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Two and three-point functions in Liouville theory

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Abstract

Based on our generalization of the Goulian-Li continuation in the power of the 2D cosmological term we construct the two and three-point correlation functions for Liouville exponentials with generic real coefficients. As a strong argument in favour of the procedure we prove the Liouville equation of motion on the level of three-point functions. The analytical structure of the correlation functions as well as some of its consequences for string theory are discussed. This includes a conjecture on the mass shell condition for excitations of noncritical strings. We also make a comment concerning the correlation functions of the Liouville field itself.

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1 Introduction

Noncritical string theory and more general conformal field theories coupled to 2D gravity require the knowledge of Liouville theory which describes the Weyl degree of freedom of the 2D metrics. In addition, as a highly nontrivial interacting 2D field theory Liouville theory deserves a lot of attention for its own. A central problem is the construction of arbitrary real powers of the exponential of the Liouville field. They are needed as gravitational dressing factors [1, 2, 3] for vertex operators in string theory or conformal models beyond the minimal series. Furthermore, generic real powers of the Liouville exponential can be used to define via differentiation the Liouville operator itself.

In the canonical approach based on free fields [4, 5, 6] (and refs. therein) a lot of important structure, including the underlying quantum group [7] has been investigated. The general exponential under discussion is given by an infinite series of operators constructed out of free fields [8]. However, the calculation of correlation functions based on this representation is still lacking.

We approach the problem of Liouville correlation functions in this paper along a less rigorous way following the idea of Goulian and Li [9]. It implies a continuation from integer values of a certain variable s to arbitrary real values. Applied to minimal conformal models the procedure reproduces the results of corresponding matrix model calculations. It has been justified by comparison of its asymptotics with that obtained for the original functional integral in the quasiclassical limit [10]. There is also a way to understand the success of the method within the canonical treatment [11].
The original method of [9] was developed only for those exponentials which are needed as dressing factors for primary fields in minimal conformal models, i.e. for special rational values of s. In the derivation explicit use has been made of constraints induced by the special degenerate kinematics in one-dimensional target space. In a previous paper [12] we have constructed a continuation free of these restrictions.

Using our continuation procedure and starting from the functional integral representation we present in section 2 a systematic construction of three and two-point functions of arbitrary powers of Liouville exponentials. In section 3 we check the Liouville equation for the three-point function. The uniqueness proof of [10] is restricted to exponentials with rational coefficients as they are necessary to dress minimal models. With a more general validity proof still lacking we understand the check of the Liouville equation as a very strong argument in favour of the correctness of our continuation procedure. Section 4 is devoted to a remark on the two and three-point functions of the Liouville field itself. Section 5 describes the analytic structure of the correlation functions derived before. Based on this we comment some related work on off-shell critical strings [13] and formulate a conjecture concerning the mass shell condition for excitations of noncritical strings.

2 Three and two-point functions of arbitrary Liouville exponentials

We define the correlation functions under discussion for $N \geq 3$ by

$$G_N(z_1,...,z_N|\beta_1,...,\beta_N) \;=\; \langle \prod_{j=1}^{N} e^{\beta_j \phi(z_j)} \rangle \;=\; \int D\phi \; e^{-S_L[\phi|\hat{g}]} \prod_{j=1}^{N} e^{\beta_j \phi(z_j)} \tag{1}$$

with

$$S_L[\phi|\hat{g}] \;=\; \frac{1}{8\pi} \int d^2 z \sqrt{\hat{g}} \Big(\hat{g}^{mn} \partial_m \phi \partial_n \phi \;+\; Q \hat{R}(z) \phi(z) \;+\; \mu^2 e^{\alpha \phi(z)} \Big) \;. \tag{2}$$

$\hat{g}$ is a classical reference metric of the 2D manifold of spherical topology, $\hat{R}$ the corresponding Ricci scalar. Q parametrizes the central charge c_L of the Liouville theory by

$$c_L \;=\; 1 \;+\; 3Q^2 \;. \tag{3}$$

If the Liouville theory describes the gravitational sector of a conformal matter theory with central charge c_M one has in addition

$$c_L \;+\; c_M \;-\; 26 \;=\; 0 \;. \tag{4}$$

The factor α in the exponential is fixed to attribute to $e^{\alpha\phi}$ a conformal $(1,1)$ dimension and the property of being a "microscopic" operator, i. e. $\alpha < \frac{Q}{2}$ [14]

$$\frac{1}{2} \alpha_\pm (Q - \alpha_\pm) \;=\; 1 \quad , \tag{5}$$

$$\alpha_\pm \;=\; \frac{Q}{2} \;\pm\; \frac{\sqrt{Q^2 - 8}}{2} \;, \qquad \alpha \;=\; \alpha_- \;. \tag{6}$$

