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Stress-Energy Tensor and Ultraviolet Behaviour in Massive Integrable Quantum Field Theories

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Abstract

The short distance behaviour of massive integrable quantum field theories is analyzed in terms of the form factor approach. We show that the on-shell dynamics is compatible with different definitions of the stress-energy tensor $T_{\mu\nu}(x)$ of the theory. In terms of form factors, this is equivalent to having a possible non-zero matrix element F_1 of the trace of $T_{\mu\nu}$ on one-particle state. Each choice of F_1 induces a different scaling behaviour of the massive theory in the ultraviolet limit.

1 Introduction

One of the most relevant steps towards the physical interpretation of a given field dynamics consists in the identification of the stress tensor $T_{\mu\nu}(x)$. It gives the local distribution of energy and momentum, and rules the response of the system under local scale transformations. In this paper we consider quantum field theories involving one scalar field $\varphi(x)$ defined in $(1+1)$ flat space-time where there is, however, a natural ambiguity in the definition of $T_{\mu\nu}(x)$. To see this, the simplest way is to initially consider the theory defined on a curved two-dimensional manifold with metric tensor $g_{\mu\nu}$ and scalar curvature R

$$\mathcal{A} = \int d^2\xi \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu\varphi\partial_\nu\varphi - V(\varphi) + \alpha R \varphi \right) , \tag{1.1}$$

In the above action, α is a free parameter that in the limit of a flat manifold $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$, labels the one-dimensional family of stress-energy tensors associated to the on-shell dynamics

$$T_{\mu\nu}(x) = \tilde{T}_{\mu\nu}(x) + \alpha (\partial_\mu\partial_\nu - g_{\mu\nu}\Box) \varphi(x) , \tag{1.2}$$

where

$$\tilde{T}_{\mu\nu}(x) = 2\pi \left[\partial_\mu\varphi \, \partial_\nu\varphi - \eta_{\mu\nu} \left(\frac{1}{2}(\partial\varphi)^2 - V(\varphi) \right) \right] . \tag{1.3}$$

Although in the flat Minkowski space the equations of motion for the field φ do not depend on the specific definition of $T_{\mu\nu}(x)$, the ultraviolet behaviour of the theory may be extremely sensitive to any modification of this operator. A well-known example is provided by the conformal invariant QFT [1, 2]: in a Coulomb Gas language, an extra derivative term in the definition of the stress-energy tensor results in a non-zero background charge that, in turn, induces a non-trivial scaling behaviour of the operators of the theory [3, 4]. It is therefore an interesting question to see what are the consequences of a redefinition of $T_{\mu\nu}(x)$ also in the case of quantum field theories which do not have scaling invariant properties. A special class of these theories are the massive integrable models where the off-shell dynamics may be completely characterized in terms of the on-shell scattering amplitudes of the massive excitations. In fact, we can take advantage of their integrability, and compute exactly the matrix elements of local operators $\mathcal{O}_k(x)$ on the asymptotic states, by means of the *Form Factor Bootstrap Approach* [5-13]. All correlation functions are then reconstructed in terms of their spectral representation. For large values of the relative distances, these correlation functions will have an exponential decay ruled by the lowest massive state appearing in their spectral decomposition. Their short distance behaviour, on the contrary, present power law singularities. In order to identify the conformal dimensions of the operators, we need to analyze the short distance singularity of the correlation functions $< T_{zz}(z,\overline{z})\, \mathcal{O}_k(0) >$. For those operators that correspond to primary fields in the conformal limit $(mR) \to 0$ $(R^2 = z\overline{z})$, we have

$$< T_{zz}(z, \overline{z})\, \mathcal{O}_k(0) > \simeq \frac{\Delta(\mathcal{O}_k)}{z^2} \quad . \tag{1.4}$$

The conformal dimensions $\Delta(\mathcal{O}_k)$ will depend in general on the definition of $T_{\mu\nu}(x)$ and therefore different definitions of this operator may induce different scaling behaviours of the theory in the deep ultraviolet region. Analogously, the central charge of the ultraviolet theory may be extracted from the short-distance singularity of the correlator

$$< T_{zz}(z, \overline{z})\, T_{zz}(0) > \simeq \frac{c}{2z^4} \quad . \tag{1.5}$$

The plan of the paper is as follows. In section 2 we discuss the general properties of the stress-energy tensor. Using entirely the formalism of the form factor approach, we derive the conditions on the matrix elements of $T_{\mu\nu}(x)$ which guarantee its conservation and locality. From this analysis, there is a one-dimensional space of possible stress-energy tensors $T_{\mu\nu}(x)$ compatible with the on-shell dynamics. In section 3 and 4 we then discuss the ultraviolet behaviours of the simplest Affine Toda Field Theories [24], i.e. the Sinh-Gordon and the Bullough-Dodd models based respectively on the simply-laced algebra $A_1^{(1)}$ and on the non-simply laced algebra $A_2^{(2)}$. In both theories, different ultraviolet limits may be reached by varying the definition of the stress-energy tensor, with the relevant conformal data simply obtained in terms of the form factors. Our conclusions are summarized in section 5.

2 Form Factors of the Stress-Energy Tensor

Let us consider an integrable two-dimensional massive QFT characterized by its elastic factorizable S-matrix [14, 15, 16]. As it is well known, the form factor approach is quite effective to characterize the operators in such a theory in terms of their matrix elements on the set of asymptotic states [5, 6]. Adopting the standard parametrization of the momenta in terms of the rapidity variable β, i.e. $p^\mu = (m \cosh\beta, m \sinh\beta)$, any local operator $\mathcal{O}_k(x)$ will be uniquely identified[1] by the set of its Form Factors (FF)

$$F_n^k(\beta, \ldots, \beta_n) = < 0|\mathcal{O}_k(0)|\beta_1, \ldots, \beta_n >_{\text{in}} . \tag{2.1}$$

In the above definition, the set $\{\beta_i\}$ is ordered as $\beta_1 > \beta_2 \ldots > \beta_n$ and our normalization is given by

$${}_{\text{in}}< \beta_1', \ldots, \beta_m' \,|\, \beta_1, \ldots, \beta_n >_{\text{in}} = \delta_{m,n} \prod_{i=1}^{n} 2\pi\, \delta(\beta_i' - \beta_i) \quad . \tag{2.2}$$

[1] This justifies the interchangeable use we make of the words "operator" or "form factors" in the rest of the paper.

Once the FF of the local operators $\mathcal{O}_k(x)$ are known, their two-point (and higher) correlation functions can be reconstructed through the unitary sum[2]

$$< \mathcal{O}_k(r)\, \mathcal{O}_k(0) > = \sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} \mid F_n^k(\beta_1 \ldots \beta_n) \mid^2 \exp\left(-mr \sum_{i=1}^{n} \cosh\beta_i\right) \tag{2.3}$$

Let us discuss then the properties of the FF, as dictated by relativistic invariance and general requirements of QFT [5, 6]. For an operator $\mathcal{O}_k(x)$ of spin s, relativistic invariance implies

$$F_n^k(\beta_1+\Lambda, \beta_2+\Lambda, \ldots, \beta_n+\Lambda) \;=\; e^{s\Lambda}\, F_n^k(\beta_1, \beta_2, \ldots, \beta_n)\;. \tag{2.4}$$

The FF of a given theory are solutions of a set of functional and recursive equations. The functional equations arise from the monodromy properties of the functions F_n^k which are ruled by the S-matrix[3]

$$\begin{aligned} F_n^k(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n) \;&=\; F_n^k(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n)\, S(\beta_i - \beta_{i+1})\;, \qquad (2.5)\\ F_n^k(\beta_1 + 2\pi i, \ldots, \beta_{n-1}, \beta_n) \;&=\; F_n^k(\beta_2, \ldots, \beta_n, \beta_1) = \prod_{i=2}^{n} S(\beta_i - \beta_1)\, F_n^k(\beta_1, \ldots, \beta_n)\;. \end{aligned}$$

The recursive equations are obtained, on the contrary, by looking at the pole structure of the matrix elements F_n^k. The first kind of poles are kinematical poles located at $\beta_{ij} = i\pi$. The corresponding residues give rise to a recursive equation between the n-particle and the $(n+2)$-particle FF

$$-i \lim_{\tilde\beta\to\beta} (\tilde\beta - \beta) F_{n+2}^k(\tilde\beta + i\pi, \beta, \beta_1, \beta_2, \ldots, \beta_n) = \left(1 - \prod_{i=1}^{n} S(\beta - \beta_i)\right)\, F_n^k(\beta_1, \ldots, \beta_n)\;. \tag{2.6}$$

If bound states are present in the spectrum, there is another set of recursive equations obtained by looking at their poles in the matrix elements. Let $\beta_{ij} = i\,u$ be the location of the pole in the two-particle scattering amplitude corresponding to the bound state. Then the corresponding residue in the FF is given by

$$-\,i \lim_{\epsilon\to 0} \epsilon\, F_{n+1}(\beta + i\,u - \frac{\epsilon}{2}, \beta + \frac{\epsilon}{2}, \beta_1, \ldots, \beta_{n-1}) \;=\; \Gamma\, F_n(\beta, \beta_1, \ldots, \beta_{n-1})\;, \tag{2.7}$$

where Γ is the on-shell three-particle vertex. This equation establishes a recursive structure between the $(n+1)$- and n-particle form factors.

The two chains of recursive equations,

$$\begin{array}{c} \ldots \to F_{n+4} \to F_{n+2} \to F_n \to F_{n-2} \to \ldots \\ \ldots \to F_{n+4} \to F_{n+3} \to F_{n+2} \to F_{n+1} \to \ldots \end{array} \tag{2.8}$$

[2]This expression holds for scalar operators. The generalization of this equation for operators of spin s is easily done by using eq. (2.4).

[3]For simplicity we consider theory with only one massive self-conjugate particle.

(and the consistency conditions associated to them) are quite effective for the explicit determination of the FF of a given theory.

