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Correlation Functions Along a Massless Flow

G. Delfino, G. Mussardo and P. Simonetti

International School for Advanced Studies,
and
Istituto Nazionale di Fisica Nucleare
34014 Trieste, Italy

Abstract

A non-perturbative method based on the Form Factor bootstrap approach is proposed for the analysis of correlation functions of 2-D massless integrable theories and applied to the massless flow between the Tricritical and the Critical Ising Models.

1. The 2-D integrable Quantum Field Theories (QFT) associated to the massless Renormalization Group (RG) flows between two different fixed points describe important physical situations and present several reasons of interest. Relevant examples are provided by the $O(3)$ non-linear sigma model with the $\theta = \pi$ topological term [1], by the massless flows between two consecutive conformal minimal models [2] and by the 2-D formulation of the Kondo problem [3]. One of the important features of these QFT (in addition to their integrability) is the absence of a mass gap, i.e. their correlation functions have a power law behaviour both in the ultraviolet and in the infrared regions (with different critical exponents, though), separated by a non trivial crossover in between. Usual perturbative methods may fail to capture the physical content of this kind of theories and it is therefore useful to develop some alternative approach to their analysis. Such an approach does exist for the massive QFT, where the powerful method of the Form Factors (FF) [4] allows us to enlighten their dynamics and provides, in particular, fast convergent series for the correlation functions [4, 5, 6]. In the light of these achievements, it is an important question to see whether or not the FF approach can be also successfully applied to the massless QFT. There are, however, several aspects that need a careful investigation, in particular the analytic structure of the FF and the convergent properties of the spectral representation series. The purpose of this letter is to study the simplest massless non-scaling invariant theory associated to the flow from the Tricritical Ising Model (TIM) to the Critical Ising Model. A suitable extension of this approach may be useful to investigate a concrete physical problem, as for instance the computation of correlators of the Kondo problem along the flow between the two fixed points.

2. The massless scattering theory has been developed in [1, 7, 8]. In a massless 2-D theory, the excitations consist in a set of left (L) and right (R) movers. These are the lowest energy states which propagate all along the RG flow and constitute the relevant degrees of freedom of the problem. Higher massive states are regarded as decoupled, their influence being eventually seen in the analytic structure of the physical amplitudes of the massless particles. The simplest way of approaching the massless integrable scattering theory is probably to think of it as a particular limit of a factorizable massive scattering theory. To avoid inessential complications, let us consider a massive integrable model whose spectrum consists of a single self-conjugate particle A of mass m which, in a suitable limit, gives rise to a massless theory with right-mover and left-mover particles. A technical important point is the different role played by the rapidity variable in the massive and in the massless theories. In the massive theory, the dispersion relation may be written in terms of the rapidity as

$$p^0 = m \cosh \theta \quad , \quad p^1 = m \sinh \theta \ . \tag{1}$$

The elastic two-particle S-matrix in the massive theories only depends on the Mandelstam variable s, given by $s = (p_1^\mu(\theta_1) + p_2^\mu(\theta_2))^2 = 2m^2(1 + \cosh\theta)$, where $\theta \equiv \theta_1 - \theta_2$. Use of the rapidity variable substantially simplifies the analytic structure of the amplitude by absorbing the threshold branch cut singularities in the original variable s. The physical sheet of the s-plane can be mapped into the strip $0 \leq \text{Im}\ \theta \leq \pi$ of the θ-plane in such a way that the two-particle S-matrix becomes a meromorphic function of the rapidity difference. In a massless theory, there are two possible scattering configurations. The first of them consists in the RL scattering, the second in the RR (LL) scattering. In the RL case, the Mandelstam variable s is the only independent relativistic invariant quantity. Since the mass gap is zero, the unitarity and crossing cuts of the S-matrix in the s-plane join at the origin, so that the Riemann surface splits into two distinct parts: the "upper" ("lower") part consists in the half of the physical (unphysical) sheet with $Im\, s > 0$ and in the half of the unphysical (physical) sheet with $Im\, s < 0$. Some care is required to introduce a parameterization in terms of rapidities since we need to send m to zero in eq. (1) but keeping the energy and the momentum finite. For this aim, it is sufficient to replace θ by $\beta \pm \beta_0/2$ and to take the limits $m \to 0, \beta_0 \to +\infty$ in such a way that the mass parameter $M \equiv m e^{\beta_0/2}$ remains finite. According to the sign in front of β_0, we have

