SISSA/10/2005/FM

Mass Generation in Perturbed Massless Integrable Models

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Abstract

We extend form-factor perturbation theory to non–integrable deformations of massless integrable models, in order to address the problem of mass generation in such systems. With respect to the standard renormalisation group analysis this approach is more suitable for studying the particle content of the perturbed theory. Analogously to the massive case, interesting information can be obtained already at first order, such as the identification of the operators which create a mass gap and those which induce the confinement of the massless particles in the perturbed theory.

Introduction. Given the large number of remarkable results obtained from the study of two–dimensional integrable quantum field theories (IQFTs), at present one of the most interesting challenges consists of developing a systematic approach to study non-integrable models, at least when they are deformations of integrable ones. For massive field theories a convenient perturbative scheme, based on the exact knowledge of the form-factors (FFs) of the original integrable theory, was suggested in [1]. Already at first order, it proved able to provide a great deal of information, such as the evolution of the particle content, the variation of the masses and the change of the ground state energy – results successfully checked by numerical studies.

The main purpose of this paper is to extend Form Factor Perturbation Theory (FFPT) to non–integrable deformations of massless IQFTs. The most fundamental question that one may ask in this context is whether a perturbation creates a gap in the excitation spectrum – a problem usually addressed via the renormalisation group (RG) equations near a fixed point [2]. Moreover, if massive particles are created, one would like to understand whether they are adiabatically related to the original massless excitations or, like in the massive case, confinement takes place. Since the RG eq.s cannot provide a complete answer to any of the above questions, it is worth exploring other alternative routes. The FFPT relies directly on the particle description of the unperturbed theory and, for this reason, it seems to be the most natural and suitable approach for studying the evolution of the particle content when the perturbation is switched on.

Our analysis is presently limited to the first order of FFPT, its extension to higher orders being, as in the massive case, an interesting but non-trivial mathematical problem. Despite the fact that one must be careful in handling results at such low order, some useful conclusions can nevertheless be reached. For instance, it will be possible to discriminate between operators which do not spoil the massless nature of the theory and those which instead induce a mass gap in the spectrum. Moreover, the confinement of the original massless excitations can be traced back to the non–local properties of the perturbing operator with respect to them. These results provide the first information on the perturbed theory and may guide a further analysis of its properties. It should be stressed that answering the above questions in their full generality is, obviously, a fairly complicated problem since it concerns the global structure of the RG flows rather than their local properties around the fixed points. It is well known, for instance, that adding a relevant perturbation to a massless action does not necessarily imply that the resulting infrared theory will be massive: indeed the perturbing operator may induce a flow into a new critical point, with some of the massive excitations decoupled from the new massless ones [3, 4]. An example even more subtle is given by the roaming trajectories discovered by Al. Zamolodchikov [5] (and further analysed in [6, 7]), i.e. an infinite cascade of massless flows finally ending in a massive phase.

FFPT for Massive Field Theories. Consider non–integrable theories obtained as a deformation of an integrable action $\mathcal{A}_{int}$

$$\mathcal{A} \,=\, \mathcal{A}_{int} + \lambda \int d^2x\, \Psi(x) \ . \tag{1}$$

The exact knowledge of the FFs of the operator $\Psi(x)$ on the asymptotic states of the unperturbed theory allows one to set up an expansion of various physical quantities of the new theory in powers of λ, the so called FFPT [1]. We present initially some known results of FFPT for massive theories in a way that is more suitable for the extension to the massless case. It is useful to recall that in most of the cases of interest, the integrable action of a massive theory can be defined in terms of a deformation of a CFT [8], $\mathcal{A}_{int} \,=\, \mathcal{A}_{CFT} + g \int d^2x\, \Phi(x)$, where $\Phi(x)$ is a relevant scalar field of conformal weights $\Delta_\Phi = \overline{\Delta}_\Phi < 1$.

