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Asymptotic factorisation of form factors in two-dimensional quantum field theory

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Abstract

It is shown that the scaling operators in the conformal limit of a two-dimensional field theory have massive form factors which obey a simple factorisation property in rapidity space. This has been used to identify such operators within the form factor bootstrap approach. A sum rule which yields the scaling dimension of such operators is also derived.

1. The solution of conformal field theories represented a crucial step in our understanding of two-dimensional quantum field theories [1, 2]. It not only amounts to a complete description of the fixed points of the renormalisation group (RG), but also provides the starting point for the study of physical systems away from criticality. For example, one can consider the theory defined by the action

$$\mathcal{A} = \mathcal{A}_{CFT} + g \int d^2x \, \varphi(x) \ , \tag{1}$$

as describing the perturbation of a conformal invariant theory by the operator $\varphi(x)$ of scaling dimension $2\Delta < 2$. Since the coupling constant g has physical dimension $m^{2-2\Delta}$, m being a mass scale, the theory (1) is no longer scale invariant; rather, it is associated to a RG trajectory flowing out of the original fixed point. In many cases an infinite number of integrals of motion survive in the perturbed theory and the resulting off-critical model is said to be integrable. A bootstrap procedure can then be applied, usually resulting in the determination of the exact particle spectrum and S-matrix of the theory [3, 4, 5]. As a consequence of integrability, the S-matrix turns out to be completely elastic and factorised. On the other hand, it is commonly believed that the knowledge of the S-matrix amounts to a complete solution of a quantum field theory. In particular, it should encode the information about the operator content of the model and should enable the computation of correlation functions. The method which has proved so far very effective in dealing with such an ambitious program is known as the *form factor bootstrap*. Form factors (FF) are matrix elements of local operators $\mathcal{O}(x)$ between asymptotic multiparticle states and will be denoted as [1]

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) = \langle 0 | \mathcal{O}(0) | A(\theta_1), \ldots, A(\theta_n) \rangle \ . \tag{2}$$

They are an interesting subject for theoretical investigation because their structure involves the particle description of the theory, an expression of the infrared dynamics, as well as the operator content, which is deeply related to the conformal structure of the ultraviolet fixed point. Moreover, if the FF are known, correlation functions can be written down in terms of the spectral sum

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(0) \rangle = \sum_n \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \ldots \frac{d\theta_n}{2\pi} F_n^{\mathcal{O}_1}(\theta_1, \ldots, \theta_n) \left[F_n^{\mathcal{O}_2}(\theta_1, \ldots, \theta_n) \right]^* e^{iP_n^\mu x_\mu} \ , \tag{3}$$

P_n^μ being the total energy-momentum of the n-particle intermediate state.

The determination of FF in integrable models proceeds through two basic steps. One first enforces the general constraints of analyticity, unitarity and crossing symmetry deriving from the standard S-matrix theory [6, 7]. Since no information about the specific

[1] In order to avoid inessential complication of the notation, we refer throughout this letter to a theory whose spectrum consists of a single species A of particles with mass m. We also adopt the standard parameterisation of the on mass shell momenta in terms of the rapidity variables θ_i: $p_i^\mu = (m \cosh\theta_i, m \sinh\theta_i)$.

nature of the operator $\mathcal{O}(x)$ is provided at this stage, the general solution of the resulting system of functional and residue equations (linear in the operator) must correspond to a complete description of the operator content of the theory. It has been shown for several models that the dimensionality of the linear space of solutions of these equations coincides with that predicted by the underlying conformal theory [8, 9, 10].

