Finite temperature correlation functions in integrable models: derivation of the semiclassical limit from the formfactor expansion

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Abstract

We propose an approach to the problem of finite temperature dynamical correlation functions in integrable one-dimensional models with a spectral gap. The approach is based on the analysis of the singularities of the operator matrix elements and is not model specific. For the long time, large distance asymptotics of the correlation functions we obtain a formula which, as a particular case, contains the expression for the dynamical susceptibility of Quantum Ising model suggested by Sachdev and Young, Phys. Rev. Lett. **78**, 2220 (1997).

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I. INTRODUCTION

In this paper we discuss a venerable problem of finite temperature time-dependent correlation functions in integrable models. For one-dimensional models with gapless linear spectrum there is a general solution: the Matsubara n-point correlation functions at finite temperature (and in a finite box, if necessary) can be obtained from the corresponding $T = 0$ correlators by the conformal transformation. For integrable models with spectral gaps the current state of affairs is as follows. At $T = 0$ two-point correlators can be evaluated using the formfactor approach (see [1], [2],[3] for reviews). This approach allows one to calculate matrix elements of the operators between the exact eigenstates (formfactors) which enter the Lehmann expansion for the correlation functions. The formfactor approach turned out to be highly effective for $T = 0$ two-point functions. However, its generalization for multi-point functions, as well as for $T \neq 0$ faces difficulties related to singularities in the formfactors[1].In this paper we propose a method of summation of these singularities. As a particular case, the results reproduce the semiclassical formulae obtained in [5].

The problem of these singularities is quite general, we start the discussion with a concrete and probably simplest example - the Quantum Ising model. After that we will extend the discussion for other integrable models. The Hamiltonian H of the Ising model is equivalent to the Hamiltonain of noninteracting massive fermions

$$H = \sum_n [-J\sigma^z_n\sigma^z_{n+1} + h\sigma^x_n] \equiv \sum_n [-h\mu^z_{n-1/2}\mu^z_{n+1/2} + J\mu^x_n] \equiv \sum \epsilon(p)F^+_pF_p$$
$$\epsilon(p) = \sqrt{(J-h)^2 + 4Jh\sin^2(p/2)} \qquad (1)$$

where $\mu^z_{n+1/2} = \prod_{j<n}\sigma^x_j$ is the disorder operator and F, F^+ are fermion annihilation and creation operators. The fermions represent domain walls between ground states with different orientation of the magnetization (solitons).

Calculations simplify in the continuum limit, when the spectral gap $M = |J-h|$ is much smaller than the bandwidth $\sim J$ and the spectrum is relativistic $\epsilon(p) = \sqrt{c^2p^2 + M^2}$, $c^2 = Jh$. In this case energy and momentum of a quasiparticle are conveniently parameterized by rapidity θ ($p = c^{-1}M\sinh\theta$). Then the eigenstates of Hamiltonian (1) are represented by sets of rapidities $\{\theta_i\}$ such that the energy and momentum of the system are equal to

$$E = M\sum_{i=1}^n \cosh\theta_i, \quad P = c^{-1}M\sum_{i=1}^n \sinh\theta_i \qquad (2)$$

Below we set $c = 1$. Operators σ^z and μ^z (we will call them σ and μ) have infinitely many matrix elements. The formfactors depend on rapidity differences (Lorentz invariance!) in the following way [4]:

$$\langle\theta_1,...\theta_n|\sigma|\theta'_1,...\theta'_m\rangle = AM^{1/8}\prod_{i<j}\tanh(\theta_{ij}/2)\prod_{p<q}\tanh(\theta'_{pq}/2)\prod_{i,p}\coth[(\theta_i-\theta'_p)/2] \quad (3)$$

where at $h < J$ (the ordered phase) $n + m$ is even for σ and odd for μ (for $h > J$ it is the other way around). Here $\theta_{ij} = \theta_i - \theta_j$, A is a known numerical constant. The singularities in (3) appear as soon as some *in* and *out* rapidities coincide. Singularities of this kind known as kinematical poles appear routinely in integrable models. There are operators however, which matrix elements are free of singularities. In the Ising model such operator is the energy density operator

$$\sigma^x(x) = \sum_k \mathrm{e}^{-\mathrm{i}qx}\gamma(k)\gamma(k-q)\hat{F}_k\hat{F}_{q-k}, \quad \hat{F}_k = \hat{F}^+_{-k}, \gamma(k) = \sqrt{1+k/\epsilon(k)} \quad (4)$$

II. THE PRIOR EFFORTS

Since in the Ising model formfactors are known explicitely, it has been a common belief that its analysis would lead to better understanding of the general problem. Several approaches to the problem of singularities have been proposed.