$$\begin{aligned} p^0 &= p^1 = \frac{M}{2} e^{\beta} \qquad \text{for right-movers} \;, \\ p^0 &= -p^1 = \frac{M}{2} e^{-\beta} \qquad \text{for left-movers} \;. \end{aligned} \tag{2}$$

In short, it is useful to regard a right-mover $A_R(\beta)$ or a left-mover $A_L(\beta)$ as formally defined by the limits $A_{R,L}(\beta) = \lim_{\beta_0 \to +\infty} A(\beta \pm \beta_0/2)$. The Mandelstam variable s is now given by $s = M^2 e^{\beta_1 - \beta_2}$. Due to the aforementioned splitting of the s-plane, in the massless case we need two meromorphic functions of the rapidity difference in order to represent a function defined on the two sheets of the Riemann surface. Let $S_{RL}(\beta)$ $(\tilde{S}_{RL}(\beta))$ be the values of the scattering amplitude on the "upper" ("lower") sheet above defined. $S_{RL}(\beta)$ $(\tilde{S}_{RL}(\beta))$ is obtained from the value $S(\theta)$ $(S(-\theta))$ of the massive amplitude as the limit $A(\theta_1) \to A_R(\beta_1)$ and $A(\theta_2) \to A_L(\beta_2)$, i.e.

$$S_{RL}(\beta) = \lim_{\beta_0 \to +\infty} S(\beta + \beta_0) \;, \quad \tilde{S}_{RL}(\beta) = \lim_{\beta_0 \to +\infty} S(-\beta - \beta_0) \;.$$

The unitarity and crossing relations can be now expressed as $S_{RL}(\beta)\, \tilde{S}_{RL}(\beta) = 1$ and $S_{RL}(\beta) = \tilde{S}_{RL}(\beta + i\pi)$. Let us now turn our attention to the RR (LL) scattering. Since in these channels the variable s is identically zero, all the usual analyticity arguments of the S-matrix theory cannot apply, a circumstance which clearly reflects the lacking of intuitive understanding of the massless scattering process in $(1+1)$ dimensions. In spite of this fact, the amplitudes S_{RR} and S_{LL} can be formally defined in terms of the massless limit of the massive theory. However, the rapidity shifts in the massless limit cancel in these cases and therefore we simply have $S_{RR}(\beta) = S_{LL}(\beta) = S(\beta)$. In view of this identity, the amplitudes S_{RR} and S_{LL} are expected to fulfill (in the rapidity variable) the equations valid for the massive theory. As in the massive case, it is useful to encode all the scattering information of the massless theories into the Faddev-Zamolodchikov algebra

$$A_{\alpha_1}(\beta_1)A_{\alpha_2}(\beta_2) = S_{\alpha_1\alpha_2}(\beta_1-\beta_2)A_{\alpha_2}(\beta_2)A_{\alpha_1}(\beta_1) \ ,$$

where $\alpha_i = R, L$, $A_R(\beta)$ and $A_L(\beta)$ are the creation operators for the R and L movers respectively and $S_{LR}(\beta_1-\beta_2) = \tilde{S}_{RL}(\beta_2-\beta_1)$.

