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On the Operator Content of the Sinh-Gordon Model

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Abstract

We classify the operator content of local hermitian scalar operators in the Sinh-Gordon model by means of independent solutions of the form-factor bootstrap equations. The corresponding linear space is organized into a tower-like structure of dimension n for the form factors F_{2n} and F_{2n-1}. Analyzing the cluster property of the form factors, a particular class of these solutions can be identified with the matrix elements of the operators $e^{kg\phi}$. We also present the complete expression of the form factors of the elementary field $\phi(x)$ and the trace of the energy-momentum tensor $\Theta(x)$.

1 Introduction

The identification of local operators in a given model and the computation of their multi-point correlators is one of the most important problems in Quantum Field Theory (QFT). Although a lot is known about two-dimensional massless QFT [1, 2, 3], our present understanding is incomplete for massive QFT. A basic principle that allows us to identify the operator content, to predict the mass spectrum of bound and resonance states and to compute the correlation functions of the local fields is still missing. Some progress can be achieved for those massive QFT where the dynamics can be analyzed to a certain extent by means of non-perturbative methods: this is the case for instance of two-dimensional integrable massive QFT, mainly characterized by the factorizable S-matrix of the asymptotic states [4, 5, 6]. In this paper we propose the use of the *form factor bootstrap approach* [7, 8, 9, 10, 11] to classify the operator content of an integrable massive QFT. The form factors (FF) $F_n^{\mathcal{O}}$ are matrix elements of a generic local operator $\mathcal{O}$ between the vacuum and a set of n particle asymptotic states[1],

$$F_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n) \;=\; \langle 0 \mid \mathcal{O}(0,0) \mid \beta_1, \beta_2, \ldots, \beta_n \rangle_{in} \quad . \tag{1.1}$$

Based on general properties of a QFT (as unitarity, analyticity and locality), the form factor bootstrap approach leads to a system of linear functional equations for the matrix elements $F_n^{\mathcal{O}}$. The important point, already noticed in [10], is that these equations do *not* refer to any specific operator. This physical arbitrariness is reflected in the existence of independent solutions to the same set of functional equations. This pleasant feature allows us to classify the operator content by associating each of these solutions to an independent quantum operator of the theory. In this paper we pursue this idea for the Sinh-Gordon model, which is particularly suitable for its simple structure as an integrable QFT.

The paper is organized as follows: in Sect. 2 we discuss general properties of the Sinh-Gordon model and the form factors of local operators. In Sect. 3 we give the solutions of the recursive equations for the form factors of hermitian scalar operators. The cluster property of some specific solutions are discussed in Sect. 4. Our conclusions are presented in Sect. 5.

2 Form-Factors in the Sinh-Gordon Model

Let us briefly review the most important properties of the FF in the Sinh-Gordon theory. For a self-consistent treatment reference is made to paper [11].

[1]In the following we will consider a theory with only one particle and we parametrize the momentum of this particle in terms of the rapidity β, $p^0 \;=\; m \cosh \beta_i \;\; , p^1 \;=\; m \sinh \beta_i \;\; .$

The Sinh-Gordon theory is defined by the action

$$\mathcal{S} \,=\, \int d^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_0^2}{g^2}\,\cosh\, g\phi(x)\right] \,. \tag{2.1}$$

It is the simplest example of the affine Toda Field Theories [12], possessing a Z_2 symmetry $\phi \to -\phi$. Its two-particle S-matrix is given by [13]

$$S(\beta, B) \,=\, \frac{\tanh \frac{1}{2}(\beta - i\frac{\pi B}{2})}{\tanh \frac{1}{2}(\beta + i\frac{\pi B}{2})} \,, \tag{2.2}$$

where B is the so-called renormalized coupling constant

$$B(g) \,=\, \frac{2g^2}{8\pi + g^2} \quad . \tag{2.3}$$

For real values of g the S-matrix has no poles in the physical sheet and hence there are no bound states. The S-matrix presents a weak-strong duality under the transformation $g \to 8\pi/g$.