- One can try to regularize the singularities by considering a finite system. Then the momenta of *in* and *out* states are quantized differently. Thus, in the Ising model they are respectively integers and half-integers of $2\pi/L$. It was suggested in [7] that one can sum up these singularities, though, to the best of our knowledge, this program has not been implemented.

- One can interchange imaginary time and space axices (the Wick rotation). Then finite T Matsubara Green's function of the infinite system becomes zero temperature correlator for a system on a circle of circumference $1/T$. The corresponding formfactor expansion is manifestly free of singularities, though one faces the problem of calculating formfactors in a system of finite size. For the Ising model this was achieved in [8]. The Green's function was obtained rigorously in the form of the series. Below we use these series in order to evaluate the dynamical correlation functions.

- At finite T and longest times one can try the "renormalized classical" approach proposed in [5]. The suggested answer for the causal correlation function of the magnetizations in the ordered phase valid for $T << M$ is

$$D(x,t) \equiv \langle \sigma(t,x)\sigma(0,0)\rangle = CM^{1/4}\exp\left[-\int \frac{\mathrm{d}p}{\pi}\mathrm{e}^{-\epsilon(p)\beta}|x - t\frac{\partial\epsilon(p)}{\partial p}|\right] \tag{5}$$

where $\epsilon = \sqrt{M^2+p^2} \approx M + p^2/2M$ and C is a numerical constant. This expression suggests that the typical correlation length scales as t. One can approximate Eq.(5) as $D(x,t) \approx C\exp[-n(T)\max(|x|, v|t|)]$ where $v = \sqrt{\pi T/2M}$ is the thermal velocity of kinks and $n(T) = \sqrt{TM/2\pi}\mathrm{e}^{-M/T}$ is the average number of solitons (fermions).

Below we derive (5) directly from the formfactor approach.

III. INTERCHANGE OF TIME AND SPACE TRICK.

The result for the Matsubara Green's function of the Ising model order parameters obtained in [8] is

$$\langle\sigma(\tau,x)\sigma(0,0)\rangle = \tag{6}$$
$$CM^{1/4}\mathrm{e}^{-2|x|n(T)}\left\{1 + \sum_{n=1}^{\infty}\frac{T^{2n}}{(2n)!}\sum_{q_1,\dots q_{2n}}\prod_{i=1}^{2n}\frac{\mathrm{e}^{-|x|\epsilon_i - \mathrm{i}\tau q_i - \eta(q_i)}}{\epsilon_i}\prod_{i>j}\left(\frac{q_i-q_j}{\epsilon_i+\epsilon_j}\right)^2\right\}$$

where τ is imaginary time, $n(T) = \sqrt{TM/2\pi}\mathrm{e}^{-M/T}$ is the soliton density and $q = 2\pi T m$ (m integer), $\epsilon(q) = \sqrt{M^2+q^2}$. The index

$$\eta(q) = \frac{2\epsilon(q)}{\pi}\int_0^{\infty}\frac{\mathrm{d}x}{\epsilon^2(q)+x^2}\ln\coth[\beta\epsilon(x)/2] \tag{7}$$

at $T << M$ is exponentially small and will be neglected in the further discussion.

The symmetry breaking transition at $T = 0$ leads to the magnetization $\langle\sigma\rangle = \pm[CM^{1/4}]^{1/2}$. This is reflected in the zeroeth term in braces in (6). As a caviat we add that formula (6) was derived for $\tau = 0$ and the time dependence was conjectured and then checked numerically for the lattice of finite size (up to N=6 in the time direction).

