theta_{12})}\right).$$

This exactly reproduces the expression for the trace operator eq. (16) above, with the new normalization constant

$$\tilde{N}_2^{\Theta}=\frac{M^2(\mathcal{C}_{p_1}\mathcal{C}_{p_2})^4}{\pi\; p_1p_2}\frac{\sin\frac{\pi p_1}{2}\sin\frac{\pi p_2}{2}}{\sin\pi\frac{(p_1+p_2)}{2}}=-<\Theta>\frac{(\mathcal{C}_{p_1}\mathcal{C}_{p_2})^4}{\pi^2\; p_1p_2}.$$

The value for the vacuum energy

$$<\Theta>=-\pi M^2\frac{\sin\frac{\pi p_1}{2}\sin\frac{\pi p_2}{2}}{\sin\pi\frac{(p_1+p_2)}{2}}, \tag{19}$$

was first obtained by Bethe ansatz in [4].
From the computation made above, one may conclude that the exponential fields $\mathcal{O}_{a_1,a_2,\pm\frac{\alpha_3}{2}}=e^{i(2a_1\varphi_1+2a_2\varphi_2\pm\alpha_3\varphi_3)}$ are associated to the p-function (18), as the expression of the trace operator is [4]:

$$\Theta\sim e^{i\alpha_3\varphi_3}\cos(\alpha_1\varphi_1+\alpha_2\varphi_2)+e^{-i\alpha_3\varphi_3}\cos(\alpha_1\varphi_1-\alpha_2\varphi_2).$$

[11] We used the fact (in particular here for two particles) that the form factors in the SG model with the p-function p_{SG}^{0} -which corresponds to the identity operator- are equal to 0.

Let us consider the case $p_1 = p_2 = 1$ and $p_3 = 0$, where we have two SG models at the free fermion point. Using the results of [22], the expression for form factors associated to this p-function gives the expected result:

$$
\begin{aligned}
F^{a_1,a_2}_{+-,+-}(\theta_1,\theta_2) &= \frac{\cos\frac{\theta_{12}}{2i}}{\sin\frac{\theta_{12}}{2i}}\,\frac{1}{\cos^2\frac{\theta_{12}}{2i}}\times \\
& e^{\left(\sqrt{2}(a_1+a_2)+\frac{1}{2}\right)\theta_{21}}\sin\sqrt{2}\pi(a_1+a_2) - e^{\left(\sqrt{2}(a_1+a_2)-\frac{1}{2}\right)\theta_{21}}\sin\sqrt{2}\pi(a_1+a_2) \\
&= \frac{2i}{\cos\frac{\theta_{12}}{2i}}\sin\sqrt{2}\pi\,(a_1+a_2)\;e^{\sqrt{2}(a_1+a_2)\theta_{21}}.
\end{aligned}
$$

For more than two particles, there should be some significant simplifications in our general formula.

Fateev and Lashkevich constructed in [18] with a very different method the form factors for the most general exponential fields of the type $e^{2ia_1\varphi_1+2ia_2\varphi_2+i(2a_3\pm\alpha_3)\varphi_3}$, which supposes that we introduce in the p-function a dependence with respect to an extra parameter a_3, that does not affect the non locality factor. For the time being, I do not see how to do this for a_3 arbitrary. The first proposal that one could imagine for the exponential fields $e^{2i(a_1\varphi_1+a_2\varphi_2\pm\frac{k}{2}\alpha_3\varphi_3)}$, where k is an odd integer, is the following p-function:

$$
\begin{aligned}
p_{2n}(\theta,u,v) = {}& p_{SG}^{\frac{a_1}{\alpha_1}}(\theta,u)\otimes p_{SG}^{\frac{a_2}{\alpha_2}+\frac{k}{2}}(\theta,v) + p_{SG}^{\frac{a_1}{\alpha_1}}(\theta,u)\otimes p_{SG}^{\frac{a_2}{\alpha_2}-\frac{k}{2}}(\theta,v) \\
& + p_{SG}^{\frac{a_1}{\alpha_1}+\frac{k}{2}}(\theta,u)\otimes p_{SG}^{\frac{a_2}{\alpha_2}}(\theta,v) + p_{SG}^{\frac{a_1}{\alpha_1}-\frac{k}{2}}(\theta,u)\otimes p_{SG}^{\frac{a_2}{\alpha_2}}(\theta,v)
\end{aligned}
\tag{20}
$$

However if one considers the free fermion point, one does not recover the expected result (unless $k=1$), so this proposal should be rejected.