Let us first assume that the theory has only one massive particle in the spectrum, $A(\beta)$, where β parameterises the dispersion relation: $p_0 \,=\, m \cosh\beta$ and $p_1 \,=\, m \sinh\beta$. The integrability of the theory allows one to compute its exact factorised scattering amplitudes [8], $S(\beta_{12})$ $(\beta_{12} = \beta_1 - \beta_2)$, and the FFs [9] of the various operators $\mathcal{O}$ on the set of asymptotic states

$$F^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n) \,=\, \langle 0 \mid \mathcal{O}(0) \mid A(\beta_1)A(\beta_2)\ldots A(\beta_n)\rangle \ . \tag{2}$$

A convenient way to study the mass correction induced by the non–integrable deformation (1) is to employ the Hamiltonian formalism, in the same spirit of standard quantum mechanics perturbation theory. The Hamiltonian associated to (1) can be written as

$$H \,=\, \frac{1}{2\pi} \int dx^1\, T_{00}(x^1, 0) \,-\, \lambda \int dx^1\, \Psi(x^1, 0) \ , \tag{3}$$

where $T_{\mu\nu}(x)$ is the stress–energy tensor of the integrable theory, $\mathcal{A}_{int}$. The operator T_{00} can be expressed in terms of the its trace, $\Theta(x)$, using the conservation law $\partial^\mu T_{\mu\nu} = 0$,

$$\partial_1^2\, \Theta(x^1, x^0) \,=\, (\partial_1^2 - \partial_0^2)\, T_{00}(x^1, x^0) \ . \tag{4}$$

In particular, for the two–particle Form Factor we have

$$\langle 0 \mid T_{00}(x^1, x^0) \mid A(\beta_i)A(\beta_j)\rangle = -\sinh^2 \frac{\beta_i + \beta_j}{2}\, \langle 0 \mid \Theta(x^1, x^0) \mid A(\beta_i)A(\beta_j)\rangle \ . \tag{5}$$

The essential results of FFPT are easily re-derived within this formalism. Let us first consider the unperturbed integrable case, $\lambda = 0$. Evaluating the matrix element of both sides of Eq. (3) on the asymptotic states $\langle A(\beta_i) \mid$ and $\mid A(\beta_j)\rangle$, and using the relation (5), one obtains the usual normalisation condition for the trace of the stress-energy tensor of the massive integrable theory,

$$\langle A(\beta) \mid \Theta(0) \mid A(\beta) \rangle \,=\, F^{\Theta}(i\pi) \,=\, 2\pi m^2 \ , \tag{6}$$

an equation which shows the relationship between the FF of this operator and the mass scales of the theory.

Repeating the same procedure for the non–integrable theory (3), one obtains instead the first order correction to the mass of the particles, as given in [1],

$$\delta m^2 \,\simeq\, 2\,\lambda\, F^{\Psi}(i\pi) \ . \tag{7}$$

If the operator $\Psi(x)$ is non–local with respect to the particles $A(\beta)$, $F^{\Psi}(\beta)$ has a pole for $\beta = i\pi$ and (7) diverges. This divergent correction to their masses implies the confinement of the particles $A(\beta)$, that are no longer excitations of the action (1) [1]. This phenomenon appears for instance in the magnetic deformation of the low–temperature phase of the Ising model [1, 10, 11] as well as in two-frequency sine-Gordon model [12].

It should be noticed that, since this is a strong coupling analysis, i.e. carried out in the infrared region (IR), if λ in (1) scales under RG, it has to be replaced in (7) by its renormalised value at energy of the order of the mass of the theory

$$\lambda \rightarrow \lambda^{eff} \simeq \lambda(m^{-1}) \ . \tag{8}$$

As a consequence, unless the RG flow is known exactly, quantitative predictions can be made only on universal mass ratios.

If the theory has n non-degenerate particles, $A_a(\beta)$, with masses m_a ($a = 1, \ldots, n$; $m_a \neq m_{a'}$) the above analysis can be easily extended and gives the following mass variation

$$\delta m_a^2 \,\simeq\, 2\,\lambda\ F^{\Psi}_{\bar{a}a}(i\pi) \ . \tag{9}$$

When some of the particles, say n' ($n' < n$), have the same mass, this equation has to be generalised like in quantum mechanics perturbation theory for degenerate levels, i.e. the perturbed masses are obtained by diagonalising the matrix $\{M_{k,l}\} = \{F^{\Psi}_{k,l}(i\pi)\}$, where indices k, l belong to the degenerate multiplet. If the symmetry of the perturbing operator is less than the symmetry of the multiplet, the perturbation will typically split it.