The correlation function of the disorder parameters can be presented as a sum over odd number of q's:

$$\langle\mu(\tau,x)\mu(0,0)\rangle = \tag{8}$$
$$CM^{1/4}\mathrm{e}^{-2|x|n(T)}\sum_{n=0}^{\infty}\frac{T^{2n+1}}{(2n+1)!}\sum_{q_1,\dots q_{2n+1}}\prod_{i=1}^{2n+1}\frac{\mathrm{e}^{-|x|\epsilon_i - \mathrm{i}\tau q_i - \eta(q_i)}}{\epsilon_i}\prod_{i>j}\left(\frac{q_i-q_j}{\epsilon_i+\epsilon_j}\right)^2$$

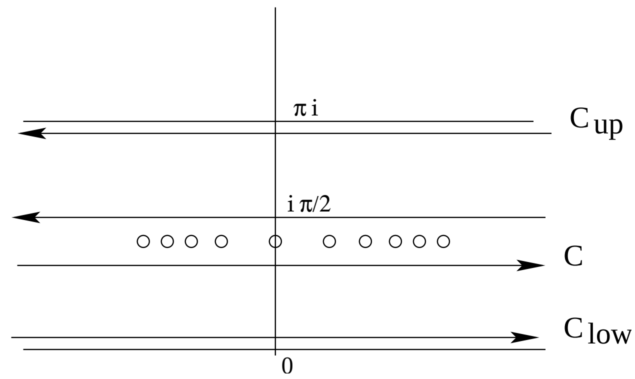

FIG. 1: The integration contours in θ plane. The circles are poles at $\theta = \mathrm{i}\pi/2 + \sinh^{-1}(2\pi n T/M)$.

Equations (6,8) are well suited to determine the static correlation functions as the expansion in powers of $\exp[-M|x|]$. To study the time dependence we rewrite these formulae in a more convenient form, namely, present the N-th terms in Eqs. (6,8) as a contour integral

$$\frac{1}{N!}\int_C \mathrm{d}\theta_1 \frac{\mathrm{e}^{\tau\cosh\theta_1 + \mathrm{i}x\sinh\theta_1}}{\mathrm{e}^{\beta\cosh\theta_1} - 1} ... \int_C \mathrm{d}\theta_N \frac{\mathrm{e}^{\tau\cosh\theta_N + \mathrm{i}x\sinh\theta_N}}{\mathrm{e}^{\beta\cosh\theta_1} - 1} \prod_{i>j} \tanh^2[(\theta_i - \theta_j)/2] \tag{9}$$

with the contour C shown on Fig.(1). One can deform this contour into the superposition $C \to C_{low} + C_{up}$, where C_{low} lies just above real axis while C_{up} corresponds to $\Im m\theta = \pi - 0$. As a result one obtains a sum of particle (lower contours) and antiparticle (upper contours) contributions:

$$\sum_{n=1}^{N} \frac{1}{n!(N-n)!} \int \prod_{i=1}^{n} \mathrm{d}\theta_i f(\epsilon_i) \mathrm{e}^{\tau\epsilon_i + \mathrm{i}xp_i} \prod_{j=1}^{N-n} \mathrm{d}\theta'_j f(\epsilon_j) \mathrm{e}^{(\beta-\tau)\epsilon_j - \mathrm{i}xp_j} \times$$
$$\prod_{i>k} \tanh^2[(\theta_i - \theta_k)/2] \prod_{j>p} \tanh^2[(\theta'_i - \theta'_p)/2] \times \prod_{i,p} \coth^2[(\theta_i - \theta'_p + \mathrm{i}0)/2] \tag{10}$$

where $f(\epsilon) = [\mathrm{e}^{\beta\epsilon} - 1]^{-1}$ is the Bose distribution function and $p = M\sinh\theta, \epsilon = M\cosh\theta$. In Eq.(10) n and $N-n$ are numbers of particles and antiparticles. Note that the singularities of the formfactors are now removed from the real axis to one side of the contour, as was suggested in [9]. Formula similar to equation (10) was proposed in [10]. However, there is a noticable difference: the distribution functions are of Bose type, although the quasiparticles of the Ising model are fermions!