The second step consists in selecting out of the general solution the FF corresponding to particular operators, the scaling operators being the objects of main physical interest. Such operators provide the natural physical basis for the space of local operators of the theory. Nevertheless, it is by no means obvious how to identify this basis among the solutions of the FF equations. A first selection rule comes from the general constraint [12]

$$\lim_{|\theta_i|\to\infty} F_n^\Phi(\theta_1,\ldots,\theta_n) \leq const.\; e^{\Delta_\Phi|\theta_i|} \;, \tag{4}$$

relating the asymptotic behaviour of the matrix elements of a scaling operator $\Phi(x)$ to its scaling dimension $2\Delta_\Phi$ [2]. When the operator space splits into different sectors distinguished by some internal symmetry of the theory, the asymptotic bound (4) is typically sufficient to fix the FF solutions. Yet, the problem remains conceptually interesting in absence of internal symmetries. We now show that in the latter case the FF of the relevant ($\Delta_\Phi < 1$) scaling operators of the theory are characterised by the following asymptotic factorisation property

$$\lim_{\alpha\to+\infty} F_{r+l}^\Phi(\theta_1+\alpha,\ldots,\theta_r+\alpha,\theta_{r+1},\ldots,\theta_{r+l}) = \frac{1}{\langle\Phi\rangle} F_r^\Phi(\theta_1,\ldots,\theta_r)\, F_l^\Phi(\theta_{r+1},\ldots,\theta_{r+l}) \;. \tag{5}$$

Such a property had been noticed in the past to be fulfilled by some FF solutions in specific models [11].

Consider the case in which the action (1) describes a massive integrable model without internal symmetries and denote by $S(\theta)$ the two-particle scattering amplitude, θ being the rapidity difference between the colliding particles. It can be argued on general grounds that

$$\lim_{\theta\to\infty} S(\theta) = 1 \;, \tag{6}$$

and that this result implies that the FF of a relevant scaling operator $\Phi(x)$ actually tend to a constant (with respect to θ_i) in the limit (4), so that the limit in eq. (5) is surely well defined. Also, due to the absence of internal symmetries, the vev $\langle\Phi\rangle$ should be nonvanishing and can be written as

$$\langle\Phi\rangle = v_\Phi\, m^{2\Delta_\Phi} \;, \tag{7}$$

v_Φ being a dimensionless constant.

[2]We will refer to scalar operators.

The basic point to be realised is that the limit in the r.h.s. of eq. (5) is actually a massless limit into the ultraviolet conformal point. In general, such limit is obtained sending to zero the mass of each particle while sending to infinity its rapidity in order to keep the energy finite [13, 14]. The resulting theory consists of right and left-moving massless particles, A_R and A_L, whose Zamolodchikov-Faddeev operators may be formally obtained from those of the original massive particle A as

$$\begin{aligned} A_R(\theta) &= \lim_{\alpha\to+\infty} A(\theta+\alpha/2) \ , \\ A_L(\theta) &= \lim_{\alpha\to+\infty} A(\theta-\alpha/2) \ . \end{aligned} \tag{8}$$

The dispersion relations are

$$p^0 = p^1 = \frac{M}{2}e^{\theta} \qquad \text{for right-movers} \ , \tag{9}$$

$$p^0 = -p^1 = \frac{M}{2}e^{-\theta} \qquad \text{for left-movers} \ , \tag{10}$$

$M = me^{\alpha/2}$ being a finite parameter. The scattering amplitudes characterising the interaction of the masseless particles are readily obtained taking the limit of the massive amplitude: $S_{RR}(\theta) = S_{LL}(\theta) = S(\theta)$ and $S_{RL}(\theta) = \lim_{\alpha\to+\infty} S(\theta+\alpha)$. In the present context the latter limit for S_{RL} gives a constant phase whose rapidity independence ensures the decoupling of the right and left sectors and the scale invariance of the massless theory [3]. In absence of internal symmetries, eq. (6) implies $S_{RL} = 1$. In summary, the massless scattering theory defined through the above limiting procedure provides a particle description of the conformal point to which the original massive model flows in the ultraviolet limit.

