

An Existence Proof for Interacting Quantum Field Theories with a Factorizing S-Matrix

Gandalf Lechner

Institut für Theoretische Physik, Universität Göttingen,
37077 Göttingen, Germany

lechner@physik.uni-goe.de

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Abstract

Within the context of quantum field theories on two-dimensional Minkowski space with a factorizing S-matrix, a new construction scheme for relativistic model theories is carried out. We show that for a large class of such S-matrices, there exists an associated local quantum field theory in the sense of Haag and Kastler. The crucial step in this approach consists in proving the existence of observables localized in bounded spacetime regions, which is accomplished by verifying the modular nuclearity condition for wedge algebras.

1 Introduction

In relativistic quantum field theory, the construction of models describing interacting particles is still one of the most important unsolved problems. Although free field theories are thoroughly understood, and many model-independent features of quantum field theories have been discovered in axiomatic field theory, the explicit construction of local quantum fields leading to non-trivial cross sections can in most cases only be accomplished in perturbation theory.

As the main difficulties arise from the requirement that the theory should be strictly local, several authors [42, 44, 36, 13, 15] have tried to circumvent this obstruction by first considering easier constructable, non-local theories, and then pass to a local formulation in a second step.

An example for such a construction scheme is the program inducted by Schroer [42], which is concerned with model theories on two-dimensional Minkowski space having a factorizing S-matrix. Using concepts from the formfactor program [45], in which such models are usually investigated, one postulates a factorizing S-matrix S and constructs certain Wightman fields on a Hilbert space similar to Fock space. It then turns out that these fields share many properties with the free field, and their scattering states reproduce the two-particle S-matrix of S [31]. The only drawback is that they are not strictly local,

but only localizable in wedges¹. These fields must therefore not be regarded as the basic fields of the model, but rather as auxiliary objects (so-called *polarization-free generators*, cf. [8]). To construct a local theory, one needs to answer the question whether such a non-local model contains fields or observables localized in bounded spacetime regions like double cones. These local objects would then constitute a well-defined quantum field theory and reproduce the S-matrix S used as an input, thus providing the solution to the inverse scattering problem for S . The question of existence of well-localized observables thus amounts to the question of the existence of a local quantum field theory with the prescribed S-matrix S .

The existence problem for local observables is most conveniently analyzed in the algebraic approach to quantum field theory [26]. Passing from the wedge-local fields to the observable algebras they generate, one obtains a net $W \mapsto \mathcal{A}(W)$ of von Neumann algebras $\mathcal{A}(W)$ acting on the Hilbert space \mathcal{H} of the model, indexed by the wedge regions $W \subset \mathbb{R}^2$. In two dimensions, any double cone \mathcal{O} is the intersection of two opposite wedges, $\mathcal{O} = W_1' \cap W_2$, where $W_1 \subset W_2$ is an inclusion of wedges and the dash denotes the causal complement. The question of existence of observables localized in \mathcal{O} is in this setup therefore equivalent to the question whether the intersection $\mathcal{A}(W_1)' \cap \mathcal{A}(W_2)$ is non-trivial [13].

The *modular nuclearity condition* [10] provides a sufficient condition for the non-triviality of this intersection, and was therefore proposed as a tool in the present context in [13]. Considering an inclusion $\mathcal{A}(W_1) \subset \mathcal{A}(W_2)$ of two wedge algebras which have the vacuum Ω as a common cyclic and separating vector, let the modular operator of $(\mathcal{A}(W_2), \Omega)$ be denoted Δ_{W_2} . If the map

$$\Xi_{W_1 \subset W_2} : \mathcal{A}(W_1) \longrightarrow \mathcal{H}, \quad \Xi_{W_1 \subset W_2} A := \Delta_{W_2}^{1/4} A \Omega, \quad (1.1)$$

is nuclear [10], then the local von Neumann algebra $\mathcal{A}(\mathcal{O}) := \mathcal{A}(W_1)' \cap \mathcal{A}(W_2)$ associated to the double cone $\mathcal{O} = W_1' \cap W_2$ is of type III₁ according to the classification of Connes (in particular, non-trivial). In view of the theorems of Bisognano and Wichmann [3] and Borchers [6], the modular operator Δ_{W_2} takes the concrete form of a boost with imaginary rapidity parameter in many models, and the task of verifying this condition seems to be more easily manageable as compared to the explicit construction of local operators.

Within the context of the models constructed in Schroer's program from a factorizing S-matrix S , the modular nuclearity condition is known to hold for two special choices of S [13, 32]. For a certain infinite set of underlying S-matrices, it was shown in [33] that the maps (1.1) are at least compact. In the present paper, we will prove that (1.1) is nuclear, thus establishing the existence of local quantum field theories with a given factorizing S-matrix within a class \mathcal{S}_0 which is defined in the body of the paper.

This article is organized as follows: In Sect. 2, we recall the basic structure of the models considered, and define the nets of wedge algebras we are working with. Sect. 3 is devoted to a study of the family \mathcal{S} of S-matrices. We define a subclass $\mathcal{S}_0 \subset \mathcal{S}$, and consider the model theories associated to S-matrices in \mathcal{S}_0 subsequently. In Sect. 4, analytic properties of rapidity space wavefunctions of wedge-localized observables are studied. The main result of that section is Prop. 4.4, being a prerequisite for the proof

¹A wedge in Minkowski space is a Poincaré transform of the right-wedge $W_R := \{x \in \mathbb{R}^d : x_1 > |x_0|\}$.

of the modular nuclearity condition, which is carried out in Sect. 5. Finally, we present our conclusions in Sect. 6.

2 Wedge-local theories with a factorizing S-matrix

In this section we briefly review the structure of the models under consideration. For a more thorough discussion and the proofs of the statements made here, we refer the reader to the publications [31, 13, 42, 44].

We consider a single species of neutral particles of mass $m > 0$ on two-dimensional Minkowski space, and the family of factorizing S-matrices (*cf.*, for example, [27]) for these particles. A factorizing S-matrix is uniquely fixed by its two-particle S-matrix, and can hence be completely described by its *scattering function* S_2 . In two dimensions, S_2 is a function of the rapidity θ only, and is defined by

$${}_{\text{out}}\langle\theta_1, \theta_2 | \theta'_1, \theta'_2\rangle_{\text{in}} =: S_2(|\theta_1 - \theta_2|) \cdot \delta(\theta_1 - \theta'_1) \delta(\theta_2 - \theta'_2) , \quad (2.1)$$

and analytic continuation to negative values of its argument. The standard assumptions of unitarity, crossing symmetry and hermitian analyticity [21] for the S-matrix imply corresponding properties of the associated scattering function, which we take as a definition.

Definition 2.1. *A scattering function is an analytic function*

$$S_2 : S(0, \pi) = \{\zeta \in \mathbb{C} : 0 < \text{Im } \zeta < \pi\} \longrightarrow \mathbb{C} \quad (2.2)$$

which is bounded and continuous on the closure of this strip and satisfies the equations

$$\overline{S_2(\theta)} = S_2(\theta)^{-1} = S_2(-\theta) = S_2(\theta + i\pi), \quad \theta \in \mathbb{R}. \quad (2.3)$$

The set of all scattering functions is denoted by \mathcal{S} .

An explicit computation of \mathcal{S} is given in Prop. 3.1 in the next section. Generalizing (2.2), the notation $S(\alpha, \beta) = \{\zeta \in \mathbb{C} : \alpha < \text{Im } \zeta < \beta\}$ will be used for arbitrary $\alpha < \beta$ in the following.

Fixing a scattering function $S_2 \in \mathcal{S}$, we now outline the construction of the quantum field theoretic model whose scattering is governed by the corresponding S-matrix. It turns out to be most convenient to work on a Hilbert space \mathcal{H} which is generated from the one-particle space $\mathcal{H}_1 := L^2(\mathbb{R}, d\theta)$ in a manner similar to the case of the familiar Bosonic Fock space. Consider the representation $D_n^{S_2}$ of the symmetric group \mathfrak{S}_n of n objects on $L^2(\mathbb{R}^n) = \mathcal{H}_1^{\otimes n}$, which represents the transpositions $\tau_k(1, \dots, n) = (1, \dots, k+1, k, \dots, n)$ as

$$(D_n^{S_2}(\tau_k)\psi_n)(\theta_1, \dots, \theta_n) := S_2(\theta_{k+1} - \theta_k) \cdot \psi_n(\theta_1, \dots, \theta_{k+1}, \theta_k, \dots, \theta_n), \quad \psi_n \in L^2(\mathbb{R}^n). \quad (2.4)$$

We define \mathcal{H}_n as the largest closed subspace of $\mathcal{H}_1^{\otimes n}$ which is invariant under $D_n^{S_2}$, and put

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_0 := \mathbb{C} \cdot \Omega. \quad (2.5)$$

Here Ω denotes the unit vacuum vector of the theory. Vectors ψ in \mathcal{H} are thus sequences of n -particle rapidity wavefunctions, $\psi = (\psi_0, \psi_1, \psi_2, \dots)$, $\psi_n \in L^2(\mathbb{R}^n)$, which are “ S_2 -symmetric”,

$$\psi_n(\theta_1, \dots, \theta_{k+1}, \theta_k, \dots, \theta_n) = S_2(\theta_k - \theta_{k+1}) \cdot \psi_n(\theta_1, \dots, \theta_n), \quad \psi_n \in \mathcal{H}_n, \quad (2.6)$$

as a generalization of total permutation symmetry or antisymmetry. Similar Hilbert space have also been studied in the context of generalized statistics [35].

On \mathcal{H} , we have a unitary representation U of the Poincaré group: The transformation consisting of a boost with rapidity parameter λ and a subsequent space-time translation along $a \in \mathbb{R}^2$ is represented by $U(a, \lambda)$,

$$(U(a, \lambda)\psi)_n(\theta_1, \dots, \theta_n) := e^{i \sum_{k=1}^n p(\theta_k) \cdot a} \cdot \psi_n(\theta_1 - \lambda, \dots, \theta_n - \lambda), \quad (2.7)$$

where $p(\theta) = m(\cosh \theta, \sinh \theta)^T$ is the usual parametrization of the upper mass shell $\{p \in \mathbb{R}^2 : p_0 = (p_1^2 + m^2)^{1/2}\}$ in two dimensions. We also write $U(a) := U(a, 0)$ for a translation without boost.

Furthermore, the Zamolodchikov-Faddeev algebra [47] with scattering function S_2 is represented by operator-valued distributions z^\dagger, z on (a dense domain in) \mathcal{H} , which act as creation- and annihilation operators. Explicitly,

$$(z(\theta)\psi)_n(\theta_1, \dots, \theta_n) := \sqrt{n+1} \cdot \psi_{n+1}(\theta, \theta_1, \dots, \theta_n), \quad (2.8)$$

and $z^\dagger(\theta) := z(\theta)^*$. The fact that these operators represent Zamolodchikov’s algebra is equivalent to the exchange relations

$$z(\theta_1)z(\theta_2) = S_2(\theta_1 - \theta_2)z(\theta_2)z(\theta_1), \quad (2.9)$$

$$z(\theta_1)z^\dagger(\theta_2) = S_2(\theta_2 - \theta_1)z^\dagger(\theta_2)z(\theta_1) + \delta(\theta_1 - \theta_2) \cdot \text{id}_{\mathcal{H}}. \quad (2.10)$$

The proper creation and annihilation operators are denoted $z(\psi) = \int d\theta \psi(\theta)z(\theta)$ and $z^\dagger(\psi) = \int d\theta \psi(\theta)z^\dagger(\theta)$, $\psi \in \mathcal{H}_1$. For later reference we note the bound, familiar from free field theory,

$$\|z^\dagger(\xi)\psi_n\| \leq \sqrt{n+1} \|\xi\| \|\psi_n\|, \quad \|z(\xi)\psi_n\| \leq \sqrt{n} \|\xi\| \|\psi_n\|, \quad \xi \in \mathcal{H}_1, \psi_n \in \mathcal{H}_n. \quad (2.11)$$

With these preparations, a quantum field may be defined as, $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\phi(f) := z^\dagger(f^+) + z(f^-), \quad f^\pm(\theta) := \frac{1}{2\pi} \int d^2x f(x) e^{\pm ip(\theta) \cdot x}. \quad (2.12)$$

In the case of the constant scattering function $S_2 = 1$, this definition reproduces the free scalar field ϕ_0 of mass m . For generic $S_2 \in \mathcal{S}$, the operator $\phi(f)$ also shares many convenient properties, like essential self-adjointness for real f , with $\phi_0(f)$, but fails to be strictly local. Nonetheless, the field ϕ can be used for the construction of a wedge-local net.

We write $W_R := \{x \in \mathbb{R}^2 : x_1 > |x_0|\}$ and $W_L := W'_R$ for its causal complement. A dash on a subalgebra of $\mathcal{B}(\mathcal{H})$ is used to denote its commutant in $\mathcal{B}(\mathcal{H})$.

Proposition 2.2. [31, 13] *Let $S_2 \in \mathcal{S}$ and define*

$$\mathcal{A}(W_L) := \{e^{i\phi(f)} : f \in \mathcal{S}_{\mathbb{R}}(W_L)\}'' , \quad \mathcal{A}(W_R) := \mathcal{A}(W_L)', \quad (2.13)$$

$$\mathcal{A}(W_{R/L} + x) := U(x)\mathcal{A}(W_{R/L})U(x)^{-1}, \quad x \in \mathbb{R}^2. \quad (2.14)$$

- a) $W \mapsto \mathcal{A}(W)$ is a local net of wedge algebras on \mathbb{R}^2 , which transforms covariantly under the adjoint action of U (2.7), and the vacuum vector Ω is cyclic and separating for each $\mathcal{A}(W)$.
- b) The modular group $\{\Delta^{it}\}_t$ and modular conjugation J of $(\mathcal{A}(W_R), \Omega)$ act according to, $\psi \in \mathcal{H}$,

$$(\Delta^{it}\psi)_n(\theta_1, \dots, \theta_n) = \psi_n(\theta_1 + 2\pi t, \dots, \theta_n + 2\pi t), \quad (2.15)$$

$$(J\psi)_n(\theta_1, \dots, \theta_n) = \overline{\psi_n(\theta_n, \dots, \theta_1)}. \quad (2.16)$$

Using techniques developed in [8], it was shown in [31] that the two-particle S-matrix of this wedge-local quantum field theory is the one corresponding to S_2 .

As mentioned in the Introduction, Schroer and Wiesbrock proposed to construct a net of *local* observable algebras by forming intersections of wedge algebras [44]. On two-dimensional Minkowski space, each double cone is a translate of

$$\mathcal{O}_x := W_R \cap (W_L + x), \quad x \in W_R. \quad (2.17)$$

Hence the definition

$$\mathcal{A}(\mathcal{O}_x) := \mathcal{A}(W_R) \cap \mathcal{A}(W_L + x), \quad x \in W_R, \quad (2.18)$$

yields a local net of double cone algebras which transforms covariantly under $\text{Ad}U$. However, these algebras might turn out to be trivial, *i.e.* it is a priori uncertain whether local observables exist in a theory with given scattering function $S_2 \in \mathcal{S}$.