Massless IQFTs. Massless non–scale invariant IQFTs are associated to RG flows between two different fixed points. With respect to their ultraviolet fixed point, such theories admit a well–defined description in terms of the corresponding CFT perturbed by a relevant operator. However, from the physical point of view of selecting the low–energy massless excitations, it is more appropriate to view them as irrelevant perturbation of their IR fixed point action

$$\mathcal{A}_{massless} = \mathcal{A}_{CFT}^{IR} + g \int d^2x\, \hat{\Phi}(x) + \cdots \quad , \tag{10}$$

where the irrelevant field $\hat{\Phi}(x)$ specifies the approaching direction to the CFT of the IR fixed point. The scattering theory of massless IQFTs is discussed in detail in [4] whereas their Form Factors in [13], so we shall outline only some basic facts below. The excitations of these theories consist of right (R) and left (L) moving particles. They are defined as $p_1 \geq 0$ and $p_1 \leq 0$ branches of the relativistic dispersion relation $p_0 = |p_1|$, which can be parameterised as $p_0 = p_1 = (M/2)\,\mathrm{e}^{\beta}$ for the R movers, $A_R(\beta)$, and $p_0 = -p_1 = (M/2)\,\mathrm{e}^{-\beta}$ for the L movers, $A_L(\beta)$, where M is a mass scale. Within this parameterisation, the Maldenstam variable for the RL scattering process is given by: $s_{RL}(\beta_{ij}) = M^2\,\mathrm{e}^{\beta_{ij}}$. Contrary to the massive case, where the threshold of the scattering process is given by $\beta_{ij} = 0$, for the RL sector of the massless scattering the threshold is reached in the limit $\beta_{ij} \to -\infty$. In the RR and LL sectors the Mandelstam variable is always zero, showing that all analyticity arguments of the S–matrix theory cannot be applied: the scattering amplitudes in these channels can be properly defined only as analytic continuation of the massive case [4]. For this purpose, in fact, it is useful to regard the massless excitations as a particular limit of the massive particles[1].

Writing the S–matrix in a compact form $A_{\alpha_1}(\beta_1)A_{\alpha_2}(\beta_2) = S_{\alpha_1,\alpha_2}(\beta_{12})\,A_{\alpha_2}(\beta_2)A_{\alpha_1}(\beta_1)$, ($\alpha_i = R, L$) the equations satisfied by the FFs can be written in analogy to the massive case [13]. For the two–particle matrix element $F^{\mathcal{O}}_{\alpha_1,\alpha_2}(\beta_{12}) = \langle 0|\mathcal{O}(0)|A_{\alpha_1}(\beta_1), A_{\alpha_2}(\beta_2)\rangle$, we have for instance

$$\begin{aligned} F^{\mathcal{O}}_{\alpha_1,\alpha_2}(\beta) &= S_{\alpha_1,\alpha_2}(\beta)\,F^{\mathcal{O}}_{\alpha_2,\alpha_1}(-\beta) \ ; \\ F^{\mathcal{O}}_{\alpha_1,\alpha_2}(\beta + 2\pi i) &= \mathrm{e}^{-2i\pi\gamma_{\mathcal{O}}}\,F^{\mathcal{O}}_{\alpha_2,\alpha_1}(-\beta) \ , \end{aligned} \tag{11}$$

where $\gamma_{\mathcal{O}}$ is the non-locality index of the operator $\mathcal{O}$ with respect to the massless particles. Their analytic structure, however, differs from the massive case. In massive theories, the multi-particle FFs are meromorphic functions in the strip $0 \leq \mathrm{Im}\,\beta < 2\pi$ and present simple pole singularities associated either to bound states or to particle-antiparticle annihilation processes. In massless theories the same kinds of singularities are expected in the RR and LL sectors, since they formally behave like the massive cases. In the RL and