To obtain the casual Green's functions, one has to replace τ with it. From now on we will be interested in the behavior of the Green's functions at longest real times $t >> M^{-1}$ and $|x| >> M^{-1}$. Only those terms in (10) which contain both particles and antiparticles are singular. The leading contribution to the Green's function in each order comes from the vicinity of each singularity. Each θ_{in} carries the thermal exponent $\mathrm{e}^{-\beta M} << 1$. It is thus sufficient to restict summation in (10) by terms with equal number of *in* and *out* rapidities, each *in* rapidity θ_i being close to one of the *out* rapidities θ'_j. The integral over the difference $\theta_i - \theta'_j$ can be evaluated as described above. Their is $n!$ ways to choose a pair for each θ_i; this factor cancels one of the $n!$ terms from the denominator of Eq.(10). The resulting series sum up to the exponent:

$$\langle\sigma(x,t)\sigma(0,0)\rangle = CM^{1/4}\theta(t)\mathrm{e}^{-2|x|n(T)}\exp[R(t,x)] \tag{11}$$

where

$$R(t, x > 0) = \frac{1}{\pi^2}\int \mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{e}^{-\beta M(1+\theta_1^2/2)}\frac{\mathrm{e}^{\mathrm{i}t(\theta_1^2-\theta_2^2)/2+\mathrm{i}x\theta_{12}}}{(\theta_{12}+\mathrm{i}0)^2} =$$
$$\frac{\mathrm{e}^{-\beta M}}{\pi}\int \mathrm{d}\theta\mathrm{e}^{-\beta M\theta^2/2}\left[|x| - |x - M\theta t|\right] \tag{12}$$

This result reproduces Eq.(5).

IV. THE GENERAL CASE

Apart from the nontrivial change of the distribution function $f(\epsilon)$ which is not important in the limit $T << m$, Eq. (10) has a form which suggests a straightforward generalization for other integrable models. Namely, for an operator $\mathcal{O}$ in a given integrable model we will use Eq.(10) with the Ising model formfactors replaced by the formfactors of $\mathcal{O}$. At temperatures much smaller than the gap we can replace the distrubution functions by the Boltzmann exponents. The asymptotics of the correlation function is then determined solely by the behavior of the formfactors in the vicinity of the kinematical poles. As we have already pointed out, existence of kinematical poles is a general consequence of the theory of formfactors [1] rather than a specific feature of the Ising model. These poles are due to annihilation processes and occur when rapidities of a particle and an anti-particle coincide. Moreover, the behavior of formfactors in the vicinity of the poles is universal for all integrable models. Accordingly the resolution of the problem of singularities is universal rather than model-specific.

Consider a generic matrix element of operator $\mathcal{O}$:

$$F^{\mathcal{O}}(\{u\},\{v\};\{\bar{a}\},\{b\}) = \langle u_1,...u_n|\hat{\mathcal{O}}|v_1,...v_m\rangle_{\bar{a}_1,...\bar{a}_n;b_1,...b_m} \tag{13}$$

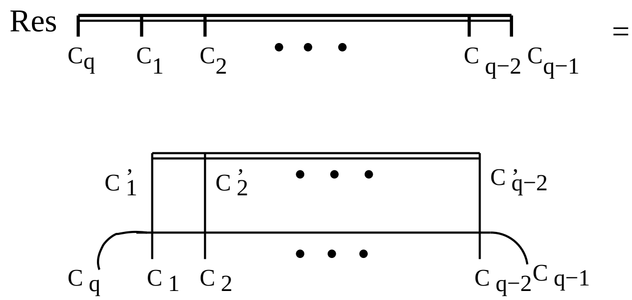

FIG. 2: A graphic representation of Eq.(16). The double line represents the formfactors. Intersections of the solid lines represent either unit matrices (as in the first term in the square brackets) or S matrices.

where u_i, v_j are rapidities and $\bar{a}_i, b_j$ are isotopic indices. One can express formfactor (13) in terms of some canonical matrix elements. Let $F^{\mathcal{O}}_{irr}$ be the irreducible part of $F^{\mathcal{O}}$ given by the connected diagrams. According to [9],[10], $F^{\mathcal{O}}_{irr}$ is equal to the matrix element between the vacuum and a certain excited state of $n+m$ particles:

$$F^{\mathcal{O}}(\{u\},\{v\};\{\bar{a}\},\{b\})_{irr} = \langle 0|\hat{\mathcal{O}}|v_1,...v_m; \mathrm{i}\pi + u_n + \mathrm{i}0, ...\mathrm{i}\pi + u_1 + \mathrm{i}0\rangle_{b_1,...b_m;\bar{a}_n,...\bar{a}_1}; \quad (14)$$

where $\bar{a}$ indices are obtained from a ones by charge conjugation.