The modular nuclearity condition [13], ensuring the non-triviality of (2.18), takes the following concrete form in this setting: If the map

$$\Xi(x) : \mathcal{A}(W_R) \ni A \mapsto \Delta^{1/4}U(x)A\Omega \in \mathcal{H}, \quad (2.19)$$

$$(\Xi(x)A\Omega)_n(\theta_1, \dots, \theta_n) = e^{i\sum_{k=1}^n p(\theta_k - \frac{i\pi}{2}) \cdot x} (A\Omega)_n(\theta_1 - \frac{i\pi}{2}, \dots, \theta_n - \frac{i\pi}{2}) \quad (2.20)$$

is nuclear², $\mathcal{A}(\mathcal{O}_x)$ is of type III_1 and thus contains non-trivial operators.

In the case of a constant underlying scattering function $S_2 = \pm 1$, the modular nuclearity condition has been verified in [13] and [32], respectively.

As (2.20) involves the continuation of $\mathbb{R}^n \ni \theta \mapsto (A\Omega)_n(\theta)$, $A \in \mathcal{A}(W_R)$, to complex values of θ , the proof of the nuclearity condition requires detailed knowledge about the analytic structure of these wavefunctions. Before gathering such information in Sect. 4, we study as a prerequisite analytic properties of the scattering function S_2 in the next section.

²For the general definition of a nuclear map, see for example [11]. In the present paper we shall only encounter trace class operators on Hilbert spaces as special examples of nuclear maps.

3 The family of scattering functions

In this section we define a subfamily $\mathcal{S}_0 \subset \mathcal{S}$ of the family of scattering functions as specified by Definition 2.1. Elements in \mathcal{S}_0 will be distinguished by a condition on the distribution of their resonances and a condition on the phase shift. To begin with, we calculate an explicit formula for the elements of \mathcal{S} . For a related result, see [38].

Proposition 3.1. (The form of the scattering function)

a) Let \mathcal{Z} denote the family of finite or infinite sequences $\{\beta_k\} \subset \mathbb{C}$ satisfying the conditions

i) $0 < \text{Im}\beta_k \leq \frac{\pi}{2}$,

ii) β_k and $-\overline{\beta_k}$ appear the same (finite) number of times in the sequence $\{\beta_k\}$,

iii) $\{\beta_k\}$ has no finite limit point,

iv) $\sum_k \text{Im} \frac{1}{\sinh \beta_k} < \infty$.

Then there holds

$$\mathcal{S} = \left\{ \zeta \mapsto \varepsilon \cdot e^{ia \sinh \zeta} \cdot \prod_k \frac{\sinh \beta_k - \sinh \zeta}{\sinh \beta_k + \sinh \zeta} : \varepsilon = \pm 1, a \geq 0, \{\beta_k\} \in \mathcal{Z} \right\}. \quad (3.1)$$

The product in (3.1) converges absolutely and uniformly in ζ on compact subsets of the strip $S(0, \pi)$.

b) Let

$$\kappa(S_2) := \min \{ \pi, \inf \{ \text{Im} \zeta : \zeta \in S(0, \pi), S_2(\zeta) = 0 \} \}. \quad (3.2)$$

Each $S_2 \in \mathcal{S}$ is analytic in the strip $S(-\kappa(S_2), \pi + \kappa(S_2))$ and non-vanishing in $S(-\kappa(S_2), \kappa(S_2))$.

If $a = 0$ in (3.1), and S_2 has only finitely many zeros in $S(0, \pi)$, S_2 is uniformly bounded on each strip $\overline{S(-\kappa, \pi + \kappa)}$ with $\kappa < \kappa(S_2)$.

The proof of this Proposition can be found in the appendix.

It is well known from the study of analytic properties of scattering amplitudes that in general, analyticity of the S-matrix in a domain larger than the physical region (which in the present setting is $S(0, \pi)$) cannot be expected [37]. However, if the theory is required to have decent thermodynamical properties, one is led to consider only models whose scattering function can be analytically continued to a slightly larger region, as the following heuristic argument suggests.

Poles of S_2 lying in the strip $S(-\pi, 0)$ are usually interpreted as evidence for unstable particles with a finite lifetime [21]. The lifetime of such a resonance becomes arbitrarily long if the corresponding pole lies sufficiently close to the real axis. In fact, it is possible to choose a sequence of zeros β_k (which is accompanied by a sequence of poles at $-\beta_k$ because of $S_2(-\theta) = S_2(\theta)^{-1}$) in accordance with the properties i)-iv) listed in Prop. 3.1 a) such that the scattering function exhibits infinitely many resonances with arbitrarily long lifetimes and “masses” m_k so that $\sum_k e^{-m_k/T}$ diverges for all temperatures $T > 0$.

But a model with these characteristics cannot be expected to have a regular thermodynamical behaviour or only a finite partition function [14, 12]. Given the relation of the modular nuclearity condition to the thermodynamically motivated energy nuclearity condition [11, 14], we therefore expect the maps $\Xi(x)$ (2.19) *not* to be nuclear in a model with the previously described distribution of poles in its scattering function (although there might still exist local observables even in this situation). To exclude such models, we require S_2 to have only a finite number of poles in $S(0, \pi)$. In this case $\kappa(S_2) > 0$, and hence all singularities of S_2 lie a finite distance off the real axis so that the lifetimes of all resonances are bounded from above. Hence regular thermodynamical properties can be expected, and the modular nuclearity condition might be satisfied.

In addition to this requirement we make a second restriction on the scattering function, which is of a more technical nature and can already be found in [28]. Namely, we require that the phase shift $\frac{1}{2i} \frac{\log S_2(\zeta)}{\sinh \zeta}$ goes to zero if $|\operatorname{Re} \zeta| \rightarrow \infty$ within in $S(0, \pi)$. In view of the factorization formula (3.1), this amounts to demanding that the exponential factor is absent, *i.e.* $a = 0$ in (3.1).

The definition of the restricted family of scattering functions \mathcal{S}_0 comprises these two requirements:

Definition 3.2. $\mathcal{S}_0 := \{S_2 \in \mathcal{S} : S_2 \text{ has finitely many zeros in } S(0, \pi), a = 0 \text{ in (3.1)}\}$.

We point out that \mathcal{S}_0 contains the scattering function of the Sinh-Gordon model, which consists of a single factor in the product (3.1), with purely imaginary β_1 [1, 24].

According to Prop. 3.1 b), each scattering function $S_2 \in \mathcal{S}_0$ is analytic and bounded in a strip $S(-\kappa, \pi + \kappa)$, $0 < \kappa < \kappa(S_2) \leq \pi$, slightly wider than the physical region $S(0, \pi)$. (The cutoff at π is introduced in (3.2) to obtain a finite value of $\kappa(S_2)$ for each $S_2 \in \mathcal{S}$.) Fixing $\kappa \in (0, \kappa(S_2))$, we denote this bound by

$$\|S_2\|_\kappa := \sup \{|S_2(\zeta)| : \zeta \in \overline{S(-\kappa, \pi + \kappa)}\} < \infty. \quad (3.3)$$

Whereas $S_2 \in \mathcal{S}$ is always tacitly assumed, we will in the rest of the paper point out explicitly when the stronger assumption $S_2 \in \mathcal{S}_0$ is used.

4 Analytic properties of wedge-local wavefunctions

As a prerequisite for the verification of the modular nuclearity condition, we will in this section study properties of state vectors $A\Omega$, $A \in \mathcal{A}(W_R)$, which reflect the localization of A in the right wedge and the boundedness $\|A\| < \infty$. In the setting of the models described in Sect. 2, $A\Omega$ is given by a sequence of n -particle rapidity wavefunctions

$$(A\Omega)_n := P_n A\Omega \in \mathcal{H}_n \subset L^2(\mathbb{R}^n), \quad n \in \mathbb{N}_0. \quad (4.1)$$

(P_n denotes the orthogonal projection $P_n : \bigoplus_{n=0}^\infty \mathcal{H}_1^{\otimes n} \rightarrow \mathcal{H}_n$.) The localization of A corresponds to a support restriction in position space and thus to certain analytic properties of the wavefunctions $(A\Omega)_n(\theta_1, \dots, \theta_n)$ and improper matrix elements $\langle \theta_1, \dots, \theta_k | A | \theta'_1, \dots, \theta'_l \rangle$ in rapidity space. In addition, the boundedness of $\|A\|$ puts bounds on the analytic continuations of these functions, and the analyticity and boundedness of the scattering

function $S_2 \in \mathcal{S}_0$ further enhance these features. After some steps of successive analytic continuation, we arrive at the main result of this section: $(A\Omega)_n$ extends to a bounded analytic function in a tube domain in \mathbb{C}^n , the shape of which depends on the underlying scattering function (Prop. 4.4).

To begin with, we study the analytic properties of matrix elements of commutators of the Zamolodchikov type creation and annihilation operators (2.9, 2.10) with $A \in \mathcal{A}(W_R)$.

Lemma 4.1. *Let $A \in \mathcal{A}(W_R)$, $n_1, n_2 \in \mathbb{N}_0$, $\Psi_1 \in \mathcal{H}_{n_1}$, $\Psi_2 \in \mathcal{H}_{n_2}$ and consider the two functionals $C, C^\dagger : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$,*

$$C(f) := \langle \Psi_1, [z(\hat{f}), A] \Psi_2 \rangle, \quad C^\dagger(f) := \langle \Psi_1, [z^\dagger(\hat{f}), A] \Psi_2 \rangle, \quad (4.2)$$

where $\hat{f}(\theta) := \tilde{f}(m \sinh \theta)$.

There exists a function $\hat{C} : \overline{S(-\pi, 0)} \rightarrow \mathbb{C}$ (depending linearly on Ψ_2 and A , and conjugate linearly on Ψ_1) with the following properties:

- a) \hat{C} is analytic in $S(-\pi, 0)$.
- b) Let $c(n_1, n_2) := \sqrt{2n_1 + 1} + \sqrt{2n_2 + 1}$. Then

$$\left[\int_{\mathbb{R}} d\theta |\hat{C}(\theta - i\lambda)|^2 \right]^{1/2} \leq c(n_1, n_2) \|\Psi_1\| \|\Psi_2\| \|A\|, \quad 0 \leq \lambda \leq \pi. \quad (4.3)$$

For $\lambda \searrow 0$ and $\lambda \nearrow \pi$, $\hat{C}(\cdot - i\lambda)$ converges in the norm topology of $L^2(\mathbb{R})$ to its boundary values.

- c) Let $f \in \mathcal{S}(\mathbb{R})$. There holds

$$C(f) = \int_{\mathbb{R}} d\theta \hat{C}(\theta) \hat{f}(\theta), \quad C^\dagger(f) = - \int_{\mathbb{R}} d\theta \hat{C}(\theta - i\pi) \hat{f}(\theta). \quad (4.4)$$

Proof. The time zero fields φ, π of ϕ (2.12) are given by, $f \in \mathcal{S}(\mathbb{R})$,

$$\varphi(f) := z^\dagger(\hat{f}) + z(\hat{f}_-), \quad \hat{f}(\theta) := \tilde{f}(m \sinh \theta), \quad (4.5)$$

$$\pi(f) := i(z^\dagger(\omega \hat{f}) - z(\omega \hat{f}_-)), \quad \hat{f}_-(\theta) := \hat{f}(-\theta). \quad (4.6)$$

Here ω denotes the one particle Hamiltonian, which acts as the (unbounded) multiplication operator $\omega(\theta) = m \cosh \theta$. To obtain information about the functionals C, C^\dagger , we will study the corresponding matrix elements of the semi-local fields φ, π first. As the creation and annihilation operators are linear combinations of these fields, the statements of the Lemma can then be deduced easily.

The operators $\varphi(f), \pi(f)$ are well-defined on the subspace $\mathcal{D} \subset \mathcal{H}$ of finite particle number and satisfy $\varphi(f)^* \supset \varphi(\bar{f})$, $\pi(f)^* \supset \pi(\bar{f})$. In particular, they are hermitian for real f . Along the same lines as in [31, Prop. 1, Prop. 2], one can prove that $\varphi(f), \pi(f)$ have selfadjoint closures (for real f) which are affiliated with $\mathcal{A}(W_R)' = \mathcal{A}(W_L)$ if $\text{supp } f \subset \mathbb{R}_-$. Hence the two functionals

$$C_-(f) := \langle \Psi_1, [\varphi(f), A] \Psi_2 \rangle, \quad C_+(f) := \langle \Psi_1, [\pi(f), A] \Psi_2 \rangle, \quad (4.7)$$

are well-defined and vanish for $\text{supp } f \subset \mathbb{R}_-$.

To derive bounds on $|C_\pm(f)|$, we first note (with $\|f\|_2 := (\int dx |f(x)|^2)^{1/2}$)

$$\|\omega^{1/2} \hat{f}\|^2 = \int d\theta m \cosh \theta |\hat{f}(\theta)|^2 = \int dp |\tilde{f}(p)|^2 = \|f\|_2^2, \quad f \in L^2(\mathbb{R}, dx). \quad (4.8)$$

Combining this equation with (2.11) and (4.5), and taking into account the Fock structure of \mathcal{H} , we obtain, $j = 1, 2$,

$$\|\varphi(\omega^{1/2} f) \Psi_j\|^2 = \|z^\dagger(\omega^{1/2} \hat{f}) \Psi_j\|^2 + \|z(\omega^{1/2} \hat{f}_-) \Psi_j\|^2 \leq (2n_j + 1) \|\Psi_j\|^2 \|f\|_2^2. \quad (4.9)$$

So the Schwarz inequality gives

$$\begin{aligned} |C_-(\omega^{1/2} f)| &\leq \|\varphi(\omega^{1/2} \bar{f}) \Psi_1\| \|A \Psi_2\| + \|A^* \Psi_1\| \|\varphi(\omega^{1/2} f) \Psi_2\| \\ &\leq (\sqrt{2n_1 + 1} + \sqrt{2n_2 + 1}) \|\Psi_1\| \|\Psi_2\| \|A\| \|f\|_2. \end{aligned}$$

Applying an analogous argument to π yields the same bound for $|C_+(\omega^{-1/2} f)|$. Thus

$$\frac{|C_\pm(\omega^{\mp 1/2} f)|}{\|f\|_2} \leq c(n_1, n_2) \|\Psi_1\| \|\Psi_2\| \|A\|, \quad c(n_1, n_2) := \sqrt{2n_1 + 1} + \sqrt{2n_2 + 1}. \quad (4.10)$$

These estimates imply that C_+ and C_- are tempered distributions, and moreover the functionals $f \mapsto C_\pm(\omega^{\mp 1/2} f)$ are given by integration with functions in $L^2(\mathbb{R})$ whose norm is bounded by $c(n_1, n_2) \|\Psi_1\| \|\Psi_2\| \|A\|$. In particular, the Fourier transforms \tilde{C}_\pm exist as well-defined functions. Since $\text{supp } C_\pm \subset \mathbb{R}_+$, these functions have an analytic continuation to the lower half plane, satisfying polynomial bounds at the boundary and at infinity [40, Thm. IX.16].