[1] For instance, if $A(\beta)$ is a massive excitation of mass m with a S–matrix equal to $S(\beta)$, the massless limit is constructed by shifting the rapidities $\beta \to \beta_{R,L} \pm \beta_0/2$ and taking the double limits $\beta_0 \to \infty$ and $m \to 0$ while $M = m e^{\beta_0}$ is kept fixed: $A_{R,L}(\beta) = \lim_{\beta_0\to\infty} A(\beta \pm \beta_0/2)$. When one considers the S-matrix in the RR and LL sectors, the rapidity shifts cancel and therefore $S_{RR}(\beta) = S_{LL}(\beta) = S(\beta)$. As functions of the rapidity variable, these amplitudes are then expected to satisfy the same equations valid for the massive case.

LR sectors instead, bound state poles are absent while kinematic poles may appear only if both particles have vanishing momentum. However, instead of producing a recursive equation like in the massive case, here the presence of kinematic poles imposes a condition on the asymptotic behaviour of the FF's. In particular, like in the massive case, a pole is present in the two-particle FF only if the operator is non-local

$$\lim_{\beta_{RL}\to-\infty} F_{RL}^{\mathcal{O}}(i\pi+\beta_{RL}) = \infty \quad \text{for } \mathcal{O} \text{ non} - \text{local} \tag{12}$$

(an analogous equation can be written for the FF in the LR sector.)

FFPT for Massless Field Theories. Suppose now that an operator $\hat{\Psi}(x)$ of the infrared CFT is added to the effective action (10), so that its integrability is broken

$$\mathcal{A} = \mathcal{A}_{massless} + \lambda \int d^2x\, \hat{\Psi}(x) \ . \tag{13}$$

Repeating initially the analysis of the previous section for the unperturbed case $\lambda = 0$, one finds[2] the following normalisation conditions for the trace of the stress-energy tensor

$$F_{RR}^{\Theta}(i\pi) \;=\; F_{LL}^{\Theta}(i\pi) = 2\pi \ ; \tag{14}$$

$$F_{RL}^{\Theta}(\beta) \;=\; 0 \ . \tag{15}$$

The last equation can be viewed as an essential property of a massless integrable theory, i.e. a non-trivial generalisation to massless non-scale invariant theories of properties of CFTs.

Consider now the case when λ is non-zero. If the perturbing operator $\hat{\Psi}(x)$ has vanishing FF on the RL (LR) sector, it is easy to see that, at the lowest order, it does not change the masslessness nature of the theory. Indeed, it does not spoil both the validity of eq. (15) and the analytic structure of the Green's functions of the original massless theory. On the contrary, if the operator $\hat{\Psi}(x)$ has non-vanishing FFs in the RL (LR) sector of the theory, this perturbation immediately generates a mass gap, a quantity which can be estimated by sandwiching Eq. (3) on R and L asymptotic states

$$\delta m \simeq 2\, \lambda^{eff} \lim_{\beta_{RL}\to-\infty} F_{RL}^{\hat{\Psi}}(i\pi+\beta_{RL}) \ . \tag{16}$$

From a kinematical point of view, the above limit is the expectation value of the perturbing operator at the (zero–energy) threshold of the crossed RL channel. In the above equation the effective coupling constant, λ^{eff}, is defined like in (8) with $m \to 0$. As a consequence,

[2] It should be kept in mind that to avoid trivial vanishing of the RR (LL) FFs in taking the massless limit of (5), one has to rescale scalar operators by their mass dimension $\mathcal{O}(x) \to \mathcal{O}(x)/m^{2\Delta_{\mathcal{O}}}$ and define their FFs $F_{\alpha,\alpha}^{\mathcal{O}}(\beta_{12}) = \lim_{m\to 0} \frac{F^{\mathcal{O}}(\beta_{12})}{m^{2\Delta_{\mathcal{O}}}}$, where $F^{\mathcal{O}}(\beta_{12})$ is the two–particle form-factor of the massive version of the theory.

if the perturbing operator is irrelevant with respect to the IR CFT, λ^{eff} scales to zero, i.e. the actual mass gap vanishes in the infrared region although it is present at intermediate scales. On the other hand, if it is relevant, it will grow to the scale of the mass being generated. At this level the relevance of an operator has to be established by scaling arguments. In summary, two conditions have to be fulfilled for generating, at lowest order, a mass gap in the theory: the perturbing operator has to be relevant with non–vanishing RL (LR) matrix elements.