It is unclear to me what the form factors of exponential fields are for $a_3 \neq \pm 2\alpha_3$. The results of [18] indicate it should be possible to construct them, but I am not capable of making any comparisons with the results of this paper: in particular the authors of [18] have not succeeded in their article into making explicit evaluations of their two-particles integrals for the simpler cases $a_i = \frac{k_i}{2}\alpha_i$, so even their two particle form factor for the trace operator is not explicitly computed.

One may think of introducing a dependence w.r.t a_3 by considering linear combination of functions of the type (15) where we would change the number of elements of T. However, such a function (15) is very close to a similar function obtained in closely related mathematical problems though different physical contexts in [30,31]; in these articles only the function with $\#T = n-1$ was considered, and it seems to me that this is what should be done also here[12].

RSOS restrictions.

- For integer values of p_2 and $p_2 \geq 3$, the QFT admits a quantum group restriction with respect to the symmetry group $U_{q_2}(sl_2)$, $q_2 = \exp(\frac{2i\pi}{p_2})$, giving the parafermionic sine-

[12] It is not even clear to the author whether the form factors constructed above really correspond to the exponential fields of the UV CFT $e^{2ia_1\varphi_1+2ia_2\varphi_2+\pm i\alpha_3\varphi_3}$ -although we have little doubt about the trace operator- and are not only a formal mathematical solution of the equations. In other words, the author has no clear understanding of the correspondence between space of states in the CFT and space of states in the corresponding integrable perturbed model, beside the sine-Gordon case.

Gordon models [19] with action:

$$\mathcal{A}_{p_1,p_2} = \mathcal{A}^{(0)}_{p_2} + \int d^2x \left[\frac{1}{16\pi}(\partial_\mu \varphi)^2 - \kappa \left(\psi\bar{\psi} e^{i\rho\varphi} + \psi^\dagger \bar{\psi}^\dagger e^{-i\rho\varphi} \right) \right], \tag{21}$$

where $\mathcal{A}^{(0)}_{p_2}$ is the action of the $\mathbb{Z}_{p_2-2}$ parafermionic CFT with central charge $c = 2 - 6/p_2$, and the fields ψ, $\psi^\dagger$, $(\bar{\psi}, \bar{\psi}^\dagger)$ are the holomorphic (antiholomorphic) parafermionic currents with spin $\Delta = 1 - \frac{1}{p_2-2}$ $(\Delta = -\bar{\Delta})$. The field φ is a scalar boson field and the parameter ρ is given by

$$\rho^2 = \frac{8\pi\ p_1}{(p_2-2)(p_1+p_2-2)}.$$

We denote the QFT (21) by $\mathcal{P}(p_1, p_2)$. The perturbing operator has conformal dimension $\Delta_{pert} = \frac{\rho^2}{8\pi} + 1 - \frac{1}{p_2-2}$. In addition to the conformal symmetry, the $\mathbb{Z}_{p_2-2}$ parafermionic CFT possesses an additional symmetry generated by the parafermionic currents ψ, $(\bar{\psi})$. Basic fields in this CFT are the order parameters σ_j, $j = 0, 1, \ldots, p_2 - 3$ with conformal dimension:

$$\delta_j = \frac{j(p_2-j-2)}{2p_2(p_2-2)}.$$

All other fields in this CFT can be obtained from the fields σ_j by application of the generators of the parafermionic symmetry [34]. We introduce as basic operators in $\mathcal{P}(p_1, p_2)$ the local fields:

$$\Phi^{(j)}_a = \sigma_j \exp(ia\varphi). \tag{22}$$

The vacuum structure, spectrum and scattering theory of the QFT (21) are described in [19]. Its S-matrix is

$$-S^{SG}_{p_1}(\theta_{12}) \otimes S^{RSOS}_{p_2}(\theta_{12}), \quad p_1 \geq 1.$$