The zero mode integration can be performed explicitly [9]. For integer

$$s_N \;=\; \frac{Q - \sum_{j=1}^{N} \beta_j}{\alpha} \tag{7}$$

also the remaining functional integral can be done: [2]

$$G_N(z_1,...,z_N|\beta_1,...,\beta_N) \;=\; \frac{\Gamma(-s_N)}{\alpha} \Big(\frac{\mu^2}{8\pi} \Big)^{s_N} \prod_{1 \leq i < j \leq N} |z_i \;-\; z_j|^{-2\beta_i \beta_j}$$

$$\cdot \int \prod_{I=1}^{s_N} \Big(d^2 w_I \prod_{j=1}^{N} |z_j \;-\; w_I|^{-2\alpha\beta_j} \Big) \prod_{1 \leq I < J \leq s_N} |w_I \;-\; w_J|^{-2\alpha^2} \;. \tag{8}$$

[2] Effectively we can handle μ^2, z_j, w_I as dimensionless quantities, since in addition to the original dimensional μ^2, z_j, w_I further dimensional parameters are involved (RG-scale, scale of the background $\hat{R}$ and an integration constant in the Liouville action) which can be used to introduce suitable quotients. A complete discussion of this point is given in the second reference of [3].

Due to the nonexistence of a SL(2,C) invariant vacuum one has to be careful with respect to the usual conformal structure of N-point functions.

Let us start with the 3-point function. Fortunately from the explicit representation (8) we can prove for integer s_3 that the standard structure of the z_j dependence is realized: At first the integral in (8) is convergent for small $|\beta_i|$ and large Q. Then with allowed substitutions of integration variables one gets first in this region and later by continuation everywhere for Möbius transformations $z_i = V(u_i)$

$$G_3(z_1, z_2, z_3|\beta_1, \beta_2, \beta_3) = \prod_{i=1}^{3} |V'(u_i)|^{-2\Delta_i} \; G_3(u_1, u_2, u_3|\beta_1, \beta_2, \beta_3) \tag{9}$$

with

$$\Delta_i = \frac{1}{2}\beta_i(Q - \beta_i) \; . \tag{10}$$

This yields enough information to establish the usual conformal structure

$$\begin{aligned} G_3(z_1, z_2, z_3|\beta_1, \beta_2, \beta_3) &= A_3(\beta_1, \beta_2, \beta_3) \; |z_1 - z_2|^{2(\Delta_3 - \Delta_1 - \Delta_2)} \\ &\cdot \; |z_1 - z_3|^{2(\Delta_2 - \Delta_1 - \Delta_3)} |z_2 - z_3|^{2(\Delta_1 - \Delta_2 - \Delta_3)} \; , \end{aligned} \tag{11}$$

$$A_3 = \lim_{u_3 \to \infty} |u_3|^{4\Delta_3} G_3(0, 1, u_3|\beta_1, \beta_2, \beta_3) \; . \tag{12}$$

With (8) this gives A_3 as

$$A_3 = \frac{\Gamma(-s_3)}{\alpha} \Big(\frac{\mu^2}{8\pi}\Big)^{s_3} \int \prod_{I=1}^{s_3} (d^2 w_I |w_I|^{-2\alpha\beta_1} |1 - w_I|^{-2\alpha\beta_2}) \prod_{1 \leq I < J \leq s_3} |w_I - w_J|^{-2\alpha^2} \; . \tag{13}$$

Using the Dotsenko-Fateev integrals [16] this can be written as [12]

$$A_3(\beta_1, \beta_2, \beta_3) = \frac{\Gamma(-s_3)}{\alpha} \; \Gamma(1 + s_3) \Big(\frac{\mu^2 \; \Gamma(1 + \frac{\alpha^2}{2})}{8 \; \Gamma(-\frac{\alpha^2}{2})} \Big)^{s_3} \prod_{i=0}^{3} F_i \tag{14}$$

with [3]

$$F_i = \exp\Big(f(\alpha\bar{\beta}_i, \frac{\alpha^2}{2}|s_3) - f(\alpha\beta_i, \frac{\alpha^2}{2}|s_3) \Big) \; , \qquad i = 1, 2, 3 \tag{15}$$

$$\bar{\beta}_i = \frac{1}{2}(\beta_j + \beta_k - \beta_i) = \frac{1}{2}(Q - \alpha s_3) - \beta_i \; , \qquad (i, j, k) = \mathrm{perm}(1, 2, 3) \; , \tag{16}$$

$$F_0 = \Big(-\frac{\alpha^2}{2}\Big)^{-s_3} \frac{1}{\Gamma(1 + s_3)} \exp\Big(f(1 - \frac{\alpha^2}{2}s_3, \frac{\alpha^2}{2}|s_3) - f(1 + \frac{\alpha^2}{2}, \frac{\alpha^2}{2}|s_3) \Big) \; , \tag{17}$$

[3] F_0 in the present paper differs from that in ref. [12] by a factor $\pi(-1)^{s_3+1}$. The reason is that, as discussed in [12] in detail, in going from integer s to noninteger s one has to give meaning to the formal quantity $\Gamma(0)\sin(\pi s)$. Instead of $\Gamma(0)\sin(\pi s) \to 1$ we use now the more natural convention $\sin(\pi s)\Gamma(0) \to \pi(-1)^{s+1}$. The sign effect will be crucial in section 3 to ensure the validity of the Liouville equation.