The form-factors of the Sinh-Gordon model satisfy the equations relative to a QFT without bound states [8] :

$$\begin{aligned} F_n(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) &= F_n(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n)\, S(\beta_i - \beta_{i+1}) \,, \\ F_n(\beta_1 + 2\pi i,\ldots,\beta_{n-1},\beta_n) &= F_n(\beta_2,\ldots,\beta_{n-1},\beta_n,\beta_1) \,, \\ -i \lim_{\tilde\beta\to\beta} (\tilde\beta - \beta) F_{n+2}(\tilde\beta + i\pi, \beta, \ldots, \beta_n) &= \left(1 - \prod_{i=1}^{n} S(\beta - \beta_i)\right) F_n(\beta_1,\ldots,\beta_n) \,. \end{aligned} \tag{2.4}$$

For an operator $\mathcal{O}(x)$ of spin s, relativistic invariance implies

$$F_n^{\mathcal{O}}(\beta_1 + \Lambda, \beta_2 + \Lambda, \ldots, \beta_n + \Lambda) \,=\, e^{s\Lambda}\, F_n^{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_n) \,. \tag{2.5}$$

In the Sinh-Gordon model a convenient parameterization of the n-particle FF, which takes into account their kinematical poles, is given by [11]

$$F_n(\beta_1,\ldots,\beta_n) \,=\, H_n\, Q_n(x_1,\ldots,x_n) \prod_{i<j} \frac{F_{\min}(\beta_{ij})}{(x_i + x_j)} \quad , \tag{2.6}$$

where $x_i \equiv e^{\beta_i}$ and $\beta_{ij} = \beta_i - \beta_j$. Let us discuss the different terms entering the above expression. $F_{\min}(\beta)$ is an analytic function given by

$$F_{\min}(\beta, B) \,=\, \prod_{k=0}^{\infty} \left| \frac{\Gamma\left(k + \frac{3}{2} + \frac{i\hat\beta}{2\pi}\right) \Gamma\left(k + \frac{1}{2} + \frac{B}{4} + \frac{i\hat\beta}{2\pi}\right) \Gamma\left(k + 1 - \frac{B}{4} + \frac{i\hat\beta}{2\pi}\right)}{\Gamma\left(k + \frac{1}{2} + \frac{i\hat\beta}{2\pi}\right) \Gamma\left(k + \frac{3}{2} - \frac{B}{4} + \frac{i\hat\beta}{2\pi}\right) \Gamma\left(k + 1 + \frac{B}{4} + \frac{i\hat\beta}{2\pi}\right)} \right|^2 \quad . \tag{2.7}$$

It satisfies the functional equations

$$\begin{aligned} F_{\min}(\beta) &= F_{\min}(-\beta)\, S(\beta, B)\ , \\ F_{\min}(i\pi - \beta) &= F_{\min}(i\pi + \beta)\ , \\ F_{\min}(i\pi + \beta, B) F_{\min}(\beta, B) &= \frac{\sinh\beta}{\sinh\beta + \sinh\frac{i\pi B}{2}}\ . \end{aligned} \tag{2.8}$$

It has a simple zero at the threshold $\beta = 0$ (since $S(0, B) = -1$) and no poles in the physical strip $0 \leq \mathrm{Im}\,\beta \leq \pi$, with an asymptotic behaviour given by

$$\lim_{\beta\to\infty} F_{\min}(\beta, B) = 1\ . \tag{2.9}$$

H_n are normalization constants, which can be conveniently chosen as

$$H_{2n+1} = H_1 \mu^{2n} \quad , \quad H_{2n} = H_2 \mu^{2n-2}\ , \tag{2.10}$$

with

$$\mu \equiv \left(\frac{4\sin(\pi B/2)}{F_{min}(i\pi, B)}\right)^{\frac{1}{2}} \tag{2.11}$$

and H_1, H_2 are two independent parameters. Finally, the functions $Q_n(x_1, \ldots, x_n)$ are symmetric polynomials in the variables x_i, which have to be fixed by the recursion equations satisfied by the form factors. For FF of spinless operators, their total degree is equal to $n(n-1)/2$. On the other hand, the partial degree of Q_n in each variable x_i is fixed by the nature and the asymptotic behaviour of the operator $\mathcal{O}$ which is considered. Exploiting the parameterization (2.6) together with the functional equations (2.4) and (2.8), the polynomials $Q_n(x_1, \ldots, x_n)$ have to satisfy the recursive equations [11]