3. Massless form factors are defined to be the matrix elements of local operators between asymptotic states. To determine their analytic properties, let us firstly consider the matrix element of a scalar operator $O(x)$ between the vacuum and the two-particle state $\mid A_R(\beta_1)A_L(\beta_2) >$. Let $F_{RL}(\beta)$ and $\tilde{F}_{RL}(\beta)$ be the functions which, for real values of $\beta = \beta_1 - \beta_2$, take values on the upper and lower edges of the unitarity cut, respectively. They are related each to the other through the equations

$$\begin{aligned} F_{RL}(\beta) &= S_{RL}(\beta)\tilde{F}_{RL}(\beta) \ ; \\ F_{RL}(\beta+i\pi) &= \tilde{F}_{RL}(\beta-i\pi) \ . \end{aligned} \tag{3}$$

Using now $F_{LR}(\beta) = \tilde{F}_{RL}(-\beta)$, and the fact that $F_{RR}(\beta)$ and $F_{LL}(\beta)$ share the same properties of the FF of the massive theory, the monodromy equations for the massless two-particle FF may be written in a compact form as

$$\begin{aligned} F_{\alpha_1\alpha_2}(\beta) &= S_{\alpha_1\alpha_2}(\beta)F_{\alpha_2\alpha_1}(-\beta) \ ; \\ F_{\alpha_1\alpha_2}(\beta+2\pi i) &= F_{\alpha_2\alpha_1}(-\beta) \ , \end{aligned} \tag{4}$$

where $\alpha_i = R, L$. These equations are easily generalized to the n-particle FF,

$$\begin{aligned} & F_{\alpha_1\ldots\alpha_i\alpha_{i+1}\ldots\alpha_n}(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) = \\ & \quad = S_{\alpha_i\alpha_{i+1}}(\beta_i-\beta_{i+1})F_{\alpha_1\ldots\alpha_{i+1}\alpha_i\ldots\alpha_n}(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n) \ ; \\ & F_{\alpha_1\alpha_2\ldots\alpha_n}(\beta_1+2\pi i,\beta_2,\ldots,\beta_n) = F_{\alpha_2\ldots\alpha_n\alpha_1}(\beta_2,\ldots,\beta_n,\beta_1) \ , \end{aligned} \tag{5}$$

where $F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n) = < 0|O(0)|A_{\alpha_1}(\beta_1),\ldots,A_{\alpha_n}(\beta_n) >$ are meromorphic functions of rapidities defined in the strips $0 \leq \text{Im}\ \beta_i < 2\pi$. In the massive case, the form factors present simple pole singularities associated either to the bound states in the scattering amplitudes or to the particle-antiparticle annihilation processes [4]. However, stable bound states are usually forbidden in massless theories due to the absence of thresholds. This leads to exclude the presence of the first kind of singularities in the RL sub-channels, which are the only ones to which standard S-matrix theory can be applied. On the other hand, the RR and LL scattering amplitudes formally behave as in the massive case. Hence, it is natural to assume that, whenever the amplitude $S_{\alpha\alpha}(\beta)$ has a simple pole for $\beta = iu$ ($u \in (0,\pi)$) with residue g, this induces a corresponding equation for the FF

$$i\, res_{\beta_{n+1}-\beta_{n+2}=iu} F_{\alpha_1\ldots\alpha_n\alpha\alpha}(\beta_1,\ldots,\beta_n,\beta_{n+1},\beta_{n+2}) = g F_{\alpha_1\ldots\alpha_n\alpha}(\beta_1,\ldots,\beta_n,\beta_{n+1}-iu) \;.$$

With reference to the annihilation poles, they may only occur in the RR or LL sub-channels. The residue equation in this case reads

$$-\, i\, res_{\beta'=\beta+i\pi} F_{\alpha\alpha\alpha_1\ldots\alpha_n}(\beta',\beta,\beta_1,\ldots,\beta_n) = \left(1 - \prod_{i=1}^{n} S_{\alpha_i\alpha}(\beta_i-\beta)\right) F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n) \;. \tag{6}$$

The invariance of the massless theory under the spatial inversion transformation will provide the additional relation $F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n) = F_{\mathcal{P}[\alpha_n]\ldots\mathcal{P}[\alpha_1]}(-\beta_n,\ldots,-\beta_1)$,, where $\mathcal{P}[R]=L$, $\mathcal{P}[L]=R$.