$$f(a,b|s) \;=\; \sum_{j=0}^{s-1} \log\,\Gamma(a+bj)\;, \qquad \text{integer } s\;. \tag{18}$$

The function f fulfills the relations

$$f(a,b|s+1) = f(a,b|s) + \log\Gamma(a+bs)\;, \tag{19}$$

$$f(a+1,b|s) = f(a,b|s) + s\log b + \log\Gamma(\frac{a}{b}+s) - \log\Gamma(\frac{a}{b})\;, \tag{20}$$

$$f(a+b,b|s) = f(a,b|s) + \log\Gamma(a+bs) - \log\Gamma(a)\;, \tag{21}$$

$$f(a+b(s-1),-b|s) = f(a,b|s)\;. \tag{22}$$

Further functional relations can be found in [12], we present here only those we need in the discussion of the present paper. There exists a continuation of $f(a,b|s)$ to arbitrary complex a, b, s given by

$$\begin{aligned} f(a,b|s) = \int_0^\infty \frac{dt}{t}\Big(s(a-1)e^{-t} + b\frac{s(s-1)}{2}e^{-t} - s\frac{e^{-t}}{1-e^{-t}} \\ +\frac{(1-e^{-tbs})e^{-at}}{(1-e^{-tb})(1-e^{-t})}\Big). \end{aligned} \tag{23}$$

It fulfills all the functional relations. Using the integral representation and the functional relations one can prove [12] that $\exp(f(a,b|s))$ is a meromorphic function. Due to (22) it is sufficient to investigate the case Re $b \geq 0$. Under this circumstance $\exp f$ has poles at

$$a \;=\; -bj \;-\; l \qquad\qquad \text{(poles)} \tag{24}$$

and zeros at

$$a \;+\; bs \;=\; -bj \;-\; l \qquad\qquad \text{(zeros)}\;. \tag{25}$$

In both cases j and l are integers ≥ 0. The order of poles and zeros is determined by the number of different realizations of the r.h.s. of eqs. (24) and (25), respectively.

We now turn to the 2-point function. Taking (1) unmodified also for $N=2$ would imply $G_2(z_1,z_2|\beta_1,\beta_2) \;=\; G_3(z_1,z_2,z_3|\beta_1,\beta_2,0)$. The unwanted z_3-dependence as usual in conformal theories drops for $\Delta_1 = \Delta_2$. However, the z-independent factor $A_3(\beta,\beta,\beta_3)$ diverges for $\beta_3 \to 0$. This can be seen from

$$\begin{aligned} \prod_{j=1}^{3} F_j \;&=\; \prod_{j=1}^{3}\Big(\exp[f(\alpha\bar\beta_j+1,\frac{\alpha^2}{2}|s_3) - f(\alpha\beta_j+1,\frac{\alpha^2}{2}|s_3)]\;\frac{\Gamma(\frac{2\bar\beta_j}{\alpha}+s_3)\Gamma(1+\frac{2\bar\beta_j}{\alpha})}{\Gamma(\frac{2\bar\beta_j}{\alpha}+s_3)\Gamma(1+\frac{2\beta_j}{\alpha})}\Big) \\ &\cdot\;\; \frac{8\beta_1\beta_2\beta_3}{(\beta_1+\beta_2-\beta_3)(\beta_1+\beta_3-\beta_2)(\beta_2+\beta_3-\beta_1)}\;, \end{aligned} \tag{26}$$

which is a consequence of (20) and (16). The reason for this divergence is the change of the situation with respect to the conformal Killing vectors (CKV). The 3-punctured sphere has no CKV's while the 2-punctured sphere has one. The (divergent) volume of the corresponding subgroup of the Möbius group SL(2,C) leaving z_1 and z_2 fixed is

$$V^{(2)}_{CKV} \;=\; \int \frac{d^2w\ |z_1-z_2|^2}{|z_1-w|^2\ |z_2-w|^2}\quad . \tag{27}$$

Having this in mind we define

$$G_2(z_1,z_2|\beta) \;=\; \langle e^{\beta\phi(z_1)}e^{\beta\phi(z_2)}\rangle \;=\; \frac{1}{V^{(2)}_{CKV}}\int D\phi\ e^{-S_L[\phi]}e^{\beta\phi(z_1)}e^{\beta\phi(z_2)}\quad . \tag{28}$$

Treating the functional integral in analogy to that for the 3-point function and choosing $\int d^2w_1$ as the cancelled integration one gets

$$\begin{aligned} G_2(z_1,z_2|\beta) \;&=\; \frac{\Gamma(-s_2)}{\alpha}\Big(\frac{\mu^2}{8\pi}\Big)^{s_2}|z_1\ -\ z_2|^{-2\beta^2}\ \frac{|z_1-w_1|^2\ |z_2-w_1|^2}{|z_1-z_2|^2} \\ &\cdot\int\prod_{I=2}^{s_2}d^2w_I\prod_{I=1}^{s_2}\Big(|z_1-w_I|\ |z_2-w_I|\Big)^{-2\alpha\beta}\prod_{1\le I<J\le s_2}|w_I-w_J|^{-2\alpha^2}. \end{aligned} \tag{29}$$