Massless form factors can be introduced along the same lines [15]. In order to prevent a trivial vanishing of the matrix elements (2) when $m \to 0$, it is convenient to refer to the rescaled operator

$$\hat{\Phi} = \frac{\Phi}{m^{2\Delta_\Phi}} \ . \tag{11}$$

Since the matrix elements of a scalar operator depend on rapidity differences only, we can write

$$\begin{aligned} \lim_{\alpha\to+\infty} F^{\hat{\Phi}}_{r+l}(\theta_1+\alpha,\ldots,\theta_r+\alpha,\theta_{r+1},\ldots,\theta_{r+l}) = \\ \lim_{\alpha\to+\infty} F^{\hat{\Phi}}_{r+l}(\theta_1+\alpha/2,\ldots,\theta_r+\alpha/2,\theta_{r+1}-\alpha/2,\ldots,\theta_{r+l}-\alpha/2) = \\ F^{\hat{\Phi}}_{r,l}(\theta_1,\ldots,\theta_r|\theta_{r+1},\ldots,\theta_{r+l}) \ , \end{aligned} \tag{12}$$

[3]When the right-left amplitude is rapidity dependent the theory describes a massless flow between two fixed points.

where

$$F^{\hat{\Phi}}_{r,l}(\theta_1,\ldots,\theta_r|\theta'_1,\ldots,\theta'_l) \equiv \langle 0|\hat{\Phi}(0)|A_R(\theta_1),\ldots,A_R(\theta_r),A_L(\theta'_1),\ldots,A_L(\theta'_l)\rangle \ , \qquad (13)$$

denotes a massless form factor. On the other hand, as a consequence of the decoupling of the right and left sectors at the conformal point, the massless form factor of a scaling operator can be written in the factorised form

$$F^{\hat{\Phi}}_{r,l}(\theta_1,\ldots,\theta_r|\theta'_1,\ldots,\theta'_l) = \mathcal{R}^{\hat{\Phi}}_r(\theta_1,\ldots,\theta_r)\,\mathcal{L}^{\hat{\Phi}}_l(\theta'_1,\ldots,\theta'_l) \ . \qquad (14)$$

Consider now a form factor in the massive theory and take its massless limit in which all the particles become right-movers. Due to Lorentz invariance, the rapidity shifts needed for the limit can be completely rescaled out and we have

$$F^{\hat{\Phi}}_r(\theta_1,\ldots,\theta_r) = F^{\hat{\Phi}}_{r,0}(\theta_1,\ldots,\theta_r|) = \mathcal{R}^{\hat{\Phi}}_r(\theta_1,\ldots,\theta_r)\,\mathcal{L}^{\hat{\Phi}}_0 \ . \qquad (15)$$

A similar equation holds when all the particles become left-movers. Since $\langle\hat{\Phi}\rangle = v_\Phi$ along the whole flow, at the critical point we have [4]

$$\mathcal{R}^{\hat{\Phi}}_0 = \mathcal{L}^{\hat{\Phi}}_0 = v_\Phi^{1/2} \ . \qquad (16)$$

Using eqs. (14), (15) and (16), eq. (12) can be rewritten as

$$\lim_{\alpha\to+\infty} F^{\hat{\Phi}}_{r+l}(\theta_1+\alpha,\ldots,\theta_r+\alpha,\theta_{r+1},\ldots,\theta_{r+l}) = \frac{1}{v_\Phi}F^{\hat{\Phi}}_r(\theta_1,\ldots,\theta_r)\,F^{\hat{\Phi}}_l(\theta_{r+1},\ldots,\theta_{r+l}) \ , \quad (17)$$

in terms of massive FF only. Going back from $\hat{\Phi}$ to Φ, eq. (5) follows.

It is worth noticing that eq. (5) can be used to determine the vev $\langle\Phi\rangle$ from the knowledge of the multiparticle FF [5]. Taking into account that only the vev of the perturbing operator can be obtained by other means (thermodynamic Bethe ansatz), this must be regarded as a remarkable circumstance.

The crucial conditions entering the above derivation of the factorisation property (5) are the nonvanishing of all the FF of the considered scaling operator (including the vev) and their constant asymptotic behaviour. These features, which are guaranteed in absence of internal symmetries, are often shared by some operators in theories with symmetry. The FF of these operators then factorise asymptotically according to eq. (5). As an example, we mention the exponential operators in lagrangian theories like the Sinh-Gordon model [11], or the linear combinations $\sigma\pm\mu$ of the order and disorder parameters in the thermal

[4] Remember that $\hat{\Phi}$ is a rescaled (dimensionless) operator; the vev (7) of the operator Φ vanishes in the massless limit, as expected in conformal field theory.