$$(-)^n\, Q_{n+2}(-x, x, x_1, \ldots, x_n) = x D_n(x, x_1, x_2, \ldots, x_n)\, Q_n(x_1, x_2, \ldots, x_n) \tag{2.12}$$

with

$$D_n(x, x_1, \ldots, x_n) = \sum_{k=1}^{n} \sum_{m=1,odd}^{k} [m]\, x^{2(n-k)+m} \sigma_k^{(n)} \sigma_{k-m}^{(n)} (-1)^{k+1}\ . \tag{2.13}$$

We have introduced the symbol $[n]$ defined by

$$[n] \equiv \frac{\sin(n\frac{B}{2})}{\sin\frac{B}{2}} \tag{2.14}$$

and the *elementary symmetric polynomials* $\sigma_k^{(n)}(x_1, \ldots, x_n)$, given by the generating function [14]

$$\prod_{i=1}^{n}(x + x_i) = \sum_{k=0}^{n} x^{n-k}\, \sigma_k^{(n)}(x_1, x_2, \ldots, x_n)\ . \tag{2.15}$$

Conventionally the $\sigma_k^{(n)}$ with $k > n$ and with $n < 0$ are zero. The explicit expressions are given as

$$\sigma_k^{(n)} = \sum_{i_1<i_2<\ldots<i_k}^{n} x_{i_1} x_{i_2} \ldots x_{i_k} \tag{2.16}$$

The $\sigma_k^{(n)}$ are linear in each variable x_i and their total degree is k. They satisfy the recursive equations

$$\sigma_k^{(n+2)}(-x, x, x_1, \ldots, x_n) \;=\; \sigma_k^{(n)}(x_1, x_2, \ldots, x_n) - x^2 \sigma_{k-2}^{(n)}(x_1, x_2, \ldots, x_n) \;. \tag{2.17}$$

As mentioned in the introduction, the linear equations satisfied by the FF do not refer to any specific operator. Hence the operator content of the model is encoded in the vector space of solutions of the recursive equations (2.12). Among them, we expect to find the FF of the (renormalized) elementary field $\phi(x)$ and the trace of the energy-momentum tensor Θ. Their normalizations are given by

$$\begin{array}{rcl} <0 \mid \phi(0) \mid \beta> & = & \frac{1}{\sqrt{2}} \;, \\ <\beta \mid \Theta(0) \mid \beta> & = & 2\pi m^2 \;. \end{array} \tag{2.18}$$

where m is the physical mass.

3 Solution of the Recursive Equations

In this section, we discuss the most general solution of the recursive equations (2.12) in the space of symmetric polynomials $\mathcal{P}$ of total degree $n(n-1)/2$. Any independent solution defines the matrix elements of a local scalar operator in the theory. A clarification is in order here. In the following we will only consider the FF of *irreducible operators*. Their form factors have the distinguished property that they cannot be further factorized in terms of the elementary symmetric polynomial $\sigma_k^{(n)}$. We will not consider for instance FF of *derivative scalar operators* of the kind $(\partial\bar{\partial})^k\mathcal{O}$. The reason for this is that they can be simply obtained by multiplying the FF $F_n^{\mathcal{O}}$ by the term $(\frac{\sigma_{n-1}\sigma_1}{\sigma_n})^k$.

3.1 General Solution for Q_3 and Q_4

To understand the structure of the linear space of the FF, it is worth considering the first polynomials $Q_1, .., Q_4$.

By relativistic invariance, Q_1 has to be a constant which we denote by $A_1^{(1)}$. For the same reason Q_2 is proportional to σ_1, since this is the only symmetric polynomial of degree 1, and we define $Q_2 = A_1^{(2)}\sigma_1$.