The above discussion holds for any FF of a local operator, in particular for those of the stress-energy tensor $T_{\mu\nu}(x)$. However, due to the special role of this operator, its FF has some distinguishing properties. Since $T_{\mu\nu}(x)$ is conserved, it may be expressed in terms of an auxiliary scalar field $A(x)$ as [6]

$$T_{\mu\nu}(x) = (\partial_\mu \partial_\nu - g_{\mu\nu} \Box) A(x) . \tag{2.9}$$

In light-cone coordinates $x^{\pm} = x^0 \pm x^1$, its components are given by

$$T_{++} = \partial_+^2 A \quad , T_{--} = \partial_-^2 A , \tag{2.10}$$

$$\Theta = T_\mu^\mu = - \Box A = -4 \partial_+ \partial_- A . \tag{2.11}$$

Introducing the variables $x_j = e^{\beta_j}$ and the elementary symmetric polynomials in n-variables $\sigma_i^{(n)}$ defined by the generating function

$$\prod_{j=1}^{n} (x + x_j) = \sum_{i=0}^{n} x^{k-i} \sigma_i^{(n)}(x_1, \ldots, x_k) , \tag{2.12}$$

it is easy to see that

$$\begin{aligned} F_n^{T_{++}}(\beta_1, \ldots, \beta_n) &= -\frac{1}{4} m^2 \left(\frac{\sigma_{n-1}^{(n)}}{\sigma_n^{(n)}} \right)^2 F_n^A(\beta_1, \ldots, \beta_n) , \\ F_n^{T_{--}}(\beta_1, \ldots, \beta_n) &= -\frac{1}{4} m^2 \left(\sigma_1^{(n)} \right)^2 F_n^A(\beta_1, \ldots, \beta_n) , \\ F_n^{\Theta}(\beta_1, \ldots, \beta_n) &= m^2 \frac{\sigma_1^{(n)} \sigma_{n-1}^{(n)}}{\sigma_k} F_n^A(\beta_1, \ldots, \beta_n) . \end{aligned} \tag{2.13}$$

Solving for F_n^A, we have

$$\begin{aligned} F_n^{T_{++}}(\beta_1, \ldots, \beta_n) &= -\frac{1}{4} \frac{\sigma_{n-1}^{(n)}}{\sigma_1^{(n)} \sigma_n^{(n)}} F_n^{\Theta}(\beta_1, \ldots, \beta_n) , \\ F_n^{T_{--}}(\beta_1, \ldots, \beta_n) &= -\frac{1}{4} \frac{\sigma_1^{(n)} \sigma_n^{(n)}}{\sigma_{n-1}^{(n)}} F_n^{\Theta}(\beta_1, \ldots, \beta_n) . \end{aligned} \tag{2.14}$$

Hence the complete knowledge of $T_{\mu\nu}$ is encoded into the form factors of the trace Θ. As any spinless operator, its form factors $F_n^{\Theta}(\beta_1, \ldots, \beta_n)$ depend only on the difference of the rapidities $\beta_{ij} = \beta_i - \beta_j$. Moreover, since the FF of T_{--} and T_{++} have the same singularity structure of the FF of Θ, $F_n^{\Theta}(\beta_1, \ldots, \beta_n)$ (for $n > 2$) has to be proportional to the combination of symmetric polynomials $\sigma_1^{(n)} \sigma_{n-1}^{(n)}$ which corresponds to the invariant total energy-momentum.

Additional constraints on F_n^Θ are obtained from the knowledge of their asymptotic behaviour in each variable β_i. This behaviour generally depends on the particular model under consideration. For the lagrangian quantum field theories discussed in this paper, we have

$$F_n^\Theta(\beta_1 + \Delta, \beta_2, \ldots, \beta_n) \stackrel{\Delta \to \infty}{\longrightarrow} o(1) \ , \tag{2.15}$$

i.e. they become constant for large values of the individual momenta. This condition can be easily checked by analyzing the asymptotic behaviour of the Feynman diagrams entering the perturbative definition of these matrix elements [6, 9].

Concerning their normalization, the recursive structure of the space of FF reduces the problem of finding the normalization of the matrix element of $\Theta(x)$ for the initial conditions of the double chain (2.8), i.e. the two-particle FF $F_2^\Theta(\beta_{12})$ and the one-particle FF $F_1^\Theta(\beta)$.

The normalization of the two-particle FF $F_2^\Theta(\beta_{12})$ can be fixed by making use of the definition of the energy operator [6]

$$E = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx^1 \, T^{00}(x) \ . \tag{2.16}$$

In fact, computing the matrix element of both terms of this equation on the asymptotic states $< \beta'|$ and $|\beta >$, for the left hand side we have

$$< \beta'| \, E \, |\beta > = \, 2\pi \, m \, \cosh \beta \, \delta(\beta' - \beta) \ . \tag{2.17}$$

On the other hand, taking into account that $T^{00} = \partial_1^2 A$ and using the relation

$$< \beta'| \, \mathcal{O}(x) |\beta > = \, e^{i(p^\mu(\beta') - p^\mu(\beta)) x_\mu} \, F_2^{\mathcal{O}}(\beta, \, \beta' \, - \, i\pi) \tag{2.18}$$

valid for any hermitian operator $\mathcal{O}$, we obtain

$$F_2^{\partial_1^2 A}(\beta_1, \beta_2) \, = \, - \, m^2 \, (\sinh \beta_1 \, + \sinh \beta_2)^2 \, F_2^A(\beta_{12}) \ . \tag{2.19}$$

Then, from eqs. (2.13) and (2.16), the normalization of F_2^Θ is given by

$$F_2^\Theta(i\pi) \, = \, 2\, \pi \, m^2 \ . \tag{2.20}$$

However, no special constraint exists for the matrix element of $\Theta(x)$ on the one-particle state

$$F_1^\Theta \, = < 0 \mid \Theta(0) \mid \beta > \ , \tag{2.21}$$

which is then a free parameter of the theory. Notice that from Lorentz invariance, it does not depend on the rapidity variable β.

Since higher FF of Θ are obtained as solutions of the recursive equations (2.6) and (2.7) (with initial condition given by the one-particle and two-particle matrix elements), the arbitrariness of F_1^Θ propagates in the recursive structure (2.8) of the FF and therefore gives rise to a one-parameter family of possible stress-energy tensor $T_{\mu\nu}$ for a given theory.

A simple example of the above discussion is provided by the free massive bosonic theory, with equation of motion

$$(\Box + m^2)\,\varphi \,=\, 0 \;. \tag{2.22}$$

The S-matrix in this case is simply $S = 1$ and therefore the FF have trivial monodromy properties and simple analytic structure. Among the local operators of the theory, the elementary field $\varphi(x)$ is identified by the set of FF

$$F_n^{\varphi}(\beta_1,\beta_2,\ldots,\beta_n) \;=< 0|\varphi(0)|\beta_1,\beta_2,\ldots,\beta_n >= \,\frac{1}{\sqrt{2}}\,\delta_{1,n} \;\;, \tag{2.23}$$

and its two-point euclidean correlator reduces to a Bessel function

$$\begin{aligned} < \varphi(R)\,\varphi(0) >_E \;&=\; \sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{+\infty}\frac{d\beta_1}{2\pi}\cdots\int_{-\infty}^{+\infty}\frac{d\beta_n}{2\pi}\,|F_n(\beta 1,\ldots,\beta_n)|^2\,e^{-mR\sum_i\cosh\beta_i} \\ &=\; \frac{1}{\pi}K_0(mR)\;. \end{aligned} \tag{2.24}$$

The absence of interaction implies that the composite operators $\varphi^k/k!$ are simply defined by the following FF

$$< 0|\frac{\varphi^k(0)}{k!}|\beta_1,\beta_2,\ldots,\beta_n >= \left(\frac{1}{\sqrt{2}}\right)^k\;\delta_{n,k}\;\;. \tag{2.25}$$

The equation of motion is compatible with a class of stress-energy tensor labelled by the free parameter Q appearing in the definition of $\Theta(x)$

$$\Theta(x) \;=\; 2\pi\left(m^2\varphi^2 + \frac{Q}{\sqrt{\pi}}\Box\varphi\right)\;\;. \tag{2.26}$$

In terms of FF we have

$$\begin{aligned} F_0^\Theta &= 0 \;, \\ F_1^\Theta &= -\sqrt{2\pi}m^2\,Q \;\;, \\ F_2^\Theta &= 2\pi m^2 \;\;, \\ F_k^\Theta &= 0 \quad, k > 2 \;. \end{aligned} \tag{2.27}$$

The meaning of Q becomes clear once we analyze the ultraviolet limit of this massive theory. The central charge of the underlying CFT which rules the ultraviolet properties of the model may be computed by using the c-theorem sum rule [18, 19]

$$c \,=\, \frac{3}{2}\int_0^\infty dR\,R^3 < \Theta(R)\Theta(0) >_E \;\;. \tag{2.28}$$

The euclidean correlator is given by

$$
\begin{aligned}
< \Theta(R)\Theta(0) >_E &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi} \cdots \int_{-\infty}^{+\infty} \frac{d\beta_k}{2\pi} \, |F_k(\beta 1, \ldots, \beta_k)|^2 \, e^{-mR \sum_i \cosh \beta_i} \\
&= m^4 \left(2\, (K_0(mR))^2 \, + \, 2Q^2 K_0(mR) \right) \quad , \qquad (2.29)
\end{aligned}
$$

and the result of the integral (2.28) is simply

$$
c \, = \, 1 \, + 12\, Q^2 \quad . \qquad (2.30)
$$

Hence the one-particle FF of Θ is related to the background charge of the CFT reached in the ultraviolet limit. We could have obtained the same result by directly analyzing the ultraviolet limit of the holomorphic component of the stress-energy tensor. Indeed

$$
\begin{aligned}
< T_{zz}(z, \bar{z}) T_{zz}(0,0) >_E &= \left(\frac{\bar{z}}{z} \right)^2 < T_{zz}(R,R) T_{zz}(0,0) >_E \\
&= m^4 \left(\frac{\bar{z}}{z} \right)^2 \left(2\, (K_2(mR))^2 + 2Q^2 K_4(mR) \right) \quad , \qquad (2.31)
\end{aligned}
$$

($R^2 = z\bar{z}$ and $z = x^0 + ix^1$, $\overline{z} = x^0 - ix^1$), and in the limit $(mR) \to 0$ we have

$$
< T_{zz}(z, \overline{z}) T_{zz}(0,0) > = \frac{c}{2z^4} \, = \, \frac{1 \, + \, 12\, Q^2}{2z^4} \quad . \qquad (2.32)
$$