[5] More precisely, what one can fix is the ratio $\frac{\langle\Phi\rangle}{F_1^\Phi}$ which, being independent from the normalisation of the operator, is an universal number.

Ising model [17, 18]. It seems reasonable to expect that a suitable generalisation of eq. (5) should apply to any scaling operator even in presence of internal symmetries, but no general pattern has been identified so far.

2. Once the FF solutions for the relevant scaling operators have been selected using the factorisation property (5), it remains to be established which scaling operator each solution corresponds to. For this purpose, it is important to be able to recover the scaling dimension $2\Delta_\Phi$ from the FF solution. In principle this can be extracted evaluating the short distance behaviour of the spectral series (3) for the correlator $\langle\Phi(x)\Phi(0)\rangle$. In practice, better quantitative results can be obtained exploiting the properties of the stress tensor as the generator of dilatations. In order to illustrate this point it is useful to start with some perturbative consideration.

The operator space of the conformal point and that of the perturbed theory have the same basic structure. In particular, a scaling operator $\Phi(x)$ in the off-critical theory can be associated to a conformal operator $\tilde{\Phi}(x)$ of scaling dimension $2\Delta_\Phi$. When doing that in a perturbative framework, however, renormalisation effects induced by ultraviolet divergences must be taken into account [16]. In fact, denote by X a generic product of operators and consider the usual perturbative expansion of the correlator

$$\langle X\,\Phi(0)\rangle = \langle X\,\tilde{\Phi}(0)\rangle_{CFT} + g\int_{\epsilon<|x|<R} d^2x\,\langle X\,\tilde{\Phi}(0)\,\tilde{\varphi}(x)\rangle_{CFT} + \mathcal{O}(g^2)\;, \qquad (18)$$

where the correlators in the right hand side are computed in the conformal theory, and ϵ and R regularise the ultraviolet and infrared divergences, respectively. The integral in eq. (18) is UV divergent only if the conformal OPE

$$\tilde{\varphi}(x)\,\tilde{\Phi}(0) = \sum_k C^k_{\varphi\Phi}\,|x|^{2(\Delta_k-\Delta_\Phi-\Delta)}\,\tilde{A}_k(0) \qquad (19)$$

contains operators $\tilde{A}_k$ with scaling dimension $2\Delta_k$ such that

$$\gamma_k \equiv \Delta_k - \Delta_\Phi - \Delta + 1 \leq 0\;\;. \qquad (20)$$

In this case we obtain a first order UV finite correlator $\langle X\,\Phi(0)\rangle$ by defining the renormalised operator as [6]

$$\Phi = \tilde{\Phi} + g\sum_k b_k\epsilon^{2\gamma_k}\,\tilde{A}_k + \mathcal{O}(g^2)\;, \qquad (21)$$

where

$$b_k = -\frac{\pi C^k_{\varphi\Phi}}{\gamma_k}\;, \qquad (22)$$

[6] Eq. (21) must be suitably modified in the case of logarithmic divergences ($\gamma_k = 0$).

and the sum runs only over the operators $\tilde{A}_k$ satisfying the condition (20). This implies that, in general, renormalisation mixes the original operator $\tilde{\Phi}$ with a finite number of operators of less scaling dimension.

Consider now the euclidean correlators of the operator Φ with the components of the stress energy tensor $T = \frac{1}{4}(T_{11} - T_{22} - 2iT_{12})$ and $\Theta = T_{11} + T_{22}$

$$\langle T(z,\bar{z})\Phi(0)\rangle = \frac{F(z\bar{z})}{z^2} \; ,$$
$$\langle \Theta(z,\bar{z})\Phi(0)\rangle_c = \frac{G(z\bar{z})}{z\bar{z}} \; , \qquad (23)$$

where z and $\bar{z}$ are complex coordinates. Conservation of the stress energy tensor espressed by the equation