In the rapidity picture, we consider

$$\hat{C}_+(\theta) := \tilde{C}_+(m \sinh \theta), \quad \hat{C}_-(\theta) := m \cosh \theta \cdot \tilde{C}_-(m \sinh \theta). \quad (4.11)$$

As $\sinh(\cdot)$ maps $S(-\pi, 0)$ to the lower half plane, \hat{C}_\pm are analytic in this strip. In view of the estimate on $|C_\pm(\omega^{\mp 1/2} f)|$, the boundary values (4.11) are functions in $L^2(\mathbb{R}, d\theta)$, with norm bounded by $c(n_1, n_2) \|\Psi_1\| \|\Psi_2\| \|A\|$. Note also the reflection symmetry about $\mathbb{R} - \frac{i\pi}{2}$:

$$\hat{C}_\pm(\theta - \frac{i\pi}{2} + i\mu) = \pm \hat{C}_\pm(-\theta - \frac{i\pi}{2} - i\mu), \quad -\frac{\pi}{2} \leq \mu \leq \frac{\pi}{2}. \quad (4.12)$$

For the proof of part b) we need an estimate on the L^2 -norm of $\theta \mapsto \hat{C}_\pm(\theta - i\lambda)$. To this end we consider the functions $\hat{C}_\pm^{(s)}(\zeta) := e^{-ims \sinh \zeta} \cdot \hat{C}_\pm(\zeta)$, $s > 0$, which decay rapidly for $|\theta| \rightarrow \infty$, $0 < \lambda < \pi$:

$$|\hat{C}_\pm^{(s)}(\theta - i\lambda)| = |e^{-ims \sinh(\theta - i\lambda)} \hat{C}_\pm(\theta - i\lambda)| = e^{-ms \sin \lambda \cosh \theta} |\hat{C}_\pm(\theta - i\lambda)|. \quad (4.13)$$

As $|\hat{C}_\pm(\theta - i\lambda)|$ is bounded by a polynomial in $\cosh \theta$ for $|\theta| \rightarrow \infty$, we have $\hat{C}_\pm^{(s)}(\cdot - i\lambda) \in L^2(\mathbb{R}, d\theta)$ for all $\lambda \in [0, \pi]$ and $s > 0$. In view of the previous estimates and (4.12), $\|\hat{C}_\pm\|_2$

and $\|\hat{C}_\pm(\cdot - i\pi)\|_2$ are bounded by $c(n_1, n_2)\|\Psi_1\|\|\Psi_2\|\|A\|$. So a variant of the three lines theorem [33, Lemma A.1] can be applied and we get

$$\|\hat{C}_\pm^{(s)}(\cdot - i\lambda)\|_2 \leq c(n_1, n_2)\|\Psi_1\|\|\Psi_2\|\|A\|, \quad 0 \leq \lambda \leq \pi. \quad (4.14)$$

But as (4.13) is monotonically increasing as $s \rightarrow 0$, this uniform bound holds also for $\hat{C}_\pm(\cdot - i\lambda)$, $0 \leq \lambda \leq \pi$. Together with the convergence of $\hat{C}_\pm(\cdot - i\lambda)$ in the sense of distributions to its boundary values, this implies that statement b) of the Lemma holds true for \hat{C}_\pm .

Having established the necessary features of \hat{C}_\pm , we now turn to the function \hat{C} from the claims of the Lemma. It is defined as

$$\hat{C}(\zeta) := \frac{1}{2}(\hat{C}_-(\zeta) + i\hat{C}_+(\zeta)), \quad \zeta \in \overline{S(-\pi, 0)}, \quad (4.15)$$

and inherits properties a) and b) from \hat{C}_\pm . To show c), we solve (4.5, 4.6) with respect to $z^\dagger(\hat{f})$, $z(\hat{f}_-)$ and get $z(\hat{f}) = \frac{1}{2}(\varphi(f_-) + i\pi(\omega^{-1}f_-))$, $f_-(x) = f(-x)$,

$$\begin{aligned} C(f) &= \frac{1}{2}(C_-(f_-) + iC_+(\omega^{-1}f_-)) = \frac{1}{2} \int dp \left(\tilde{C}_-(p) + \frac{i\tilde{C}_+(p)}{\sqrt{p^2 + m^2}} \right) \tilde{f}(p) \\ &= \frac{1}{2} \int d\theta (\hat{C}_-(\theta) + i\hat{C}_+(\theta)) \hat{f}(\theta) = \int d\theta \hat{C}(\theta) \hat{f}(\theta). \end{aligned}$$

For the creation operator we have $z^\dagger(\hat{f}) = \frac{1}{2}(\varphi(f) - i\pi(\omega^{-1}f))$ and, by taking into account $\hat{C}_\pm(\theta - i\pi) = \pm\hat{C}_\pm(-\theta)$ (4.12),

$$\begin{aligned} C^\dagger(f) &= \frac{1}{2}(C_-(f) - iC_+(\omega^{-1}f)) = \frac{1}{2} \int d\theta (\hat{C}_-(-\theta) - i\hat{C}_+(-\theta)) \hat{f}(\theta) \\ &= -\frac{1}{2} \int d\theta (\hat{C}_-(\theta - i\pi) + i\hat{C}_+(\theta - i\pi)) \hat{f}(\theta) = - \int d\theta \hat{C}(\theta - i\pi) \hat{f}(\theta). \end{aligned}$$

These calculations complete the proof of the Lemma. \square

Lemma 4.1 can be used to derive analytic properties of single particle wavefunctions $(A\Omega)_1$ corresponding to operators $A \in \mathcal{A}(W_R)$. In fact, by putting $\Psi_1 = \Psi_2 = \Omega$ in (4.2) we obtain, $\hat{f} \in L^2(\mathbb{R}, d\theta)$,

$$\int d\theta \hat{f}(\theta) \hat{C}(\theta) = \langle \Omega, [z(\hat{f}), A] \Omega \rangle = \langle \bar{\hat{f}}, A\Omega \rangle = \int d\theta \hat{f}(\theta) (A\Omega)_1(\theta).$$

Hence the single particle wavefunctions of operators localized in W_R enjoy the same analytic properties as the function \hat{C} specified in Lemma 4.1. Using the relations of Zamolodchikov's algebra and the analyticity of the scattering function, this result will be generalized to a statement about joint analyticity in n complex variables for the n -particle functions $(A\Omega)_n$ in the following.

To save space, we will throughout use the shorthand notations

$$z_k := z(\theta_k), \quad z_k^\dagger := z^\dagger(\theta_k), \quad S_{r,k} := S_2(\theta_r - \theta_k), \quad \delta_{r,k} := \delta(\theta_r - \theta_k). \quad (4.16)$$

Improper matrix elements of $A \in \mathcal{A}(W_R)$ are defined as the kernels of

$$\mathcal{S}(\mathbb{R}^{k+l}) \ni f_1 \otimes \dots \otimes f_k \otimes f'_1 \otimes \dots \otimes f'_l \longmapsto \langle z^\dagger(\overline{f_1}) \dots z^\dagger(\overline{f_k}) \Omega, A z^\dagger(f'_1) \dots z^\dagger(f'_l) \Omega \rangle$$

and denoted $\langle \theta_1, \dots, \theta_k | A | \theta'_1, \dots, \theta'_l \rangle := \langle z^\dagger_1, \dots, z^\dagger_k \Omega, A z^\dagger_{l'}, \dots, z^\dagger_{l'} \Omega \rangle$. Taking into account $\|A\| < \infty$ and the bounds (2.11), it follows from the nuclear theorem that these are well-defined tempered distributions of order zero in $\mathcal{S}(\mathbb{R}^{k+l})'$. The normalization of the creation operators z^\dagger is chosen in such a way that there holds [31]

$$(A\Omega)_n(\theta_1, \dots, \theta_n) = \frac{1}{\sqrt{n!}} \langle \theta_1, \dots, \theta_n | A\Omega \rangle = \frac{1}{\sqrt{n!}} \langle z^\dagger_1 \dots z^\dagger_n \Omega, A\Omega \rangle. \quad (4.17)$$

Lemma 4.2. *Let $A \in \mathcal{A}(W_R)$, $n \in \mathbb{N}_0$, $k \in \{0, \dots, n\}$ and consider*

$$\begin{aligned} \mathcal{D}_{n,k}(\theta_1, \dots, \theta_n) &:= \sum_{N=0}^k \sum_{1 \leq k_1 < \dots < k_N \leq k} (-1)^N \prod_{j=1}^N \prod_{r_j=k_j+1}^k S_{r_j, k_j} \times \\ &\quad \times \langle z_{k_N} \dots z_{k_1} z_{k_1+1}^\dagger \dots z_n^\dagger \Omega, A z_k^\dagger \dots \widehat{z_{k_N}^\dagger} \dots \widehat{z_{k_1}^\dagger} \dots z_1^\dagger \Omega \rangle, \end{aligned} \quad (4.18)$$

where the hats on $\widehat{z_{k_1}^\dagger}, \dots, \widehat{z_{k_N}^\dagger}$ indicate omission of the corresponding creation operators.

a) $\mathcal{D}_{n,k}$ is a tempered distribution on $\mathcal{S}(\mathbb{R}^n)$. For $k=0$ and $k=n$ we have

$$\mathcal{D}_{n,0}(\theta_1, \dots, \theta_n) = \langle \theta_1, \dots, \theta_n | A\Omega \rangle, \quad (4.19)$$

$$\mathcal{D}_{n,n}(\theta_1, \dots, \theta_n) = \langle A^* \Omega | \theta_n, \dots, \theta_1 \rangle. \quad (4.20)$$

b) $\mathcal{D}_{n,k}$ has an analytic continuation in the variable θ_{k+1} to the strip $S(-\pi, 0)$ for $k \leq n-1$, with boundary values in the sense of distributions. At $\text{Im}(\theta_{k+1}) = -\pi$ we find

$$\mathcal{D}_{n,k}(\theta_1, \dots, \theta_k, \theta_{k+1} - i\pi, \theta_{k+2}, \dots, \theta_n) = \mathcal{D}_{n,k+1}(\theta_1, \dots, \theta_n). \quad (4.21)$$

c) For $0 \leq \lambda \leq \pi$, $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R})$, there holds the bound

$$\left| \int d^n \theta \mathcal{D}_{n,k}(\theta_1, \dots, \theta_{k+1} - i\lambda, \dots, \theta_n) \prod_{j=1}^n f_j(\theta_j) \right| \leq \sqrt{2} \sqrt{n!} 2^{n+1} \|A\| \prod_{j=1}^n \|f_j\|_2. \quad (4.22)$$

Before presenting the proof of this Lemma, we point out that the analytic properties of $(A\Omega)_n$ established here strenghten what can be deduced from modular theory: By successive application of b) to $\mathcal{D}_{n,0}$, it follows that (along a certain path) $\mathcal{D}_{n,0}$ has an analytic continuation from \mathbb{R}^n to $\mathbb{R}^n - i(\pi, \dots, \pi)$. The corresponding boundary value is according to a) given by

$$\begin{aligned} (A\Omega)_n(\theta_1 - i\pi, \dots, \theta_n - i\pi) &= \frac{1}{\sqrt{n!}} \mathcal{D}_{n,0}(\theta_1 - i\pi, \dots, \theta_n - i\pi) = \frac{1}{\sqrt{n!}} \mathcal{D}_{n,n}(\theta_1, \dots, \theta_n) \\ &= \overline{(A^* \Omega)_n(\theta_n, \dots, \theta_1)} = (JA^* \Omega)_n(\theta_1, \dots, \theta_n), \end{aligned} \quad (4.23)$$

where J denotes the modular conjugation of $(\mathcal{A}(W_R), \Omega)$ (2.16). The very left hand side of (4.23) is $(\Delta^{1/2} A \Omega)_n(\theta_1, \dots, \theta_n)$ (cf. (2.15)). By modular theory, this coincides with $(J A^* \Omega)_n(\theta_1, \dots, \theta_n)$, in accordance with the results of the Lemma.

But whereas the analyticity of $\zeta \mapsto \Delta^{i\zeta} A \Omega$ in $S(-\frac{1}{2}, 0)$ implies analyticity of $(A \Omega)_n$ only in the single variable $n^{-1}(\theta_1 + \dots + \theta_n)$ in the strip $S(-\pi, 0)$, the stronger results of Lemma 4.2 lead to a statement about analyticity of $(A \Omega)_n$ in the n complex variables $\theta_1, \dots, \theta_n$ in a certain tube domain, which is formulated as a corollary below (Cor. 4.3).

Proof of Lemma 4.2. a) As the matrix elements in (4.18) are distributions of order zero, and the products of scattering functions are bounded and continuous, $\mathcal{D}_{n,k} \in \mathcal{S}(\mathbb{R}^n)'$ is well-defined.

To prove (4.19), note that for $k = 0$, the sum in (4.18) consists only of the term corresponding to $N = 0$, and the product of scattering functions also drops out. So

$$\mathcal{D}_{n,0}(\theta_1, \dots, \theta_n) = \langle z_1^\dagger \cdots z_n^\dagger \Omega, A \Omega \rangle = \langle \theta_1, \dots, \theta_n | A \Omega \rangle.$$

For $k = n$, all terms in (4.18) with $N > 0$ are zero because the left hand side of the scalar product vanishes: $z_{k_N} \cdots z_{k_1} \Omega = 0$. Thus we are again left with the term corresponding to $N = 0$, which is

$$\mathcal{D}_{n,n}(\theta_1, \dots, \theta_n) = \langle \Omega, A z_n^\dagger \cdots z_1^\dagger \Omega \rangle = \langle A^* \Omega | \theta_n, \dots, \theta_1 \rangle.$$

For the proof of b), we rely on an alternative formula for $\mathcal{D}_{n,k}$, namely

$$\begin{aligned} \mathcal{D}_{n,k}(\theta_1, \dots, \theta_n) &= \sum_{\mathbf{k}} (-1)^N \prod_{\substack{1 \leq j \leq N \\ k_j + 1 \leq r_j \leq k}} S_{r_j, k_j} \cdot \prod_{t=1}^N S_{k_t, k+1} \times \\ &\quad \times \langle z_{k_N} \cdots z_{k_1} z_{k+2}^\dagger \cdots z_n^\dagger \Omega, [z_{k+1}, A] z_k^\dagger \cdots \widehat{z_{k_N}^\dagger} \cdots \widehat{z_{k_1}^\dagger} \cdots z_1^\dagger \Omega \rangle. \end{aligned} \quad (4.24)$$

Here $\sum_{\mathbf{k}}$ abbreviates the sum over all subsets $\{k_1, \dots, k_N\}$ of $\{0, \dots, k\}$, which is also present in (4.18). The identity of (4.18) and (4.24) is proven in the appendix by a calculation based on Zamolodchikov's algebra.