At level 3, the most general symmetric polynomial of total degree 3 is given by

$$\mathcal{P}_3 = A_1^{(3)}\sigma_3 + A_2^{(3)}\sigma_1\sigma_2 + A_3^{(3)}\sigma_1^3 \;, \tag{3.1}$$

and in order to be identified as a form factor of the theory it should satisfy the recursive equation

$$-\mathcal{P}_3(-x,x,x_1) = xD_1(x)Q_1(x_1) = -x^2\sigma_1^{(1)}A_1^1 \ . \tag{3.2}$$

The solution of (3.2) gives rise to the most general form factor Q_3

$$Q_3 \ = \ (A_1^{(1)} - A_2^{(3)})\sigma_3 + A_2^{(3)}\sigma_1\sigma_2 \ . \tag{3.3}$$

Given $A_1^{(1)}$, the linear space of FF at level 3 is a one-dimensional manifold parametrized by $A_2^{(3)}$, for instance.

An analogous result holds for Q_4. Starting with the most general symmetric polynomial of total degree 6

$$\begin{aligned}\mathcal{P}_4 \ &= \ A_1^{(4)}\sigma_4\sigma_2 + A_2^{(4)}\sigma_3\sigma_2\sigma_1 + A_3^{(4)}\sigma_4\sigma_1^2 + A_4^{(4)}\sigma_3^2 + A_5^{(4)}\sigma_3\sigma_1^3 \\ &+ \ A_6^{(4)}\sigma_2^3 + A_7^{(4)}\sigma_2^2\sigma_1^2 + A_8^{(4)}\sigma_2\sigma_1^4 + A_9^{(4)}\sigma_1^6 \ ,\end{aligned} \tag{3.4}$$

and imposing the recursive equation (2.12) with initial condition $A_1^{(2)}$, the final form of Q_4 is given by

$$Q_4 = A_2^{(4)}\sigma_3\sigma_2\sigma_1 + A_3^{(4)}(\sigma_4\sigma_1^2 + \sigma_3^2) \quad , \tag{3.5}$$

with

$$A_3^{(4)} + A_2^{(4)} \ = \ A_1^{(2)} \ .$$

Also in this case the linear space of FF at level 4 is a one-dimensional manifold.

It is interesting to note that both solutions can be written as a sum of determinants[2]

$$\begin{aligned} Q_3 \ &= \ A_1^{(3)} \left\| \begin{array}{cc} 0 & \sigma_3 \\ -1 & 0 \end{array} \right\| + A_2^{(3)} \left\| \begin{array}{cc} \sigma_1 & [2]\sigma_3 \\ 0 & \sigma_2 \end{array} \right\| \quad , \\ Q_4 \ &= \ A_1^{(4)} \left\| \begin{array}{ccc} \sigma_1 & [2]\sigma_3 & 0 \\ 0 & \sigma_2 & [2]\sigma_4 \\ 0 & 0 & \sigma_3 \end{array} \right\| + A_2^{(4)} \left\| \begin{array}{ccc} [2]\sigma_1 & [3]\sigma_3 & 0 \\ 1 & [2]\sigma_2 & [3]\sigma_4 \\ 0 & \sigma_1 & [2]\sigma_3 \end{array} \right\| \quad . \end{aligned}$$

Let us discuss some properties of $Q_1, \ldots, Q_4$. First of all at any step of the recursion process a new free parameter enters the solution. Secondly, the partial degree[3] of Q_n ($n = 1, \ldots, 4$) does not exceed $n-1$. Hence, all these FF will be at the most constant for $\beta_i \to \infty$. Finally, seeking solutions of the FF equations which vanish as $\beta_i \to \infty$, there is only a *unique* class of polynomials, i.e. Q_2 and Q_4 identically zero and $Q_3 \sim \sigma_3$. In [11] it has been shown that these are the first form-factors of the elementary field ϕ.

[2] We use the notation $||A|| \equiv \mathrm{detA}$.

[3] Note that the partial degree in each x_i determines the asymptotic behaviour of the FF and is fixed by the total number of $\sigma^{(n)}$'s in each term of the sum.

3.2 Properties of the General Solution Q_n

Important properties of the polynomials Q_n can be easily obtained by analyzing the recursive equations (2.12). As a first step let us show that the partial degree of the polynomials Q_n satisfies

$$\deg(Q_n) \leq n-1 \quad . \tag{3.6}$$

We have seen above that this property is true for $Q_1, .., Q_4$. To prove that this persists for higher polynomials, the cases (a) $Q_n \neq 0$ and (b) $Q_n = 0$ have to be considered separately.