To complete our discussion on the free theory, let us compute the conformal dimensions $\Delta(\alpha)$ characterizing the scaling properties of the exponential operators $V_a = e^{\alpha\varphi}$ in the ultraviolet limit. This will be identified as the coefficient of the most singular term in the ultraviolet limit of the correlator $< T_{zz}(z, \bar{z})\, V_\alpha(0) >$. Using the FF of $V_\alpha(0)$ given by

$$
< 0 | V_\alpha(0) | \beta_1, \ldots, \beta_n > = \left(\frac{\alpha}{\sqrt{2}} \right)^n \quad , \qquad (2.33)
$$

we have

$$
\begin{aligned}
< T_{zz}(z, \bar{z})\, V_\alpha(0) >_E &= \left(\frac{\bar{z}}{z} \right) < T_{zz}(R,R)\, V_\alpha(0) >_E \\
&= m^2 \left(\frac{\bar{z}}{z} \right) \left(-\frac{\alpha^2}{2\pi} \, (K_1(mR))^2 \, + \, \alpha \frac{Q}{\sqrt{\pi}} K_2(mR) \right) \quad ,
\end{aligned}
$$

and for $(mR) \to 0$

$$
< T_{zz}(z, \bar{z})\, V_\alpha(0) >_E \sim \frac{1}{z^2} \left(-\frac{\alpha^2}{8\pi} \, + \, \frac{\alpha\, Q}{\sqrt{4\pi}} \right) \quad , \qquad (2.34)
$$

i.e.

$$
\Delta\,(\alpha) \; = \, -\frac{\alpha^2}{8\pi} \, + \, \frac{\alpha\, Q}{\sqrt{4\pi}} \quad . \qquad (2.35)
$$

Due to the background charge, they differ from the gaussian value $\Delta(\alpha) = -\alpha^2/8\pi$.

3 The Sinh-Gordon Model

The free theory provides an easy example of non-trivial ultraviolet behaviours induced by a non-zero value of F_1^{Θ}. It is interesting to see if similar occurrences are also present for interacting theories. In this section we will analyze the Sinh-Gordon model and in the next one the Bullough-Dodd model.

3.1 Basic Properties

The Sinh-Gordon theory is a classical integrable model defined by the equation of motion

$$\Box\varphi \;=\; -\frac{m_0^2}{2g}\left(e^{g\varphi} - e^{-g\varphi}\right) \;. \tag{3.1}$$

The theory is invariant under a Z_2-symmetry $\varphi \to -\varphi$. The integrability of the model also persists at the quantum level and the exact two-body elastic S-matrix involving the asymptotic particles created by the operator φ is given by [17]

$$S(\beta, B) = \frac{\tanh\frac{1}{2}(\beta - i\frac{\pi B}{2})}{\tanh\frac{1}{2}(\beta + i\frac{\pi B}{2})} \;. \tag{3.2}$$

The coupling constant dependence of the model is encoded into the function

$$B(g) = \frac{g^2}{4\pi}\frac{1}{(1+g^2/8\pi)} \;. \tag{3.3}$$

Since there is no pole on the physical sheet $0 < \mathrm{Im}\,\beta < \pi$ for real values of the coupling constant g, the Sinh-Gordon model presents no bound states. Notice that the S-matrix is invariant under the duality transformation $B \to 2 - B$, which establishes a mapping of the theory between the weak coupling and strong coupling regimes, i.e. $g \to 8\pi/g$. This symmetry will be respected by all FF of manifestly self-dual operators.

3.2 Space of the Form Factors

The form factors of the Sinh-Gordon model have been investigated in [9, 10]. In this subsection we briefly recall the basic results obtained in refs. [9, 10] which are relevant for our subsequent considerations.

In order to compute the FF of this theory, the first step is to take into account their monodromy properties dictated by the S-matrix. To this aim, let $F_{\mathrm{min}}^{\mathrm{SG}}(\beta, B)$ be the solution of the functional equations

$$\begin{aligned} F_{\mathrm{min}}^{\mathrm{SG}}(\beta, B) &= F_{\mathrm{min}}^{\mathrm{SG}}(-\beta, B)\, S(\beta, B) \,, \\ F_{\mathrm{min}}^{\mathrm{SG}}(i\pi - \beta, B) &= F_{\mathrm{min}}^{\mathrm{SG}}(i\pi + \beta, B) \,, \end{aligned} \tag{3.4}$$

which has no poles and zeros in the physical sheet $0 < \text{Im}\,\beta \leq \pi$ and with an asymptotic behaviour given by

$$\lim_{\beta\to\infty} F_{\text{min}}^{\text{SG}}(\beta, B) = 1 \ . \tag{3.5}$$

Its explicit expression reads

$$\begin{aligned} F_{\text{min}}^{\text{SG}}(\beta, B) &= \mathcal{N}(B) \exp\left[8 \int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right) \sinh\left(\frac{x}{2}(1-\frac{B}{2})\right) \sinh\frac{x}{2}}{\sinh^2 x} \sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)\right] \ , \\ \mathcal{N}(B) &= \exp\left[-4 \int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right) \sinh\left(\frac{x}{2}(1-\frac{B}{2})\right) \sinh\frac{x}{2}}{\sinh^2 x}\right] \ , \end{aligned} \tag{3.6}$$

where $\hat{\beta} = i\pi - \beta$. Equivalently

$$F_{\text{min}}^{\text{SG}}(\beta, B) = \prod_{k=0}^{\infty} \left| \frac{\Gamma\left(k+\frac{3}{2}+\frac{i\hat{\beta}}{2\pi}\right) \Gamma\left(k+\frac{1}{2}+\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right) \Gamma\left(k+1-\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)}{\Gamma\left(k+\frac{1}{2}+\frac{i\hat{\beta}}{2\pi}\right) \Gamma\left(k+\frac{3}{2}-\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right) \Gamma\left(k+1+\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)} \right|^2 \ . \tag{3.7}$$

Since $S(0, B) = -1$ (for $B \neq 0$ and 2), $F_{\text{min}}^{\text{SG}}(\beta, B)$ vanishes at the two-particle threshold value $\beta = 0$. As discussed in [11], this generally induces a suppression of all higher thresholds in the spectral representation of the correlation functions and gives rise to very fast convergent series.

In terms of $F_{\text{min}}^{\text{SG}}(\beta, B)$, a convenient parameterization of the n-particle FF of the Sinh-Gordon model is given by

$$F_n(\beta_1, \ldots, \beta_n) = H_n\, Q_n(x_1, \ldots, x_n) \prod_{i<j} \frac{F_{\text{min}}^{\text{SG}}(\beta_{ij}, B)}{(x_i + x_j)} \ , \tag{3.8}$$

Here $x_i \equiv e^{\beta_i}$ and H_n are normalization constants, which can be conveniently chosen as

$$H_{2n+1} = H_1 \mu^{2n}(B) \ , \quad H_{2n} = H_2 \mu^{2n-2}(B) \ , \tag{3.9}$$

with

$$\mu(B) \equiv \left(\frac{4\sin(\pi B/2)}{\mathcal{N}(B)}\right)^{\frac{1}{2}} \ . \tag{3.10}$$

The functions $Q_n(x_1, \ldots, x_n)$ are symmetric polynomials in the variables $x_1, \ldots, x_n$, solutions of the recursion equations

$$(-)^n\, Q_{n+2}(-x, x, x_1, \ldots, x_n) = x D_n(x, x_1, x_2, \ldots, x_n)\, Q_n(x_1, x_2, \ldots, x_n) \ , \tag{3.11}$$

with

$$D_n(x, x_1, \ldots, x_n) = \sum_{k=1}^{n} \sum_{m=1,odd}^{k} [m]\, x^{2(n-k)+m} \sigma_k^{(n)} \sigma_{k-m}^{(n)} (-1)^{k+1} \ , \tag{3.12}$$

and

$$[n] \equiv \frac{\sin(n\pi\frac{B}{2})}{\sin(\pi\frac{B}{2})} \quad . \tag{3.13}$$

For FF of spinless operators, their total degree is equal to $n(n-1)/2$ whereas their partial degree in each variable x_i is fixed by the asymptotic behaviour of the operator $\mathcal{O}_k$ which is under investigation.

As shown in [10], the problem to classify all possible scalar operators of the Sinh-Gordon model reduces to find the most general class of solutions of eq. (3.11). Since this is a homogeneous equation, its solutions span a linear space whose basis may be written in terms of the so-called *elementary solutions* given by

$$\mathcal{Q}_n(k) \,=\, \det M_{ij}(k) \;\; , \tag{3.14}$$

where $M_{ij}(k)$ is an $(n-1) \times (n-1)$ matrix with entries

$$M_{ij}(k) = \sigma_{2i-j}\,[i-j+k] \;\; . \tag{3.15}$$

These polynomials depend on an arbitrary integer k and satisfy

$$\mathcal{Q}_n(k) \,=\, (-1)^{n+1}\mathcal{Q}_n(-k) \;\; . \tag{3.16}$$

Therefore the structure of the FF of the SGM consists in a sequence of finite linear spaces whose dimensions grow linearly as n increasing the number $2n-1$ or $2n$ of external particles. The reason is that, at each level of the recursive process, the space of the FF is enlarged by including the kernel solutions of the recursive equation (3.11), i.e. $Q_n(-x,x,x_1,\ldots,x_{n-2}) \,=\, 0$. With the constraint that the total order of the polynomials is $\frac{n(n-1)}{2}$, the kernel is unique and given by $\Sigma_n(x_1,\ldots,x_n) \,=\, \det\sigma_{2i-j}$. These solutions then gives rise to the half-infinite chains under the recursive pinching $x_1 = -x_2 = x$

$$\begin{array}{ccccccccccccc}
\ldots & \rightarrow & Q^{(1)}_{n+4} & \rightarrow & Q^{(1)}_{n+2} & \rightarrow & Q^{(1)}_{n} & \rightarrow & Q^{(1)}_{n-2} & \rightarrow & \ldots & \rightarrow & Q^{(1)}_{3} \;\rightarrow\; 1 \\
\ldots & \rightarrow & Q^{(2)}_{n+4} & \rightarrow & Q^{(2)}_{n+2} & \rightarrow & Q^{(2)}_{n} & \rightarrow & Q^{(2)}_{n-2} & \rightarrow & \ldots & \rightarrow & \Sigma_2 \\
 & & & & . & & . & & . & & . & & . \\
 & & & & . & & . & & . & & . & & . \\
\ldots & \rightarrow & Q^{(n-2)}_{n+4} & \rightarrow & Q^{(n-2)}_{n+2} & \rightarrow & Q^{(n-2)}_{n} & \rightarrow & \Sigma_{n-2} & & & & \\
\ldots & \rightarrow & Q^{(n)}_{n+4} & \rightarrow & Q^{(n)}_{n+2} & \rightarrow & \Sigma_{n} & & & & & & \\
\ldots & \rightarrow & Q^{(n+2)}_{n+4} & \rightarrow & \Sigma_{n+2} & & & & & & & &
\end{array} \tag{3.17}$$

The explicit expressions of such solutions can be found by determining the linear combination of $\mathcal{Q}_n(k)$ which reduces to Σ_n at the level n.