For $p_2 = 3$, $\psi \equiv 1$ and the QFT $\mathcal{P}(p_1, 3)$ is the usual Sine-Gordon model with coupling constant $\rho^2 = \beta^2 = \frac{8\pi\ p_1}{1+p_1}$.
For $p_2 = 4$, the parafermionic current $\psi(z)$ is a Majorana fermion and the theory $\mathcal{P}(p_1, 4)$ describes the N=1 supersymmetric sine-Gordon model with a coupling constant $\rho^2 = \frac{8\pi\ p_1}{2(2+p_1)}$.
In order to obtain form factors of the trace of the energy momentum tensor in this family of models directly from the formula for form factors of the trace in the SS-model, we use the RSOS procedure inspired from the SG case, and we propose as p-function of the operator $\psi\bar{\psi}e^{i\rho\varphi} + \psi^\dagger\bar{\psi}^\dagger e^{-i\rho\varphi}$:

$$p^{\Theta}_{2n}(\theta, u, v) = p^{\Theta}_{SG}(\theta, u) \otimes p^{\frac{1}{2}}_{SG}(\theta, v),$$

with the modification of the Bethe ansatz state $\Psi^{p_2}_{\epsilon_1\epsilon_2\ldots\epsilon_{2n}}$ like in equation (7). The two-particles form factor for the trace operator reads explicitly:

$$\begin{aligned} F^{\Theta}(\theta_1, \theta_2) &= \frac{4i\pi M^2}{p_1 p_2} \cosh\frac{\theta_{12}}{2}\ f_{ss}(\theta_{12}) \\ &\times \left(\frac{(s_1 \otimes \bar{s}_2 + \bar{s}_1 \otimes s_2) \otimes (e^{\frac{i\pi}{2p_2}}\ s_1 \otimes \bar{s}_2 + e^{-\frac{i\pi}{2p_2}}\ \bar{s}_1 \otimes s_2)}{\sinh\frac{1}{2p_1}(i\pi - \theta_{12}) \sinh\frac{1}{p_2}(i\pi - \theta_{12})} \right). \end{aligned} \tag{23}$$

It is normalized such that

$$F^{\Theta}(\theta_1 + i\pi, \theta_1) = 2\pi M^2\ (s_1 \otimes \bar{s}_2 + \bar{s}_1 \otimes s_2) \otimes (e^{\frac{i\pi}{2p_2}}\ s_1 \otimes \bar{s}_2 + e^{-\frac{i\pi}{2p_2}}\ \bar{s}_1 \otimes s_2).$$

The expression (23) for the two particles form factor of the trace of the energy momentum tensor correctly reproduces in a more or less accurate way the known value for the UV central charge $c = 3 - 6/p_2$: numerical checks are presented in Table 3. One sees that increasing p_2 with p_1 fixed, the conformal dimension of the perturbing operator gets closer and closer to one, so one may expect a badly convergent integral (9), which would be responsible for the observed lack of precision in the numerical tests. For p_2 fixed and increasing p_1, the conformal dimension again gets close to one: we do not have any explanation for the fact that for $p_2 \neq 3$, the accuracy becomes better for p_1 bigger.

Again, if we use the expression (6) for p^{Θ}_{SG}, we find the following value for the normalization constant $\tilde{N}^{\Theta}_2$:

$$\tilde{N}^{\Theta}_2 = \frac{M^2(\mathcal{C}_{p_1}\mathcal{C}_{p_2})^4}{\pi p_1 p_2} \tan\frac{\pi p_1}{2}.$$

It is surprising to note that had we identified the normalization of the trace operator as we previously did:

$$\tilde{N}^{\Theta}_2 = - < \Theta > \frac{(\mathcal{C}_{p_1}\mathcal{C}_{p_2})^4}{\pi^2 p_1 p_2},$$

this would lead to the following prediction for $< \Theta >$:

$$< \Theta > = -\pi M^2 \tan\frac{\pi p_1}{2},$$

which is certainly true for $p_2 = 3, 5, 7, \ldots$, but not for $p_2 = 4, 6, 8, \ldots$, where in the latter case the v.e.v of $< \Theta >$ should be vanishing, according to formula (19)! Consequently, in this case, the normalization of the two particles form factor of the trace operator does not reproduce the correct answer for the vacuum expectation value of the trace operator.

