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Form-factors of exponential fields in the affine $A_{N-1}^{(1)}$ Toda model

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Abstract

A free field representation for form-factors of exponential operators $e^{\vec{a}\vec{\varphi}}$ in the affine $A_{N-1}^{(1)}$ Toda model is proposed. The one and two particle form-factors are calculated explicitly.

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An effective tool for form-factor calculations is so called "free field representation" approach [1]. Its essentials [2] were borrowed from Baxter's corner transfer matrix method [3], [4], [5]. The approach is based on the examination of dynamical symmetry algebras acting in an angular quantization space (AQS) of a massive integrable model. AQS is a field-theoretic analog of the space, where the lattice corner transfer matrix acts, and inherits remarkable features of the later one. AQS can be typically identified with a representation space of some "deformed algebra" taken in a proper "scaling limit". At the moment relations between deformed algebras and quantum field theory (QFT) integrable models are mainly studied on the following examples:

$U_q(\widehat{sl_2})$ at the level one [6], [7] $\longleftrightarrow$ $SU(2)$ invariant Thirring model [4], [5], [1], [8];

Elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl_2})$ [9] $\longleftrightarrow$ sin-Gordon model [1], [10], [11];

Deformed Virasoro algebra [12], [13], [14], [15] $\longleftrightarrow$ sin(h)-Gordon model [16], [10].

In the last example the dynamical symmetry algebra manifests itself in a very simple way; A current of the deformed Virasoro algebra coincides with the Zamolodchikov-Faddeev operator of the basic scalar particle, acting in AQS [16]. In spite of an apparent triviality of this observation, it can form the basis for derivation of form-factors of local operators. Indeed, using the bosonization of the deformed Virasoro algebra [14], [15] we can represent form-factors as traces over free Fock spaces [1]. Then, the well known technique [17] makes possible to reduce the calculation of the traces to a vacuum averaging of products of vertex operators. This program have been implemented in the work [18] and the form-factors of exponential fields in the sinh-Gordon model [19] were recovered. The final rules for the free field representation appear to be very simple and one can try to generalize them to more complicated QFT models. The best candidate to do this is related to the deformed W-algebra introduced in the works [15], [20]. It would appear reasonable that the bosonization of this algebra provide the free field representation for the form-factors in the affine Toda field theory. An analysis of the results of Refs. [15], [20] shows that currents of the deformed W-algebra in a proper scaling limit coincide with the Zamolodchikov-Faddeev operators acting in AQS of the affine $A^{(1)}_{N-1}$ Toda QFT. Therefore, the form-factors of the model can be given by traces over AQS and then rewritten as vacuum averages. The purpose of this letter is to present the final rules for the calculation of the form-factors of exponential fields. Using the free field representation, we derive the one and two particle form-factors and discuss their properties.

1. Preliminaries

The affine $A^{(1)}_{N-1}$ Toda model [21], [22] describes dynamics of the $N-1$ component real scalar field

$$\vec{\varphi}(x) = \big(\, \varphi^{(1)}, ..., \varphi^{(N-1)} \,\big) \ , \tag{1.1}$$

governed by the Euclidean action

$$\mathcal{A}_T = \int d^2x \left\{ \frac{1}{8\pi} \left(\partial_\nu \vec{\varphi}\right)^2 + \mu \left(\sum_{k=1}^{N-1} \ e^{b\vec{\alpha}_k\vec{\varphi}} + e^{b\vec{\alpha}_0\vec{\varphi}} \right) \right\} . \tag{1.2}$$

Here we denote by

$$\vec{\alpha}_k \in \mathbf{R}^{N-1} \,, \qquad \vec{\alpha}_k^2 = 2 \qquad (k = 1, ..., N-1)$$

a set of the simple positive roots and

$$-\vec{\alpha}_0 = \sum_{k=1}^{N-1} \vec{\alpha}_k$$

is the maximal root of the simple Lie algebra A_{N-1}. The action (1.2) is invariant under the finite group generated by the elements,

$$\begin{aligned} &\hat{\mathcal{C}}: \ \vec{\alpha}_k\vec{\varphi} \to \vec{\alpha}_{N-k}\vec{\varphi} \,, \qquad\qquad \vec{\alpha}_0\vec{\varphi} \to \vec{\alpha}_0\vec{\varphi} \ ; \\ &\hat{\Omega}: \ \vec{\alpha}_1\vec{\varphi} \to \vec{\alpha}_2\vec{\varphi} \to ... \to \vec{\alpha}_{N-1}\vec{\varphi} \to \vec{\alpha}_0\vec{\varphi} \to \vec{\alpha}_1\vec{\varphi} \ . \end{aligned} \tag{1.3}$$