3.3 Cluster Operators and Fundamental Exponentials

The *fundamental exponential operators* $\Phi_{\pm}(x) = e^{\pm g\varphi(x)}$ define the Sinh-Gordon model and in general appear in the expression of the stress-energy tensor. In order to calculate their matrix elements, let us consider initially those FF which satisfy the requirements

- To be asymptotically constant for $\beta_i \rightarrow \infty$, i.e.

$$F_n(\beta_1 + \Delta, \beta_2, \ldots, \beta_n) \stackrel{\Delta\to\infty}{\longrightarrow} o(1) \ .$$

- To be proportional to the combination[4] $\sigma_1 \sigma_{n-1}$ (for $n > 2$).

- To be the solution of the *cluster equations*

$$\lim_{\Delta\to+\infty} F_{k+l}\left(\beta_1 + \Delta, \ldots, \beta_k + \Delta, \beta_{k+1}, \ldots, \beta_{k+l}\right) = F_k\left(\beta_1, \ldots, \beta_k\right) F_l\left(\beta_{k+1}, \ldots, \beta_{k+l}\right)$$

  with initial condition $F_0 = 1$.

There are two classes of FF which fulfill the three above conditions. Their expressions are given by

$$F_n^{(\pm)}(\beta_1, \ldots, \beta_n) = H_n^{(\pm)}(B)\, Q_n(1) \prod_{i<j}^{n} \frac{F_{\rm min}^{\rm SG}(\beta_{ij}, B)}{(x_i + x_j)} \ , \tag{3.18}$$

where

$$H_n^{(+)}(B) = (\mu(B))^n \quad , \quad H_n^{(-)} = (-1)^n\,(\mu(B))^n \ . \tag{3.19}$$

The corresponding operators, which are self-dual by construction, will be called *cluster operators* and denoted as $V_{\pm}(x, B)$. We conjecture that the fundamental exponentials of the Sinh-Gordon model may be written as[5]

$$\begin{aligned} \Phi_+(x, B) &\equiv \theta(1 - B)\, V_+(x, B) + \theta(B - 1)\, V_-(x, B) \ , \\ \Phi_-(x, B) &\equiv \theta(1 - B)\, V_-(x, B) + \theta(B - 1)\, V_+(x, B) \ . \end{aligned} \tag{3.20}$$

Postponing a non-trivial check of this position until when we will study the UV-behaviour of the model, let us in the meantime discuss the properties of the operators $\Phi_{\pm}(x, B)$ so defined.

First of all, they satisfy the cluster property by construction, in agreement with the perturbative analysis for the matrix elements of the operators $e^{\pm g\varphi(x)}$. Secondly, the FF of $\Phi_{\pm}(x, B)$ are not individually invariant under the duality transformation but each operator is mapped onto the other under the mapping $B \rightarrow 2 - B$, i.e.

$$\Phi_{\pm}(x, B) = \Phi_{\mp}(x, 2 - B) \ . \tag{3.21}$$

[4] As discussed in sect. 2, this factorization property is shared by the general FF of Θ.

[5] For the value of the step function at the origin we use $\theta(0) = 1/2$.

Therefore they form a bidimensional representation of the duality symmetry. However, this mapping becomes degenerate at the self-dual point $B = 1$ where an identification occurs between the two exponential operators $\Phi_\pm(x, B)$. Namely, at $B = 1$ the matrix elements of the two fundamental exponentials become indistinct and denoting by $\Phi(x)$ the resulting operator, its FF are given by

$$F_n^{\Phi}(\beta_1, \ldots, \beta_2) \;=\; \begin{cases} (\mu(1))^n \; Q_n(1) \prod_{i<j}^{n} F_{\rm min}^{\rm SG}(\beta_{ij})/(x_i + x_j) & n \text{ even } , \\ 0 & n \text{ odd } . \end{cases} \tag{3.22}$$

The identification of $\Phi_\pm(x)$ at the self-dual point has the additional consequence that the resulting field $\Phi(x)$ is an even operator under the Z_2 parity of the Sinh-Gordon model, as it is evident from the vanishing of its matrix elements on all $2n + 1$ particle states.

Using the FF of the fundamental exponentials and those of the elementary field given by

$$F_n^{\varphi}(\beta_1, \ldots, \beta_n) \;=\; H_n^{\varphi}\, Q_n(0) \prod_{i<j}^{n} \frac{F_{\rm min}^{\rm SG}(\beta_{ij})}{x_i + x_j} \tag{3.23}$$

$$H_{2n+1}^{\varphi} \;=\; \frac{1}{\sqrt{2}}\,(\mu(B))^n \quad , \quad H_{2n}^{\varphi} = 0 \;\; ,$$

the quantum version of the equation of motion may be written as

$$\begin{aligned} \Box\varphi(x) \;&=\; \frac{m^2}{2\sqrt{2}\mu(B)}\,\left(\theta(1-B) \;-\; \theta(B-1)\right)\left(e^{-g\varphi(x)} - e^{g\varphi(x)}\right) \\ &=\; \frac{m^2}{2\sqrt{2}\mu(B)}\,\left(V_-(x,B) - V_+(x,B)\right) \;\; . \end{aligned} \tag{3.24}$$

This equation has to be understood as an identity satisfied by the FF of the operators appearing on the left and right sides of this relation.

3.4 Class of Stress-Energy Tensors

The quantum equation of motion (3.24) is compatible with a one-dimensional space of stress-energy tensors given by

$$\Theta(x) \;=\; F_0^{\Theta}(B)\,\left(a\,\Phi_+(x,B) + (1 \;-\; a)\,\Phi_-(x,B)\right) \;\; . \tag{3.25}$$

The normalization constant $F_0^{\Theta}(B)$ may be fixed by means of the Thermodynamical Bethe Ansatz [20]

$$F_0^{\Theta}(B) \;=\; \frac{\pi m^2}{2\sin(\pi B/2)} \;\; . \tag{3.26}$$

The variable a, on the contrary, is a free parameter. Varying its value, we can weight differently the two fundamental exponentials in the trace and, consequently, we can interpolate between different scaling regimes of the Sinh-Gordon model in its ultraviolet limit.

3.4.1 The case $a = 1$

The trace of the stress-energy tensor is given in this case by

$$\Theta(x) \,=\, F_0^{\Theta}(B)\,\Phi_+(x,B) \ . \tag{3.27}$$

With such definition of Θ, we expect that the massive theory will flow in the ultraviolet regime to a CFT defined by the bare action

$$\mathcal{S}_- \,=\, \int d^2x \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m_0^2}{2g^2}\,e^{-g\varphi}\right] \ . \tag{3.28}$$

In order to support this conclusion, let us compare the central charge associated to the CFT (3.28) with the central charge obtained, on the contrary, in terms of the FF by using the c-theorem sum rule (2.28).

Assuming eq. (3.28) as definition of the ultraviolet theory, the corresponding central charge is given by (see, for instance [21])

$$c(g) \,=\, 1 \,+\, 12\,Q_-^2(g) \tag{3.29}$$

where

$$Q_-(g) \,=\, -\left(\frac{\sqrt{4\pi}}{g} + \frac{g}{2\sqrt{4\pi}}\right) \ . \tag{3.30}$$

Using eq. (3.3), it may be written as

$$c(B) \,=\, 1 \,+\, 6\left(\frac{2-B}{B} + \frac{B}{2-B} + 2\right) \ . \tag{3.31}$$

Notice that this is a self-dual function of the coupling constant, i.e. invariant under $B \to 2 - B$.

On the other hand, we may compute the central charge associated to the ultraviolet limit of the massive theory in terms of the second moment of the two-point function of the trace $\Theta(x)$ [18, 19]

$$c(B) \,=\, \frac{3}{2}\int_0^\infty dR\,R^3 < \Theta(R)\Theta(0) >_E \ . \tag{3.32}$$

According to eq. (3.27), the two-point function of the trace $\Theta(x)$ has to be computed in terms of the FF of the operator $\Phi_+(x,B)$ defined in eq. (3.20). The data reported in Table 1 and plotted in Fig. 1 show that the first two FF of $\Phi_+(x)$ are sufficient to saturate the sum-rule (3.32) and to reproduce with high percentage of precision the expression (3.31).

An additional check that the ultraviolet limit induced by this choice of Θ is ruled by CFT (3.28), is given by the computation of the conformal dimensions of the fundamental exponentials. This can be done in two different ways, using directly CFT method or analyzing the ultraviolet behaviour of massive correlators.