Although it might seem *a priori* most natural to propose for p-function for the operators $\psi\bar{\psi}\exp ia\varphi(x)$[13] (and $\psi^\dagger\bar{\psi}^\dagger \exp ia\varphi(x)$) :

$$p_{2n}(\theta, u, v) = p^{\frac{a}{\rho}}_{SG}(\theta, u) \otimes p^{\frac{1}{2}}_{SG}(\theta, v),$$

with the modified Bethe ansatz state $\tilde{\Psi}^{p_2}_{\epsilon_1\epsilon_2\ldots\epsilon_{2n}}$ as defined as in equation (7), we are not quite certain that this identification is correct: the problem not being a technical one, but a problem of *existence* of such operators[14] .

[13] We certainly expect this to be true for $p = 3$, in which case $\psi, \bar{\psi} \equiv 1$, and we should recover the form factors for the exponential fields in the sine-Gordon model.

[14] The author has already experienced a similar problem within the context of the massless N=1 super sinh-Gordon model [33], where she came to the conclusion that it might be meaningless to consider form factors of $e^{\alpha\varphi}, \sigma e^{\alpha\varphi}$ for α generic.

	p_1	$c^{(2)}_{\mathrm{num}}$	c_{exact}
$p_2 = 3$	1	1	1
	2	0.9924	
	3.6	0.9881	
	4	0.9878	
	5.2	0.9873	
	20	0.9869	
$p_2 = 4$	1	1.3976	1.5
	2	1.4404	
	3.6	1.4469	
	4	1.4473	
	5.2	1.4477	
	20	1.4480	
$p_2 = 5$	1	1.5945	1.8
	2	1.6819	
	3.6	1.7016	
	4	1.7029	
	5.2	1.7048	
	20	1.706	
$p_2 = 20$	1	1.9488	2.7
	2	2.17	
	3.6	2.26	
	4	2.27	
	5.2	2.29	
	20	2.3	

Table 3: Parafermionic Sine-Gordon model

p_2	$c^{(2)}_{\mathrm{num}}$	c_{exact}
3	0.9924	1
3.5	1.2542	1.2857...
4	1.4404	1.5
7.3	1.9422	2.1780...
10	2.063	2.4
50.3	2.211	2.8807...
$+\infty$	2.217	3

Table 4: N=2 supersymmetric restricted Sine-Gordon model

Let us mention that another interesting case is obtained for $p_1 = 2$ and p_2 arbitrary integer. The QFT $\mathcal{P}(2, p_2)$ possesses N=2 supersymmetry and is known as the N=2 supersymmetric restricted sine-Gordon model. The UV central charge is the same as before. The numerical checks on the central charge can be found in Table 4. We have found interesting to present in the table above three cases where p_2 is not an integer: as one may expect, the observables depend continuously on the parameter p_2.

- For integer values of p_1, the previous QFT $\mathcal{P}(p_1, p_2)$ admits an additional restriction with respect to the second quantum group $U_{q_1}(sl_2)$, $q_1 = \exp(\frac{2i\pi}{p_1})$. The resulting QFT [19], which corresponds to integrable perturbations with an operator Φ_C of conformal dimension $\Delta = 1 - \frac{2}{p_1+p_2-2}$ of the coset model [20] $su(2)_{p_1-2} \otimes su(2)_{p_2-2}/su(2)_{p_1+p_2-4}$, is denoted by $\mathfrak{C}(p_1, p_2)$. The S-matrix is [19, 35]