4. A perturbation of the TIM by its sub-leading energy operator ε' of conformal dimensions $(3/5,3/5)$ induces a massless flow that ends to the Critical Ising Model [2, 10]. This flow takes place along the self-dual line of the phase diagram analysed in [9, 10] and a low energy effective Lagrangian of the corresponding QFT is given by

$$\mathcal{L}_{\text{eff}} = \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} - \frac{4}{M^2}(\psi\partial\psi)(\bar{\psi}\bar{\partial}\bar{\psi}) + \ldots \;. \tag{7}$$

This Lagrangian describes the critical Ising model (free massless Majorana fermion) perturbed by the irrelevant operator $(\psi\partial\psi)(\bar{\psi}\bar{\partial}\bar{\psi}) \sim T\bar{T}$, which is the lowest dimension nonderivative field invariant under Z_2 and duality transformations. Discrete symmetries of the theory are $\psi \to \psi, \bar{\psi} \to -\bar{\psi}$ and $\psi \to -\psi, \bar{\psi} \to -\bar{\psi}$ which can be identified as the duality transformation and the spin reversal, respectively. The factorized massless scattering theory of this flow has been proposed in [7]. The basic assumption is that the massless neutral fermions appearing in (7) are the only stable particles of the theory, i.e. the spectrum is the same as the infrared fixed point theory. Since S_{RR} and S_{LL} cannot vary along the flow, they are given by the commutation relations of the Ising fermionic fields, i.e. $S_{RR}(\beta) = S_{LL}(\beta) = -1$, whereas for S_{RL} we have $S_{RL}(\beta) = \tanh\left(\frac{\beta}{2} - \frac{i\pi}{4}\right)$. Hence, on the "upper" sheet of the s-plane we have $S(s) = \frac{s-iM^2}{s+iM^2}$. Using the unitarity equation, this expression implies that, while the physical sheet is free of poles, the unphysical one contains two poles at $s = \pm iM^2$ which can be interpreted as resonances. Let us turn now to the computation of the FF[1] of a local operator $\mathcal{O}$. They can be parameterized

[1]In the sequel, $F_{r,l}(\beta_1,..,\beta_r;\beta'_1,..,\beta'_l) =< 0|O(0)|A_R(\beta_1)\ldots A_R(\beta_r)A_L(\beta'_1)\ldots A_L(\beta'_l)>$. When $\beta_1 > \beta_2 > \ldots > \beta_r$ and $\beta'_1 > \beta'_2 > \ldots > \beta'_l$, these functions coincide with the physical matrix elements computed on the *in*-asymptotic states.

as

$$F_{r,l}(\beta_1,\beta_2,\ldots,\beta_r;\beta'_1,\beta'_2,\ldots,\beta'_l) \;=\; H_{r,l}\, Q_{r,l}(x_1,x_2,\ldots,x_r;y_1,y_2,\ldots,y_l)\times$$
$$\times\prod_{1\le i<j\le r}\frac{f_{RR}(\beta_i-\beta_j)}{x_i+x_j}\prod_{i=1}^{r}\prod_{j=1}^{l} f_{RL}(\beta_i-\beta'_j)\prod_{1\le i<j\le l}\frac{f_{LL}(\beta'_i-\beta'_j)}{y_i+y_j}\;, \qquad (8)$$

where $x_i \equiv e^{\beta_i}$, $y_i \equiv e^{-\beta'_i}$ and $H_{r,l}$ are normalization constants. The auxiliary functions f_{RR}, f_{LL} and f_{RL} completely take into account the monodromy properties of the FF (8). They are the minimal solutions of the equations $f_{\alpha_1\alpha_2}(\beta) \;=\; S_{\alpha_1\alpha_2}(\beta) f_{\alpha_1\alpha_2}(\beta+2\pi i)$, with neither poles nor zeroes in the strip $0<\mathrm{Im}\beta<2\pi$. Explicitly, $f_{RR}(\beta) \;=\; f_{LL}(\beta) \;=\; \sinh\frac{\beta}{2}$, and