For the CFT defined by eq. (3.28), the conformal dimensions of the primary fields corresponding to the operators $e^{\alpha\varphi}$ are given by

$$\Delta_-(\alpha) \;=\; -\frac{\alpha^2}{8\pi} + \frac{\alpha Q_-(g)}{\sqrt{4\pi}} \quad , \tag{3.33}$$

and therefore, for the fundamental exponentials we have

$$\begin{aligned} \Delta_-(\Phi_-) \;&=\; 1 \;\; , \\ \Delta_-(\Phi_+) \;&=\; -1 - \tfrac{g^2}{4\pi} \;\; . \end{aligned} \tag{3.34}$$

On the other hand, we may compute the conformal dimensions of the fundamental exponentials by investigating the UV-limit of the correlators

$$\begin{aligned} < T_{zz}(z,\bar{z})\, \Phi_\pm(0) >_E \;\;=\;\; & \frac{\bar{z}}{z} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi} \ldots \int_{-\infty}^{+\infty} \frac{d\beta_n}{2\pi} \left(F_n^{T++}(\beta 1, \ldots, \beta_n) \right)^* \\ & \times F_n^{\Phi\pm}(\beta 1, \ldots, \beta_n) e^{-mR\sum_i \cosh\beta_i} \quad . \end{aligned} \tag{3.35}$$

At order $O(g^4)$, it is sufficient to truncate the series to the first two terms and also use the perturbative expansion

$$\mathcal{N}(B) = 1 - \frac{g^2}{8\pi^2} + O(g^4) \;\; . \tag{3.36}$$

Since for small values of g

$$F_n^{T++}(x_1, \ldots, x_n) \;=\; -F_0^{\Theta}(B) \, \frac{\sigma_{n-1}^{(n)}}{4\sigma_1^{(n)}\sigma_n^{(n)}} \, F_n^{V+}(x_1, \ldots, x_n) \;\; , \tag{3.37}$$

and

$$\begin{aligned} < T_{zz}(z,\bar{z})\, \Phi_\pm(0) >_E \;\;\simeq\;\; & \frac{\bar{z}}{z} \Big\{ - F_0^{\Theta}(B)\, F_1^{V+} F_1^{V\pm} \frac{1}{4\pi} K_2(mR) \\ & - \frac{F_0^{\Theta}(B)}{2\pi^2} \int_{-\infty}^{+\infty} d\beta \; K_2(2mR\cosh\beta)\, F_2^{V+}(2\beta) F_2^{V\pm}(2\beta) \Big\} \;\; , \end{aligned} \tag{3.38}$$

in the limit $mR \to 0$, we have

$$< T_{zz}(z,\bar{z})\, \Phi_\pm(0) >_E \;\simeq\; \frac{\Delta_-(\Phi_\pm)}{z^2} \;\; , \tag{3.39}$$

with

$$\begin{aligned} \Delta_-(\Phi_-) \;&=\; 1 + O(g^4) \;\; , \\ \Delta_-(\Phi_+) \;&=\; -1 - \tfrac{g^2}{4\pi} + O(g^4) \;\; , \end{aligned} \tag{3.40}$$

in agreement with eq. (3.34).

3.4.2 The case $a = 0$

The trace of stress-energy tensor is given in this case by

$$\Theta(x) \,=\, F_0^{\Theta}(B)\,\Phi_-(x,B) \;. \tag{3.41}$$

and we expect that the massive model will flow in the ultraviolet limit to a CFT defined by the bare action

$$\mathcal{S}_+ \,=\, \int d^2x \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2g^2}\, e^{g\varphi}\right] \;. \tag{3.42}$$

The corresponding background charge is given by [21]

$$Q_+(g) \,=\, \left(\frac{\sqrt{4\pi}}{g} + \frac{g}{2\sqrt{4\pi}}\right) \;. \tag{3.43}$$

This CFT differs from that analyzed in the previous subsection for the exchange of the role of the two fundamental exponentials.

According to the CFT defined by the bare lagrangian (3.42), the anomalous dimensions of the primary fields corresponding to the operators $e^{\alpha\phi}$ are given by

$$\Delta_+(\alpha) \,=\, -\frac{\alpha^2}{8\pi} + \frac{\alpha Q_+(g)}{\sqrt{4\pi}} \;, \tag{3.44}$$

and for the fundamental exponentials we have in this case

$$\begin{aligned} \Delta_+(\Phi_+) \;&= 1 \;, \\ \Delta_+(\Phi_-) \;&= -1 - \tfrac{g^2}{4\pi} \;. \end{aligned} \tag{3.45}$$

Repeating the same kind of computations of the previous subsection, it is easy to check that these expressions coincide, at order $O(g^4)$, with the conformal dimensions extracted from the ultraviolet behaviour of the correlators $< T_{zz}(z,\overline{z})\,\Phi_\pm(0) >$.

For what concerns the central charge, since it depends quadratically on $Q_\pm$, its value is given as before by the self-dual function (3.31). Analogous computations for the second moment of the Θ computed in terms of the FF of $\Phi_-(x,B)$ (which is the dual operator of $\Phi_+(x,B)$), lead therefore to the same results of Table 1 (see also Fig. 1).

3.4.3 General case

We are now able to give the conformal dimension $\Delta(\alpha)$ of the exponential operator $e^{\alpha\varphi}$ and the central charge of the CFT reached in the ultraviolet regime for generic value of the parameter a.

Since $\Delta(\alpha)$ is the coefficient of the most singular term obtained in the UV-limit of the correlation function $< T_{zz}(z,\overline{z})\, e^{\alpha\varphi(0)} >$, and the FF of $T_{zz}(z,\overline{z})$ depends linearly on the parameter a, the conformal dimension is given by

$$\begin{aligned}\Delta(\alpha) &= a\,\Delta_-(\alpha) + (1-a)\,\Delta_+(\alpha) = \\ &= -\frac{\alpha^2}{8\pi} + \frac{\alpha}{\sqrt{4\pi}}(1-2a)\left(\frac{\sqrt{4\pi}}{g} + \frac{g}{2\sqrt{4\pi}}\right) \quad ,\end{aligned} \tag{3.46}$$

with $\Delta_\pm(\alpha)$ defined in eqs. (3.33) and (3.44). The coefficient in front of the linear term in α in eq. (3.46) identifies the background charge and therefore the central charge of the CFT reached in the ultraviolet limit is given by

$$c = 1 + 24\,\frac{(1-2a)^2}{B(2-B)} \quad . \tag{3.47}$$

We have checked the validity of this result with the computation of the central charge in terms of the first FF of the operator (3.25). The comparison between them is shown in Fig. 3, varying a at fixed B.

As last example of possible choices of the stress-energy tensor, observe that for $a = 1/2$ we have $c = 1$, independent of the coupling constant. The corresponding expression of Θ is given by [9]

$$\Theta(x) = \frac{F_0^\Theta(B)}{2}\left(e^{g\varphi} + e^{-g\varphi}\right) \quad . \tag{3.48}$$

This operator is manifestly self-dual and Z_2-even. With this choice of a, the anomalous dimensions of the fundamental exponentials coincide, at lowest order, with their gaussian values

$$\Delta(\pm g) = -\frac{g^2}{8\pi} + o(g^4) \quad . \tag{3.49}$$

The check of the central charge obtained in this case has been done in the two-particle approximation in ref. [9] and is reported here in table 2.

4 The Bullough-Dodd massive boson

Different ultraviolet scaling regimes induced by different choice of the stress-energy tensor can be easily discussed for another integrable theory involving an interacting bosonic field, the so-called Bullough-Dodd (BD) model.

4.1 Basic Properties

The Bullough-Dodd (BD) model is defined by the equation of motion [22, 23, 24]

$$\Box\varphi = \frac{m_0^2}{3\lambda}\left(e^{-2\lambda\varphi} - e^{\lambda\varphi}\right) \quad . \tag{4.1}$$

At the quantum level, the integrability of the model leads to the elasticity and factorization of the scattering processes. The spectrum of the model consists of a massive particle state A created by the elementary field φ. This particle appears as bound state of itself in the scattering process

$$A \times A \to A \to A \times A \ . \tag{4.2}$$

The corresponding S-matrix is given by [17]

$$S(\beta,\mathcal{B}) \,=\, f_{\frac{2}{3}}(\beta)\, f_{\frac{\mathcal{B}}{3}-\frac{2}{3}}(\beta)\, f_{-\frac{\mathcal{B}}{3}}(\beta) \ . \tag{4.3}$$

Here

$$f_x(\beta) \,=\, \frac{\tanh\frac{1}{2}(\beta+i\pi x)}{\tanh\frac{1}{2}(\beta-i\pi x)} \ , \tag{4.4}$$

and the coupling constant dependence of the model is encoded into the function

$$\mathcal{B}(\lambda) \,=\, \frac{\lambda^2}{2\pi}\frac{1}{1+\frac{\lambda^2}{4\pi}} \ . \tag{4.5}$$

Like the Sinh-Gordon model, the S-matrix of the BD model is invariant under the mapping

$$\mathcal{B}(\lambda) \,\to\, 2-\mathcal{B}(\lambda) \ , \tag{4.6}$$

i.e. under the weak-strong coupling constant duality $\lambda \to 4\pi/\lambda$.

The minimal part of the S-matrix, i.e. the term $f_{\frac{2}{3}}(\beta)$, contains the physical pole $\beta = 2\pi i/3$ of the bound state and, as matter of fact, it coincides with the S-matrix of the Yang-Lee model [25]. Taking into account the coupling constant dependence of the S-matrix, the residue at the pole is given by

$$\Gamma^2(\mathcal{B}) \,=\, 2\sqrt{3}\frac{\tan\left(\frac{\pi\mathcal{B}}{6}\right)}{\tan\left(\frac{\pi\mathcal{B}}{6}-\frac{2\pi}{3}\right)}\frac{\tan\left(\frac{\pi}{3}-\frac{\pi\mathcal{B}}{6}\right)}{\tan\left(\frac{\pi\mathcal{B}}{6}+\frac{\pi}{3}\right)} \ . \tag{4.7}$$

This function, that corresponds to the three-particle vertex on mass-shell, vanishes for $\mathcal{B} = 0$ and $\mathcal{B} = 2$ (which are the free theory limits) with the corresponding scattering amplitude $S = 1$. However, it also vanishes at the self-dual point $\mathcal{B} = 1$, with the corresponding scattering amplitude $S(\beta) = f_{-2/3}$. This coincides with the S-matrix of the Sinh-Gordon model computed at $B = 2/3$. As analyzed in [26], this equality between the S-matrices of the two models implies that at the self-dual point the BD model dynamically develops a Z_2-symmetry which is a non-perturbative property of the model.