$$-S_{p_1}^{RSOS}(\theta_{12}) \otimes S_{p_2}^{RSOS}(\theta_{12}),$$

with p_1, p_2 integers, $p_1, p_2 \geq 3$.
For $p_1 = 3$, the QFT $\mathfrak{C}(3, p_2)$ describes minimal CFT models perturbed by the operator $\Phi_{1,3}$ [25].
For $p_1 = 4$, the QFT $\mathfrak{C}(4, p_2)$ describes N=1 supersymmetric minimal CFT models perturbed by the supersymmetry preserving operator Φ_C. In particular, the model $\mathfrak{C}(4, 4)$ possesses N=2 supersymmetry. The corresponding CFT has central charge $c = 1$. It describes the special point at the critical line of the Ashkin-Teller ($\mathbb{Z}_4$) model. At the N=2 supersymmetric point the thermal operator ϵ coincides with the operator Φ_C.
The basic fields in the coset CFT can be represented in terms of fields σ_j from the $\mathbb{Z}_{p_2-2}$ CFT and a scalar bosonic field φ as [36]:

$$\Phi_{l,k}^{(j)} = \sigma_j e^{i a_{lk} \varphi}, \tag{24}$$

where:

$$2a_{lk} = (k-1)\rho + \frac{(1-l)}{(p_2-2)\rho}, \quad \rho^2 = \frac{8\pi\, p_1}{(p_2-2)(p_1+p_2-2)},$$

and the integers $k < p_1$ and $l < p_1 + p_2 - 2$ satisfy the condition

$$|l - k - n(p_2 - 2)| = j; \quad n \in \mathbb{Z}_{p_2-2}.$$

We propose as p-function of the trace of the stress energy tensor:

$$p_{2n}^{\Theta}(\theta, u, v) = -\left(\sum_{i=1}^{2n} e^{-\theta_i} \sum_{i=1}^{n} e^{u_i}\right) \otimes p_{SG}^{\frac{1}{2}}(\theta, v), \tag{25}$$

then both Bethe ansatz states Ψ^{p_1,p_2} should be modified as in equation (7). Explicitly for two particles:

$$F^{\Theta}(\theta_1, \theta_2) = \frac{8i\pi M^2}{p_1 p_2} \cosh\frac{\theta_{12}}{2}\, f_{ss}(\theta_{12})$$
$$\times \left(\frac{(e^{\frac{i\pi}{2p_1}}\, s_1 \otimes \bar{s}_2 + e^{-\frac{i\pi}{2p_1}}\, \bar{s}_1 \otimes s_2) \otimes (e^{\frac{i\pi}{2p_2}}\, s_1 \otimes \bar{s}_2 + e^{-\frac{i\pi}{2p_2}}\, \bar{s}_1 \otimes s_2)}{\sinh\frac{1}{p_1}(i\pi - \theta_{12}) \sinh\frac{1}{p_2}(i\pi - \theta_{12})}\right).$$

	p_2	$c^{(2)}_{\rm num}$	$c_{\rm exact}$
$p_1=3$	3	0.5	0.5
	4	0.6988	0.7
	5	0.7972	0.8
	20	0.9744	0.9857...
$p_1=4$	4	0.9924	1
	5	1.1429	1.1571...
	20	1.4268	1.4727...
$p_1=5$	5	1.3240	1.35
$p_1=20$	20	2.2	2.5578...

Table 5: Coset model $su(2)_{p_1-2} \otimes su(2)_{p_2-2}/su(2)_{p_1+p_2-4}$

It is normalized such that

$$F^{\Theta}(\theta_1+i\pi,\theta_1) = 2\pi M^2\ (e^{\frac{i\pi}{2p_1}}\ s_1\otimes\bar{s}_2 + e^{-\frac{i\pi}{2p_1}}\ \bar{s}_1\otimes s_2)\otimes(e^{\frac{i\pi}{2p_2}}\ s_1\otimes\bar{s}_2 + e^{-\frac{i\pi}{2p_2}}\ \bar{s}_1\otimes s_2).$$

Let us note that we certainly expect for symmetry reasons the p-function

$$p^{\Theta}_{2n}(\theta,u,v) = -p^{\frac{1}{2}}_{SG}(\theta,u)\otimes\left(\sum_{i=1}^{2n} e^{-\theta_i}\sum_{i=1}^{n} e^{v_i}\right) \tag{26}$$

to lead to the same form factors as the p-function in eq. (25). Also we would expect the p-functions

$$p^{\Theta}_{2n}(\theta,u,v) = p^{\frac{1}{2}}_{SG}(\theta,u)\otimes p^{1}_{SG}(\theta,v), \tag{27}$$

$$p^{\Theta}_{2n}(\theta,u,v) = p^{1}_{SG}(\theta,u)\otimes p^{\frac{1}{2}}_{SG}(\theta,v), \tag{28}$$

to give the same result for the form factors.