$$f_{RL}(\beta) \;=\; \exp\left(\frac{\beta}{4}-\int_0^\infty \frac{dt}{t}\frac{\sin^2\left(\frac{(i\pi-\beta)t}{2\pi}\right)}{\sinh t\,\cosh\frac{t}{2}}\right)\;.$$

The function $f_{RL}(\beta)$ also satisfies the equation $f_{RL}(\beta\pm i\pi) f_{RL}(\beta) \;=\; \frac{i\gamma}{1\pm i e^{-\beta}}$, where $\gamma \;=\; \sqrt{2}e^{2G/\pi}$, G being the Catalan constant. Since S_{RR} and S_{LL} in this theory are free of poles, the FF only present kinematical poles which are explicitly inserted in (8) through the factors x_i+x_j and y_i+y_j in the denominator. With the requirement that the FF are power bounded in the momentum variables, $Q_{r,l}$ have to be rational functions, separately symmetric in the $\{x_i\}$ and $\{y_i\}$ with at most poles located at $x_i \;=\; 0$ or $y_i \;=\; 0$. Inserting the parameterization (8) into the residue equations (6) and using $H_{r,l} = -i^{-r}2^{-2r-1}\gamma^l H_{r+2,l} = -i^{-l}2^{-2l-1}\gamma^r H_{r,l+2}$, the recursive equations for $Q_{r,l}$ are given by

$$Q_{r+2,l}(-x,x,x_1,\ldots,x_r;y_1,\ldots,y_l) = x^{r-l+1}\frac{\rho_r}{\lambda_l}{\sum_{k=0}^{l}}'(-ix)^k\lambda_k Q_{r,l}(x_1,\ldots,x_r;y_1,\ldots,y_l) \quad (9)$$
$$Q_{r,l+2}(x_1,\ldots,x_r;y_1,\ldots,y_l,y,-y) = y^{l-r+1}\frac{\lambda_l}{\rho_r}{\sum_{k=0}^{r}}'(-iy)^k\rho_k Q_{r,l}(x_1,\ldots,x_r;y_1,\ldots,y_l)\;,$$

where the primed sums run over odd indices if $(r+l)$ is even and vice versa, and ρ_k (λ_l) are the elementary symmetric polynomials in the variables $\{x_i\}$ ($\{y_i\}$).

5. Eqs. (9) are quite general since they were obtained without reference to any particular operator. Here we will restrict our attention to the FF of some operators of particular physical relevance, namely the trace of the stress-energy tensor $\Theta(x) = T_\mu^\mu(x)$, the order and the disorder operators $\Phi(x)$ and $\tilde{\Phi}(x)$ of conformal dimensions $(3/80,3/80)$ in the ultraviolet limit and $(1/16,1/16)$ in the infrared one. Since these fields are spinless, under a Lorentz transformation they satisfy $Q_{r,l}(\{e^\Lambda x_i\};\{e^{-\Lambda}y_i\}) = e^{\left(\frac{r(r-1)}{2}-\frac{l(l-1)}{2}\right)\Lambda} Q_{r,l}(\{x_i\};\{y_i\})$. Selection rules for the FF are obtained by assigning the transformation properties of the massless particles under the discrete symmetries of the theory. According to the invariances of the fermionic Lagrangian (7), we assign odd Z_2-parity to both right and