4.2 Form Factors

Taking into account the bound state pole in the two-particle channel at $\beta_{ij} = 2\pi i/3$ and the one-particle pole in the three-particle channel at $\beta_{ij} = i\pi$, the general form factors of the BD model can be parameterized as

$$F_n^k(\beta_1,\ldots,\beta_n) \,=\, Q_n(x_1,\ldots,x_n)\prod_{i<j}\frac{F_{\rm min}^{\rm BD}(\beta_{ij})}{(x_i+x_j)(\omega x_i+x_j)(\omega^{-1}x_i+x_j)}\;, \tag{4.8}$$

where we have introduced the variables

$$x_i \,=\, e^{\beta_i}\;,\;\omega \,=\, e^{i\pi/3}\;. \tag{4.9}$$

$F_{\rm min}^{\rm BD}(\beta)$ is an analytic function without zeros and poles in the physical sheet, whose explicit expression is given by

$$F_{\rm min}^{\rm BD}(\beta,\mathcal{B}) \,=\, \prod_{k=0}^{\infty}\left|\frac{\Gamma\left(k+\frac{3}{2}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{7}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{4}{3}+\frac{i\hat\beta}{2\pi}\right)}{\Gamma\left(k+\frac{1}{2}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{5}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{2}{3}+\frac{i\hat\beta}{2\pi}\right)}\right. \tag{4.10}$$

$$\times\left.\frac{\Gamma\left(k+\frac{5}{6}-\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{1}{2}+\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+1-\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{2}{3}+\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)}{\Gamma\left(k+\frac{7}{6}+\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{3}{2}-\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+1+\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{4}{3}-\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)}\right|^2$$

($\hat\beta = i\pi-\beta$). Its normalization is fixed by requiring the asymptotic behaviour

$$\lim_{\beta\to\infty} F_{\rm min}^{\rm BD}(\beta,\mathcal{B}) = 1\quad. \tag{4.11}$$

Notice that at the self-dual point $\mathcal{B}=1$, the above function coincides with the $F_{\rm min}^{\rm SG}\left(\beta,\frac{2}{3}\right)$ of the Sinh-Gordon model, i.e.

$$F_{\rm min}^{\rm BD}(\beta,1) \,=\, F_{\rm min}^{\rm SG}\left(\beta,\frac{2}{3}\right)\quad. \tag{4.12}$$

The functions $Q_n(x_1,\ldots,x_n)$ are symmetric polynomials in the variables $x_1,\ldots,x_n$. Using the functional relations satisfied by $F_{\rm min}(\beta,\mathcal{B})$

$$\begin{aligned} F_{\rm min}^{\rm BD}(i\pi+\beta,\mathcal{B})\,F_{\rm min}^{\rm BD}(\beta,\mathcal{B}) \;&=\; \frac{\sinh\beta\left(\sinh\beta+\sinh\frac{i\pi}{3}\right)}{\left(\sinh\beta+\sinh\frac{i\pi\mathcal{B}}{3}\right)\left(\sinh\beta+\sinh\frac{i\pi(1+\mathcal{B})}{3}\right)}\;,\\ F_{\rm min}^{\rm BD}(\beta+\frac{i\pi}{3},\mathcal{B})\,F_{\rm min}^{\rm BD}(\beta-\frac{i\pi}{3},\mathcal{B}) \;&=\; \frac{\cosh\beta+\cosh\frac{2i\pi}{3}}{\cosh\beta+\cosh\frac{i\pi(2+\mathcal{B})}{3}}\,F_{\rm min}^{\rm BD}(\beta,\mathcal{B})\;, \end{aligned} \tag{4.13}$$

the kinematical and bound state residue conditions give rise to the following recursive equations satisfied by the functions $Q_n(x_1,\ldots,x_n)$ [26]

$$(-1)^n Q_{n+2}(-x,x,x_1,x_2,\ldots,x_n) \,=\, \frac{1}{F_{\rm min}^{\rm BD}(i\pi,\mathcal{B})}\,x^3\,U(x,x_1,x_2,\ldots,x_n)Q_n(x_1,x_2,\ldots,x_n)\;, \tag{4.14}$$

where

$$U(x,x_1,\ldots,x_n) = 2\sum_{k_1,\ldots,k_6=0}^{n}(-1)^{k_2+k_3+k_5}\,x^{6n-(k_1+\cdots+k_6)}\,\sigma_{k_1}^{(n)}\sigma_{k_2}^{(n)}\ldots\sigma_{k_6}^{(n)} \times \sin\left[\frac{\pi}{3}\left[2(k_2+k_4-k_1-k_3)+\mathcal{B}(k_3+k_6-k_4-k_5)\right]\right] , \tag{4.15}$$

and

$$Q_{n+2}(\omega x,\omega^{-1}x,x_1,\ldots,x_n) = -\frac{\sqrt{3}}{F_{\rm min}^{\rm BD}\left(\frac{2\pi i}{3},\mathcal{B}\right)}\,\Gamma(\mathcal{B})\,x^3 D(x,x_1,\ldots,x_n)\,Q_{n+1}(x,x_1,\ldots,x_n) , \tag{4.16}$$

where

$$D(x,x_1,\ldots,x_n) = \prod_{i=1}^{n}(x+x_i)(x\omega^{2+\mathcal{B}}+x_i)(x\omega^{-\mathcal{B}-2}+x_i) = \sum_{k_1,k_2,k_6=0}^{n} x^{3n-(k_1+k_2+k_6)}\,\omega^{(2+\mathcal{B})(k_2-k_3)}\,\sigma_{k_1}^{(n)}\sigma_{k_2}^{(n)}\sigma_{k_3}^{(n)} . \tag{4.17}$$

4.3 Cluster Operators and Fundamental Exponentials

Unlike the Sinh-Gordon model, we do not know presently a close solution for the recursive equations satisfied by the FF of the BD model at generic value of the coupling constant. However, as it will become clear in the following, all we need for our consideration is the explicit computation of the first representative FF of the elementary field φ and of the so-called cluster operators $\mathcal{V}_{\pm}(x)$.

The elementary field φ is identified as that operator that creates one-particle state. Therefore its lowest matrix element is given by

$$F_1^{\varphi} = <0|\varphi(0)|\beta> = \frac{1}{\sqrt{2}} . \tag{4.18}$$

With such normalization, the next FF are given by

$$F_2^{\varphi}(\beta) = -\frac{\Gamma(\mathcal{B})}{\sqrt{2}}\,\frac{\sin\frac{\pi}{6}(2+\mathcal{B})}{\sqrt{F_{\rm min}^{\rm BD}(i\pi,\mathcal{B})}}\,\frac{F_{\rm min}^{\rm BD}(\beta,\mathcal{B})}{\cosh\beta\,+\frac{1}{2}} , \tag{4.19}$$

and

$$F_3^{\varphi}(\beta_1,\beta_2,\beta_3) = \left(\prod_{i<j}^{3}\frac{F_{\rm min}^{\rm BD}(\beta_i-\beta_j,\mathcal{B})}{(x_i+x_j)(e^{i\pi/3}x_i+x_j)(e^{-i\pi/3}x_i+x_j)}\right)\frac{2\sqrt{2}}{F_{\rm min}^{\rm BD}(i\pi,\mathcal{B})}\sigma_3 \times \Big\{2\sin^2\frac{\pi}{6}(2+\mathcal{B})\,\Gamma^2(\mathcal{B})\,\left(\cos\frac{\pi}{3}(2+\mathcal{B})\,-1\right)\,\sigma_3\sigma_2\sigma_1 - W(\mathcal{B})\,\left(\sigma_3\sigma_1^3+\sigma_2^3\right)\,+\,\left(\sin^2\frac{\pi}{6}(2+\mathcal{B})\,\Gamma^2(\mathcal{B})\,+W(\mathcal{B})\right)\sigma_2^2\,\sigma_1^2\Big\} , \tag{4.20}$$

where

$$W(\mathcal{B}) \;=\; 2\sqrt{3}\,\sin\left(\frac{\pi\mathcal{B}}{6}\right)\,\sin\left(\frac{\pi}{6}(2-\mathcal{B})\right)\;. \tag{4.21}$$

In terms of them, we can easily obtain the first FF of the operator $\Box\varphi$

$$F_n^{\Box\varphi} \;=\; -\,m^2\,\frac{\sigma_1\sigma_{n-1}}{\sigma_n}\,F_n^{\varphi}\;. \tag{4.22}$$

In order to define the FF of the two fundamental exponential operators $\Phi_1(x,\mathcal{B}) \equiv e^{\lambda\varphi(x)}$ and $\Phi_2(x,\mathcal{B}) \equiv e^{-2\lambda\varphi(x)}$ of the Bullough-Dodd model, let us consider initially the definition of the cluster operators $\mathcal{V}_{\pm}(x,\mathcal{B})$. As for the Sinh-Gordon model, we are looking for these operators in the class of FF which are asymptotically constant for $x_i \to \infty$ and proportional to the invariant combination of symmetric polynomials $\sigma_1\sigma_{n-1}$ for $n > 2$. The first representative of such FF are given in Appendix A. The important point is that all the higher FF obtained by solving the recursive equations will depend on the constants H_1 and H_2 appearing in the equations (A.1) and (A.2), which play the role of arbitrary initial conditions of the recursive structure. Their relative value can be fixed though, if we require that the above FF satisfy an additional condition, i.e. the cluster property

$$\lim_{\Delta\to+\infty} F_{k+l}\left(\beta_1+\Delta,\ldots,\beta_k+\Delta,\beta_{k+1},\ldots,\beta_{k+l}\right) \;=\; F_k\left(\beta_1,\ldots,\beta_k\right)F_l\left(\beta_{k+1},\ldots,\beta_{k+l}\right) \tag{4.23}$$

with $F_0 \,=\, 1$. In this case we have

$$\begin{aligned} H_1^{\pm}(\mathcal{B}) &= \frac{1}{\sqrt{F_{\rm min}^{\rm BD}(i\pi,\mathcal{B})}}\left\{-\sin\left(\frac{\pi}{6}(\mathcal{B}+2)\right)\Gamma(\mathcal{B}) \pm \sqrt{\sin^2\left(\frac{\pi}{6}(\mathcal{B}+2)\right)\Gamma^2(\mathcal{B})+4W(\mathcal{B})}\right\}\;, \\ H_2^{\pm}(\mathcal{B}) &= (H_1^{\pm})^2(\mathcal{B})\;. \end{aligned} \tag{4.24}$$