Moreover, like in the massive case, the mass correction δm may be a finite or a divergent quantity, depending on the locality properties of the perturbing operator with respect to the fields that generate the massless particles. If the operator $\hat{\Psi}(x)$ is local, δm is finite and the massive excitations of the perturbed theory are adiabatically related to the massless particles of the original one. If, instead, $\hat{\Psi}(x)$ is a non-local operator, it follows from Eq. (12) and (16) that δm diverges: in this case, the original massless excitations are confined as soon as λ is switched on. In other words, the massive particles of the perturbed theory are not associated, in this case, to the operators that create the original massless ones. The examples discussed below should help in clarifying these two situations.

If the theory contains more than one type of massless particle $A_{a,R/L}$ $(a = 1,\ldots,n)$ the previous approach has to be generalised in analogy with perturbation theory for degenerate levels. Since the particles $A_{a,\alpha}$ form a complete basis for the scattering theory, by using FFPT it should be possible, in principle, to predict whether *any* massless excitations survive, and this is a clear advantage with respect to the RG. Although a complete answer to this question involves the entire series in λ, nevertheless the first order of FFPT may provide useful hints on the decoupling of massive and massless modes of the theory under investigation.

Massless flows between minimal models. Let us now apply the above methods to some specific examples, starting from the massless flow between the Tricritical Ising Model (TIM) and the Critical Ising Model (CIM). The quantum field theory associated to this RG flow can either be seen as TIM perturbed by its sub-leading energy operator ϵ' of conformal dimensions $\Delta_{\epsilon'} = \bar{\Delta}_{\epsilon'} = 3/5$ or as CIM perturbed by the irrelevant operator $T\bar{T}$ (see [4] and references therein). The factorised scattering theory for this massless flow was first proposed in [4] and the basic FFs calculated in [13]. The spectrum consists of massless neutral fermions, with S-matrix $S_{RR}(\beta) = S_{LL}(\beta) = -1$, while $S_{RL}(\beta) = \tanh(\beta/2 - i\pi/4)$. As well as studying the non-integrable theory obtained by the insertion of the energy operator $\epsilon(x)$ of the CIM, we will also consider the deformation of the massless action by the disorder operator $\mu(x)$. The latter is non-local, $\gamma_\mu = 1/2$, with respect to the massless fermion excitations. The energy operator has two particle FFs only in the RL sector of the form [13]

$$F^{\epsilon}_{RL}(\beta) = Z_{\epsilon} \exp\left(\frac{\beta}{4} - \int \frac{\mathrm{d}t}{t} \frac{\sin^2\left(\frac{(i\pi-\beta)}{2\pi}\right)}{\sinh t \cosh \frac{t}{2}}\right) \quad , \tag{17}$$

where Z_ϵ is a normalisation constant. The disorder operator, on the other side, has also FFs in the RR and LL sectors, of the form $F^{\mu}_{RR}(\beta) = Z'_{\mu} \tanh(\beta/2)$. For the FFs in the RL sector we find

$$F^{\mu}_{RL}(\beta) = Z_{\mu} \exp\left(-\frac{\beta}{4} - \int \frac{\mathrm{d}t}{t} \frac{\sin^2\left(\frac{(i\pi-\beta)}{2\pi}\right)}{\sinh t \cosh \frac{t}{2}}\right) \quad . \tag{18}$$

The above results agree with the roaming limit of the FFs of the sinh-Gordon theory [15] – a limit in which the sinh-Gordon model corresponds to the above massless flow [5, 13, 7].