- In the case (a) the proof is done by induction. Let us assume $\deg(Q_n) \leq n-1$. Since D_n is bilinear in $\sigma^{(n)}$ (see eq. (2.13)), the partial degree of $Q_{n+2}(-x, x, x_1, \ldots x_n)$ in the variables $x_1, \ldots x_n$ is smaller or equal to $n+1$. But the partial degree of $Q_{n+2}(x_1, x_2, \ldots x_{n+2})$ is equal to $Q_{n+2}(-x, x, x_1 \ldots, x_n)$, therefore the partial degree of Q_{n+2} must be less or equal to $n+1$.

- In the case (b), the space of the solutions is given by the kernel of eq. (2.12), i.e.

  $$Q_{n+2}(-x, x, \ldots, x_{n+2}) = 0 \quad . \tag{3.7}$$

  In the space of polynomials $\mathcal{P}$ of total degree $\frac{(n+2)(n+1)}{2}$, there is only one solution of this equation, i.e.

  $$Q_{n+2} = \prod_{i<j}^{n+2} (x_i + x_j) \quad . \tag{3.8}$$

  This polynomial has partial degree $n+1$ and coincides with the denominator of eq. (2.6).

We have therefore shown that the partial degree of Q_n must be less or equal to $(n-1)$ for any spinless irreducible operator. A first consequence of this statement is that the FF of such operators cannot diverge in the limit $\beta_i \to \infty$. Secondly, it is now easy to understand the appearance of one additional parameter at each step of the recursion process. This is simply because the total dimension of the space of the polynomials Q_n is given by the dimension of the space of the polynomial Q_{n-2}, summed with the dimension of the kernel. Since the kernel is a one-dimensional manifold, the dimension of the space of solutions increases exactly by one at each step of the recursion. With the initial conditions $\dim(Q_1) = \dim(Q_2) = 1$, we finally obtain

$$\dim(Q_{2n-1}) = \dim(Q_{2n}) = n \quad . \tag{3.9}$$

Therefore the most general FF at level n of irreducible scalar operators span a finite linear space which can be expressed in terms of a basis Q_n^k

$$\begin{aligned} Q_{2n}(A_1^{(2n)},\ldots,A_n^{(2n)}) &= \sum_{p=1}^{n} A_p^{(2n)} Q_{2n}^p \\ Q_{2n-1}(A_1^{(2n-1)},\ldots,A_n^{(2n-1)}) &= \sum_{p=1}^{n} A_p^{(2n-1)} Q_{2n-1}^p \ . \end{aligned} \tag{3.10}$$

Each of these polynomials defines the matrix elements of a quantum operator of the Sinh-Gordon model. Note that the dimension of the linear space of the FF grows exactly as the number of powers of the elementary field $\phi(x)$, i.e. ϕ^k, $k < n$. Therefore we expect to identify the tower structure of the FF with the space of the matrix elements of the composite operators ϕ^k. The precise correspondence will be discussed in a forthcoming publication [15].

3.3 Elementary Solutions

An interesting class of solutions of the recursive equations (2.12) from which we can extract a basis for the FF space is given by[4]

$$Q_n(k) = ||M_{ij}(k)|| \quad , \tag{3.11}$$

where $M_{ij}(k)$ is an $(n-1)\times(n-1)$ matrix with entries

$$M_{ij}(k) = \sigma_{2i-j}\,[i-j+k] \quad . \tag{3.12}$$

These polynomials, which we call *elementary solutions*, depend on an arbitrary integer k and satisfy

$$Q_n(k) \;=\; (-1)^{n+1} Q_n(-k) \quad . \tag{3.13}$$

Although all $Q_n(k)$ are solutions of (2.12) not all of them can be linearly independent. The reason is that the dimension of space of the solutions at level $N = 2n$ (or $N = 2n-1$) is n at most. The first non trivial $Q_n(k)$ are given by the determinant

$$Q_3(k) \;=\; \left\| \begin{array}{cc} [k]\sigma_1 & [k+1]\sigma_3 \\ \text{[k-1]} & [k]\sigma_2 \end{array} \right\| \quad . \tag{3.14}$$

Using the trigonometrical identity $[n]^2 - [n-1][n+1] = 1$, it is easy to see that they satisfy eq. (2.12) (with $A_0^1 = 1$) for any integer value of k.