The distribution (4.24) consists of a product of scattering functions and a matrix element of $[z_{k+1}, A]$. The product depends on θ_{k+1} only through the factors $S_2(\theta_{k_t} - \theta_{k+1})$, which are analytic in this variable in the strip $S(-\pi, 0)$ if the other rapidities are held fixed and real. In view of the crossing symmetry (2.3), the boundary value at $\text{Im} \theta_{k+1} = -\pi$ is $S_2(\theta_{k_t} - (\theta_{k+1} - i\pi)) = S_2(\theta_{k+1} - \theta_{k_t})$.

The matrix element in (4.24) depends on θ_{k+1} only through the commutator $[z(\theta_{k+1}), A]$. If the variables $\theta_1, \dots, \theta_k, \theta_{k+2}, \dots, \theta_n$ are smeared out with testfunctions, the matrix element distribution is the boundary value of a function analytic in $S(-\pi, 0)$, and its boundary value at $\mathbb{R} - i\pi$ is given by the same expression, with the commutator $[z_{k+1}, A]$ replaced by $-[z_{k+1}^\dagger, A]$ (Lemma 4.1). Consequently (4.24) is the boundary value of a function analytic

in $\theta_{k+1} \in S(-\pi, 0)$, and the boundary value on the other side of the strip is

$$\begin{aligned}
& - \sum_{\mathbf{k}} (-1)^N \prod_{\substack{1 \leq j \leq N \\ k_j+1 \leq r_j \leq k}} S_{r_j, k_j} \cdot \prod_{t=1}^N S_{k+1, k_t} \cdot \langle z_{k_N} \cdots z_{k_1} z_{k+2}^\dagger \cdots z_n^\dagger \Omega, [z_{k+1}^\dagger, A] z_k^\dagger \cdots \widehat{z_{k_N}^\dagger} \cdots \widehat{z_{k_1}^\dagger} \cdots z_1^\dagger \Omega \rangle \\
& = \sum_{\mathbf{k}} (-1)^N \prod_{j=1}^N \prod_{r_j=k_j+1}^{k+1} S_{r_j, k_j} \cdot \left(\langle z_{k_N} \cdots z_{k_1} z_{k+2}^\dagger \cdots z_n^\dagger \Omega, A z_{k+1}^\dagger \cdots \widehat{z_{k_N}^\dagger} \cdots \widehat{z_{k_1}^\dagger} \cdots z_1^\dagger \Omega \rangle \right. \\
& \quad \left. - \langle z_{k+1} z_{k_N} \cdots z_{k_1} z_{k+2}^\dagger \cdots z_n^\dagger \Omega, A z_k^\dagger \cdots \widehat{z_{k_N}^\dagger} \cdots \widehat{z_{k_1}^\dagger} \cdots z_1^\dagger \Omega \rangle \right). \quad (4.25)
\end{aligned}$$

We now compare (4.25) with the claimed expression (4.18) for $k+1$, where the sum $\sum_{\mathbf{k}}$ runs over all subsets of $\{1, \dots, k+1\}$ with N elements $k_1 < \dots < k_N$, $N \in \{0, \dots, k+1\}$. The first term in (4.25) gives those summands of (4.18) in which $k_N < k+1$. (In particular the term corresponding to $N=0$ is, and the term corresponding to $N=k+1$ is *not* included here.) The second term of (4.25) gives those summands of (4.18) for which $k_N = k+1$. So part b) of the Lemma has been proven.

To show c), we again consider $\mathcal{D}_{n,k}$ as given by (4.24). Note that for $\theta_1, \dots, \theta_n \in \mathbb{R}$ and $0 \leq \lambda \leq \pi$, the product of scattering functions in $\mathcal{D}_{n,k}(\theta_1, \dots, \theta_{k+1} - i\lambda, \dots, \theta_n)$ is bounded by unity as a consequence of the three lines theorem. The matrix element in (4.24), smeared with $\mathbf{f}(\theta_1, \dots, \theta_n) := f_1(\theta_1) \cdots f_n(\theta_n)$, $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R})$, can be estimated with the bound from Lemma 4.1 c). With the notations used there, we put

$$\Psi_1(\mathbf{f}) := z(\overline{f_{k_N}}) \cdots z(\overline{f_{k_1}}) z^\dagger(\overline{f_{k+2}}) \cdots z^\dagger(\overline{f_n}) \Omega, \quad (4.26)$$

$$\Psi_2(\mathbf{f}) := z^\dagger(f_k) \cdots \widehat{z^\dagger(f_{k_N})} \cdots \widehat{z^\dagger(f_{k_1})} \cdots z^\dagger(f_1) \Omega. \quad (4.27)$$

Thus $\Psi_1 \in \mathcal{H}_{n-k-N-1}$ and $\Psi_2 \in \mathcal{H}_{k-N}$. The bounds (2.11) lead to

$$\|\Psi_1(\mathbf{f})\| \cdot \|\Psi_2(\mathbf{f})\| \leq \sqrt{(n-1)!} \|f_1\|_2 \cdots \|f_n\|_2,$$

and the constant from Lemma 4.1 c) can be estimated as $c(n-k-N-1, k-N) \leq 2\sqrt{2n}$. According to (4.3),

$$|\langle \Psi_1(\mathbf{f}), [z^\dagger(f_{k+1}), A] \Psi_2(\mathbf{f}) \rangle| \leq 2\sqrt{2} \sqrt{n!} \|f_1\|_2 \cdots \|f_n\|_2 \|A\|. \quad (4.28)$$

As $\sum_{\mathbf{k}}$ is a sum of 2^n terms and the scattering functions are bounded by 1, we arrive at

$$\left| \int d^n \theta \mathcal{D}_{n,k}(\theta_1, \dots, \theta_{k+1} - i\lambda, \dots, \theta_n) \prod_{j=1}^n f_j(\theta_j) \right| \leq \sqrt{2} \sqrt{n!} 2^{n+1} \|f_1\|_2 \cdots \|f_n\|_2 \|A\|,$$

which is the claimed bound (4.22). \square

By appeal to the Malgrange-Zerner ('flat tube') theorem, the separate analytic continuations in the single variables $\theta_1, \dots, \theta_n$, as specified in part b) of the preceding Lemma, can be joined to an analytic function of n complex variables. In the following, we use the same symbol $(A\Omega)_n$ for the analytic continuation of the wavefunction (4.1) in order not to overburden our notation.

Corollary 4.3. *Let $A \in \mathcal{A}(W_R)$ and define*

$$\mathcal{T}_n := \mathbb{R}^n - i\Lambda_n, \quad \Lambda_n := \{\boldsymbol{\lambda} \in \mathbb{R}^n : \pi > \lambda_1 > \lambda_2 > \dots > \lambda_n > 0\}. \quad (4.29)$$

- a) $(A\Omega)_n$ has an analytic continuation to the tube \mathcal{T}_n . The wavefunctions (4.1) are recovered from $(A\Omega)_n(\cdot - i\boldsymbol{\lambda})$ as a limit in $\mathcal{S}(\mathbb{R}^n)'$ for $\boldsymbol{\lambda} \rightarrow 0$ in Λ_n .
- b) Let $d_\infty(\boldsymbol{\lambda}, \partial\Lambda_n)$ denote the distance of $\boldsymbol{\lambda}$ from the boundary of $\overline{\Lambda_n}$, measured in maximum norm. There holds the bound, $\boldsymbol{\theta} \in \mathbb{R}^n$, $\boldsymbol{\lambda} \in \Lambda_n$,

$$|(A\Omega)_n(\boldsymbol{\theta} - i\boldsymbol{\lambda})| \leq 2\sqrt{2} (4\sqrt{2}\pi^{-1})^n d_\infty(\boldsymbol{\lambda}, \partial\Lambda_n)^{-n/2} \cdot \|A\|. \quad (4.30)$$

Proof. a) Let $\boldsymbol{f} \in \mathcal{S}(\mathbb{R}^n)$. We claim that the convolution $(A\Omega)_n * \boldsymbol{f}$, considered as a function of $\theta_1, \dots, \theta_k$, with $\theta_{k+1}, \dots, \theta_n \in \mathbb{R}$ fixed, is analytic in the tube $\mathbb{R}^k - i\Lambda_k$ and continuous on its closure. Our proof is based on induction in $k \in \{1, \dots, n\}$. In view of $\sqrt{n!}(A\Omega)_n = \mathcal{D}_{n,0}$ (4.19) and $\mathbb{R}^1 - i\Lambda_1 = S(-\pi, 0)$, the claim for $k = 1$ follows immediately from Lemma 4.2 b).

So assume analyticity of $(\theta_1, \dots, \theta_k) \mapsto ((A\Omega)_n * \boldsymbol{f})(\theta_1, \dots, \theta_n)$ in $\mathbb{R}^k - i\Lambda_k$. According to Lemma 4.2, the boundary value at $\text{Im}\theta_1 = \dots = \text{Im}\theta_k = -\pi$ is given by $\mathcal{D}_{n,k} * \boldsymbol{f}/\sqrt{n!}$, which in turn has an analytic continuation in $\theta_{k+1} \in S(-\pi, 0)$. By application of the flat tube theorem (*cf.*, for example, [23]), we conclude that $(A\Omega)_n * \boldsymbol{f}$, considered as a function of the first $k+1$ variables, has an analytic continuation to the convex closure of the set

$$\mathbb{R}^{k+1} - i(\{(\lambda_1, \dots, \lambda_k, 0) : (\lambda_1, \dots, \lambda_k) \in \Lambda_k\} \cup \{(\pi, \dots, \pi, \lambda_{k+1}) : \pi > \lambda_{k+1} > 0\}), \quad (4.31)$$

which coincides with $\mathbb{R}^{k+1} - i\Lambda_{k+1}$. So by induction, $(A\Omega)_n * \boldsymbol{f}$ is analytic on \mathcal{T}_n and continuous on $\overline{\mathcal{T}_n}$. As \boldsymbol{f} was arbitrary, statement a) follows.

To prove b), let $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R})$ and put $\boldsymbol{f} := f_1 \otimes \dots \otimes f_n$. Lemma 4.2 c) implies that at points $\boldsymbol{\theta} - i\boldsymbol{\lambda} \in \overline{\mathcal{T}_n}$ whose imaginary part vector is of the form $\boldsymbol{\lambda} = (\pi, \dots, \pi, \lambda_{k+1}, 0, \dots, 0)$, $0 \leq \lambda_{k+1} \leq \pi$, there holds the bound

$$|((A\Omega)_n * \boldsymbol{f})(\boldsymbol{\theta} - i\boldsymbol{\lambda})| \leq 2^{n+1}\sqrt{2} \|A\| \prod_{j=1}^n \|f_j\|_2. \quad (4.32)$$

By a standard argument (*cf.*, for example the proof of [33, Lemma A.2]), this bound can be seen to hold for arbitrary $\boldsymbol{\lambda} \in \overline{\Lambda_n}$.

To establish (4.30), let $\boldsymbol{\theta} \in \mathbb{R}^n$, $\boldsymbol{\lambda} \in \Lambda_n$ and consider a polydisc $\mathcal{D}_n(\boldsymbol{\zeta})$ with center $\boldsymbol{\zeta} = \boldsymbol{\theta} - i\boldsymbol{\lambda}$ and sufficiently small radius ρ such that $\mathcal{D}_n(\boldsymbol{\zeta}) \subset \boldsymbol{\zeta} + [-\rho, \rho]^{\times n} + i[-\rho, \rho]^{\times n} \subset \mathcal{T}_n$. Let χ_ρ denote a smoothed characteristic function of $[-\rho, \rho]$. By the mean value property for analytic functions and application of (4.32), we have

$$\begin{aligned} |(A\Omega)_n(\boldsymbol{\theta} - i\boldsymbol{\lambda})| &\leq (\pi\rho^2)^{-n} \int_{\mathcal{D}_n(\boldsymbol{\zeta})} d^n\boldsymbol{\theta}' d^n\boldsymbol{\lambda}' |(A\Omega)_n(\boldsymbol{\theta}' + i\boldsymbol{\lambda}')| \\ &\leq (\pi\rho^2)^{-n} \int_{[-\rho, \rho]^{\times n}} d^n\boldsymbol{\lambda}' |((A\Omega)_n * \chi_\rho \otimes \dots \otimes \chi_\rho)(\boldsymbol{\theta} + i(\boldsymbol{\lambda}' - \boldsymbol{\lambda}))| \\ &\leq 2\sqrt{2} \left(\frac{4}{\pi\rho}\right)^n \|A\| \|\chi_\rho\|_2^n \end{aligned}$$

Letting χ_ρ tend to the characteristic function of $[-\rho, \rho]$ and

$$\rho \rightarrow d_\infty(\boldsymbol{\lambda}, \partial\Lambda_n) = \max_{k=1, \dots, n} \sup\{|\lambda_k - \mu_k| : \boldsymbol{\mu} \in \overline{\partial\Lambda_n}\} \quad (4.33)$$

yields (4.30). \square

To further enlarge the domain of analyticity of $(A\Omega)_n$ and maintain a similar bound in the enlarged region, more detailed information about the form of the scattering function S_2 is needed. In contrast to the preceding results, the assumption $S_2 \in \mathcal{S}_0$ will be essential for our further analysis. Recall that for $S_2 \in \mathcal{S}_0$, there exists $\kappa \in (0, \kappa(S_2))$ such that S_2 is analytic and bounded by $\|S_2\|_\kappa$ on $S(-\kappa, \pi + \kappa)$. With the help of these properties, much more can be said about the analytic structure of $(A\Omega)_n$.

Proposition 4.4. *Consider a scattering function $S_2 \in \mathcal{S}_0$. Let $A \in \mathcal{A}(W_R)$, and*

$$\mathcal{B}_n(\kappa) := \{\boldsymbol{\lambda} \in \mathbb{R}^n : 0 < \lambda_1, \dots, \lambda_n < \pi, \lambda_k - \lambda_l < \kappa, 1 \leq l < k \leq n\}. \quad (4.34)$$

a) *The wavefunction $(A\Omega)_n$ is analytic in the tube $\mathbb{R}^n - i\mathcal{B}_n(\kappa(S_2))$.*

b) *Let $d(\boldsymbol{\lambda}) := \min\{\lambda_k, \pi - \lambda_k : k = 1, \dots, n\}$. There holds the bound, $\kappa \in (0, \kappa(S_2))$,*

$$|(A\Omega)_n(\boldsymbol{\theta} - i\boldsymbol{\lambda})| \leq \left(\frac{c \|S_2\|_\kappa}{d(\boldsymbol{\lambda})^{1/2}} \right)^n \cdot \|A\|, \quad \boldsymbol{\theta} \in \mathbb{R}^n, \boldsymbol{\lambda} \in \mathcal{B}_n(\kappa). \quad (4.35)$$

Here c is a numerical constant, and $\|S_2\|_\kappa = \sup\{|S_2(\zeta)| : \zeta \in \overline{S(-\kappa, \pi + \kappa)}\}$ (3.3).