Using now (16) and (17), it is easy to see that the perturbation by $\epsilon(x)$ induces a finite mass in the system, as it could have been expected on different grounds. At the critical point, in fact, $\epsilon(x)$ is bilinear in the fermionic operators that generate the massless particles, $\epsilon \sim \bar{\psi}\psi$, and therefore the perturbed theory describes massive fermions in the presence of an irrelevant perturbation $T\bar{T}$.

Consider now the perturbation of the massless action by the non-local operator $\mu(x)$. By computing the limit (16) of the two–particle FF (18) of this operator, one sees that in this case δm diverges, i.e. the initial excitations can no longer propagate as asymptotic states in the new vacuum of the theory created by the insertion of this field. Like in the massive case, there is a simple explanation of this confinement phenomenon in terms of the LG effective description of the theory [1]. Indeed, in the unperturbed theory the elementary excitations can be equivalently considered as massless kinks interpolating between two degenerate minima of the LG potential [14]. However the insertion of the disorder magnetic operator $\mu(x)$ lifts the degeneracy between the minima, thus making the kinks unstable.

As a matter of fact, the flow between the TIM and the CIM is the simplest example of a one-parameter family of RG trajectories interpolating between the conformal minimal models $\mathcal{A}_p$, with central charge $c_p = 1 - 6/p(p+1)$ ($p = 3, 4$ describe the CIM and TIM respectively). The flows start from $\mathcal{A}_\infty$ and pass close to all the other minimal models, remaining massless all the way down to the very last fixed point, $p = 3$, after which they become massive [5, 6, 7]. The trajectories going out from each critical point are described as $\mathcal{A}_p$ perturbed by the operator ϕ_{13} [3]: $\mathcal{A}_p^{eff} = \mathcal{A}_p + \lambda \int \mathrm{d}^2x \phi^p_{13}$ (where the upper index in ϕ_{13} indicates the relative CFT) and the excitations are massless kinks interpolating between the $(p-2)$ degenerate vacua of the effective LG potential [14]. An interesting problem consists of predicting the evolution of the spectrum along these flows, in particular the successive decoupling of the massless modes in the cascade of massless RG flows, by applying FFPT. The analysis of this problem is however beyond the scope of the present letter.

Spinon confinement in sigma models. Another important application of FFPT is in the study of the mass spectrum of the $O(3)$ non-linear sigma-model with a topological term

$$\mathcal{A}_\theta = \frac{1}{2f^2}\int \mathrm{d}^2x\,(\partial_\mu n_\alpha)^2 + i\,\theta\,T, \quad (\alpha = 1,2,3\ ;\ n_\alpha^2 = 1) \tag{19}$$

where f and θ are dimensionless coupling constants and T is the integer-valued topological term related to the instanton solutions of the model. The two values $\theta = (0,\pi)$ are the only ones for which the action (19) is known to be integrable. At $\theta = 0$ the excitations form a massive $O(3)$ triplet whose scattering theory was constructed in [8, 16]. At $\theta = \pi$ the theory is instead massless [17, 18, 19, 20] and corresponds to the RG flow between the $c = 2$ CFT and the $SU(2)_1$ Wess-Zumino-Witten (WZW) model. The factorised scattering theory was suggested in [17]: it consists of right and left doublets, $A_{a,R}$ and $A_{a,L}$ $(a = 1,2)$, that transform according to the $s = 1/2$ representation of $SU(2)$ (spinons). However, as soon as one moves away from $\theta = \pi$, the spinons confine [21, 22] and the actual spectrum of the theory in the vicinity of this point has been determined in [22]. Let us discuss in some detail how the spinon confinement takes place. In the FFPT this amounts to show that the topological term is non–local wrt the fields that create the spinons, a property that can be easily checked by looking at the CFT limit of these operators.