left-movers, and even (odd) parity to right (left) movers under duality transformation. The trace of the energy-momentum tensor is expressed in terms of the ultraviolet perturbing field and the conjugate coupling constant λ by the relation $\Theta(x) = \frac{8}{5}\pi\lambda\varepsilon'(x)$. Since the subenergy ε' is even under both spin reversal and duality transformation, Θ will have nonvanishing form factors $F_{r,l}$ only for even r and l, starting from $r = l = 2$ (the vacuum expectation value $F_{0,0}$ must be identically zero since it vanishes in the infrared limit). The conservation of the energy-momentum tensor implies the factorization $Q_{r,l}(\{x_i\};\{y_i\}) = \rho_1\lambda_1 T_{r,l}(\{x_i\};\{y_i\})$. The leading infrared behaviour of $F_{2,2}$ is easily computed by using the Lagrangian (7)

$$F_{2,2}(\beta_1,\beta_2;\beta_1',\beta_2') \to -4\pi M^2 \sinh\frac{\beta_1-\beta_2}{2}\sinh\frac{\beta_1'-\beta_2'}{2}\, e^{\beta_1+\beta_2-\beta_1'-\beta_2'} , \tag{10}$$

and, with this extra piece of information, we can completely fix its exact expression

$$F_{2,2}(\beta_1,\beta_2;\beta_1',\beta_2') = \frac{4\pi M^2}{\gamma^2}\sinh\frac{\beta_1-\beta_2}{2}\prod_{i,j=1,2} f_{RL}(\beta_i-\beta_j')\sinh\frac{\beta_1'-\beta_2'}{2} . \tag{11}$$

The recursive equations (9) are then iteratively solved by using (11) as initial condition. Using $H_{2n,2m} = \pi M^2 i^{n(n+1)+m(m+1)} 2^{2(n^2+m^2)-n-m}\gamma^{-2nm}$, the first right chains are determined to be

$$\begin{aligned} T_{2n,2}(\{x_i\};\{y_i\}) &= (i)^{n^2-1}\left(\rho_{2n}\frac{\lambda_1}{\lambda_2}\right)^{n-1} , \\ T_{2n,4}(\{x_i\};\{y_i\}) &= (i)^{n^2-2n}\left(\frac{\rho_{2n}}{\lambda_4}\right)^{n-2}\sum_{k=0}^{n-1}\rho_{2k+1}\lambda_1^{n-1-k}\lambda_3^k , \\ &\vdots \end{aligned} \tag{12}$$

Using now the right-left symmetry relation $T_{r,l}(\{x_i\};\{y_i\}) = T_{l,r}(\{y_i\};\{x_i\})$, one can immediately obtain the solution for the corresponding left chains. Let us now consider the magnetization operator. The invariance of the theory under spin reversal implies that Φ has nonvanishing FF only on an odd number of particles whereas $\tilde{\Phi}$ only on an even one. Taking $F_{1,0} = F_{0,1} = 1$ as initial conditions for the recursive equations for Φ and $F_{0,0} = F_{1,1} = 1$ for $\tilde{\Phi}$, we find[2]

$$\begin{aligned} Q_{r,0} &= \rho_r^{(r-1)/2} , \\ Q_{r,1} &= \frac{\rho_r^{r/2-1}}{\lambda_1^{r/2}} , \\ Q_{r,2} &= \rho_r^{(r-3)/2}\sum_{k=0}^{r}{}'\rho_k\lambda_2^{(k-r+1)/2} , \end{aligned} \tag{13}$$

[2]As explained in the first reference of [5], the recursive equations for the FF of the disorder operator $\tilde{\Phi}$ are given by (6) but with a plus sign in front of the product of S-matrices.

$$Q_{r,3} = \frac{\rho_r^{(r/2-2)}}{\lambda_3^{r/2-1}} \sum_{k=0}^{r}{}' \rho_k \lambda_2^{k/2} \ ,$$
$$\vdots$$

where the prime denotes a sum only over even indices and r should be chosen such that $r+l$ is odd for Φ and even for $\tilde{\Phi}$.