This means (note $s_2=\frac{Q-2\beta}{\alpha}=1+s_3(\beta,\beta,\alpha)$)

$$G_2(z_1,z_2|\beta) \;=\; -\frac{\mu^2}{8\pi s_2}G_3(z_1,z_2,w_1|\beta,\beta,\alpha)\ |z_1-z_2|^{-2}\ |z_1-w_1|^2\ |z_2-w_1|^2\ . \tag{30}$$

From (11) we see that the w_1 dependence on the r.h.s. cancels. For this result $\Delta_1=\Delta_2$ is crucially. Altogether we find

$$G_2(z_1,z_2|\beta) \;=\; \frac{A_2(\beta)}{|z_1-z_2|^{4\Delta}}\quad , \tag{31}$$

with

$$A_2(\beta) \;=\; -\frac{\mu^2}{8\pi s_2}\ A_3(\beta,\beta,\alpha)\quad . \tag{32}$$

By the use of (22) and (21) one can eliminate the function f completely and derive the simple result

$$A_2(\beta) \;=\; -\ \frac{1}{\alpha\pi s_2}\Big(\frac{\mu^2\ \Gamma(\frac{\alpha^2}{2})}{8\ \Gamma(1-\frac{\alpha^2}{2})}\Big)^{s_2}\ \frac{\Gamma(1-\frac{\alpha^2}{2}s_2)\ \Gamma(1-s_2)}{\Gamma(\frac{\alpha^2}{2}s_2)\ \Gamma(s_2)}\quad . \tag{33}$$

Due to (7) and (5) s_2 expressed by α and β is

$$s_2=1+\frac{2}{\alpha^2}-\frac{2\beta}{\alpha}\ .$$

This formula for A_2 coincides up to an irrelevant factor $-\frac{\pi\alpha^{11}}{32}$ with the expression presented in [9] for the integrated 2-point function in gravitationally dressed minimal models, i.e. for rational s_2. For A_2 describing the gravitational dressing of the two point function in minimal models the form (33) fits into the “leg-factor” structure known for higher correlation functions ($N \geq 3$), [19].

The extension of the procedure to the one and zero-point function (partition function) is straightforward. One has to cancel $V_{CKV}^{(1)}$ and $V_{CKV}^{(0)}$ related to the one-punctured and unpunctured sphere, respectively. However, in the one-point case the method yields inconsistent results, the dependence on the fixed unintegrated w_1, w_2 does not cancel for generic β. This is a reflection of the absence of a SL(2,C) invariant vacuum, which prevents looking at the one-point function as a scalar product of two physical states. We come back to this point in section 4. For the partition function Z we get

$$\begin{aligned} Z \;=\; G_0 &= -\frac{\mu^6}{512\pi^3 s_0(1-s_0)(2-s_0)}\; A_3(\alpha,\alpha,\alpha) \\ &= -\frac{\mu^2}{8\pi^3(\alpha+\frac{2}{\alpha})}\;\Big(\frac{\mu^2\;\Gamma(\frac{\alpha^2}{2})}{8\;\Gamma(1-\frac{\alpha^2}{2})}\Big)^{\frac{2}{\alpha^2}}\;\frac{\Gamma(-\frac{2}{\alpha^2})}{\Gamma(\frac{2}{\alpha^2}-1)}. \end{aligned} \tag{34}$$

After this discussion of G_2 and G_0 we want to mention a modified argumentation, which avoids the use of the w_I integral representation (A similar argument for the string S-matrix elements, i.e. integrated correlation functions of matter $\otimes$ Liouville has been applied in [9]). From the formal functional integral (1) one gets by differentiation with respect to μ^2 that $-8\pi\frac{d}{d\mu^2}G_2(z_1,z_2|\beta)$ has to be equal to the once integrated 3-point function $\int d^2z_3 G_3(z_1,z_2,z_3|\beta,\beta,\alpha)$. To complete this to an exact statement we must divide by $V_{CKV}^{(2)}$, i.e.

$$-\,8\pi\frac{d}{d\mu^2}G_2(z_1,z_2|\beta) \;=\; \frac{1}{V_{CKV}^{(2)}}\int d^2z_3\; G_3(z_1,z_2,z_3|\beta,\beta,\alpha)\;\;. \tag{35}$$

Taking into account $G_N \propto \mu^{2s_N}$ one arrives at (31), (32), immediately.

Closing this section we want to make a comment on the quasiclassical approximation $Q\to\infty$. In this limit one has $\alpha\cdot Q\to 2$. To get Q^2 as an overall factor in front of the action one should make the usual rescaling $\phi\to\frac{Q}{2}\phi$. Hence a sensible quasiclassical limit is

$$\alpha\to 0,\quad \beta\;=\;\alpha b,\quad \mu^2\;=\;\frac{2m^2}{\alpha^2},\quad b, m^2 \text{ fixed.}$$