There are $N-1$ particles in the spectrum of the model and we will denote them by $\big\{B_n\big\}_{n=1}^{N-1}$. Acting on asymptotic states, (1.3) corresponds to the transformations

$$\begin{aligned} &\hat{\mathcal{C}}: \ B_n \to B_{N-n} \ ; \\ &\hat{\Omega}: \ B_n \to \omega^n \, B_n \ , \end{aligned} \tag{1.4}$$

with $\omega = e^{i\frac{2\pi}{N}}$. The masses of the particles are simple related [21], [22]

$$M_n = M \ [n] \ . \tag{1.5}$$

Here and bellow we use the conventional notation

$$[n] = \frac{\sin(\frac{\pi n}{N})}{\sin(\frac{\pi}{N})} \ . \tag{1.6}$$

The model (1.2) is an integrable QFT. Its two-body S-matrix, describing $B_n B_m \to B_m B_n$ scattering, was proposed in [22]. If

$$(p_n^\nu + p_m^\nu)^2 = M_n^2 + M_m^2 + 2M_n M_m \cosh(\theta) \ , \tag{1.7}$$

is the usual relativistic invariant [23], the S-matrix reads

$$\mathcal{S}_{nm}(\theta) = \prod_{p=1}^{n}\prod_{q=1}^{m} S\big(\theta + \frac{i\pi}{N}(n-2p) - \frac{i\pi}{N}(m-2q)\big) \ , \tag{1.8}$$

where

$$S(\theta) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi}{N})\, \sinh(\frac{\theta}{2} - \frac{i\pi b}{NQ})\, \sinh(\frac{\theta}{2} - \frac{i\pi}{NQb})}{\sinh(\frac{\theta}{2} - \frac{i\pi}{N})\, \sinh(\frac{\theta}{2} + \frac{i\pi b}{NQ})\, \sinh(\frac{\theta}{2} + \frac{i\pi}{NQb})} \tag{1.9}$$

and

$$Q = b^{-1} + b \ . \tag{1.10}$$

To analyze an analytical structure of the two-body S-matrix, it is useful to rewrite (1.8) in the form [24]

$$\mathcal{S}_{nm}(\theta) = \prod_{q=1}^{\min(m,n)} \frac{F\big(\theta - \frac{i\pi}{N} - \frac{i\pi}{N}(n+m-2q)\big)}{F\big(\theta + \frac{i\pi}{N} + \frac{i\pi}{N}(n+m-2q)\big)} \ , \tag{1.11}$$

where

$$F\big(\theta \pm i\frac{\pi}{N}\big) = \frac{\sinh(\frac{\theta}{2} \pm \frac{i\pi b}{NQ})\sinh(\frac{\theta}{2} \pm \frac{i\pi}{NQb})}{\sinh(\frac{\theta}{2})\sinh(\frac{\theta}{2} \pm \frac{i\pi}{N})} \ . \tag{1.12}$$

In the physical strip $0 < \Im m\, \theta < \pi$ the amplitude $\mathcal{S}_{nm}(\theta)$ possesses two simple poles at $\theta = i\frac{\pi}{N}\,(N - |N-n-m|)$ and $\theta = i\frac{\pi}{N}\,|n-m|$. The first pole corresponds to the particle B_{n+m} (here $n+m$ should be taken modulo N) in the direct channel $B_n B_m \to B_m B_n$ scattering while the second pole corresponds to the particle $B_{|n-m|}$ in the cross-channel. The S-matrix (1.8), (1.11) has also double poles located at

$$\theta = i\frac{\pi}{N}\,\big(\,N - |N-n-m+2q|\,\big) \qquad q = 1, 2, ..., \min(m,n) - 1 \ . \tag{1.13}$$

Simple origin of such singularities has long been known [25].