Proof. Let \mathfrak{S}_n denote the group of permutations of n objects and consider the “permuted wavefunctions”

$$(A\Omega)_n^\rho(\boldsymbol{\theta}) := (A\Omega_n)(\rho^{-1}\boldsymbol{\theta}) = (A\Omega)_n(\theta_{\rho(1)}, \dots, \theta_{\rho(n)}), \quad \rho \in \mathfrak{S}_n,$$

which by Cor. 4.3 are analytic in the permuted tubes $\mathcal{T}_n^\rho := \mathbb{R}^n - i\Lambda_n^\rho$,

$$\Lambda_n^\rho := \rho\Lambda_n = \{\boldsymbol{\lambda} \in \mathbb{R}^n : \pi > \lambda_{\rho(1)} > \dots > \lambda_{\rho(n)} > 0\}.$$

Starting from the exchange relation (2.6), it can be shown by induction in the number of transpositions necessary to generate a permutation, that $(A\Omega)_n^\rho$ is related to $(A\Omega)_n$ by

$$(A\Omega)_n(\boldsymbol{\theta}) = S^\rho(\boldsymbol{\theta}) \cdot (A\Omega)_n^\rho(\boldsymbol{\theta}), \quad S^\rho(\boldsymbol{\theta}) := \prod_{\substack{1 \leq l < k \leq n \\ \rho(l) > \rho(k)}} S_2(\theta_k - \theta_l). \quad (4.36)$$

As S_2 is analytic in $S(-\kappa(S_2), \pi + \kappa(S_2))$, all the functions S^ρ , $\rho \in \mathfrak{S}_n$, are analytic in the tube $\mathbb{R}^n + i\mathcal{B}'_n(\kappa(S_2))$ with base

$$\mathcal{B}'_n(\kappa(S_2)) := \{\boldsymbol{\lambda} \in \mathbb{R}^n : -\kappa(S_2) < \lambda_k - \lambda_l < \pi + \kappa(S_2), \quad 1 \leq l < k \leq n\}.$$

Hence the right hand side of the first equation in (4.36) can be analytically continued to the tube based on $\mathcal{B}'_n(\kappa(S_2)) \cap (-\Lambda_n^\rho)$. But the left hand side of (4.36) is analytic in $\mathbb{R}^n - i\Lambda_n$, and both sides converge in the sense of distributions to the same boundary

values on the real subspace. So we may apply Epstein's generalization of the Edge of the Wedge Theorem [22] and the tube theorem to conclude that $(A\Omega)_n$ has an analytic continuation to the tube whose base is the convex closure of

$$\bigcup_{\rho \in \mathfrak{E}_n} \mathcal{B}'_n(\kappa(S_2)) \cap (-\Lambda_n^\rho).$$

Since the convex closure of $\bigcup_{\rho} \Lambda_n^\rho$ is the cube $(0, \pi)^{\times n}$, it follows that $(A\Omega)_n$ is analytic in the tube based on $-(0, \pi)^{\times n} \cap \mathcal{B}'_n(\kappa(S_2)) = -\mathcal{B}_n(\kappa(S_2))$.

b) Let $\kappa \in (0, \kappa(S_2))$. The functions S^ρ are bounded on $\mathbb{R}^n - i\overline{\mathcal{B}_n(\kappa)}$ because each factor $S_2(\zeta_k - \zeta_l)$ is bounded (Prop. 3.1 b)). Since $S_2(\zeta_k - \zeta_l)$ attains its supremum at $\text{Im}(\zeta_k - \zeta_l) = -\kappa$, and S^ρ is invariant with respect to translations along the total diagonal in \mathbb{C}^n , the supremum of $|S^\rho(\zeta)|$, $\zeta \in \mathbb{R}^n - i\overline{\mathcal{B}_n(\kappa)}$, equals the supremum of $|S^\rho(\zeta)|$ when ζ varies over the tube based on

$$\mathcal{B}_n''(\kappa) := \{\lambda \in \mathbb{R}^n : \lambda_1 + \dots + \lambda_n = 0, |\lambda_k - \lambda_l| \leq \kappa, k, l = 1, \dots, n\}. \quad (4.37)$$

By the multidimensional analogue of the three lines theorem [5], the supremum of S^ρ over $\mathbb{R}^n + i\mathcal{B}_n''(\kappa)$ is attained on a subspace of the form $\mathbb{R}^n + i\xi$, where ξ is a vertex of $\mathcal{B}_n''(\kappa)$, *i.e.* $\xi_k - \xi_l = \pm\kappa$. At most $(n-1)$ of the differences $\xi_k - \xi_l$ can equal $-\kappa$ simultaneously, and in view of $|S_2(0)| = 1$, $|S_2(\kappa)| \leq 1$, we arrive at $|S^\rho(\zeta)| \leq \|S_2\|_\kappa^{n-1}$, $\zeta \in \mathbb{R}^n - i\overline{\mathcal{B}_n(\kappa)}$.

Together with (4.30), this gives a bound on $(A\Omega)_n(\zeta)$. By repeating the argument from the proof of Cor. 4.3 b), it follows that the (non-analytic) function defined by $\mathbb{R}^n - i(0, \pi)^{\times n} \ni \zeta \mapsto (A\Omega)_n^\rho(\zeta)$ for $\text{Im}\zeta \in -\Lambda_n^\rho$ satisfies the same bound as (4.30), with $d_\infty(\lambda, \partial\Lambda_n)$ replaced by $d_\infty(\lambda, \partial(0, \pi)^{\times n}) = \max\{\lambda_k, \pi - \lambda_k : k = 1, \dots, n\} = d(\lambda)$. Putting $c := 16\pi^{-1}$ we thus get, $\theta \in \mathbb{R}^n$, $\lambda \in \mathcal{B}_n(\kappa)$,

$$|(A\Omega)_n(\theta - i\lambda)| \leq 2\sqrt{2} \left(\frac{4\sqrt{2}\|S_2\|_\kappa}{\pi d(\lambda)^{1/2}} \right)^n \|A\| \leq \left(\frac{c\|S_2\|_\kappa}{d(\lambda)^{1/2}} \right)^n \|A\|, \quad (4.38)$$

and the proof is complete. \square

5 Proof of the Nuclearity Condition

In this section we state and prove our main results, namely we show that the modular nuclearity condition holds for inclusions of sufficiently far separated wedge regions if the underlying scattering function S_2 is in the family \mathcal{S}_0 (Theorem 5.4), and for arbitrary inclusions of wedges if $S_2 \in \mathcal{S}_0$ and $S_2(0) = -1$ (Theorem 5.6).

First note that it is sufficient to consider the maps

$$\Xi(0, s) : \mathcal{A}(W_R) \rightarrow \mathcal{H}, \quad \Xi(0, s)A := \Delta^{1/4}U(0, s)A\Omega \quad (5.1)$$

corresponding to wedge inclusions of the type

$$W_R + (0, s) \subset W_R, \quad s > 0. \quad (5.2)$$

As W_R is stable under boosts, a more general inclusion $W_R + x \subset W_R$, $x \in W_R$, of right wedges can be transformed to (5.2) (with $s = \sqrt{x \cdot x} > 0$) by a velocity transformation with appropriately chosen rapidity parameter. Since the boosts commute with the modular operator Δ of $(\mathcal{A}(W_R), \Omega)$, it follows that $\Xi(x)$ is nuclear if and only if $\Xi(0, \sqrt{x \cdot x})$ is nuclear, and in this case³ $\|\Xi(x)\|_1 = \|\Xi(0, \sqrt{x \cdot x})\|_1$. Therefore we only consider inclusions of the form (5.2), and use the shorthand notation $\Xi(s) := \Xi(0, s)$.

The map $\Xi(s)$ will be studied in terms of its n -particle projections, defined as

$$\Xi_n(s) : \mathcal{A}(W_R) \rightarrow \mathcal{H}_n, \quad \Xi_n(s)A := P_n \Xi(s)A, \quad s > 0. \quad (5.3)$$

Taking into account that $\Delta^{1/4}$ leaves the particle number invariant (cf. (2.15)), $\Xi_n(s)$ is seen to act as

$$\Xi_n(s)A = \Delta^{1/4}(A(s)\Omega)_n, \quad A(s) := U(0, s)AU(0, s)^{-1}. \quad (5.4)$$

In order to profit from the properties of the wavefunctions $(A\Omega)_n$ established in the previous section, we will treat $\Xi_n(s)$ as the concatenation of two maps as depicted in the following commutative diagram.

$$\begin{array}{ccc} & \mathcal{A}(W_R) & \\ \Sigma_n(s, \kappa) \downarrow & \searrow \Xi_n(s) & \\ H^2(\mathcal{T}_n(\kappa)) & \xrightarrow{\Delta_n(s, \kappa)} & \mathcal{H}_n \end{array}$$

By taking advantage of Prop. 4.4, it will be shown that $\Sigma_n(s, \kappa)A := (A(\frac{s}{2})\Omega)_n$ may be considered as a bounded map from $\mathcal{A}(W_R)$ into a certain Banach space $H^2(\mathcal{T}_n(\kappa))$ of analytic functions, defined below (Lemma 5.2). Viewed as a map from $H^2(\mathcal{T}_n(\kappa))$ into \mathcal{H}_n , the operator $\Delta_n(s, \kappa) = \Delta^{1/4}U(\frac{s}{2})$ is nuclear (Lemma 5.3). As the nuclear operators between two Banach spaces form an ideal in the algebra of bounded operators between the same spaces, this structure implies the nuclearity of $\Xi_n(s)$. By calculating bounds on $\|\Xi_n(s)\|_1$, we then proceed to nuclearity statements for $\Xi(s)$.

Definition 5.1. *Let $\kappa > 0$ and*

$$\lambda_0 := (-\frac{\pi}{2}, \dots, -\frac{\pi}{2}) \in \mathbb{R}^n, \quad \mathcal{C}_n(\kappa) := (-\frac{\kappa}{2}, \frac{\kappa}{2})^{\times n}, \quad \mathcal{T}_n(\kappa) := \mathbb{R}^n + i(\lambda_0 + \mathcal{C}_n(\kappa)). \quad (5.5)$$

For functions $F : \mathcal{T}_n(\kappa) \rightarrow \mathbb{C}$ we introduce the notations

$$F_\lambda(\theta) := F(\theta + i\lambda), \quad \lambda - \lambda_0 \in \mathcal{C}_n(\kappa), \quad (5.6)$$

$$\|F\| := \sup \left\{ \left[\int_{\mathbb{R}^n} d^n \theta |F_\lambda(\theta)|^2 \right]^{1/2} : \lambda - \lambda_0 \in \mathcal{C}_n(\kappa) \right\}, \quad (5.7)$$

and define the Hardy space over the tube $\mathcal{T}_n(\kappa)$ as

$$H^2(\mathcal{T}_n(\kappa)) := \{F : \mathcal{T}_n(\kappa) \rightarrow \mathbb{C} \text{ analytic} : \|F\| < \infty\}. \quad (5.8)$$

³We write $\|X\|_1$ to denote the nuclear norm of X [11].

It is a well-known fact that the Hardy space $(H^2(\mathcal{T}_n(\kappa)), \|\cdot\|)$ is a Banach space [30]. Recalling the definition of the space-time translations U on \mathcal{H} (2.7), it becomes apparent that the n -particle wavefunctions of $A(s)$ take the form

$$(A(s)\Omega)_n(\boldsymbol{\theta}) = \prod_{k=1}^n e^{-ims \sinh \theta_k} \cdot (A\Omega)_n(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \mathbb{R}^n. \quad (5.9)$$

Lemma 5.2. *Let $s > 0$, $\kappa \in (0, \kappa(S_2))$. The map*

$$\Sigma_n(s, \kappa)A := (A(\frac{s}{2})\Omega)_n \quad (5.10)$$

is a bounded operator between the Banach spaces $(\mathcal{A}(W_R), \|\cdot\|_{\mathcal{B}(\mathcal{H})})$ and $(H^2(\mathcal{T}_n(\kappa)), \|\cdot\|)$. Its norm can be estimated as

$$\|\Sigma_n(s, \kappa)\| \leq \left(\frac{2c \|S_2\|_\kappa K_0(ms \cos \kappa)^{1/2}}{(\pi - \kappa)^{1/2}} \right)^n =: \sigma(s, \kappa)^n, \quad (5.11)$$

where K_0 denotes the Bessel function $K_0(\xi) := \int_0^\infty e^{-\xi \cosh t} dt$ and c is the numerical constant from Prop. 4.4.