The whole set[5] of FF of the elementary field $\phi(x)$ and the trace of the energy-momentum tensor $\Theta(x)$ can be easily expressed in terms of the $Q_n(k)$. In fact the FF of

[4] For simplicity we suppress the dependance of $Q_n(k)$ on the variables x_i.

[5] The first representatives of them were computed in [11].

the field $\phi(x)$ are given by $Q_n(0)$. Note that the FF of $\phi(x)$ are automatically zero for even n, in agreement with the Z_2 parity of the model. On the contrary, for odd n they vanish asymptotically when $\beta_i \to \infty$, as follows from the LSZ reduction formula [11]. Concerning the FF of $\Theta(x)$, these are given by the even polynomials $Q_{2n}(1)$, which tend to a constant when $\beta_i \to \infty$.

4 Cluster-Property and Exponential Operators

We can associate a quantum operator $\Psi_k(x)$ to any elementary solution $Q_n(k)$ $(k \neq 0)$. To identify such operators it is interesting to analyze the cluster property of their FF. By cluster transformation we generally mean the behaviour of a form factor under the shift of a subset of the rapidities, i.e.

$$F_n^{\mathcal{O}_a}(\beta_1 + \Delta, \ldots, \beta_m + \Delta, \beta_{m+1}, \ldots, \beta_n) \ . \tag{4.1}$$

Taking the limit $\Lambda \to \infty$, $F_n^{\mathcal{O}_a}$ can be decomposed into two functions of m and $(n-m)$ variables respectively. It is easy to prove that both functions satisfy all the set of axioms (2.4). Therefore they can be considered as FF of some operators $\mathcal{O}_b$ and $\mathcal{O}_c$

$$\lim_{\Lambda\to\infty} F_n^{\mathcal{O}_a}(\beta_1+\Delta, \ldots, \beta_m+\Delta, \beta_{m+1}, \ldots, \beta_n) = F_m^{\mathcal{O}_b}(\beta_1, \ldots, \beta_m) F_{n-m}^{\mathcal{O}_c}(\beta_{m+1}, \ldots, \beta_n) \tag{4.2}$$

Shortly,

$$\mathcal{O}_a \ \to \ \mathcal{O}_b \times \mathcal{O}_c \ .$$

We will prove that the operators Ψ_k are mapped onto themselves under the cluster transformation, i.e.

$$\Psi_k \ \to \ \Psi_k \times \Psi_k \ . \tag{4.3}$$

To this aim, let us introduce some notations.

It is useful to define the cluster-operator $\mathcal{C}_m$ (acting on the symmetric functions) by means of

$$\mathcal{C}_m \left(f(x_1, \ldots, x_n) \right) \equiv f(x_1 e^{\Delta}, x_2 e^{\Delta}, \ldots, x_m e^{\Delta}, x_{m+1}, \ldots, x_n) \ \ m < n \ . \tag{4.4}$$

For example,

$$\mathcal{C}_1(\sigma_1^{(n)}) = e^{\Delta} x_1 + x_2 \ldots + x_n = e^{\Delta} \sigma_1^1 + \hat{\sigma}_1^{(n-1)} \quad , \tag{4.5}$$

where we have defined

$$\hat{\sigma}_i^{(n-k)} \equiv \sigma_i^{(n-k)}(x_{n-k+1}, x_{n-k+2}, \ldots, x_n) \quad .$$

One easily finds that

$$\mathcal{C}_m(\sigma_k^{(n)}) = \sum_{i=1}^{k} \sigma_{k-i}^{(m)} e^{(k-i)\Delta} \hat{\sigma}_i^{(n-m)} \quad . \tag{4.6}$$

Since the cluster properties are fixed by the leading term of this sum, we have

$$\begin{aligned} \mathcal{C}_m(\sigma_k^{(n)}) &\sim \sigma_m^{(m)} e^{m\Delta} \hat{\sigma}_{k-m}^{(n-m)} & m \leq k \quad , \\ \mathcal{C}_m(\sigma_k^{(n)}) &\sim \sigma_k^{(m)} e^{k\Delta} & m \geq k \quad . \end{aligned} \tag{4.7}$$