$$\bar{\partial}T + \frac{1}{4}\partial\Theta = 0 \; , \qquad (24)$$

leads to the differential relation

$$\dot{D} = \frac{1}{4}G \; , \qquad (25)$$

where $D \equiv F + \frac{1}{4}G$ and the dot stays for $z\bar{z}\frac{d}{dz\bar{z}}$. Since the trace of the stress tensor is related to the perturbing field as

$$\Theta(x) = 4\pi g\,(1-\Delta)\,\varphi(x) \; , \qquad (26)$$

the short distance behaviour of the function G is determined by the OPE (19)

$$G(x) \simeq 2\pi g(2-2\Delta)C^0_{\varphi\Phi}\,|x|^{2\gamma_0}\,\langle A_0\rangle \; , \qquad x \to 0 \qquad (27)$$

where we denoted by A_0 the most relevant operator appearing in (19). We can now distinguish two basic cases:

a) $\gamma_0 > 0$. In this case G vanishes as $x \to 0$ (conformal limit) and we conclude that the function D is stationary and coincides with F at the fixed point. Since the operator $\Phi(x)$ does not mix under renormalisation, its conformal OPE with $T(x)$ can be safely used in a neighbourhood of the fixed point

$$F(x) \simeq \Delta_\Phi\langle\Phi\rangle \; , \qquad x \to 0 \; . \qquad (28)$$

If the theory described by the action (1) corresponds to a massless flow between two fixed points, a similar analysis can be repeated in the neighbourhood of the infrared fixed point. Then, integrating eq. (25) over all distance scales, one finds

$$\Delta_\Phi^{UV} - \Delta_\Phi^{IR} = -\frac{1}{4\pi\langle\Phi\rangle}\int d^2x\,\langle\Theta(x)\,\Phi(0)\rangle_c \; . \qquad (29)$$

In a massive theory $\Delta_\Phi^{IR} = 0$. We will illustrate in a moment with few examples the effectiveness of this sum rule within the FF approach.

b) $\gamma_0 \le 0$. The function G no longer vanishes at $x=0$ and an attempt to use the sum rule (29) would be frustrated by the divergence of the integral. This is a consequence of the fact that the operator $\Phi(x)$ now mixes under renormalisation (see eq. (21)) so that the function F no longer behaves as in (28) at short distances. The correct behaviour can instead be obtained integrating eq. (25). One finds

$$F(x) \simeq \pi g\,(1-\Delta) C^0_{\varphi\Phi} \frac{1-\gamma_0}{\gamma_0}\,|x|^{2\gamma_0}\langle A_0\rangle\ , \qquad \gamma_0 < 0 \tag{30}$$

$$F(x) \simeq 2\pi g\,(1-\Delta) C^0_{\varphi\Phi} \log|x| \langle A_0\rangle\ , \qquad \gamma_0 = 0\ . \tag{31}$$

Actually, eq. (29) is nothing but the Ward identity expressing the fact that Θ is the generator of scale transformations, and as such a suitable generalisation is valid in arbitrary dimension. This may be derived simply by modifying the arguments presented, for example, in ref. [19]. Consider the effect of making a nonuniform infinitesimal RG transformation, which generates the coordinate change $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$, where $\delta x^\mu = \epsilon x^\mu$, corresponding to uniform scale transformation, in the region between two hyperspheres, of radii R_1 and R_2, while $\delta x^\mu = 0$ in the regions $|x| < R_1' < R_1$ and $|x| > R_2' > R_2$. Within the thin shells $R_1' < |x| < R_1$ and $R_2 < |x| < R_2'$, δx^μ is chosen to be an arbitrary differentiable function which matches smoothly onto the other regions. From the definition of the stress tensor, the change in the hamiltonian (action) is