Proof. We first have to show that $\Sigma_n(s, \kappa)$ maps $\mathcal{A}(W_R)$ into $H^2(\mathcal{T}_n(\kappa))$. To this end, note that the tube $\mathcal{T}_n(\kappa)$ is contained in the analyticity domain of $(A\Omega)_n$, specified in Prop. 4.4, because $\kappa < \kappa(S_2) < \pi$. Since the exponential factor in (5.9) is entire, the analyticity of $(A(\frac{s}{2})\Omega)_n$ in $\mathcal{T}_n(\kappa)$ follows. To compute the claimed bound, note that in view of (4.35) there holds $(\boldsymbol{\mu} \in \mathcal{C}_n(\kappa))$

$$|(A(\frac{s}{2})\Omega)_n(\boldsymbol{\theta} + i\boldsymbol{\lambda}_0 + i\boldsymbol{\mu})|^2 \leq \left(\frac{c \|S_2\|_\kappa}{d(-\boldsymbol{\lambda}_0 - \boldsymbol{\mu})^{1/2}} \right)^{2n} \prod_{k=1}^n e^{-ms \cos \mu_k \cosh \theta_k} \cdot \|A\|^2, \quad (5.12)$$

with $d(-\boldsymbol{\lambda}_0 - \boldsymbol{\mu}) = \min\{\frac{\pi}{2} - \mu_k, \frac{\pi}{2} + \mu_k : k = 1, \dots, n\} \geq \frac{\pi - \kappa}{2}$. Since $|\mu_k| < \frac{\kappa}{2} < \frac{\pi}{2}$, the right hand side in (5.12) is rapidly decreasing for $|\theta_k| \rightarrow \infty$. Hence $(A(\frac{s}{2})\Omega)_n \in H^2(\mathcal{T}_n(\kappa))$. Integration over $\boldsymbol{\theta} \in \mathbb{R}^n$ and the estimate $K_0(ms \cos \mu_k) \leq K_0(ms \cos \kappa)$ give (5.11). \square

With the notation introduced above, we have $\Xi_n(s) = \Delta^{1/4}U(\frac{s}{2}) \circ \Sigma_n(s, \kappa)$, where $\Delta^{1/4}U(\frac{s}{2})$ has to be considered as an operator from $H^2(\mathcal{T}_n(\kappa))$ to \mathcal{H}_n . To clarify the action of this operator, recall that in the models under consideration the modular group of $(\mathcal{A}(W_R), \Omega)$ acts like the (rescaled) group of Lorentz boosts (2.15), i.e. there holds

$$(\Delta^\mu(A(s)\Omega)_n)(\boldsymbol{\theta}) = (A(s)\Omega)_n(\theta_1 - 2\pi i\mu, \dots, \theta_n - 2\pi i\mu), \quad 0 \leq \mu \leq \frac{1}{2}. \quad (5.13)$$

Setting $\mu = \frac{1}{4}$, we obtain

$$(\Xi_n(s)A)(\boldsymbol{\theta}) = (A(s)\Omega)_n(\boldsymbol{\theta} + i\boldsymbol{\lambda}_0) = \prod_{k=1}^n e^{-\frac{ms}{2} \cosh \theta_k} \cdot (A(\frac{s}{2})\Omega)_n(\boldsymbol{\theta} + i\boldsymbol{\lambda}_0). \quad (5.14)$$

Therefore $\Delta^{1/4}U(\frac{s}{2})$ acts on $H^2(\mathcal{T}_n(\kappa))$ as the map

$$\begin{aligned} \Delta_n(s, \kappa) : H^2(\mathcal{T}_n(\kappa)) &\longrightarrow \mathcal{H}_n, \\ (\Delta_n(s, \kappa)F)(\boldsymbol{\theta}) &:= \prod_{k=1}^n e^{-\frac{ms}{2} \cosh \theta_k} \cdot F_{\boldsymbol{\lambda}_0}(\boldsymbol{\theta}). \end{aligned} \quad (5.15)$$

The operator $\Delta_n(s, \kappa)$ is studied in the following Lemma.

Lemma 5.3. *Let $s > 0$, $\kappa > 0$, and $\Delta_n(s, \kappa)$ be defined as in (5.15).*

- a) $\Delta_n(s, \kappa)$ is a nuclear map between the Banach spaces $(H^2(\mathcal{T}_n(\kappa)), \|\cdot\|)$ and $(\mathcal{H}_n, \|\cdot\|)$.
- b) Define $T_{s, \kappa}$ as the integral operator on $L^2(\mathbb{R}, d\theta)$ with kernel

$$T_{s, \kappa}(\theta, \theta') = \frac{e^{-\frac{ms}{2} \cosh \theta}}{i\pi (\theta' - \theta - \frac{i\kappa}{2})}. \quad (5.16)$$

$T_{s, \kappa}$ is of trace class, and there holds the bound

$$\|\Delta_n(s, \kappa)\|_1 \leq \|T_{s, \kappa}\|_1^n < \infty. \quad (5.17)$$

Proof. We will need the following two features of functions $F \in H^2(\mathcal{T}_n(\kappa))$, which can be deduced by standard arguments from the mean value property. The first property is $F_{\boldsymbol{\lambda}}(\boldsymbol{\theta}) \rightarrow 0$ for $|\theta_k| \rightarrow \infty$, $\boldsymbol{\lambda} - \boldsymbol{\lambda}_0 \in \mathcal{C}_n(\kappa)$. Secondly we remark that $F_{\boldsymbol{\lambda}}$ converges in the norm topology of $L^2(\mathbb{R}^n)$ as $\boldsymbol{\lambda} - \boldsymbol{\lambda}_0$ approaches the boundary of $\mathcal{C}_n(\kappa)$.

Bearing these features of F in mind, we fix $\boldsymbol{\theta} \in \mathbb{R}^n$ and a polydisc $\mathcal{D} \subset \mathcal{T}_n(\kappa)$ with center $\boldsymbol{\theta} + i\boldsymbol{\lambda}_0$. By virtue of Cauchy's integral formula, we can represent $F(\boldsymbol{\theta} + i\boldsymbol{\lambda}_0)$ as a contour integral over \mathcal{D} ,

$$F(\boldsymbol{\theta} + i\boldsymbol{\lambda}_0) = \frac{1}{(2\pi i)^n} \oint_{\mathcal{D}} d^n \zeta' \frac{F(\zeta')}{\prod_{k=1}^n (\zeta'_k - \theta_k + \frac{i\pi}{2})}.$$

Taking advantage of the two properties of F mentioned above, we can deform the contour of integration to the boundary of $\mathcal{T}_n(\kappa)$. After multiplication with the exponential factor (5.15) we arrive at

$$(\Delta_n(s, \kappa)F)(\boldsymbol{\theta}) = \frac{1}{(2\pi i)^n} \sum_{\boldsymbol{\varepsilon}} \int_{\mathbb{R}^n} d^n \boldsymbol{\theta}' \prod_{k=1}^n \frac{\varepsilon_k e^{-\frac{ms}{2} \cosh \theta_k}}{(\theta'_k - \theta_k - \frac{i\varepsilon_k \kappa}{2})} \cdot F_{\boldsymbol{\lambda}_0 - \frac{\kappa}{2} \boldsymbol{\varepsilon}}(\boldsymbol{\theta}'),$$

where the summation runs over $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_1, \dots, \varepsilon_n = \pm 1$. Expressed through the integral operator $T_{s, \kappa}$, this equation reads

$$\Delta_n(s, \kappa)F = 2^{-n} \sum_{\boldsymbol{\varepsilon}} \varepsilon_1 \cdots \varepsilon_n (T_{s, \varepsilon_1 \kappa} \otimes \dots \otimes T_{s, \varepsilon_n \kappa}) F_{\boldsymbol{\lambda}_0 - \frac{\kappa}{2} \boldsymbol{\varepsilon}}. \quad (5.18)$$

The L^2 -convergence of F to its boundary values implies that $F \mapsto F_{\boldsymbol{\lambda}_0 - \frac{\kappa}{2} \boldsymbol{\varepsilon}}$ is a bounded operator from $H^2(\mathcal{T}_n(\kappa))$ into $L^2(\mathbb{R}^n)$ with norm not exceeding one.

The integral operator $T_{n, \kappa}$ can be expressed in terms of the quantum mechanical position and momentum operators X and P on $L^2(\mathbb{R})$ as

$$T_{s, \kappa} = 2 e^{-\frac{ms}{2} \cosh X} \chi_{[0, \infty)}(P) e^{-\frac{\kappa}{2} P}.$$

According to [41, Thm. XI.21], such a representation implies that $T_{s, \kappa}$ is of trace class on $L^2(\mathbb{R})$, and consequently $\Delta_n(s, \kappa)$ is a nuclear map.

As $T_{s, -\kappa}$ is unitary equivalent to $T_{s, \kappa}$ (the equivalence being implemented by V , $(Vf)(\theta) := i \cdot f(-\theta)$), it is of trace class, too, and $\|T_{s, -\kappa}\|_1 = \|T_{s, \kappa}\|_1$. Taking into account that the sum in (5.18) runs over 2^n terms, and $\|F \mapsto F_{\boldsymbol{\lambda}_0 - \frac{\kappa}{2} \boldsymbol{\varepsilon}}\| \leq 1$, the claimed bound $\|\Delta_n(s, \kappa)\|_1 \leq \|T_{s, \kappa}\|_1^n$ follows. \square

Theorem 5.4. *Consider a model theory with scattering function $S_2 \in \mathcal{S}_0$. Then there exists a distance $s_0 < \infty$ such that $\Xi(s)$ is nuclear for all $s > s_0$.*

Hence for each double cone $\mathcal{O}_{a,b} = (W_R + a) \cap (W_L + b)$ with $b - a \in W_R$ and $(a - b)^2 > s_0^2$, the corresponding observable algebra $\mathcal{A}(\mathcal{O}_{a,b})$ is isomorphic to the hyperfinite type III_1 factor, and there exist non-trivial operators localized in $\mathcal{O}_{a,b}$.

Proof. Let $\kappa \in (0, \kappa(S_2))$. We have $\Xi_n(s) = \Delta_n(s, \kappa) \circ \Sigma_n(s, \kappa)$, and by (5.11, 5.17)

$$\|\Xi_n(s)\|_1 \leq \|\Sigma_n(s, \kappa)\| \cdot \|\Delta_n(s, \kappa)\|_1 \leq (\sigma(s, \kappa) \cdot \|T_{s, \kappa}\|_1)^n. \quad (5.19)$$

For $s \rightarrow \infty$, $\|T_{s, \kappa}\|_1$ and $\sigma(s, \kappa)$ converge monotonously to zero (cf. (5.16) and (5.11)). So there exists $s_0 < \infty$ such that $\sigma(s, \kappa)\|T_{s, \kappa}\|_1 < 1$ for all $s > s_0$. But for these values of s , there holds $\sum_{n=0}^{\infty} \|\Xi_n(s)\|_1 < \infty$. Hence the series $\sum_{n=0}^{\infty} \Xi_n(s)$ of nuclear operators converges in nuclear norm to $\Xi(s)$. Since the ideal of nuclear operators between two Banach spaces is closed in nuclear norm, the nuclearity of $\Xi(s)$ follows.

The algebraic structure of $\mathcal{A}(W_R \cap (W_L + (0, s)))$ is a consequence of the modular nuclearity condition [13], and the corresponding statement about the transformed double cones $\mathcal{O}_{a,b}$ follows from the Poincaré covariance of the theory, as was pointed out at the beginning of this section. \square

Theorem 5.4 establishes the existence of local observables in double cones having a minimal size. Below we discuss a second theorem which improves the previous one under an additional assumption on the underlying scattering function, namely we establish the nuclearity of $\Xi(s)$ without restriction on the splitting distance s if $S_2(0) = -1$.

The previous estimate on $\|\Xi_n(s)\|_1$ was rather crude because we did not fully take into account the effects of the “ S_2 -statistics”, i.e. the symmetry structure (2.6) of the wavefunctions. This can most easily be accomplished in the two extremal cases of the constant scattering functions $S_2 = \pm 1$, where the Hilbert space \mathcal{H} of the model coincides with the Bosonic Fock space $\mathcal{H}^+ = \bigoplus_{n \geq 0} \mathcal{H}_n^+$ and Fermionic Fock space $\mathcal{H}^- = \bigoplus_{n \geq 0} \mathcal{H}_n^-$ over \mathcal{H}_1 , respectively. Here the combinatorial problems appearing in estimating the nuclear norms of $\Xi(s)$ have been settled in [14] and [32], respectively. To use these results also in the context of a generic scattering function S_2 , we map the S_2 -symmetric Hilbert space \mathcal{H} to \mathcal{H}^+ or \mathcal{H}^- with a distinguished unitary $Y^\pm : \mathcal{H} \rightarrow \mathcal{H}^\pm$ which will be constructed below. The choice between \mathcal{H}^+ and \mathcal{H}^- is made by the behaviour of the scattering function at the origin, where it can take only the values $S_2(0) = \pm 1$.

All objects associated with the scattering functions $S_2 = \pm 1$ will be tagged with an index “ \pm ”: z_\pm^\dagger, z_\pm are the creation and annihilation operators on the Fock spaces \mathcal{H}^\pm , which are left invariant by the usual representations D_n^\pm of the group \mathfrak{S}_n of permutations of n objects, acting on $\mathcal{H}_1^{\otimes n} = L^2(\mathbb{R}^n)$. The orthogonal projections $\bigoplus_{n \geq 0} \mathcal{H}_1^{\otimes n} \rightarrow \mathcal{H}_n^\pm$ will be denoted by P_n^\pm . Correspondingly, $D_n^{S_2}$ stands for the representation of \mathfrak{S}_n which leaves \mathcal{H}_n invariant (2.4).

In preparation for the construction of the unitaries Y_n^\pm , recall that each scattering function $S_2 \in \mathcal{S}_0$ is analytic and nonvanishing in the strip $S(-\kappa(S_2), \kappa(S_2))$ (Prop. 3.1 b)). So there is an analytic function $\delta : S(-\kappa(S_2), \kappa(S_2)) \rightarrow \mathbb{C}$ (the phase shift) such that

$$S_2(\zeta) = S_2(0)e^{2i\delta(\zeta)}, \quad \zeta \in S(-\kappa(S_2), \kappa(S_2)). \quad (5.20)$$

Since S_2 has modulus one on the real line, δ takes real values on \mathbb{R} , and we fix it uniquely by the choice $\delta(0) = 0$. Note that in view of $S_2(-\theta) = \overline{S_2(\theta)}$, $\theta \in \mathbb{R}$, δ is odd.

Lemma 5.5. *Let $S_2 \in \mathcal{S}_0$ with $S_2(0) = \pm 1$ and $\delta : S(-\kappa(S_2), \kappa(S_2)) \rightarrow \mathbb{C}$ be defined as above. Consider the function*

$$Y_n^\pm(\zeta) := \prod_{1 \leq k < l \leq n} (\pm e^{i\delta(\zeta_k - \zeta_l)}) , \quad (5.21)$$

and the corresponding multiplication operator (denoted by the same symbol Y_n^\pm).

- a) Let $\kappa \in (0, \kappa(S_2))$. Viewed as an operator on $H^2(\mathcal{T}_n(\kappa))$, Y_n^\pm is a bounded map with $\|Y_n^\pm\|_{\mathcal{B}(H^2(\mathcal{T}_n(\kappa)))} \leq \|S_2\|_\kappa^{n/2}$.
- b) Viewed as an operator on $L^2(\mathbb{R}^n, d^n\theta)$, Y_n^\pm is a unitary intertwining the representations D_n and D_n^\pm of the symmetric group \mathfrak{S}_n and hence mapping the subspace \mathcal{H}_n onto the subspace \mathcal{H}_n^\pm .