The goal of this letter is to propose the free field representation for the form-factors [1]

$$\langle\, 0\, |\, e^{\vec{a}\vec{\varphi}}\, |\, B_{n_p}(\theta_p)...B_{n_1}(\theta_1)\,\rangle$$

[1] Our convention for the normalization of the asymptotic states is $\langle B^m(\theta)\,|\,B_n(\theta')\,\rangle = 2\pi\, \delta_n^m\, \delta(\theta-\theta')$.

of exponential fields $e^{\vec{a}\vec{\varphi}}$ in the affine $A_{N-1}^{(1)}$ Toda QFT. We assume that the exponential fields are normalized in accordance with short distance behavior of the two point correlation function

$$\langle\, 0\,|\, e^{\vec{a}\vec{\varphi}(x)}\, e^{-\vec{a}\vec{\varphi}(x)}\,|\, 0\,\rangle \to |x-y|^{2\vec{a}^2}\,, \qquad \text{as} \;\; |x-y|\to 0\;. \tag{1.14}$$

Notice that $|\vec{a}|$ should be sufficiently small number, in order to (1.14) be a leading asymptotic. Under the normalization (1.14), the vacuum expectation values

$$\mathcal{G}_{\vec{a}} = \langle\, 0\,|\, e^{\vec{a}\vec{\varphi}}\,|\, 0\,\rangle\;, \tag{1.15}$$

have been recently calculated by V.A. Fateev [26][2]. In this work we do not use their explicit form. Just note that due to the unbroken discreet symmetry (1.3), (1.4),

$$\mathcal{G}_{\vec{a}} = 1 + O(\vec{a}^2)\;, \quad \text{as} \qquad |\vec{a}|\to 0\;. \tag{1.16}$$

2. Free field representation for form-factors

The construction given bellow was initiated by the results of Refs. [15] and [20].

To describe the free field representation for the form-factors, we need to introduce a set of vertex operators $\{\Lambda_s(\theta)\}_{s=1}^{N}$ associated with the weights $\vec{h}_s$ of the first fundamental (vector) representation π_1 of A_{N-1}

$$\{\vec{h}_s\}_{s=1}^{N}: \qquad \vec{h}_s\,\vec{\alpha}_k = \delta_{s,k}-\delta_{s,k+1}\,, \qquad k=1,...,N-1\;. \tag{2.1}$$

The averaging of the products of these vertex operators will be performed by Wick's theorem using the rules,

$$\begin{aligned} &\langle\langle\, \Lambda_s(\theta)\,\rangle\rangle = 1\;,\\ &\langle\langle\, \Lambda_s(\theta_2)\Lambda_s(\theta_1)\,\rangle\rangle = R(\theta_1-\theta_2)\;,\\ &\langle\langle\, \Lambda_s(\theta_2)\Lambda_{s'}(\theta_1)\,\rangle\rangle = R(\theta_1-\theta_2)\; F\big(\theta_1-\theta_2+\mathrm{sgn}(s'-s)\frac{i\pi}{N}\big)\,, \qquad s\neq s'. \end{aligned} \tag{2.2}$$

Here the function $F(\theta)$ is given by (1.12) and $R(\theta)$ for $-2\pi < \Im m\,\theta < 0$ reads

$$R(\theta) = \exp\bigg\{-4\int_0^\infty \frac{dt}{t}\; \frac{\sinh\big(t(1-\frac{1}{N}\big)\,\sinh\big(\frac{tb}{NQ}\big)\,\sinh\big(\frac{t}{NQb}\big)}{\sinh^2(t)}\;\cosh\big(t(1-\frac{i\theta}{\pi})\big)\bigg\} \tag{2.3}$$

[2] For $N=2$ the vacuum expectation values were found in [27], [28].

and is defined through an analytic continuation outside this domain. $R(\theta)$ satisfies the relations,

$$R(-\theta)=S(\theta)\,R(\theta)\,, \qquad R(-2\pi i-\theta)=R(\theta)\ , \tag{2.4}$$

with $S(\theta)$ (1.9). From the formulas (2.2) and (2.4) it follows immediately that $\Lambda_s(\theta)$ obey the commutation relations

$$\Lambda_s(\theta_1)\Lambda_{s'}(\theta_2)=S(\theta_1-\theta_2)\ \Lambda_s(\theta_1)\Lambda_{s'}(\theta_2)\ . \tag{2.5}$$