$$\delta S = -(1/S_d)\int \partial^\mu \delta x^\nu T_{\mu\nu} d^dx, \tag{32}$$

where S_d, the area of a d-dimensional sphere, is conventionally included. This gives a contribution $\int_{R_1<|x|<R_2} \Theta\, d^dx$ from this region. Within the two shells, integrate by parts: the bulk term vanishes by the conservation $\partial^\mu T_{\mu\nu} = 0$ of the stress tensor, leaving two surface integrals on $|x| = R_1$ and R_2 of the form $\int x^\mu T_{\mu\nu} dS^\nu$. Now consider evaluating $\langle\Phi(0)\rangle$ with respect to this modified hamiltonian. Since no change has been made for $|x| < R_1$, this quantity is in fact invariant. Thus the first order changes from the bulk and boundary terms should cancel. Expanding these out from the hamiltonian to $O(\epsilon)$, the bulk term gives

$$-\frac{1}{S_d}\int_{R_1<|x|<R_2} \langle\Theta(x)\Phi(0)\rangle_c\, d^dx, \tag{33}$$

while the boundary terms involve integrals over $\langle T_{rr}(x)\Phi(0)\rangle$. In the limits $R_1 \to 0$ and $R_2 \to \infty$ these may be evaluated using the operator product expansion formulae [20] appropriate to the UV or the IR conformal theories respectively:

$$T_{\mu\nu}(x)\Phi(0) = \frac{dx_\Phi}{d-1}\frac{x_\mu x_\nu - (1/d)x^2 g_{\mu\nu}}{|x|^{d+2}}\Phi(0) + \cdots, \tag{34}$$

where x_Φ is the scaling dimension of Φ. This then leads to the result

$$(x_\Phi^{UV} - x_\Phi^{IR})\langle\Phi\rangle = -\frac{1}{S_d}\int \langle\Theta(x)\Phi(0)\rangle_c\, d^dx, \tag{35}$$

which is the generalisation of (29), with x_Φ twice the complex dimension Δ_Φ.

In the case when the right hand side diverges, it is not permissible to take the limit $(R_1 \to 0, R_2 \to \infty)$, and, as above, this may be seen through the operator mixing which is exhibited in the operator product expansion (34).

3. The Ising model provides an interesting example for testing the ideas discussed in this letter. The situation is particularly simple in the purely thermal case, which in terms of the action (1) corresponds to perturbing the Ising critical point by the energy density operator $\varepsilon(x)$ of scaling dimension $2\Delta_\varepsilon = 1$ ($g \sim T - T_c$). The only other relevant operator in the model is the local magnetisation $\sigma(x)$ of scaling dimension $2\Delta_\sigma = 1/8$. The theory can be described in terms of a free majorana fermion of mass m, corresponding to a scattering amplitude $S(\theta) = -1$. The only nonvanishing FF of the components of the stress tensor T and $\Theta = 2\pi g \varepsilon$ (bilinear in the fermions) are

$$\begin{aligned} F_2^\Theta(\theta_1, \theta_2) &= -2i\pi m^2 \sinh \frac{\theta_1 - \theta_2}{2} \,, \\ F_2^T(\theta_1, \theta_2) &= \frac{i\pi}{2} m^2 e^{\theta_1 + \theta_2} \sinh \frac{\theta_1 - \theta_2}{2} \,. \end{aligned}$$

The correlators of σ with T and Θ vanish for symmetry reasons at $T > T_c$. Alternatively, one can work at $T < T_c$, or exploit duality and refer instead to the disorder operator μ. Here we only need its two-particle FF

$$F_2^\mu(\theta_1, \theta_2) = i\langle\mu\rangle \tanh \frac{\theta_1 - \theta_2}{2} \,. \tag{36}$$