Proof. a) Since δ is analytic in $S(-\kappa, \kappa)$, so is the function Y_n^\pm in $S(-\frac{\kappa}{2}, \frac{\kappa}{2})^{\times n}$. Depending only on differences of rapidities, Y_n^\pm is also analytic in $S(-\frac{\kappa}{2}, \frac{\kappa}{2})^{\times n} + i\lambda_0 = \mathcal{T}_n(\kappa)$. By application of the same argument as in the proof of the bounds in Prop. 4.4 b), it follows that

$$|Y_n^\pm(\zeta)| \leq \|S_2\|_\kappa^{n/2}, \quad \zeta \in \mathcal{T}_n(\kappa). \quad (5.22)$$

So $\|Y_n^\pm \cdot F\| \leq \|S_2\|_\kappa^{n/2} \cdot \|F\|$, $F \in H^2(\mathcal{T}_n(\kappa))$, which proves a).

b) Considered as a multiplication operator on $L^2(\mathbb{R}^n)$, Y_n^\pm multiplies with a phase and is hence unitary. Let $\tau_j \in \mathfrak{S}_n$ denote the transposition exchanging j and $j+1$, $j \in \{1, \dots, n\}$, and pick arbitrary $\psi_n \in L^2(\mathbb{R}^n)$, $\theta \in \mathbb{R}^n$.

$$\begin{aligned} (D_n^\pm(\tau_j)Y_n^\pm\psi_n)(\theta) &= \pm \prod_{\substack{1 \leq k < l \leq n \\ (k,l) \neq (j,j+1)}} (\pm e^{i\delta(\theta_k - \theta_l)}) \cdot (\pm e^{i\delta(\theta_{j+1} - \theta_j)}) \psi_n(\theta_1, \dots, \theta_{j+1}, \theta_j, \dots, \theta_n) \\ &= \prod_{1 \leq k < l \leq n} (\pm e^{i\delta(\theta_k - \theta_l)}) \cdot S_2(\theta_{j+1} - \theta_j) \psi_n(\theta_1, \dots, \theta_{j+1}, \theta_j, \dots, \theta_n) \\ &= (Y_n^\pm D_n^{S_2}(\tau_j)\psi_n)(\theta) \end{aligned}$$

As the transpositions τ_j generate \mathfrak{S}_n , this calculation shows that Y_n^\pm intertwines D_n^\pm and $D_n^{S_2}$. In particular, Y_n^\pm restricts to a unitary mapping \mathcal{H}_n onto \mathcal{H}_n^\pm . \square

Defining Y_0^\pm and Y_1^\pm as the identity on $\mathcal{H}_0 = \mathbb{C}$ and $\mathcal{H}_1 = L^2(\mathbb{R})$, respectively, we thus obtain a unitary $Y^\pm := \bigoplus_{n=0}^\infty Y_n^\pm$ mapping \mathcal{H} to \mathcal{H}^\pm , where the choice between Bosonic and Fermionic Fock space is made by the sign of $S_2(0) = \pm 1$. The same construction has been carried out in the context of Fock spaces with generalized statistics [35]. But whereas in that work, the essential quality of Y_n^\pm was property b) of the preceding Lemma, here also the preservation of the Hardy space structure, as stated in part a) of Lemma 5.5, is important. For it allows us to use a splitting of $\Xi_n(s)$ in a bounded and a

nuclear operator as before, and at the same time work with a simpler symmetry structure.

Consider a model theory whose scattering function $S_2 \in \mathcal{S}_0$ satisfies $S_2(0) = \pm 1$, and define the maps

$$\Xi_n^\pm(s) := Y_n^\pm \Xi_n(s) : \mathcal{A}(W_R) \longrightarrow \mathcal{H}_n^\pm, \quad \Xi_n^\pm(s) := Y_n^\pm \Xi_n(s), \quad \Xi^\pm(s) := Y^\pm \Xi(s).$$

Since $Y^\pm : \mathcal{H} \rightarrow \mathcal{H}^\pm$ is unitary, $\Xi(s)$ is nuclear if and only if $\Xi^\pm(s)$ is, and in this case $\|\Xi(s)\|_1 = \|\Xi^\pm(s)\|_1$. Taking into account that the functions Y_n^\pm depend only on differences of rapidities, and $U(s)$ acts as a multiplication operator, we have

$$\Xi_n^\pm(s)A = \Delta^{1/4}U(\frac{s}{2})Y_n^\pm(A(\frac{s}{2})\Omega)_n = \Delta_n^\pm(s, \kappa) \circ Y_n^\pm \Sigma_n(s, \kappa). \quad (5.23)$$

Here $\Sigma_n(s, \kappa)$ is defined as in (5.10) and $\Delta_n^\pm(s, \kappa)$ acts as $\Delta_n(s, \kappa)$ (5.15), but is now viewed as a map from the totally (anti-) symmetrized Hardy space $P_n^\pm H^2(\mathcal{T}_n(\kappa))$ to \mathcal{H}_n^\pm . By Lemma 5.2 and Lemma 5.5 a), $Y_n^\pm \Sigma_n(s, \kappa)$ is a continuous map from $\mathcal{A}(W_R)$ to $P_n^\pm H^2(\mathcal{T}_n(\kappa))$, $\kappa \in (0, \kappa(S_2))$. Its norm is bounded by

$$\|Y^\pm \Sigma_n(s, \kappa)\| \leq \|S_2\|_\kappa^{n/2} \cdot \sigma(s, \kappa)^n =: \sigma'(s, \kappa)^n. \quad (5.24)$$

In the case $S_2(0) = -1$ the Pauli principle becomes effective and the set $\Xi^-(s)\mathcal{A}(W_R)$ is much smaller than in the case of $S_2(0) = +1$. Taking advantage of this effect, we find the following Theorem.

Theorem 5.6. *Consider a model theory with scattering function $S_2 \in \mathcal{S}_0$ and $S_2(0) = -1$. Then the maps $\Xi(s)$ are nuclear for every $s > 0$, and there holds the bound*

$$\|\Xi(s)\|_1 < \sum_{n=0}^{\infty} \frac{(\sigma'(s, \kappa) \|T_{s, \kappa}\|_1)^n}{\sqrt{n!}} < \infty. \quad (5.25)$$

Here $\sigma'(s, \kappa)$ denotes the constant defined in (5.24), and $T_{s, \kappa}$ is the trace class integral operator on $L^2(\mathbb{R})$ defined in Lemma 5.3.

Consequently, all double cone algebras $\mathcal{A}(\mathcal{O})$ are isomorphic to the hyperfinite type III_1 factor and in particular contain non-trivial observables.

Proof. Proceeding along the same lines as in the proof of Lemma 5.3, we infer that $\Delta_n^-(s, \kappa)$ is nuclear and can be represented as in (5.18). With the notation introduced there, we have, $A \in \mathcal{A}(W_R)$,

$$\Xi_n^-(s)A = 2^{-n} \sum_{\epsilon} \varepsilon_1 \cdots \varepsilon_n (T_{s, \varepsilon_1 \kappa} \otimes \dots \otimes T_{s, \varepsilon_n \kappa}) (Y_n^- \Sigma_n(s, \kappa)A)_{\lambda_0 - \frac{\kappa}{2} \epsilon}. \quad (5.26)$$

Consider the positive operator $\hat{T}_{s, \kappa} := (|T_{s, \kappa}|^2 + |T_{s, -\kappa}|^2)^{1/2}$, which is of trace class on $L^2(\mathbb{R})$ and satisfies $\|\hat{T}_{s, \kappa}\|_1 \leq \|T_{s, \kappa}\|_1 + \|T_{s, -\kappa}\|_1 = 2 \|T_{s, \kappa}\|_1$ [29]. We choose an orthonormal basis $\{\psi_k\}_k$ of $L^2(\mathbb{R})$, consisting of eigenvectors ψ_k of $\hat{T}_{s, \kappa}$, with eigenvalues $t_k \geq 0$. So $\hat{T}_{s, \kappa}$ acts as $\hat{T}_{s, \kappa} \xi = \sum_{k=1}^{\infty} t_k \langle \psi_k, \xi \rangle \psi_k$, $\xi \in L^2(\mathbb{R})$, and its trace norm is $\|\hat{T}_{s, \kappa}\|_1 = \sum_{k=1}^{\infty} t_k < \infty$. As a consequence of the Pauli principle, the vectors

$$\psi_{\mathbf{k}} := z_-^\dagger(\psi_{k_1}) \cdots z_-^\dagger(\psi_{k_n}) \Omega = \sqrt{n!} P_n^-(\psi_{k_1} \otimes \dots \otimes \psi_{k_n}) \quad (5.27)$$

form an orthonormal basis of \mathcal{H}_n^- if $\mathbf{k} = (k_1, \dots, k_n)$ varies over $k_1 < k_2 < \dots < k_n$, $k_1, \dots, k_n \in \mathbb{N}$.

Expanding $\Xi_n^-(s)A$ in this basis, we thus have

$$\Xi^-(s)A = \sum_{n=0}^{\infty} \sum_{k_1 < \dots < k_n} \langle \psi_{\mathbf{k}}, \Xi_n^-(s)A \rangle \psi_{\mathbf{k}}$$

and consequently

$$\|\Xi^-(s)\|_1 \leq \sum_{n=0}^{\infty} \sum_{k_1 < \dots < k_n} \sup_{A \in \mathcal{A}(W_R)} \frac{|\langle \psi_{\mathbf{k}}, \Xi_n^-(s)A \rangle|}{\|A\|}. \quad (5.28)$$

To estimate $|\langle \psi_{\mathbf{k}}, \Xi_n^-(s)A \rangle|$, we use (5.26, 5.27, 5.24) and get

$$\begin{aligned} |\langle \psi_{\mathbf{k}}, \Xi_n^-(s)A \rangle| &\leq 2^{-n} \sqrt{n!} \sum_{\varepsilon} \left| \langle T_{s, \varepsilon_1 \kappa} \psi_{k_1} \otimes \dots \otimes T_{s, \varepsilon_n \kappa} \psi_{k_n}, (Y_n^- \Sigma_n(s, \kappa) A)_{\lambda_0 - \frac{\kappa}{2} \varepsilon} \rangle \right| \\ &\leq \sqrt{n!} \left(\frac{\sigma'(s, \kappa)}{2} \right)^n \sum_{\varepsilon} \|T_{s, \varepsilon_1 \kappa} \psi_{k_1}\| \cdots \|T_{s, \varepsilon_n \kappa} \psi_{k_n}\| \cdot \|A\|. \end{aligned} \quad (5.29)$$

Taking into account $\|T_{s, \pm \kappa} \psi_{k_j}\| \leq \|\hat{T}_{s, \kappa} \psi_{k_j}\| = t_{k_j}$, we find from (5.28) the following upper bound on the nuclear norm of $\Xi^-(s)$:

$$\begin{aligned} \|\Xi^-(s)\|_1 &\leq \sum_{n=0}^{\infty} \sqrt{n!} \sigma'(s, \kappa)^n \sum_{k_1 < \dots < k_n} t_{k_1} \cdots t_{k_n} \\ &< \sum_{n=0}^{\infty} \frac{\sigma'(s, \kappa)^n}{\sqrt{n!}} \sum_{k_1, \dots, k_n=0}^{\infty} t_{k_1} \cdots t_{k_n} = \sum_{n=0}^{\infty} \frac{(\sigma'(s, \kappa) \|T_{s, \kappa}\|_1)^n}{\sqrt{n!}}. \end{aligned}$$

This series converges for each $\sigma'(s, \kappa) \cdot \|T_{s, \kappa}\|_1$, *i.e.* for each $s > 0$, and provides the claimed bound (5.25). \square

In the models with constant scattering functions $S_2 = \pm 1$, the nuclearity of $\Xi(s)$ has already been shown for arbitrary $s > 0$ in [13] and [32], respectively. The bound on $\|\Xi(s)\|_1$ obtained in the case $S_2 = -1$ is smaller than (5.25), suggesting that our estimates are not optimal. The case $S_2 = +1$ could not be treated with the methods developed here, and it is presently not known to us whether $\Xi(s)$ is nuclear for arbitrary short splitting distances s in models with scattering function $S_2 \in \mathcal{S}_0$, $S_2(0) = +1$.

6 Conclusion

With the proof of the nuclearity condition in the family of models considered here, a crucial step in the construction of quantum field theories from a scattering function has been accomplished. For the possibility that the abstract definition of the theory via intersections of wedge algebras leads to the physically unacceptable situation of absence of observables localized in bounded spacetime regions can now be ruled out in the family of

models considered here. So at least within the framework of quantum field theories exhibiting a factorizing S-matrix, the program initiated by Schroer is successful and provides a way to construct interacting models without having to rely on quantization procedures or perturbative methods.

The present article shows that for proving the existence of certain theories, the complicated task of explicitly constructing strictly local objects can be avoided and replaced by an analysis of wedge-local wavefunctions. In contrast to such investigations, information about specific local fields and observables in a given model may be obtained in the framework of the formfactor program [45]. Here matrix elements of local operators, which are now known to exist, are the main objects of interest, and interesting results have been found for many kinds of different models [2, 18, 24].

As the models considered here were constructed in the spirit of inverse scattering theory, the question emerges whether the S-matrix of such a model is really given by the postulated scattering function S_2 . Whereas two-particle scattering states can be formed with the help of the wedge-local field ϕ , and are known to reproduce S_2 [31], the construction of collision states of more than two particles involves creation operators with sharper localization properties (*cf.* for example the recent review [16]). The existence of such creation operators follows from the Reeh-Schlieder property, *i.e.* the cyclicity of the vacuum vector for the local algebras $\mathcal{A}(\mathcal{O})$.

The model corresponding to the scattering function $S_2 = 1$ is the theory of a free field, and hence the Reeh-Schlieder property holds in this case. Buchholz and Summers proved that also in the model given by the scattering function $S_2 = -1$, the Reeh-Schlieder property is satisfied [17]. In fact, it can be shown that the cyclicity of the vacuum is a consequence of the modular nuclearity condition and Haag duality for the wedge algebras [34], and thus holds at least in the class of models with scattering function $S_2 \in \mathcal{S}_0$, $S_2(0) = -1$. With this additional information, the analysis from [8] may be applied to deduce that also the full S-matrix of the model constructed from S_2 coincides with the scattering operator associated to S_2 . According to Schroer [43], it is furthermore possible to argue that this model is uniquely determined by its scattering function.

In addition to properties of the scattering states, also something about the thermodynamics of models with a factorizing S-matrix can be learned from our analysis. By a slight generalization of our arguments, and following the reasoning in [11], it can be shown that the maps $\Theta_\beta(s) : \mathcal{A}(\mathcal{O}_s) \rightarrow \mathcal{H}$, $\Theta_\beta(s)A := e^{-\beta H} A \Omega$, where H denotes the Hamiltonian, are nuclear if $\Xi(s)$ is, and an estimate on their nuclear norms can be calculated. As the quantity $\|\Theta_\beta(s)\|_1$ is to be interpreted as the partition function of the restriction of the underlying theory to the “relativistic box” \mathcal{O}_s at inverse temperature β [14], such estimates provide information about gross thermodynamical properties of the system.

Besides these more specific aspects of models with a factorizing S-matrix, we mention that the general idea of constructing interacting quantum field theories by first considering nets of wedge algebras and then analyzing their relative commutants is applicable to higher-dimensional spacetimes as well. However, the modular nuclearity condition cannot be satisfied if the spacetime dimension is larger than two. Finding an adequate condition, applicable in physical spacetime and ensuring the non-triviality of intersections of wedge algebras, might therefore lead to considerable progress in the construction of interacting quantum field theories.