They also satisfy the condition

$$:\Lambda_1(\theta+\frac{i\pi(N-1)}{N})\,\Lambda_2(\theta+\frac{i\pi(N-3)}{N})\ldots\Lambda_N(\theta-\frac{i\pi(N-1)}{N}):=1\ , \tag{2.6}$$

which is compatible with (2.2) due to simple properties of the functions $F(\theta)$ and $R(\theta)$. The normal order dots in (2.6) : ... : mean that we do not need to couple vertex operators inside the normal ordered group under Wick's averaging. Weights of the n-fundamental representation π_n of A_{N-1} are given by the vectors

$$\vec{h}_{s_1}+\ldots+\vec{h}_{s_n}\,, \qquad s_1<s_2<\ldots<s_n\ . \tag{2.7}$$

We associate the vertex operator

$$:\Lambda_{s_1}(\theta+\frac{i\pi(n-1)}{N})\,\Lambda_{s_2}(\theta+\frac{i\pi(n-3)}{N})\ldots\Lambda_{s_n}(\theta-\frac{i\pi(n-1)}{N}): \tag{2.8}$$

with each of these weights.

Let $\vec{\rho}$ is the half sum of the positive roots of the Lie algebra A_{N-1},

$$\vec{\rho}\,\vec{\alpha}_k=1\,, \quad k=1,\ldots,N-1\ . \tag{2.9}$$

Introduce the operators

$$\begin{aligned}\mathbf{B}_n(\theta)=Q\sqrt{\frac{NZ_n}{2\pi}}&\sum_{1\le s_1<\ldots<s_n\le N}\omega^{(\vec{\rho}-\frac{\vec{a}}{Q})(\vec{h}_{s_1}+\ldots+\vec{h}_{s_n})}\ \times\\ &:\Lambda_{s_1}(\theta+\frac{i\pi(n-1)}{N})\,\Lambda_{s_2}(\theta+\frac{i\pi(n-3)}{N})\ldots\Lambda_{s_n}(\theta-\frac{i\pi(n-1)}{N}):\ .\end{aligned} \tag{2.10}$$

The constants Z_n here will be fixed later (see (4.16)). For real θ these operators satisfy the commutation relation,

$$\mathbf{B}_n(\theta_1)\mathbf{B}_m(\theta_2)=\mathcal{S}_{nm}(\theta_1-\theta_2)\ \mathbf{B}_m(\theta_2)\mathbf{B}_n(\theta_1)\ , \tag{2.11}$$

where $\mathcal{S}_{nm}(\theta)$ is given by (1.8). We propose the following free field representation for the form-factors of the exponential fields in the affine $A^{(1)}_{N-1}$ Toda QFT,

$$\langle\,0\,|\,e^{\vec{a}\vec{\varphi}}\,|\,B_{n_p}(\theta_p)\ldots B_{n_1}(\theta_1)\,\rangle=\mathcal{G}_{\vec{a}}\ \langle\langle\,\mathbf{B}_{n_p}(\theta_p)\ldots\mathbf{B}_{n_1}(\theta_1)\,\rangle\rangle\ . \tag{2.12}$$

This formula reduces the calculation of the form-factors to a combinatoric procedure. For $N=2$ (2.12) turns out to be the free field representation proposed in the work [18] and reproduces the results of [19].

3. One particle form-factors

From the formula (2.12) we immediately have the expression for the one particle form-factors,

$$\langle\, 0\,|\, e^{\vec{a}\vec{\varphi}}\,|\, B_n\,\rangle = \mathcal{G}_{\vec{a}}\ Q\sqrt{\frac{NZ_n}{2\pi}}\ \chi_n\big(\frac{\vec{a}}{Q}\big)\ . \tag{3.1}$$

The function $\chi_n(\vec{\lambda})$ coincides with the character of the n-fundamental representation π_n,

$$\chi_n(\vec{\lambda}) = \mathrm{tr}_{\pi_n}\Big[\,\omega^{(\vec{\rho}-\vec{\lambda})\vec{H}}\,\Big]\ . \tag{3.2}$$

Here $\vec{H} = (H^{(1)}, ..., H^{(N-1)})$ is a basis in the Cartan subalgebra of A_{N-1}, normalized with respect the Killing form $\langle\, ,\,\rangle : \langle\, H^{(\sigma)}, H^{(\sigma')}\,\rangle = \delta^{\sigma\sigma'}$. One can show that

$$\chi_n(\lambda\vec{\alpha}_j) = 4\ \sin\big(\frac{\pi\lambda}{N}\big)\ \sin\big(\frac{\pi}{N}(\lambda-1)\big)\,\omega^{-jn}\ [n] \qquad j = 0, ..., N-1\ . \tag{3.3}$$