The numerical checks provide results for the central charge in Table 5, to be compared with the exact result $c = 3-6(p_1^{-1}+p_2^{-1}-(p_1+p_2-2)^{-1})$. For $p_1=3$, they are identical to those of Table 1. Again, increasing p_1,p_2 make the conformal dimension of the perturbing operator closer and closer to one, leading to a badly converging integral (9), which may be responsible for the decrease of accuracy of the numerical results. Let us note again that the normalization constants $\tilde{N}^{\Theta}_2$ do not reproduce the correct v.e.v for the trace: indeed, with the p-function (27), we would obtain

$$<\Theta> = -\pi M^2\tan\frac{\pi p_2}{2},$$

and with the p-function (28)

$$<\Theta> = -\pi M^2\tan\frac{\pi p_1}{2},$$

whereas the correct value for $<\Theta>$ is given by (19).

$k+2$	$c_{\text{num}}^{(2)}$	c_{exact}
3	0.9869	1
4	1.4480	1.5
5	1.706	1.8
10	2.1	2.4
100	2.3	2.94
$+\infty$	2.3	3

Table 6: $SU(2)_k$ WZNW model

Certainly, if $p_2 = 3$ we recover the form-factors of the operators $e^{ia_{1k}\varphi(x)}$ in the minimal model M_{p_1} :

$$p_{2n}(\theta, u, v) = p_{SG}^{\frac{a_{1k}}{\rho}}(\theta, u) \otimes p_{SG}^{\frac{1}{2}}(\theta, v),$$

with the modification of the two Bethe ansatz states Ψ^{p_1, p_2}.

- A common limiting case of these two RSOS restrictions is the Polyakov-Wiegmann model [8], which is the same as the perturbation of the $SU(2)$ WZNW model at level k with action

$$S = S_{\text{WZNW}_k} + \lambda \int d^2x J^a \bar{J}^a,$$

where J^a ($\bar{J}^a$) are the holomorphic (antiholomorphic) $SU(2)$ currents. The S-matrix of the model is given by:

$$-S_{+\infty}^{SG}(\theta_{12}) \otimes S_{k+2}^{RSOS}(\theta_{12}).$$

We get the Principal Chiral Field model [8, 9] in the extreme limit $k = +\infty$. Whether one considers $S_{+\infty}^{RSOS}(\theta_{12})$ or $S_{+\infty}^{SG}(\theta_{12})$ amounts to the same result. The numerical estimations for the central charge with two particles contribution are given in Table 6.
The Polyakov-Wiegmann model is asymptotically free, and the trace operator is of conformal dimension one for any k; from the results above we deduce that the approximation with two particles leads to a conformal dimension of the trace operator which is less than one in the case of the $SU(2)$ Thirring model ($k = 1$), whereas when one increases k, it becomes closer and closer to one already at the level of two particles. Numerically, the value for the central charge is given by an integral that becomes logarithmically divergent when the conformal dimension of the perturbing operator is one. It is not clear why for small k the results are quite good, and poor for big k.

4 Conclusion

A few remarks should be made concerning the formula (15), which is one important result obtained in this paper.

- The function $\mathcal{M}$ is universal as it does not depend on the scattering of the theory, and appears in models where the S-matrix is given by a tensor product of different S-matrix. Typically, such a construction (with suitable modifications) is encountered in integrable massless flows [30, 33, 37], which contain three different types of scattering, between right/right movers, left/left movers, and right/left movers. It has also been used in a different physical context in [31].

- Other mathematical solutions are obtained by changing in formula (15) the number of elements of the set T. It is not clear to what they could correspond.
- Of course it is crucial to make a comparison with the results of [18].
- The so-called one to one correspondence between operators of the UV CFT and operators in the massive/massless theory is a issue which deserves, to the opinion of the author, better understanding.

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