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Appendix

Proof of Proposition 3.1

a) We first show that any $S_2 \in \mathcal{S}$ has a representation of the form (3.1). To this end, let $\varepsilon := S_2(0)$, and define $\{\beta_k\}$ as the sequence of zeros of S_2 in $\overline{S(0, \frac{\pi}{2})}$, repeated according to multiplicity. In view of (2.3), $\varepsilon = \pm 1$, and because of S_2 has modulus unity on the real line, and $\overline{S_2(-\bar{\zeta})} = S_2(\zeta)$ holds for $\zeta \in \overline{S(0, \pi)}$, the sequence $\{\beta_k\}$ has the properties *i*) and *ii*).

In the derivation of (3.1), we may restrict to scattering functions without purely imaginary zeros. For if S_2 vanishes at $i\alpha_1, \dots, i\alpha_K$, $0 < \alpha_k \leq \frac{\pi}{2}$ (There can be only finitely many of such points in view of the analyticity and continuity of S_2 and the fact that $|S_2(0)| = 1$), consider

$$R(\zeta) := \prod_{k=1}^K \frac{\sinh(i\alpha_k) - \sinh \zeta}{\sinh(i\alpha_k) + \sinh \zeta}. \quad (6.1)$$

It is readily verified that $R \in \mathcal{S}$, and its only zeros in $\overline{S(0, \frac{\pi}{2})}$ are $i\alpha_1, \dots, i\alpha_K$. But as the equations (2.3) are stable under taking products and reciprocals, it follows that $S_2(\zeta)/R(\zeta)$ is a scattering function as well, without imaginary zeros. This implies that formula (3.1) holds for scattering functions with imaginary zeros if it can be proven for those without.

The zeros of a scattering function without purely imaginary zeros occur in pairs of the form $(\beta_k, -\bar{\beta}_k)$. We denote the zeros with positive real part by γ_k , *i.e.* $\{\beta_k : k \in \mathbb{N}\} = \{\gamma_k, -\bar{\gamma}_k : k \in \mathbb{N}\}$, and assume that there are infinitely many of them. (The case of finitely many zeros leads only to simplifications in the proof.)

The hyperbolic sine is a biholomorphic map between the strip $S(0, \frac{\pi}{2})$ and the upper half plane with a cut along $i[1, \infty)$. The points $\frac{i\pi}{2} \pm \theta$, $\theta \in \mathbb{R}$, are mapped onto opposite sides of the cut. Since (2.3) implies $S_2(\frac{i\pi}{2} + \theta) = S_2(\frac{i\pi}{2} - \theta)$, the function $\hat{S}_2(z) := S_2(\operatorname{arsinh}(z))$ is analytic in the upper half plane and bounded and continuous on its closure. The zeros of \hat{S}_2 are precisely $g_k := \sinh \gamma_k$ and $-\bar{g}_k = \sinh(-\bar{\gamma}_k)$, $k \in \mathbb{N}$. Taking into account $|S_2(\theta)| = |S_2(\theta + i\pi)| = 1$, $\theta \in \mathbb{R}$, we conclude $|\hat{S}_2(z)| \leq 1$ for $\operatorname{Im} z \geq 0$ from the three lines theorem.

In the context of Hardy spaces, it is well-known (*cf.*, for example [20, Thm. 11.3]) that a function like \hat{S}_2 admits a factorization of the form $\hat{S}_2(z) = H(z)B(z)$, where H is an

analytic and nonvanishing function bounded by unity on the upper half plane, and B is the Blaschke product with zeros $\{g_k, -\overline{g_k}\}$,

$$B(z) = \prod_{k=1}^{\infty} \left(\frac{|g_k^2 + 1|}{g_k^2 + 1} \frac{z - g_k}{z - \overline{g_k}} \cdot \frac{|\overline{g_k}^2 + 1|}{\overline{g_k}^2 + 1} \frac{z + \overline{g_k}}{z + g_k} \right) = \prod_{k=1}^{\infty} \left(\frac{g_k - z}{g_k + z} \cdot \frac{-\overline{g_k} - z}{-\overline{g_k} + z} \right). \quad (6.2)$$

Because of the boundedness of \hat{S}_2 , the Blaschke condition for the upper half plane, $\sum_k \operatorname{Im} \sinh \beta_k / (1 + |\sinh \beta_k|^2) < \infty$, holds [20]. This condition is equivalent to the convergence (absolute and uniform as z varies over compact subsets of the upper halfplane) of the product (6.2). In view of the continuity of \hat{S}_2 on the real axis, and $|S_2(\theta)| = 1$, $\theta \in \mathbb{R}$, we also note that $\{\beta_k\}$ has no finite limit point (that is, property *iii*) of $\{\beta_k\}$ holds), and the Blaschke condition simplifies to property *iv*). In particular, B extends continuously to the real axis [25].

As H is nonvanishing, we find a function h analytic in the upper halfplane such that $H(z) = \varepsilon e^{ih(z)}$. Since $B(0) = 1$, we may choose $h(0) = 0$ to fix h uniquely. Moreover, using the well-known factorization theorem for bounded analytic functions on the upper half plane and the fact that \hat{S}_2 has modulus one on the real line, we conclude that there is a constant $a \geq 0$ such that $\operatorname{Im} h(z) = a \cdot \operatorname{Im} z$ [4, Thm. 6.5.4]. Hence $h(z) = a \cdot z + h(0) = a \cdot z$. Mapping back to the strip we get the claimed expression

$$S_2(\zeta) = \varepsilon e^{ia \sinh \zeta} \prod_{k=1}^{\infty} \frac{\sinh \beta_k - \sinh \zeta}{\sinh \beta_k + \sinh \zeta}, \quad \zeta \in S(0, \frac{\pi}{2}). \quad (6.3)$$

By reflection about $\mathbb{R} + \frac{i\pi}{2}$, this formula extends to $S(0, \pi)$.

To establish the converse direction, assume $a \geq 0$, $\varepsilon = \pm 1$ and a sequence $\{\beta_k\}$ satisfying *i*)-*iv*) are given. The product (6.3) converges to a bounded analytic function due to the Blaschke condition *iv*), and the absence of finite limit points *iii*) of $\{\beta_k\}$ implies continuous boundary values. Taking into account $a \geq 0$ and *ii*), the boundedness and the relations (2.3) can be checked by direct computation.

b) As the zeros of S_2 have no finite limit point and $S_2(-\zeta) = S_2(\zeta)^{-1}$, this function has a meromorphic continuation to $S(-\pi, \pi)$ with poles at $-\beta_k$, $\beta_k - i\pi$, $\overline{\beta_k}$, and $-\overline{\beta_k} - i\pi$. The boundary value at $\mathbb{R} - i\pi$ is

$$S_2(\theta - i\pi) = S_2(-\theta + i\pi)^{-1} = S_2(-\theta) = S_2(\theta + i\pi), \quad \theta \in \mathbb{R}. \quad (6.4)$$

Hence S_2 continues to a $(2\pi i)$ -periodic, meromorphic function on all of \mathbb{C} , with poles at $-\beta_k + 2\pi iN$, $\beta_k + (2N - 1)\pi i$, $\overline{\beta_k} + 2\pi iN$, $-\overline{\beta_k} + (2N - 1)\pi i$, $k, N \in \mathbb{N}$. In particular, S_2 is analytic in the strip $S(-\kappa(S_2), \pi + \kappa(S_2))$, where $\kappa(S_2)$ is defined in (3.2), and has no zeros in $S(-\kappa(S_2), \kappa(S_2))$.

To establish the bound, we put $a = 0$ in (6.3) and consider a scattering function with finitely many zeros in $S(0, \pi)$. Then $\kappa(S_2) > 0$, and since $|S_2(\theta + i\lambda)| \rightarrow 1$ for $\theta \rightarrow \pm\infty$, $\lambda \in \mathbb{R}$ fixed, we conclude that S_2 is bounded on each strip $S(-\kappa, \pi + \kappa)$, $0 < \kappa < \kappa(S_2)$, as claimed. \square

Proof of Formula (4.24)

Here we prove that the distributions $\mathcal{D}_{n,k}$ defined in (4.18) and (4.24) coincide. The shorthand notations introduced in (4.16) will be used throughout.

To begin with, we recall two exchange formulae for the Zamolodchikov algebra, which follow by repeated straightforward application of the defining relations (2.9,2.10). There holds

$$\begin{aligned} z_{k_N} \cdots z_{k_1} z_{k+1}^\dagger &= \sum_{s=1}^N \delta_{k+1,k_s} \prod_{t=1}^{s-1} S_{k+1,k_t} \cdot z_{k_N} \cdots \widehat{z_{k_s}} \cdots z_{k_1} + \prod_{t=1}^N S_{k+1,k_t} \cdot z_{k+1}^\dagger z_{k_N} \cdots z_{k_1} \quad (6.5) \\ z_{k+1} z_k^\dagger \cdots \widehat{z_{k_N}^\dagger} \cdots \widehat{z_{k_1}^\dagger} \cdots z_1^\dagger \Omega &= \sum_{\substack{u=1 \\ u \neq k_1, \dots, k_N}}^k \delta_{k+1,u} \prod_{\substack{w=u+1 \\ w \neq k_1, \dots, k_N}}^k S_{w,k+1} \cdot z_k^\dagger \cdots \widehat{z_{k_N}^\dagger} \cdots \widehat{z_u^\dagger} \cdots \widehat{z_{k_1}^\dagger} \cdots z_1^\dagger \Omega. \end{aligned}$$

With the help of these equalities and the unitarity relation $\overline{S_2(\theta_1 - \theta_2)} = S_2(\theta_2 - \theta_1)$, we can rewrite the defining formula (4.18) for $\mathcal{D}_{n,k}$ in the following way. The sum over all subsets of $\{1, \dots, k\}$, appearing in (4.18), is abbreviated as $\sum_{\mathbf{k}}$.

$$\begin{aligned} \mathcal{D}_{n,k}(\theta_1, \dots, \theta_n) &= \sum_{\mathbf{k}} (-1)^N \prod_{j=1}^N \prod_{r_j=k_j+1}^k S_{r_j,k_j} \times \quad (6.6) \\ &\times \left(\sum_{s=1}^N \delta_{k+1,k_s} \prod_{t=1}^{s-1} S_{k_t,k+1} \langle z_{k_N} \cdots \widehat{z_{k_s}} \cdots z_{k_1} z_{k+2}^\dagger \cdots z_n^\dagger \Omega, A z_k^\dagger \cdots \widehat{z_{k_N}^\dagger} \cdots \widehat{z_{k_1}^\dagger} \cdots z_1^\dagger \Omega \rangle \right. \\ &+ \prod_{t=1}^N S_{k_t,k+1} \langle z_{k_N} \cdots z_{k_1} z_{k+2}^\dagger \cdots z_n^\dagger \Omega, [z_{k+1}, A] z_k^\dagger \cdots \widehat{z_{k_N}^\dagger} \cdots \widehat{z_{k_1}^\dagger} \cdots z_1^\dagger \Omega \rangle \\ &+ \prod_{t=1}^N S_{k_t,k+1} \sum_{\substack{u=1 \\ u \neq k_1, \dots, k_N}}^k \delta_{k+1,u} \prod_{\substack{w=u+1 \\ w \neq k_1, \dots, k_N}}^k S_{w,k+1} \times \\ &\left. \times \langle z_{k_N} \cdots z_{k_1} z_{k+2}^\dagger \cdots z_n^\dagger \Omega, A z_k^\dagger \cdots \widehat{z_{k_N}^\dagger} \cdots \widehat{z_u^\dagger} \cdots \widehat{z_{k_1}^\dagger} \cdots z_1^\dagger \Omega \rangle \right) \end{aligned}$$

The second major term, involving the commutator, coincides with the claimed expression (4.24). So it remains to prove that the sum of the first and third term vanishes. We will show that the partial sum of the third term, involving N annihilation operators z_{k_1}, \dots, z_{k_N} , cancels against the partial sum of the first term with $N+1$ annihilators $z_{k'_1}, \dots, z_{k'_{N+1}}$.

Consider the third term in (4.25). The sum \sum_u runs over $u \in \{1, \dots, k\} \setminus \{k_1, \dots, k_N\}$. So $k_{s-1} < u < k_s$ for some $s \in \{1, \dots, N+1\}$, with $k_0 := 0, k_{N+1} := k+1$, *i.e.* we may put

$$k'_1 := k_1, \dots, k'_{s-1} := k_{s-1}, k'_s := u, k'_{s+1} := k_s, \dots, k'_{N+1} := k_N,$$

and end up with an ordered set $1 \leq k'_1 < k'_2 < \dots < k'_{N+1} \leq k$. In terms of the primed indices, the various products over scattering functions, appearing in the contribution with N annihilation operators to the third term, read

$$\prod_{t=1}^N S_{k_t,k+1} \cdot \prod_{\substack{w=u+1 \\ w \neq k_1, \dots, k_N}}^k S_{w,k+1} = \prod_{t=1}^{s-1} S_{k'_t,k+1} \cdot \prod_{w=u+1}^k S_{w,k+1} \quad (6.7)$$

and hence

$$\delta_{k+1,u} \left(\prod_{\substack{1 \leq j \leq N \\ k_j + 1 \leq r_j \leq k}} S_{r_j, k_j} \right) \left(\prod_{t=1}^N S_{k_t, k+1} \right) \left(\prod_{\substack{w=u+1 \\ w \neq k_1, \dots, k_N}}^k S_{w, k+1} \right) = \delta_{k+1, k'_s} \left(\prod_{\substack{1 \leq j \leq N+1 \\ k'_j + 1 \leq r_j \leq k}} S_{r_j, k'_j} \right) \left(\prod_{t=1}^{s-1} S_{k'_t, k+1} \right).$$

So the partial sum of the third term which involves N annihilators can also be written as

$$\begin{aligned} & (-1)^N \sum_{1 \leq k'_1 < \dots < k'_{N+1} \leq k} \sum_{s=1}^{N+1} \delta_{k+1, k'_s} \left(\prod_{\substack{1 \leq j \leq N+1 \\ k'_j + 1 \leq r_j \leq k}} S_{r_j, k'_j} \right) \left(\prod_{t=1}^{s-1} S_{k'_t, k+1} \right) \\ & \times \langle z_{k'_N} \cdots \widehat{z_{k'_s}} \cdots z_{k'_1} z_{k+2}^\dagger \cdots z_n^\dagger \Omega, A z_k^\dagger \cdots \widehat{z_{k'_N}^\dagger} \cdots \widehat{z_{k'_1}^\dagger} \cdots z_1^\dagger \Omega \rangle \end{aligned}$$

and is seen to cancel precisely against the partial sum of the first term, involving $N + 1$ annihilation operators. The two extremal cases corresponding to $N = 0$ in the first term and to $N = k$ in the third term remain. But in these cases, the sums over s resp. u drop out (*cf.* (6.6)) and the contributions from the first and third term vanish separately. \square

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