Eqs. (3.3), (1.16) allow one to specify the one particle form-factors of the field $\vec{\varphi}$ by taking the small $|\vec{a}|$ limit of (3.1),

$$\langle\, 0\,|\, \vec{\alpha}_j\vec{\varphi}\,|\, B_n\,\rangle = -2\sqrt{\frac{2\pi}{N}}\ \omega^{-jn}\ \sin\big(\frac{\pi n}{N}\big)\,\sqrt{Z_n}\,, \qquad j = 0, ..., N-1\ . \tag{3.4}$$

In writing (3.4) we assume that the field (1.1) is normalized in accordance with the short distance limiting behavior,

$$\langle\, 0\,|\, \varphi^{(\sigma)}(x)\varphi^{(\sigma')}(y)\,|\, 0\,\rangle \to -2\delta^{\sigma\sigma'}\ \log(M|x-y|) + O(1)\ , \quad \text{as}\ \ |x-y| \to 0\ . \tag{3.5}$$

Hence Z_n in the definition (2.10) are the wave-function renormalization constants in the affine $A^{(1)}_{N-1}$ Toda QFT.

With (3.1) and (3.3) we can check that the one particle form-factors agree with the quantum equations of motion

$$\partial_\nu\partial^\nu\big(\,\vec{\alpha}_j\vec{\varphi}\,\big) = \mathcal{M}\,\Big(2e^{b\vec{\alpha}_j\vec{\varphi}} - e^{b\vec{\alpha}_{j-1}\vec{\varphi}} - e^{b\vec{\alpha}_{j+1}\vec{\varphi}}\Big)\ , \tag{3.6}$$

with

$$\mathcal{M} = \frac{\pi M^2}{4QN\mathcal{G}\ \sin\big(\frac{\pi}{NbQ}\big)\sin\big(\frac{\pi b}{NQ}\big)\sin\big(\frac{\pi}{N}\big)}\ ,$$

and $\mathcal{G} \equiv \mathcal{G}_{b\vec{\alpha}_j}$ $(j = 0, ..., N-1)$.

Eq. (3.6) can be used in derivation of the specific free energy $f(\mu)=-\frac{1}{V}\log Z$ in the affine $A_{N-1}^{(1)}$ Toda QFT. According to (1.2), the constant $\mathcal{M}$ should coincide with $4\pi\mu$. If we combine this requirement with the obvious relation

$$N\,\mathcal{G}=-\partial_\mu f(\mu)\ , \tag{3.7}$$

we get the formula

$$bQ\ \mu\,\partial_\mu f(\mu)=-\frac{M^2(\mu)}{16\sin\left(\frac{\pi}{NbQ}\right)\sin\left(\frac{\pi b}{NQ}\right)\sin\left(\frac{\pi}{N}\right)}\ . \tag{3.8}$$

Due to (1.14), $M^2(\mu)\sim\mu^{\frac{1}{bQ}}$ up to dimensionless constant [3] and we reproduce the result of C. Destri and H. de Vega [30]

$$f(\mu)=-\frac{M^2(\mu)}{16\sin\left(\frac{\pi}{NbQ}\right)\sin\left(\frac{\pi b}{NQ}\right)\sin\left(\frac{\pi}{N}\right)}\ . \tag{3.9}$$

4. Two particle form-factors

In order to simplify the notation in this section we assume that integers $m\le n,\ m+n\le N$ without loss in generality.

By applying the free field representation (2.12), the two particle form-factors were derived. The computations are essentially based on the interesting trigonometric identity

$$\sum_{1\le k_1\le\ldots\le k_m\le n+1}\ \prod_{p=1}^{m}F\big(\theta+\frac{i\pi}{N}(n-m-2k_p+2p+1)\big)=\sum_{k=0}^{m}C_k^{m+n}\ \mathcal{K}_{m-k}^{n-k}(\theta)\ , \tag{4.1}$$

where C_k^{m+n} are the binomial coefficients and the functions $\mathcal{K}_m^n(\theta)$ are defined by the following recursion

$$\begin{aligned}&\mathcal{K}_0^n(\theta)\equiv 1\ ,\\ &\mathcal{K}_m^n(\theta)=-\frac{\sin\left(\frac{\pi b}{NQ}\right)\sin\left(\frac{\pi}{NQb}\right)\ [m+n]}{\sinh\left(\frac{\theta}{2}+i\frac{\pi(m+n)}{2N}\right)\sinh\left(\frac{\theta}{2}-i\frac{\pi(m+n)}{2N}\right)}\ \sum_{k=1}^{m}\mathcal{K}_{m-k}^{n-k}(\theta)\ \frac{[k]\,[m+n-k]}{[m]\,[n]}\ .\end{aligned} \tag{4.2}$$