Now let us consider separately the cluster property of each term entering their parametrization

$$F_n^k(\beta_1, \ldots, \beta_n) = H_n^k \, Q_n(k) \prod_{i<j}^{n} \frac{F_{\min}(\beta_{ij})}{(x_i + x_j)} \quad . \tag{4.8}$$

From eq. (2.9), we have

$$\prod_{i<j}^{n} F_{\min}(\beta_{ij}) \longrightarrow \prod_{i<j}^{m} F_{\min}(\beta_{ij}) \prod_{i<j=m+1}^{n} F_{\min}(\beta_{ij}) \quad . \tag{4.9}$$

Using eq. (4.7), the cluster property of the elementary solution $Q_n(k)$ is given by

$$\mathcal{C}_m(Q_n(k)) \sim h(n,m)\, [k]\, Q_m(k)\, Q_{n-m}(k) \quad , \tag{4.10}$$

where

$$h(n,m) = e^{\Delta m (n - \frac{m+1}{2})}\, (\sigma_m^{(m)})^{n-m} \quad .$$

Concerning the denominator in (4.8), we write it initially as

$$\prod_{i<j}^{n} (x_i + x_j) = ||\sigma_{2i-j}^{(n)}||^{\langle n-1 \rangle} \quad , \tag{4.11}$$

where the index $\langle n-1 \rangle$ indicates the dimension of the matrix. Using eq. (4.7), we obtain

$$\mathcal{C}_m \left(||\sigma_{2i-j}^{(n)}||^{\langle n-1 \rangle} \right) \sim h(n,m)\, ||\sigma_{2i-j}^{(m)}||^{\langle m-1 \rangle} \times ||\hat{\sigma}_{2i-j}^{(n-m)}||^{\langle n-m-1 \rangle} \quad . \tag{4.12}$$

Choosing the normalization constants H_1^k and H_2^k as

$$H_1^k = \mu[k] \quad , \quad H_2^k = \mu^2\, [k] \quad , \tag{4.13}$$

and using eqs. (4.9), (4.10) and (4.12), we conclude that the FF of Ψ_k are mapped onto themselves under the cluster transformation. Since this is a distinguished property of exponential operators [8], it is natural to identify the operators Ψ_k with the fields $e^{kg\phi}$. A non-trivial check is given by the computation of their anomalous dimensions $\Delta_k(g)$ obtained in the ultraviolet limit. The quantity $\Delta_k(g)$ can be computed by analyzing the limit $x \to 0$ of the correlation functions

$$\begin{aligned} G_{k,m}(x) &= \langle \Psi_k(x)\, \Psi_m(0) \rangle = \\ &= \sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n! (2\pi)^n} F_n^{\Psi_k}(\beta_1 \ldots \beta_n) F_n^{\Psi_m}(\beta_n \ldots \beta_1) \exp\left(-mr \sum_{i=1}^{n} \cosh \beta_i \right) \quad . \end{aligned} \tag{4.14}$$

At first order in g, we find $\Delta_k(g) = -k^2 g^2 / 8\pi$ which coincides with the anomalous dimensions of the exponential operators $e^{k\, g\, \phi(x)}$, computed from conformal perturbation theory.

5 Conclusions

The computation of Green functions and the classification of the operator content are key problems in Quantum Field Theory. For integrable models, a promising development has been achieved by the bootstrap approach.

In this paper we have given the general form factors of scalar irreducible operators in the Sinh-Gordon model. We have found an infinite number of elementary solutions $Q_n(k)$. As far as the renormalized coupling constant $B(g)$ is irrational, they define the FF of the independent operators Ψ_k. This is no longer true if we take rational values of $B(g)$. In these cases an identification of different operators occurs due to the symmetry of the quantum-symbol $[n]$ and the space of the fields Ψ_k is drastically reduced. This reflects the situation occurring in conformal field theory and confirms the hope that one may find similar finite-dimensional representations for the symmetry algebra of massive integrable QFT at rational values of the coupling constant.

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