Now let us consider formfactors of the type

$$F^{\mathcal{O}}(0,\theta) = \langle 0|\hat{\mathcal{O}}|\theta_1, ...\theta_q\rangle_{c_1,...c_q} \quad (15)$$

As was shown in [1] (see also [3] for review) these formfactors have poles at $\theta_i - \theta_j = \mathrm{i}\pi$ and the residues satisfy the following relation (see Fig.(2)):

$$\begin{aligned} &\mathrm{iRes}_{\theta_{q-1,q}=-\mathrm{i}\pi} F^{\mathcal{O}}_{c_1,...c_q}(\theta_1,\theta_2,\ldots,\theta_q) = \\ &F^{\mathcal{O}}_{c'_1,\ldots,c'_{q-2}}(\theta_1,\ldots,\theta_{q-2})\delta_{c_q,c'_{q-1}}\left[I - \mathrm{e}^{2\pi\mathrm{i}\varphi_{c_q,\mathcal{O}}}\prod_{i=1}^{q-2} S^{\tau_{i-1},c'_i}_{\tau_i,c_i}(\theta_{q-1}-\theta_i)\right], \end{aligned} \quad (16)$$

where S is the two-particle scattering matrix and $\varphi_{c,\mathcal{O}}$ is the *semi-locality index* between the particle creation operator A^+_c and $\mathcal{O}$. We will define this index a little bit later. Combining (13) and (16) we obtain the following behavior of two- and three-particle formfactors at the poles:

$$\langle\langle u|\mathcal{O}|v\rangle\rangle_{a,b} = -\frac{\mathrm{i}(1-\mathrm{e}^{2\pi\mathrm{i}\varphi_{a,\mathcal{O}}})}{u-v+\mathrm{i}0}\langle 0|\mathcal{O}|0\rangle\delta_{a,b} + \text{regular part} \quad (17)$$

$$\langle\langle u|\mathcal{O}|v_1,v_2\rangle\rangle_{a,b_1,b_2} = \frac{\mathrm{i}\left[I - \mathrm{e}^{2\pi\mathrm{i}\varphi_{a,\mathcal{O}}} S^{\bar{a},b_1'}_{b_2,b_1}(u - v_1 + \mathrm{i}\pi)\right]}{u - v_2 + \mathrm{i}0}\langle 0|\mathcal{O}|v_1\rangle_{b_1'} + (1 \to 2)$$
$$+\text{regular part} \tag{18}$$

Let us now define the index φ. Two operators, $\mathcal{O}_1$ and $\mathcal{O}_2$, are called mutually non-local if the Euclidean correlator $\langle \ldots \mathcal{O}_1(\mathbf{r})\mathcal{O}_2(0)\ldots\rangle$ is a multi-valued function of $\mathbf{r}$. In particular, if under analytic continuation $z \to z\mathrm{e}^{2\pi\mathrm{i}}$, $\bar{z} \to \bar{z}\mathrm{e}^{-2\pi\mathrm{i}}$ ($z = \tau + \mathrm{i}\ x$ and $\bar{z} = \tau - \mathrm{i}\ x$), the correlator acquires only a phase, $2\pi\varphi_{1,2}$, the two operators are called *semi-local*. The remarkable advantage of this definition is that the index $\varphi_{1,2}$ can be evaluated at small distances, where all correlation functions have a simple power law form.

From Eq.(17) it follows that any the operator $\mathcal{O}$ which vacuum expectation is finite, has annihilation poles only if $\varphi_{a,\mathcal{O}}$ is not an integer. This means that the operator $\mathcal{O}$ is non-local. For instance, σ and μ in the Ising model are non-local with respect to the fermions (they acquire phase π after circling around the fermions in τ, x plane). However, if the operator can create an odd number of particles, the annihilation poles appear provided the S matrix is non-trivial (see Eq.(18)). Thus the vector field $\mathbf{n}$ in the O(3) nonlinear sigma model belongs to the later category. Its formfactors do have annihilation poles [1].