The following (euclidean) correlators are then obtained using eq. (3)

$$\begin{aligned} \langle T(x) T(0)\rangle &= \frac{m^4}{16}\left(\frac{\bar{z}}{z}\right)^2 \left[K_1(m|x|)K_3(m|x|) - K_2^2(m|x|)\right] \,, \\ \langle T(x)\Theta(0)\rangle &= \frac{m^4}{4}\frac{\bar{z}}{z}\left[K_1^2(m|x|) - K_0(m|x|)K_2(m|x|)\right] \,, \\ \langle T(x)\mu(0)\rangle &= \frac{\langle\mu\rangle}{16 z^2} e^{-2m|x|} \,, \\ \langle\Theta(x)\Theta(0)\rangle_c &= m^4\left[K_1^2(m|x|) - K_0^2(m|x|)\right] \,, \\ \langle\Theta(x)\mu(0)\rangle_c &= -m^2\langle\mu\rangle\left[\frac{e^{-2m|x|}}{2m|x|} + \mathrm{E}i(-2m|x|)\right] \,, \end{aligned}$$

where K_ν are the modified Bessel functions and $\mathrm{E}i$ the exponential integral function $\mathrm{E}i(-x) = -\int_x^{+\infty} dt \exp(-t)/t$. One can easily check that the above correlators satisfy the differential equation (25). Comparison with the short distance expectations

$$\langle T(x)T(0)\rangle \simeq \frac{c}{2z^4} \,,$$

$$
\begin{aligned}
\langle T(x)\varepsilon(0)\rangle &\simeq \frac{2\pi g}{z^2}(1-\Delta_\varepsilon)\log|x| \;, \\
\langle T(x)\mu(0)\rangle &\simeq \frac{\Delta_\mu}{z^2}\langle\mu\rangle \;, & |x|\to 0 \\
\langle\varepsilon(x)\varepsilon(0)\rangle &\simeq |x|^{-4\Delta_\varepsilon} \;, \\
\langle\varepsilon(x)\mu(0)\rangle &\simeq C^\mu_{\varepsilon\mu}\langle\mu\rangle|x|^{-2\Delta_\varepsilon} \;,
\end{aligned}
$$

gives $c=1/2$, $\Delta_\mu=1/16$, $\Delta_\varepsilon=1/2$, $m=2\pi g$ and $C^\mu_{\varepsilon\mu}=-1/2$. In writing the second equation above we took into account that $\gamma_0=0$ for ε and used eq. (31) with $A_0=I$. Of course Δ_μ can also be obtained inserting the exact correlator $\langle\Theta(x)\mu(0)\rangle_c$ into the sum rule (29).

Less simple is the case of the purely magnetic perturbation of the Ising critical point ($\varphi(x)=\sigma(x)$ in the action (1)). The theory is again integrable but the spectrum now consists of eight massive particles [4]. Since the presence of the magnetic field breaks the invariance under spin reversal, no internal symmetries are left in the model. The FF bootstrap program described in the first part of this letter was carried out for this model in ref. [12, 21]. In particular, the factorisation equation (5) was used in ref. [21] to identify the FF solutions for the two relevant scaling operators σ and ε. Their scaling dimensions were also effectively estimated using the sum rule (29). Applications of the factorisation property (5) to other models are presented in ref. [22].

Finally, as an example of the massless case, let us consider the flow from the tricritical to the critical Ising points (the minimal models $\mathcal{M}_{4,5}$ and $\mathcal{M}_{3,4}$ of conformal field theory, respectively). The model is integrable and can be described in terms of a single species of massless particles whose scattering is characterised by the amplitudes $S_{RR}=S_{LL}=-1$ and $S_{RL}(\theta)=\tanh\left(\frac{\theta}{2}-\frac{i\pi}{4}\right)$ [13]. The correlator $\langle\Theta(x)\mu(0)\rangle_c$ can be computed through the spectral series (3) using the results of ref. [15], where FF of several operators of the theory were determined. Here we again consider for symmetry reasons the disorder parameter μ instead of the magnetisation σ. Both operators have scaling dimensions 3/40 in the ultraviolet limit and 1/8 in the infrared limit. Being the most relevant operator in the theory, μ does not mix under renormalisation and the sum rule (29) can be safely used to evaluate the total variation in its scaling dimension along the flow. The integration of the first (four-particle) contribution gives -0.0255; the addition of the second (six-particle) contribution leads to the result -0.0249, showing that the FF series rapidly converges to the expected result $\Delta_\mu^{UV}-\Delta_\mu^{IR}=3/80-1/16=-0.025$ in spite of the massless nature of the theory.

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