Under these conditions we find (C Euler constant) from (33), (34)

$$\begin{aligned} A_2 &= -\frac{e^{4b-2-2C}}{\pi\alpha^3(2b-1)\sin(\pi(2b-\frac{2}{\alpha^2}))}\;\Big(\frac{m^2e^2}{8}\Big)^{1-2b+\frac{2}{\alpha^2}}\;(1+O(\alpha))\;, \\ Z &= -\frac{e^{-2-2C}}{\pi^3\alpha^3\sin(-\frac{2}{\alpha^2}\pi)}\;\Big(\frac{m^2e^2}{8}\Big)^{1+\frac{2}{\alpha^2}}\;(1+O(\alpha))\;. \end{aligned} \tag{36}$$

The normalized 2-point function fulfills in the limit

$$\frac{G_2}{Z} \;=\; \frac{\pi^2}{2b-1}\Big(-\;\frac{m^2}{8}\Big)^{-2b}\;|z_1-z_2|^{-4b}\;. \tag{37}$$

Here we have used $\frac{\sin(-\frac{2\pi}{\alpha^2})}{\sin(\pi(2b-\frac{2}{\alpha^2}))} \;\rightarrow\; \exp(\pm 2\pi b i) \;=\; (-1)^{2b}$, which is valid of course only for a limit performed a little bit off the real axis, i.e. $\alpha^2 = |\alpha^2|e^{\pm i\epsilon}$. Eq. (37) gives the structure presented in ref. [14] for the quasiclassical limit in a more explicit form.

3 Liouville equation of motion

The Liouville equation in our parametrization is the equation of motion for the action (2) in the limit of flat $\hat{g}$

$$\partial^2\;\phi\;-\;\frac{\alpha\mu^2}{2}\;e^{\alpha\phi}\;=\;0\;. \tag{38}$$

As a partial check we want to prove

$$\langle\partial^2\phi(z_1)\;e^{\beta_2\phi(z_2)}\;e^{\beta_3\phi(z_3)}\rangle\;=\;\frac{\alpha\mu^2}{2}\langle e^{\alpha\phi(z_1)}\;e^{\beta_2\phi(z_2)}\;e^{\beta_3\phi(z_3)}\rangle \tag{39}$$

up to contact terms.
The l.h.s. of (39) is given by

$$4\partial_{z_1}\partial_{\bar{z}_1}\;\lim_{\beta_1\rightarrow 0}\frac{\partial}{\partial\beta_1}\;G_3(z_1,z_2,z_3|\beta_1,\beta_2,\beta_3)\;.$$

Using (10), (11) the differentiation with respect to β_1 is straightforward. In the generic case $\beta_2\neq\beta_3,\;\beta_j\neq 0,\;\;j=2,3$ one has $A_3(0,\beta_2,\beta_3)=0$ as can be seen from eqs. (14), (17), (26). Therefore, the contribution of terms with logarithms $\log|z_i-z_j|$ generated by the β_1 dependence of Δ_1 drops out in the limit $\beta_1\rightarrow 0$, i. e. [4]

$$\lim_{\beta_1\rightarrow 0}\frac{\partial}{\partial\beta_1}G_3(\;z_j\;|\;\beta_j\;)\;=\;\frac{\frac{\partial}{\partial\beta_1}A_3(\beta_1,\beta_2,\beta_3)|_{\beta_1=0}}{|z_2-z_1|^{2(\Delta_2-\Delta_3)}|z_1-z_3|^{2(\Delta_3-\Delta_2)}|z_2-z_3|^{2(\Delta_3+\Delta_2)}}\;. \tag{40}$$

After differentiation with respect to z_1 this yields

$$\langle\partial^2\phi(z_1)e^{\beta_2\phi(z_2)}e^{\beta_3\phi(z_3)}\rangle\;=\;\frac{4(\Delta_2-\Delta_3)^2\frac{\partial}{\partial\beta_1}A_3(\beta_1,\beta_2,\beta_3)|_{\beta_1=0}}{|z_2-z_1|^{2(1+\Delta_2-\Delta_3)}|z_1-z_3|^{2(1+\Delta_3-\Delta_2)}|z_2-z_3|^{2(\Delta_3+\Delta_2-1)}} \tag{41}$$

up to contact terms.
Since the r.h.s. of (39) is a special case of the 3-point function, eq. (39) is valid iff

$$4(\Delta_2-\Delta_3)^2\frac{\partial}{\partial\beta_1}A_3(\beta_1,\beta_2,\beta_3)|_{\beta_1=0}\;=\;\frac{\alpha\mu^2}{2}\;A_3(\alpha,\beta_2,\beta_3)\;. \tag{42}$$

[4]The β_1 derivative of A_3 is of course a total one refering also to the implicit β_1 dependence in $\bar{\beta}_j$ and s_3.