We proceed with the calculation of the most singular parts of the formfactors which include equal number of particles and antiparticles. We can do it explicitely for operators with $\langle 0|\mathcal{O}|0\rangle \neq 0$. To calculate the residue we apply Eq.(16) step by step. At first we obtain the residue in the two-particle formfactor. This yeilds us the expression (17) diagonal in the isotopic indices. Then we use this result to calculate the coefficient at $[(u_1 - v_1)(u_2 - v_2)]^{-1}$ in the four-particle formfactor $F_{c_2,c_1,\bar{c}_1',\bar{c}_2'}(u_1, u_2, \mathrm{i}\pi + v_2, \mathrm{i}\pi + v_1)$. As follows from Eq.(16), the S-matrices enter into the expression for this coefficient in the combination

$$S^{ab}_{c_2,c_1}(u_{12}) S^{c_2'\bar{b}}_{a,\bar{c}_1'}(\mathrm{i}\pi + u_{12}) = S^{ab}_{c_2,c_1}(u_{12}) S^{c_1',c_2'}_{ba}(-u_{12}) = \delta^{c_1'}_{c_1}\delta^{c_2'}_{c_2} \tag{19}$$

Here we used the crossing symmetry and unitarity of the S-matrices. As we see, this residue is again diagonal in the isotopic indices and equal to $\mathrm{i}(1 - \mathrm{e}^{2\pi\mathrm{i}\varphi})^2$. It is clear that a multiple application of formula (16) yields for general N the main singularity as

$$F^{\mathcal{O}}_{sing}(u_1, \ldots u_n; v_1 \ldots v_n) = \mathrm{i}\sum_{P_i}(1 - \mathrm{e}^{2\pi\mathrm{i}\varphi_{a,\mathcal{O}}})^n \prod_{i=1}^{n}(u_i - v_{P_i} + \mathrm{i}0)^{-1}\langle 0|\mathcal{O}|0\rangle \tag{20}$$

where the sum includes all permutations of $\{1, \ldots n\}$. Since the derivation uses the most general properties of S-matrices, the result holds for all integrable models. It differs from the result for the Ising model σ operator (3) only by a factor. Therefore the long time limit of the two-point correlation function of $\mathcal{O}$ operators with *finite vacuum expectation value* is essentially the same as (5) with

$$\exp[-\beta\epsilon] \to \sum_j \sin^2(\pi\varphi_j)\exp[-\beta\epsilon_j] \tag{21}$$

where the sum runs over isotopic indices of the solitons. We arrive to the conclusion that low energy dynamics of such operators is indeed semiclassical for all integrable models as long as the operator in question is *nonlocal* with respect to the kink creation operators ($\varphi \neq 0$). As we have stated above, the condition of nonlocality does not apply to operators which can create only odd number of particles (see Eq.(18)). Their dynamics may be semiclassical even if they are local with respect to the soliton creation operators.

V. LOW TEMPERATURE CHARGE DYNAMICS IN THE SINE-GORDON MODEL

To demonstrate how the developed method works, we consider a concrete example: the problem of charge counting statistics in the sine-Gordon model. It is a familiar integrable model with the Lagrangian density

$$\mathcal{L} = \frac{1}{16\pi}(\partial_\mu\Phi)^2 - m^2\cos(\gamma\Phi) \tag{22}$$

The cosine term is relevant at $\gamma < 1$. The conserved soliton charge is defined as

$$Q = \frac{\gamma}{2\pi}\int \mathrm{d}x\partial_x\Phi \tag{23}$$

Therefore $Q(x,t) = (\gamma/2\pi)[\Phi(x,t) - \Phi(x,0)]$ is the charge passed through a point x during the time t. One can calculate the probability distribution function of $Q(x,t)$ using the results for the correlation function of operators $\exp[\mathrm{i}\eta\Phi]$. In this case $\varphi = \eta/\gamma$ and the semiclassical correlation function, as follows from (5,21), is

$$\langle \mathrm{e}^{\mathrm{i}\eta\Phi(x,t)}\mathrm{e}^{-\mathrm{i}\eta\Phi(0,0)}\rangle = C^2(\gamma,\eta)\exp\left[-2\sin^2(\pi\eta/\gamma)\int\frac{\mathrm{d}p}{\pi}\mathrm{e}^{-\epsilon(p)\beta}|x - t\frac{\partial\epsilon(p)}{\partial p}|\right] \tag{24}$$