6. The spectral representation of the two-point functions is given by

$$< O(x)O(0) >= \sum_{r,l=0}^{\infty} \frac{1}{r!\, l!} \int_{-\infty}^{+\infty} \frac{d\beta_1 \ldots d\beta_r d\beta'_1 \ldots d\beta'_l}{(2\pi)^{r+l}} |F_{r,l}(\beta_1, \ldots, \beta_r; \beta'_1 \ldots \beta'_l)|^2$$
$$\times \exp\left[-\frac{Mr}{2} \left(\sum_{j=1}^{r} e^{\beta_j} + \sum_{j=1}^{l} e^{-\beta'_j} \right) \right] \ , \tag{14}$$

where we used euclidean invariance to set $x = (ir, 0)$. This expression clearly shows that, contrarily to the massive case, the convergence in the infrared limit $\beta_i \to -\infty, \beta'_i \to +\infty$ is no longer guaranteed by the exponential factor inside the integrals and completely relies on the behaviour of the form factors $F_{r,l}$ in this limit. Let's first analyze the 2-point function of Θ. As explicitly shown by eq. (10), $F_{2,2}$ goes exponentially to zero in the infrared limit so that the 4-particle contribution of $G_\Theta(r) =< \Theta(r)\Theta(0) >$ is convergent. It is also easy to check that the terms with higher number of particles are convergent and sub-leading with respect to the 4-particle contribution. Then, plugging expression (10) into the integral (14), we find for $r \to \infty$, $G_\Theta(r) \simeq \frac{16}{\pi^2 M^4 r^8}$, which perfectly matches with the expected infrared power law behaviour. On the other hand, the behaviour of this function in the ultraviolet limit $r \to 0$ is fixed by the conformal OPE, i.e. $G_\Theta(r) \simeq M^4 \left(\frac{8}{5}\pi\alpha\right)^2 (Mr)^{-12/5}$, where $\alpha = 0.148695516$ [7]. A logarithmic plot of $G_\Theta(r)$ obtained by including the first two contributions in the spectral representation (14) is given in fig. 1. The slope of the curve interpolates between the two values -8 and $-12/5$ relative to the infrared and ultraviolet fixed points; the figure also shows a very fast ultraviolet convergent pattern, analogously to the massive cases. The fast convergent behaviour of the series is also confirmed by the c-theorem sum rule $\Delta c = \frac{3}{2}\int dr\, r^3 < \Theta(r)\Theta(0) >$ [11]. In fact, we obtain $\Delta c^{(4)} = 0.19600 \pm .00007$ only including the 4-particle contribution and $\Delta c^{(6)} = 0.1995 \pm .0004$ adding the 6-particle one, where the expected value is $\Delta c = 7/10 - 1/2 = 0.2$. The fast convergent pattern of the spectral series also occurs for the 2-point function of the energy operator ε, with conformal dimensions $(1/10, 1/10)$ in the ultraviolet limit and $(1/2, 1/2)$ in the infrared one, as shown in fig. 2. All its FF can be computed by the recursive equations (9) with the initial condition $F_{1,1}(\beta) = f_{RL}(\beta)$.

A more subtle situation occurs for the correlation functions of the magnetization operators. The leading infrared behaviour of the FF of Φ and $\tilde{\Phi}$ can be determined by specializing the FF bootstrap equations to the Ising critical point where $S_{RL} = -1$ and $f_{RL}(\beta-\beta') = e^{(\beta-\beta')/2}$. With the initial conditions $F_{1,0} = F_{0,1} = F_{0,0} = F_{1,1} = 1$, the limiting expressions of the FF are given by

$$F_{r,l}^{IR}(\beta_1,\ldots,\beta_r;\beta_1',\ldots,\beta_l') = \prod_{i<j} \tanh\frac{\beta_i-\beta_j}{2}\tanh\frac{\beta_i'-\beta_j'}{2}\;, \tag{15}$$