To prove (42) one first observes that for general $\beta_2,\ \beta_3$ but $\beta_1 = 0$ all factors in A_3 except F_1 are regular and different from zero. With (15), (20) we get ($s = s_3(0,\beta_2,\beta_3) = \frac{Q-\beta_2-\beta_3}{\alpha}$)

$$\begin{aligned}
\Big(F_1\Big)_{\beta_1=0} &= 0 \qquad (43)\\
\Big(\frac{\partial}{\partial\beta_1}F_1\Big)_{\beta_1=0} &= \alpha\Big(\frac{\alpha^2}{2}\Big)^{s-1}\ \Gamma(s)\ \exp\Big(f(\frac{\alpha}{2}(\beta_2+\beta_3),\frac{\alpha^2}{2}|s)\ -\ f(1,\frac{\alpha^2}{2}|s)\Big)\ .
\end{aligned}$$

Therefore, to calculate $\frac{\partial A_3}{\partial\beta_1}$ for $\beta_1 \to 0$ we need only the derivative of F_1

$$\Big(\frac{\partial A_3}{\partial\beta_1}\Big)_{\beta_1=0} = \frac{\Gamma(-s)}{\alpha}\Gamma(1+s)\Big(\frac{\mu^2\ \Gamma(1+\frac{\alpha^2}{2})}{2\ \Gamma(-\frac{\alpha^2}{2})}\Big)^s F_0F_2F_3\frac{\partial}{\partial\beta_1}F_1|_{\beta_1=0}\ . \qquad (44)$$

Having reached this level, the proof of eq. (42) is completed by a repeated application of the functional relations (19), (20), (21).

4 Remark on the three and two-point function of the Liouville field

As mentioned in the introduction and practised at an intermediate stage in the previous section already, we relate the Liouville operator $\phi(z)$ to $\partial_\beta e^{\beta\phi(z)} = \phi e^{\beta\phi(z)}$. For reasons becoming clear in a moment we still do not specify the value of β after differentiation. We only require to treat all β_j on an equal footing. From (11), (10), (6) this yields

$$\begin{aligned}
\partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3}G_3(\ z_j\ |\ \beta_j\ ) &= |z_1-z_2|^{2(\Delta_3-\Delta_1-\Delta_2)}|z_1-z_3|^{2(\Delta_2-\Delta_1-\Delta_3)}|z_2-z_3|^{2(\Delta_1-\Delta_2-\Delta_3)}\\
&\cdot\ \Big(\ \partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3}A_3\ -\ \sqrt{Q^2-8}\ \partial_{\beta_1}\partial_{\beta_2}A_3(l_{12}\ +\ l_{13}\ +\ l_{23})\\
&+\ (Q^2-8)\ \partial_{\beta_1}A_3\ (L_{13}\ L_{12}\ +\ L_{23}\ L_{12}\ +\ L_{23}\ L_{13})\\
&+\ (Q^2-8)^{\frac{3}{2}}A_3\ L_{12}\ L_{13}\ L_{23}\ \Big)\ ,
\end{aligned} \qquad (45)$$

with $l_{ij} = \log|z_i - z_j|$, $L_{ij} = l_{ij} - l_{ki} - l_{kj}$, $(i,j,k) = \mathrm{perm}(1,2,3)$.

Use has been made of the equality of derivatives with respect to different β_j if after differentiation a symmetric point in β-space is choosen.
The natural choice $\beta_j = 0$ removes the power-like z-dependence in (45), but as can be seen from eqs. (17) and (26) A_3 is singular at this point: The value of $F_1F_2F_3$ depends on how the origin in β-space is approached. In addition F_0 has a pole (note $s_3 \to \frac{Q}{\alpha} = 1+\frac{2}{\alpha^2}$, (24), (25)).

Our functional integral yields the correlation function of Liouville exponentials directly, there is no interpretation as a vacuum expectation value with respect to a SL(2,C) invariant vacuum [14]. Therefore, operator insertions are well-defined only in the presence

of at least two Liouville exponentials playing the role of spectators. This concept worked perfectly in the previous section where we constructed $\langle\phi(z_1)e^{\beta_2\phi(z_2)}e^{\beta_3\phi(z_3)}\rangle$. To get in the same sense 2 and 3-fold insertions of ϕ one has to start with the 4 and 5-point functions of exponentials. Unfortunately, these higher correlation functions are not available up to now.
There is still another possibility by seeking for an alternative choice of the β-value realized after differentiation. The choice $\beta_j = \alpha$ leads to regular A_3. If one interpretes the remaining power-like z-dependent factor ($\Delta_j = 1$) as the density of the Möbius volume, which has to be cancelled since the ϕ-insertions no longer correspond to punctures, one arrives at an expression for $\langle\phi\phi\phi\rangle$ which is polynomial in l_{ij} up to the third order. In a similar way one can produce a two-point function containing constant, linear and quadratic terms in l_{12}. However, we have no deeper understanding of such a procedure motivated purely by technical reasons.

5 Pole-zero spectrum of the correlation functions

The spectrum of poles and zeros of the 3-point function and two special degenerate cases of 4-point functions as well as related problems for the interpretation of non-critical strings have been discussed in [12], [17]. We add in this section observations concerning the 3-point and 2-point function which are relevant in connection with some recent work on off shell critical strings [13] and which shed some light on the question of mass shell conditions for non-critical strings.