where ϵ is the soliton energy and the factor $C(\gamma,\eta)$ equal to the vacuum expectation value of $\exp[\mathrm{i}\eta\Phi]$. The latter quantity was calculated in [11]:

$$C = \left[\frac{M\sqrt{\pi}\Gamma(1/(2-2\gamma^2))}{2\Lambda\Gamma(\gamma^2/(2-2\gamma^2))}\right]^{\eta^2} \times$$

$$\exp\left\{\int_0^\infty \frac{\mathrm{d}y}{y}\left[\frac{\sinh^2(2\eta\gamma y)}{2\sinh(\gamma^2 y)\sinh y\cosh[(1-\gamma^2)y]} - 2\eta^2\mathrm{e}^{-2y}\right]\right\} \tag{25}$$

Here $M \sim m^{1/(1-\gamma^2)}$ is the soliton mass and Λ is the ultraviolet cut-off. In the field theory limit $M << \Lambda$, the η dependence in (25) is dominated by the factor $(M/\Lambda)^{\eta^2}$. Thus the magnitude of the propagator has a strong maximum at small η. At $\eta << 1$ we have

$$\langle \exp[\mathrm{i}\omega Q(x,t)]\rangle \approx A(\gamma)(\bar{M}/\Lambda)^{(\gamma\omega/2\pi)^2}\exp[-(\omega/2)^2(t/\tau_0)] \tag{26}$$

where $1/\tau_0 = (4T/\pi M)\mathrm{e}^{-M/T}$, $A(\gamma)$ is a constant and

$$\bar{M} = M\frac{\sqrt{\pi}\Gamma(1/(2-2\gamma^2))}{2\Gamma(\gamma^2/(2-2\gamma^2))}\exp\left\{2\int_0^\infty \frac{\mathrm{d}y}{y}\left[\frac{y^2\gamma^2}{2\sinh(\gamma^2 y)\sinh y\cosh[(1-\gamma^2)y]} - \mathrm{e}^{-2y}\right]\right\} \tag{27}$$

Fourier transform of (26) yields the Gaussian distribution function of Q:

$$P[Q(t)] \sim \sigma^{-1/2}\exp\left[-Q^2(t)/2\sigma\right], \quad 2\sigma = (t/\tau_0) + \pi^{-2}\gamma^2\ln(\Lambda/\bar{M}) \tag{28}$$

Not all operators have singularities resulting in semiclassical behavior of the type described above. We have already mentioned that the energy density operator for the Ising model (4) is singularity free. For the sine-Gordon model the singularity-free operator is the conserved current $J_\mu = (\gamma/2\pi)\epsilon_{\mu\nu}\partial_\nu\Phi$. The correlation function of $J \equiv J_x$ can be obtained from (24) by taking $\eta \to 0$ limit and differentiating twice with respect to t:

$$\langle\langle J(x,t)J(0,0)\rangle\rangle = \int \frac{\mathrm{d}p}{\pi}v^2(p)\delta\left(x - tv(p)\right)\mathrm{e}^{-\beta\epsilon(p)} \tag{29}$$

where $v(p) = \partial\epsilon/\partial p$. The corresponding conductivity is ballistic with a finite Drude weight

$$\sigma(\omega) = 2\pi D(T)\delta(\omega), \quad D(T) = \int\frac{\mathrm{d}p}{\pi}v^2(p)\mathrm{e}^{-\beta\epsilon(p)} \tag{30}$$

The fact that the Drude weight is finite for the sine-Gordon model ([3]), as well as for the conserved current in the O(3) nonlinear sigma model ([12],[13]) was used as an argument against the semiclassical treatment of [5]. As we see, these two statements do not contradict each other, they are just related to different correlation functions. Some operators do have ballistic dynamics, some follow the semiclassical one.

VI. CONCLUSIONS AND ACKNOWLEDGEMENTS

We conclude that there are two types of dynamics in integrable systems. Correlation functions of some operators, such as the energy density (that is σ^x) operator in the Ising model, the current and stress-energy density operators for the sine-Gordon model *etc.*, display ballistic dynamics. There are other operators, however, whose finite temperature dynamics at longest scales is semiclassical. In this paper we described the criteria for such fields and established a link between the semiclassical description and the Bethe ansatz.

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