With this choice of $H_1^{\pm}(\mathcal{B})$ and $H_2^{\pm}(\mathcal{B})$, the infinite tower of FF with (A.1) and (A.2) as first representatives, define two cluster operators $\mathcal{V}_{\pm}(x,\mathcal{B})$. By construction, the matrix elements of such operators are invariant under the duality transformation $\mathcal{B} \to 2-\mathcal{B}$. Also in this case, we conjecture that the fundamental exponential operators are given by

$$\begin{aligned} \Phi_1(x,\mathcal{B}) &=\equiv\; \theta(1-B)\,\mathcal{V}_+(x,\mathcal{B}) \,+\, \theta(B-1)\,\mathcal{V}_-(x,\mathcal{B})\;, \\ \Phi_2(x,\mathcal{B}) &=\equiv\; \theta(1-B)\,\mathcal{V}_-(x,\mathcal{B}) \,+\, \theta(B-1)\,\mathcal{V}_+(x,\mathcal{B})\;. \end{aligned} \tag{4.25}$$

This definition is in agreement with the perturbative analysis of the matrix elements of the two exponential operators $e^{\lambda\varphi(x)}$ and $e^{-2\lambda\varphi(x)}$. Concerning their properties under the duality mapping, as far as $\mathcal{B} \neq 1$, the fundamental exponentials are mapped each into the other under the mapping $\mathcal{B} \to 2-\mathcal{B}$, i.e.

$$\Phi_{1,2}(x,\mathcal{B}) \;=\; \Phi_{2,1}(x,2-\mathcal{B})\;. \tag{4.26}$$

However, this mapping becomes degenerate at the self-dual point $\mathcal{B}=1$ where, similarly to the Sinh-Gordon model, the two operators $\Phi_1(x,\mathcal{B})$ and $\Phi_2(x,\mathcal{B})$ collapse into a single operator $\Phi(x)$. Moreover, as already noticed in [26], at the self-dual point a Z_2-symmetry is dynamically implemented in the BD model. Due to the fact that at $\mathcal{B}=1$ the three-particle vertex $\Gamma(\mathcal{B})$ vanishes and $H_1^+ = -H_1^-$, the resulting operator $\Phi(x)$ will have non-zero matrix elements only on the $2n$ particle states. Its FF are entirely expressed in terms of the FF of the Sinh-Gordon model at $B=2/3$

$$F_{2n}^{\Phi} = \left(\mu\left(\frac{2}{3}\right)\right)^{2n} Q_n(1) \prod_{i<j} \frac{F_{\rm min}^{SG}\left(\beta_{ij}, \frac{2}{3}\right)}{x_i + x_j} \,, \tag{4.27}$$

with $\mathcal{N}(B)$ and $\mu(B)$ defined in eqs. (3.6) and (3.10).

Comparing the form factors of $\Box\varphi$ and the form factors of the fundamental exponentials the quantum equation of motion can be cast in the form

$$\begin{aligned}
\Box\varphi(x) &= \frac{m^2}{2\sqrt{2}}\sqrt{\frac{F_{\rm min}^{\rm BD}(i\pi,\mathcal{B})}{\sin^2\left(\frac{\pi}{6}(\mathcal{B}+2)\right)\Gamma^2(\mathcal{B})+4W(\mathcal{B})}}\,(\theta(1-\mathcal{B}) - \theta(\mathcal{B}-1))\left(e^{-2\lambda\varphi(x)} - e^{\lambda\varphi(x)}\right) \\
&= \frac{m^2}{2\sqrt{2}}\sqrt{\frac{F_{\rm min}^{\rm BD}(i\pi,\mathcal{B})}{\sin^2\left(\frac{\pi}{6}(\mathcal{B}+2)\right)\Gamma^2(\mathcal{B})+4W(\mathcal{B})}}\,(V_-(x,\mathcal{B}) - V_+(x,\mathcal{B})) \ . \qquad (4.28)
\end{aligned}$$

4.4 Class of Stress-Energy Tensor

The most general expression of the trace of the stress-energy tensor compatible with the (quantum) equation of motion of the BD model can be expressed in terms of the fundamental exponentials as

$$\Theta(x) = F_0^{\Theta}(\mathcal{B})\,(a\,\Phi_1(x,\mathcal{B}) + (1-a)\,\Phi_2(x,\mathcal{B})) \ , \tag{4.29}$$

where $F_0^{\Theta}(\mathcal{B})$ is its vacuum expectation value

$$F_0^{\Theta}(\mathcal{B}) = \frac{\pi m^2}{2W(\mathcal{B})} \ , \tag{4.30}$$

(as computed by the Thermodynamical Bethe Ansatz [20]), whereas a is a free parameter. Varying the value of a, we may reach different ultraviolet limit of the BD model. Before considering the general case, let us analyze separately the two cases $a=1$ and $a=0$.

4.4.1 The case $a=1$

For this value of a, the trace of the stress-energy tensor is given entirely by the operator $\Phi_1(x,\mathcal{B})$ and therefore we expect that the ultraviolet behaviour will be described by a CFT with bare action given by

$$\mathcal{S}_2 = \int d^2x \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{6\lambda^2}\,e^{-2\lambda\varphi}\right] \ , \tag{4.31}$$

and background charge [21]

$$Q_2(\lambda) = -\left(\frac{\sqrt{\pi}}{\lambda} + \frac{\lambda}{2\sqrt{\pi}}\right) . \tag{4.32}$$

Using eq. (4.5), the corresponding central charge is given by

$$c(\mathcal{B}) = 1 + 12Q_2^2(\lambda) = 1 + 12\left(\frac{2-\mathcal{B}}{4\mathcal{B}} + \frac{\mathcal{B}}{2-\mathcal{B}} + 1\right) , \tag{4.33}$$

This is confirmed by the computation of the central charge in terms of the c-theorem by using the FF of the operator $\Phi_1(x,\mathcal{B})$ which defines in this case the trace of the stress-energy tensor. The result of this computation is reported in Table 3 (see also Fig. 2) and the sum rule turns out to be saturated with high percentage of precision by using just the first two FF of $\Phi_1(x,\mathcal{B})$.

According to the CFT (4.31), the conformal dimensions of the primary fields $e^{\alpha\varphi}$ are given by

$$\Delta_2(\alpha) = -\frac{\alpha^2}{8\pi} + \frac{\alpha Q_2(\lambda)}{\sqrt{4\pi}} . \tag{4.34}$$

and for the fundamental exponential operators of the BD model we have

$$\begin{aligned} \Delta_2(\Phi_2) &= 1 \\ \Delta_2(\Phi_1) &= -\frac{1}{2} - \frac{3}{8\pi}\lambda^2 . \end{aligned} \tag{4.35}$$

It is easy to see that these expressions are in agreement with those extracted by looking at the short-distance behaviour of the correlators $< T_{zz}(z,\overline{z})\,\Phi_1(0) >$ and $< T_{zz}(z,\overline{z})\,\Phi_2(0) >$. The computation are similar to that of the Sinh-Gordon model, the only difference being the perturbative expansion

$$\begin{aligned} F_{\rm min}^{\rm BD}(i\pi,\mathcal{B}) &= \exp\left[-8\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{x\mathcal{B}}{6}\right)\sinh\left(\frac{x}{6}(2-\mathcal{B})\right)\,\sinh\frac{x}{2}\cosh\frac{x}{6}}{\sinh^2 x}\right] \sim \\ &\sim 1 - \left(\frac{1}{\pi} + \frac{1}{6\sqrt{3}}\right)\frac{\lambda^2}{6} + o(\lambda^4) , \end{aligned} \tag{4.36}$$

and therefore we will not repeat them here.

4.4.2 The case $a = 0$

Since the trace of the stress-energy tensor is given in this case by the operator $\Phi_2(x,\mathcal{B})$, the ultraviolet limit will be ruled by a CFT with a bare action given by

$$\mathcal{S}_1 = \int d^2x\left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{3\lambda^2}\,e^{\lambda\varphi}\right] , \tag{4.37}$$

and a background charge given by

$$Q_1(\lambda) \,=\, \left(\frac{\sqrt{4\pi}}{\lambda} + \frac{\lambda}{2\sqrt{4\pi}}\right) \,. \tag{4.38}$$

For the corresponding value of the central charge we have

$$\begin{aligned} c(\mathcal{B}) &= 1 + 12\, Q_1^2(\lambda) \\ &= 1 \,+\, 12\left(\frac{2-\mathcal{B}}{\mathcal{B}} + \frac{1}{4}\frac{\mathcal{B}}{2-\mathcal{B}} + 1\right) \,, \end{aligned} \tag{4.39}$$

whereas for the conformal dimension of the primary operators $e^{\alpha\varphi}$

$$\Delta_1(\alpha) \,=\, -\frac{\alpha^2}{8\pi} + \frac{\alpha Q_1(\lambda)}{\sqrt{4\pi}} \,.. \tag{4.40}$$

Hence, for the fundamental exponential operators we have in this case

$$\begin{aligned} \Delta_1(\Phi_1) &= 1 \\ \Delta_1(\Phi_2) &= -2 - \frac{3}{4\pi}\lambda^2 \,. \end{aligned} \tag{4.41}$$

These conformal data are again confirmed by the FF approach, as shown for instance in Fig. 2 for the central charge.