Consider $\mathcal{A}_\pi$ as our unperturbed IQFT. Close to the IR fixed point the massless flow can be described as a $SU(2)_1$ WZW model perturbed by the marginally irrelevant perturbation $(\mathrm{Tr}\,g)^2$ [17, 18, 19], $\mathcal{A}_\pi^{eff} = \mathcal{A}_{SU(2)_1} + \gamma\int \mathrm{d}^2x(\mathrm{Tr}\,g)^2 \quad (\gamma > 0)$, where g is the SU(2) matrix field. In terms of this formulation the perturbation that moves the topological term away from $\theta = \pi$ is proportional to $\mathrm{Tr}\,g$ [18], i.e. to the only relevant $SU(2)$ invariant operator in the theory that breaks parity. Thus, in the vicinity of $\theta = \pi$, the model is described by the effective action

$$\mathcal{A}^{eff} = \mathcal{A}_\pi^{eff} + \eta\int \mathrm{d}^2x\mathrm{Tr}\,g\,, \tag{20}$$

where η is a function[3] of $(\theta - \pi)$ that vanishes when $\theta = \pi$.

As discussed in Ref. [25], the spinons are created by the primary operator $\phi^{\pm}(z)$ $(\bar{\phi}^{\pm}(\bar{z}))$ with scaling dimension $h = (1/4,0)$ $(\bar{h} = (0,1/4))$. They enter the operator

[3]The form of this function determines the dependence on $(\theta - \pi)$ of the mass gap m, since it scales as $m \sim \eta^{2/3}$, up to logarithmic correction [23]. In a recent paper [24] it has been suggested that the gap behaves like $(\theta - \pi)^{1/2}$, which would imply a dependence of η on $(\theta - \pi)$ that it is not linear, as usually assumed.

product expansion (OPE)

$$\begin{aligned}\phi^\alpha(z)\phi^\beta(w) = (-)^q (z-w)^{-1/2}\epsilon^{\alpha\beta}\left(1+1/2(z-w)^2 T(w)\ldots\right)\\ -(-)^q(z-w)^{1/2}(t_a)^{\alpha\beta}\left(J^a(w)+1/2(z-w)\partial J^a(w)\ldots\right),\end{aligned} \tag{21}$$

where $\epsilon^{+-} = -\epsilon^{-+} = 1$, $(t_a)^{\alpha\beta}$ are the generators of the algebra, and q takes the values $q=0$ for states that are created by an even number of spinons and $q=1$ if the number of spinons is odd. The OPE between the spinon operator and the $SU(2)$ currents $J^a(z)$ is standard: $J^a(z)\phi^\alpha(w) = (t^a)^\alpha{}_\beta \phi^\beta(w)/(z-w)+\ldots$. From these OPEs it follows that J^a and ϕ^α are mutually local while ϕ^α and ϕ^β are not. In fact taking $\phi^\alpha(z)$ around $\phi^\beta(w)$ by sending $z \to z\,\mathrm{e}^{2\pi i}$ it produces a factor $\mathrm{e}^{2\pi i\gamma_\phi}$ with $\gamma_\phi = 1/2$.

Therefore $(\mathrm{Tr}\, g)^2 \simeq \bar{J}^a J^a$ is local with respect to the spinons and this explains why they are the fundamental excitations of $\mathcal{A}_\pi$, regardless of whether the perturbation is marginally relevant or irrelevant. Since $\mathrm{Tr}\, g$ is proportional to $(\phi^+\bar{\phi}^- + \phi^-\bar{\phi}^+)$, the OPE (21) implies that this operator is instead non-local with respect to the spinons. Hence they get confined as soon as the operator $\mathrm{Tr}\, g$ is added to $\mathcal{A}_\pi$, i.e. the perturbed model (20) has no longer spin 1/2 excitations. As discussed in [22], the actual massive excitations of the O(3) sigma model with θ–term consists of a triplet of particles and a singlet, the former stable for all value of θ whereas the latter stable only in an interval of values of θ near $\theta = \pi$.

We are grateful to G. Delfino for important discussions and A. Nichols for reading the manuscript. D.C would also like to thank A. Tsvelik and K. Schoutens for useful discussions. This work is within the activity of the European Commission TMR program HPRN-CT-2002-00325 (EUCLID).

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