where $r+l$ is odd for Φ and even for $\tilde{\Phi}$. Expression (15) clearly shows that the integrals in the spectral representation of $G_{\Phi}(r) =< \Phi(r)\Phi(0) >$ and $G_{\tilde{\Phi}}(r) =< \tilde{\Phi}(r)\tilde{\Phi}(0) >$ are infrared divergent, i.e. these correlators require the resummation of the whole (suitably regularized) spectral series. Although, in general, it is an interesting open problem to develop mathematical tools that allows us to deal with such series, here we want to show that an exact resummation can be performed in the low energy limit so that the exact infrared conformal dimensions $\Delta_{\Phi} = \Delta_{\tilde{\Phi}} = 1/16$ can be extracted. In fact, using the FF (15) and exploiting their complete factorization in a left and a right part we can write

$$\tilde{G}^{IR}(t) \equiv G_{\Phi}^{IR}(t) + G_{\tilde{\Phi}}^{IR}(t) = \left[\sum_{r=1}^{\infty}\frac{1}{r!}\int_{-\Lambda}^{\infty}\prod_{i=1}^{r}\frac{d\beta_i}{2\pi}\left(\prod_{i<j}\tanh\frac{\beta_i-\beta_j}{2}\right)^2 e^{-t\sum_{i=1}^{r}e^{\beta_i}}\right]^2\;,$$

where $t \equiv Mr/2$ and Λ is an infrared cutoff. Shifting the rapidities, it can be expressed as

$$\tilde{G}^{IR}(t) \;=\; \left[\sum_{r=1}^{\infty}\frac{1}{r!}\int_{0}^{\infty}\prod_{i=1}^{r}\frac{d\beta_i}{2\pi}\left(\prod_{i<j}\tanh\frac{\beta_i-\beta_j}{2}\right)^2 e^{-te^{-\Lambda}\sum_{i=1}^{r}e^{\beta_i}}\right]^2\;. \tag{16}$$

As in [5], this can be considered as the square of the partition function of a classical gas of particles living on a semi-infinite line and subject to the pairwise interaction $V(\beta_i-\beta_j) = -\ln\left(\tanh\frac{\beta_i-\beta_j}{2}\right)^2$. The activity of such a gas is given by $z(\beta) = \frac{1}{2\pi}e^{-te^{\beta-\Lambda}}$. This function has a plateau $z_0 = 1/2\pi$ inside a box of length $L \sim \ln\frac{e^{\Lambda}}{t}$. Hence, removing the cutoff Λ is equivalent to take the thermodynamic limit $L\to\infty$ of a gas with a constant activity z_0. Standard relation between the partition function and the bulk-free energy per unit length $f(z_0)$ allows us to write $\tilde{G}^{IR}(t) \sim e^{2f(\frac{1}{2\pi})L} \sim \left(\frac{e^{\Lambda}}{t}\right)^{2f(\frac{1}{2\pi})}$. The bulk-free energy $f(z_0)$ can exactly be computed for generic values of z_0 [5], $f(z_0) = \frac{1}{2\pi}\arcsin(2\pi z_0) - \frac{1}{2\pi^2}\arcsin^2(2\pi z_0)$, so that $f(1/2\pi) = 1/8$, which coincides with the expected infrared anomalous dimension of the fields $\Phi(x)$ and $\tilde{\Phi}(x)$.

As explicitly shown in this paper, the FF of massless theories can be exactly computed and they constitute useful non-perturbative information of the RG flow. The above analysis also shows that there are sectors in the theory where the spectral series presents a fast rate of convergence in the entire domain of definition while for others the power law behaviours are reproduced by summing the whole series of FF. We expect that this pattern is also present in other massless QFT and therefore, to take full advantage of the non-perturbative knowledge of FF, it would be interesting to develop new resummation methods of the spectral series.

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Figure Captions

Figure 1 . Logarithmic plot of $G_\Theta(r)$ including the 4 and 6 particle contributions. Dashed line: leading OPE of the ultraviolet fixed point.

Figure 2 . Logarithmic plot of the 2-point function of the energy operator ε including the 2 and 4 particle contributions.