4.4.3 The general case

It is now easy to write down the conformal dimension $\Delta(\alpha)$ of the exponential operator $e^{\alpha\varphi}$ and the central charge of the CFT reached in the ultraviolet regime for generic value of the parameter a appearing in the definition of the stress-energy tensor of the BD model. The argument is similar to that already employed in the Sinh-Gordon model. Since $\Delta(\alpha)$ is the coefficient of the most singular term obtained in the UV-limit of the correlation function $< T_{zz}(z,\overline{z})\, e^{\alpha\varphi(0)} >$ and the FF of $T_{zz}(z,\overline{z})$ depends linearly on the parameter a, the conformal dimension is given by

$$\begin{aligned} \Delta(\alpha) &= a\,\Delta_2(\alpha) + (1-a)\,\Delta_1(\alpha) \,= \\ &= -\frac{\alpha^2}{8\pi} + \frac{\alpha}{\sqrt{4\pi}}\left[\sqrt{\frac{2-\mathcal{B}}{\mathcal{B}}}\left(1 - \frac{3}{2}a\right) + \sqrt{\frac{\mathcal{B}}{2-\mathcal{B}}}\left(\frac{1}{2} - \frac{3}{2}a\right)\right] \,, \end{aligned} \tag{4.42}$$

where $\Delta_1(\alpha)$ and $\Delta_2(\alpha)$ are given in eq. (4.34) and (4.40) respectively. The linear term in α in (4.42) identifies the background charge of the corresponding Coulomb gas. Therefore the central charge of the CFT reached in the ultraviolet limit is given by

$$c \,=\, 1 \,+\, 3\frac{(\mathcal{B} + 6a - 4)^2}{\mathcal{B}(2-\mathcal{B})} \,. \tag{4.43}$$

The check of this formula in terms of the FF approach is shown in Fig. 3.

Observe that, with the choice

$$a = \frac{4-\mathcal{B}}{6} \ , \tag{4.44}$$

we have identically $c = 1$, a result that is confirmed by the FF approach within the usual accuracy of few percents. The corresponding trace of the stress-energy tensor is given by

$$\Theta(x) = F_0^{\Theta}(\mathcal{B}) \left(\frac{4-\mathcal{B}}{6} \, e^{\lambda\varphi(x)} + \frac{2+\mathcal{B}}{6} \, e^{-2\lambda\varphi(x)} \right) \ . \tag{4.45}$$

This operator is manifestly self-dual. In the limit $\lambda \to 0$, it reduces to

$$\Theta(x) \;=\; \frac{2\pi m^2}{3\lambda^2} \left(2e^{\lambda\varphi(x)} + e^{-2\lambda\varphi(x)} \right) \ , \tag{4.46}$$

which is the classical expression of $\Theta(x)$ for the Bullough-Dodd model.

5 Conclusions

The ultraviolet behaviour of a two-dimensional QFT is generally characterized by a scaling behaviour described by a CFT. The main features of a CFT are encoded in the definition of the stress-energy tensor $T_{\mu\nu}(x)$. As shown by the form factor approach, associated to an on-shell dynamics, there is a one-parameter family of possible operators $T_{\mu\nu}(x)$ that induces different scaling behaviour of the massive theory in the ultraviolet limit. In light of their simple spectrum, we have analyzed in detail the Sinh-Gordon and the Bullough-Dodd Models, computing the relevant CFT data (central charge and conformal dimensions) in terms of the FF of the fundamental exponential operators.

The ultraviolet properties of these models are strictly related to the duality symmetry of their S-matrix. This symmetry has far-reaching consequences. In fact, since the FF are computed in terms of the S-matrix, this symmetry may also be extended off-shell. In particular, it gives rise to a bidimensional representation in the space of the fundamental exponential operators of both theories. Moreover, from the self-duality of the two theories, we have an identification of the two operators at the self-dual point. This remarkable property is based on our definitions (3.20) and (4.25) of the fundamental exponentials of the two theories in terms of their respective cluster operators. The validity of these formulas has been checked by analyzing the ultraviolet behaviour of the massive theories. An important difference between the BD and the SG models is in the expressions of the central charge associated to their fundamental exponentials. In fact, whereas in the SG model we have the self-dual function (3.31) which is common to both the exponentials $\Phi_{\pm}(x)$, in the BD model we have the different expressions (4.33) and (4.39), related, however by duality

$$c(a=0,\mathcal{B}) \;=\; c(a=1, 2-\mathcal{B}) \ . \tag{5.1}$$

It would be interesting to extend our results to other Affine Toda Field Theories and to study in more detail the interplay between the massive and conformal data of the models.

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Appendix A

Solving the recursive equations (4.14) and (4.16), the first FF which have the properties to be asymptotically constant and proportional to the invariant combination $\sigma_1\sigma_{n-1}$ $(n>2)$ are given by

$$F_2(\beta) \;=\; F_{\text{min}}^{\text{BD}}(\beta,\mathcal{B})\left\{H_2 \;-\; H_1\,\frac{\sin\frac{\pi}{6}(\mathcal{B}+2)\,\Gamma(\mathcal{B})}{\sqrt{F_{\text{min}}^{\text{BD}}(i\pi,\mathcal{B})}}\,\frac{1}{\cosh\beta\;+\frac{1}{2}}\right\}\;, \tag{A.1}$$

and

$$\begin{aligned} F_3(\beta_1,\beta_2,\beta_3) \;&=\; \left(\prod_{i<j}^{3}\frac{F_{\text{min}}^{\text{BD}}(\beta_i-\beta_j,\mathcal{B})}{(x_i+x_j)(\omega\,x_i+x_j)(\omega^{-1}\,x_i+x_j)}\right)\frac{4}{F_{\text{min}}^{\text{BD}}(i\pi,\mathcal{B})}\sigma_1\sigma_2\,\times \\ &\Big\{q_1(\mathcal{B})\sigma_3^2\;+\;q_2(\mathcal{B})\sigma_3\sigma_2\sigma_1\,- \\ &H_1\,W(\mathcal{B})\,\left(\sigma_3\sigma_1^3+\sigma_2^3\right)\;+\;q_3(\mathcal{B})\sigma_2^2\sigma_1^2\Big\}\;, \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} q_1(\mathcal{B}) \;&=\; 2\Gamma(\mathcal{B})\,\sin\frac{\pi}{6}(2+\mathcal{B})\left[\cos\frac{\pi}{3}(2+\mathcal{B})\,-1\right]\left[\Gamma(\mathcal{B})H_1\,\sin\frac{\pi}{6}(2+\mathcal{B})\;+\sqrt{F_{\text{min}}^{\text{BD}}(i\pi)}\frac{H_2}{2}\right] \\ q_2(\mathcal{B}) \;&=\; \Gamma(\mathcal{B})\,\sin\frac{\pi}{6}(2+\mathcal{B})\left[\Gamma(\mathcal{B})H_1\,\sin\frac{\pi}{6}(2+\mathcal{B})\,-\sqrt{F_{\text{min}}^{\text{BD}}(i\pi)}H_2\,\left(\cos\frac{\pi}{3}(2+\mathcal{B})\,-\frac{3}{2}\right)\right] \\ q_3(\mathcal{B}) \;&=\; H_1\,W(\mathcal{B})\,-\sqrt{F_{\text{min}}^{\text{BD}}(i\pi)}\,\frac{H_2}{2}\,\Gamma(\mathcal{B})\,\sin\frac{\pi}{6}(2+\mathcal{B})\;. \end{aligned} \tag{A.3}$$

H_1 and H_2 are two arbitrary parameters.

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Table Caption

Table 1 . The first two-particle term entering the sum rule of the c-theorem for the Sinh-Gordon model with the choice $a = 1$ (second column) compared with the central charge (3.31) of CFT.

Table 2 . The first two-particle term entering the sum rule of the c-theorem for the Sinh-Gordon model with the choice $a = 1/2$. It must be compared with the claimed free boson $C_{UV} = 1$ UV-behaviour.

Table 3 . The first two-particle term entering the sum rule of the c-theorem for the Bullough-Dodd model with the choice $a = 1$ (second column) compared with the central charge (4.33) of CFT.

B	C_{UV}^{num}	C_{UV}^{Liouv}
$\frac{1}{10}$	127.28994	127.31579
$\frac{1}{5}$	67.61695	67.66667
$\frac{3}{10}$	47.98763	48.05882
$\frac{2}{5}$	38.40998	38.5
$\frac{1}{2}$	32.89395	33.
$\frac{3}{5}$	29.45222	29.57143
$\frac{7}{10}$	27.24418	27.37363
$\frac{4}{5}$	25.86323	26.
$\frac{9}{10}$	25.10126	25.24242
1	24.85738	25.

Table 1

B	C_{UV}^{num}
$\frac{1}{500}$	0.9999995
$\frac{1}{100}$	0.9999878
$\frac{1}{10}$	0.9989538
$\frac{3}{10}$	0.9931954
$\frac{2}{5}$	0.9897087
$\frac{1}{2}$	0.9863354
$\frac{2}{3}$	0.9815944
$\frac{7}{10}$	0.9808312
$\frac{4}{5}$	0.9789824
1	0.9774634

Table 2

B	C_{UV}^{num}	C_{UV}^{Liouv}
$\frac{1}{10}$	70.63001	70.63158
$\frac{1}{5}$	41.32883	41.33333
$\frac{3}{10}$	32.10844	32.11765
$\frac{2}{5}$	27.98391	28.
$\frac{1}{2}$	25.97441	26.
$\frac{3}{5}$	25.10474	25.14286
$\frac{7}{10}$	24.97886	25.03297
$\frac{4}{5}$	25.42607	25.5
$\frac{9}{10}$	26.38691	26.48485
1	27.87364	28.
$\frac{11}{10}$	29.96195	30.12121
$\frac{6}{5}$	32.80360	33.
$\frac{13}{10}$	36.66406	36.90110
$\frac{7}{5}$	42.00619	42.28571
$\frac{3}{2}$	49.67928	50.
$\frac{8}{5}$	61.39520	61.75
$\frac{17}{10}$	81.15833	81.52941
$\frac{9}{5}$	120.98370	121.33333
$\frac{19}{10}$	240.90584	241.15789

Table 3

Figure Caption

Figure 1 . The first two-particle term entering the sum rule of the c-theorem for the Sinh-Gordon model with the choice $a = 1, 0$ (dots), compared with the selfdual central charge (3.31) of CFT (solid line).

Figure 2 . The first two-particle term entering the sum rule of the c-theorem for the Bullough-Dodd model with the choice $a = 1, 0$ (dots and crosses, resp.), compared with the non selfdual central charges (4.33), (4.39) of CFT (solid thin line and solid thick line, resp.).

Figure 3 . The first two-particle term entering the sum rule of the c-theorem for the Sinh-Gordon model and the Bullough-Dodd model with the coupling constant fixed at $B = 1/2$ for different values of a (crosses and dots, resp.), compared with the central charges (3.47), (4.43) of CFT (solid thin line and solid thick line, resp.).