For applications to noncritical strings we are interested in the case Re $(\alpha^2) > 0$. This of course is realized for $c_M < 1$ but higher dimensional target space $D > 1$ made possible by the presence of a linear dilaton background [17] i.e.

$$c_M = D - 3P^2. \tag{46}$$

It is even valid for $1 \leq c_M < 13$ if (6) is taken seriously also in between $1 \leq c_M \leq 25$, since then with (3) and (4) we have

$$\alpha^2 \ = \ \frac{13 \ - \ c_M \ - \ \sqrt{(25 - c_M)(1 - c_M)}}{6} \, . \tag{47}$$

On the other side in ref. [13] off shell critical strings are constructed for $c_M = 26$ by enforcing the otherwise violated condition of conformal (1,1) dimension by the dressing with suitable Liouville exponentials. Clearly, for this application one needs $\alpha^2 < 0$.
In the first situation Re $(\alpha^2) > 0$ eqs. (24) and (25) lead to the following pole-zero pattern of $\prod_{j=1}^3 F_j$ [12]

$$
\begin{aligned}
\alpha\,\bar{\beta}_j &= \frac{\alpha^2}{2}\,k_j \;+\; l_j \quad \text{(poles)}\\
\alpha\,\beta_j &= \frac{\alpha^2}{2}\,k_j \;+\; l_j \quad \text{(zeros)}
\end{aligned}
$$

$$
\mathrm{Re}\,(\alpha^2)\; > 0, \qquad \text{integer } k_j,\; l_j, \qquad \text{both } \le 0 \quad \text{or both } > 0\;. \tag{48}
$$

While the position of zeros depends on the value of single β_j, the pole position is given by a combination out of all β_j involved. Only in applications to dressings of minimal models also the pole position factorizes (leg poles).
In the second situation Re $\alpha^2 < 0$, since (24), (25) require Re $b > 0$, one must first use the functional relation (22) to get

$$
F_j \;=\; \exp\Big(f(1-\alpha\beta_j,-\frac{\alpha^2}{2}|s_3) \;-\; f(1-\alpha\bar{\beta}_j,-\frac{\alpha^2}{2}|s_3)\Big)\;. \tag{49}
$$

Now (24), (25) are applicable again. Up to a trivial shift β_j and $\bar{\beta}_j$ change their role

$$
\begin{aligned}
\alpha\beta_j \;-\; \frac{\alpha^2}{2} &= -\,\frac{\alpha^2}{2}\,k_j \;+\; l_j \quad \text{(poles)}\\
\alpha\,\bar{\beta}_j \;-\; \frac{\alpha^2}{2} &= -\,\frac{\alpha^2}{2}\,k_j \;+\; l_j \quad \text{(zeros)}
\end{aligned}
$$

$$
\mathrm{Re}\,(\alpha^2)\; < \;0, \qquad \text{integer } k_j,\; l_j, \qquad \text{both } \le 0 \quad \text{or both } > 0\;. \tag{50}
$$

Now the position of poles of $\prod_{j=1}^{3} F_j$ is determined by the single β_j.

The remaining factors in A_3 depend on the β_j via s_3 only. For their combined pole-zero spectrum arising from the Γ-functions and F_0 one finds for Re $\alpha^2 > 0$ no zeros but poles at

$$
\frac{\alpha}{2}\sum_{i=1}^{3}\beta_i \;-\; \frac{\alpha^2}{2} \;-\; 1 = \frac{\alpha^2}{2}k \;+\; l \quad \text{(poles)}
$$

$$
\mathrm{Re}\,(\alpha^2)\; > 0, \qquad \text{integer } k,\; l, \qquad \text{both } \le 0 \quad \text{or both } > 0\;. \tag{51}
$$

In the other case Re $\alpha^2 < 0$ one has instead

$$
\begin{aligned}
\frac{\alpha}{2}\sum_{i=1}^{3}\beta_i &= \frac{\alpha^2}{2}(1-j) \;-\; 1 \quad \text{(poles)}\\
\frac{\alpha}{2}\sum_{i=1}^{3}\beta_i &= \frac{\alpha^2}{2}(k+2) \;-\; l \quad \text{(zeros)}\\
\text{or}\quad \frac{\alpha}{2}\sum_{i=1}^{3}\beta_i &= \frac{\alpha^2}{2}(1-k) \;+\; l+2 \quad \text{(zeros)}
\end{aligned}
$$

$$
\mathrm{Re}\,(\alpha^2)\; < 0, \qquad \text{integer } k,\; j,\; l \ge 0\;. \tag{52}
$$

Altogether we find a drastic change in the analytic structure with respect to the β_j in going from Re $\alpha^2 > 0$ to Re $\alpha^2 < 0$.

Let us turn to the 2-point function. From (33) we obtain immediately for arbitrary α^2

$$\begin{aligned} \alpha\beta &= \frac{\alpha^2}{2} - l \ \text{ or } \ \alpha\beta = 1 - l\,\frac{\alpha^2}{2}\,, \ \text{ integer } l \geq 0 \ \text{ (poles)} \\ \alpha\beta &= \frac{\alpha^2}{2} + j \ \text{ or } \ \alpha\beta = 1 + j\,\frac{\alpha^2}{2} \ \text{ or } \ \beta = \frac{Q}{2}, \ \text{ integer } j \geq 2 \ \text{ (zeros).} \end{aligned} \tag{53}$$

From this pole-zero pattern we can derive an interesting conjecture concerning the mass shell condition for noncritical strings. For instance the coefficient β in a gravitationally dressed vertex operator for tachyons [1, 2, 3]

$$e^{ik_\mu X^\mu(z)}\ e^{\beta\phi(z)}$$

is related to k_μ by the requirement of total conformal dimension (1,1), i.e.

$$\frac{1}{2}\beta(Q-\beta) + \frac{k(k-P)}{2} = 1\ ,$$

or equivalently ((3), (4), (46))

$$(\beta - \frac{Q}{2})^2 - (k - \frac{P}{2})^2 = \frac{1-D}{12}\ . \tag{54}$$

In contrast to the critical string, where the demand of dimension (1,1) delivers the mass shell condition $\frac{k(k-P)}{2} = 1$, eq. (54) implies no restriction for the target space momentum.

A condition on k_μ can arise only due to an additional restriction on the allowed values of β. The β dependent factor A_2 discussed above appears as the dressing factor in the 2-point S-matrix element for the tachyon excitation of the string. From the point of view of field theory in target space this object is an inverse propagator. Hence it should vanish as soon as the tachyon momentum approaches its mass shell. [5] For generic β the dressing factor $A_2(\beta)$ is different from zero and it is natural to associate its zeros with the mass shell. For $c_M < 1$ i. e. $0 < \alpha^2 < 2$ the spectrum of zeros is unbounded from above. The lowest zero is $\beta = \frac{Q}{2}$. The resulting spectrum for the mass of the gravitational dressed tachyon, i.e. $m_T^2 = \frac{1-D}{12} - (\beta - \frac{Q}{2})^2$, is not bounded from below. However, since all zeros, except that at $\beta = \frac{Q}{2}$, obey $\beta > \frac{Q}{2}$ they correspond to operators $e^{\beta\phi}$ describing states with wave functions in mini-superspace approximation $\propto e^{(\beta-\frac{Q}{2})\phi}$. These states are not normalizable in the infrared $\phi \to +\infty$ and have to be excluded [14]. On the other hand $\beta = \frac{Q}{2}$ sits just on the border to the "microscopic" states describing local insertions with wave functions peaked in the ultraviolet and "macroscopic" states with imaginary

[5] We assume that this standard reasoning makes sense also in the presence of a linear dilaton background $P \neq 0$.

exponents. [6]
$\beta = Q/2$ then leads to

$$(k - \frac{P}{2})^2 = \frac{D-1}{12}\ . \qquad (55)$$

The generalization to higher string excitations is straightforward. For instance in the graviton case an additional term $+1$ on the l.h.s. of (54) leads to $(k - \frac{P}{2})^2 = \frac{D-25}{12}$.

We stress that in contrast to discussions based on a $(D+1)$-dimensional point of view (Liouville field as additional coordinate) on the l.h.s. of (55) we have the D-dimensional k^2 only. Although the $(D+1)$-dimensional point of view is quite natural in the language of generalized σ-model actions [18], for S-matrix elements, which require the notion of asymptotic states, the situation is more involved. The $(D+1)$-dimensional concept works perfectly well for vanishing Liouville mass μ [19]. However, for $\mu \neq 0$ the singularity structure of the special cases of 4-point functions we investigated in [12, 17] seems to be a serious obstruction for a $(D+1)$-dimensional interpretation of (54). These special cases ($\beta_4 = 0$ or α) turned out to be unsatisfactory from the D-dimensional point of view too. But since both do not fit into the mass shell condition conjectured in the present paper the issue remains open and requires more work on the 4-point function.

Closing this section we add an observation on the corresponding 3-point string S-matrix element. Due to (48) for $\beta_1, \beta_2, \beta_3$ taking arbitrary values out of the zero table of (53), corresponding to candidates for mass shell values, the function A_3 has three zeros, each for every F_j. A nonvanishing A_3 is possible only if there would be in addition three coinciding poles according to (48). By explicit inspection of all possible cases one can exclude this situation. This property of A_3 is crucial for further work on the factorization of higher string S-matrix elements.

6 Concluding remarks

With this paper we contributed to the construction of correlation functions in Liouville theory. This construction is a long standing problem relevant for various aspects of string theory and general conformal field theory. We were able to calculate the 2 and 3-point functions of Liouville exponentials of arbitrary real power. The method of continuation in the parameter s passed a very crucial test. The Liouville equation of motion is fulfilled, hence we are sure that the derived correlation functions indeed reflect some essential features of quantized Liouville theory. What concerns applications to noncritical string theory an interesting conjecture on mass shell conditions emerged. Keeping the standard picture that 2-point S-matrix elements vanish on shell, we related the on shell condition to the spectrum of zeros of the 2-point function of Liouville exponentials. The further check of both the s-continuation itself as well as the spectrum conjecture requires the knowledge of the higher ($N \geq 4$) correlation functions. Unfortunately, at present the

[6] More correctly $\beta = Q/2$ corresponds to the puncture operator which requires some additional care.

necessary integral formulas are not available. However, judging this unsatisfactory state of affairs in the s-continuation approach one should take into account that the canonical operator approach still faces problems with the explicit calculation of the 2 and 3-point function.

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