Eq. (4.1) along with simple properties of symmetric polynomials [4], makes it possible to derive the formula ($m\le n,\ m+n\le N$)

$$\begin{aligned}&\langle\,0\,|\,e^{\vec a\vec\varphi}\,|\,B_m(\theta_2)B_n(\theta_1)\,\rangle=\mathcal{G}_{\vec a}\ \frac{Q^2N}{2\pi}\ \sqrt{Z_nZ_m}\times\\ &\mathcal{R}_{nm}(\theta_1-\theta_2)\ \sum_{k=0}^{m}\chi_k\big(\frac{\vec a}{Q}\big)\chi_{n+m-k}\big(\frac{\vec a}{Q}\big)\ \mathcal{K}_{m-k}^{n-k}(\theta_1-\theta_2)\ .\end{aligned} \tag{4.3}$$

[3] This constant was calculated in the work [29].

[4] The characters (3.2) are symmetric polynomials with respect of $\left\{\omega^{(\vec\rho-\frac{\vec a}{Q})\vec h_s}\right\}_{s=1}^{N}$.

Here $\chi_n(\vec{\lambda})$ are the characters of the n-fundamental representations (3.2) ($\chi_0(\vec{\lambda})\equiv 1$) and

$$\mathcal{R}_{nm}(\theta)=\prod_{p=1}^{n-1}\prod_{s=1}^{m}F(\theta+i\frac{\pi}{N}(n-m-1-2p+2s))\ \prod_{p=1}^{n}\prod_{s=1}^{m}R(\theta+i\frac{\pi}{N}(n-m-2p+2s))\ . \quad (4.4)$$

Notice that the function $\mathcal{R}_{nm}(\theta)$ is so called "minimal form-factor" [30]. It admits similar to (2.3) representation

$$\mathcal{R}_{nm}(\theta)=$$
$$\exp\Big\{-4\int_0^\infty\frac{dt}{t}\ \frac{\sinh\big(\frac{tm}{N}\big)\ \sinh\big(t(1-\frac{n}{N})\big)\ \sinh\big(\frac{tb}{NQ}\big)\ \sinh\big(\frac{t}{NQb}\big)}{\sinh\big(\frac{t}{N}\big)\sinh^2(t)}\ \cosh\big(t(1-\frac{i\theta}{\pi})\big)\Big\}\ . \quad (4.5)$$

For $m+n>N$ the form-factors can be obtained from (4.3) by the use of the charge conjugation $\hat{\mathcal{C}}$ (1.3), (1.4). Taking $|\vec{a}|\to 0$ limit of (4.3), we derive the two particle form-factors of the field $\vec{\varphi}$

$$\begin{aligned}\langle\,0\,|\,\vec{\varphi}\,|\,B_m(\theta_2)B_n(\theta_1)\,\rangle=\langle\,0\,|\,\vec{\varphi}\,|\,B_{m+n}\,\rangle\ \sqrt{\frac{Z_nZ_m}{Z_{n+m}}}\ Q\ \sqrt{\frac{N}{2\pi}}\times\\ \mathcal{R}_{nm}(\theta_1-\theta_2)\ \mathcal{K}^n_m(\theta_1-\theta_2)\ .\end{aligned} \quad (4.6)$$

A simple check based on (4.2) and (3.3) shows that Eqs. (4.3), (4.6) conform with the quantum equations of motion (3.6).

Let us discussed basic properties of the two particle form-factors. First of all, we consider their leading behavior in the large $|\theta_1-\theta_2|$ limit. Using the formulas

$$R(\theta)\to 1\,,\qquad F(\theta)\to 1\ ,\quad \text{as}\quad \theta\to\infty\ , \quad (4.7)$$

we obtain the cluster property [19], [31]

$$\langle\,0\,|\,e^{\vec{a}\vec{\varphi}}\,|\,B_m(\theta_2)B_n(\theta_1)\,\rangle\to\ \frac{\langle\,0\,|\,e^{\vec{a}\vec{\varphi}}\,|\,B_m\,\rangle\langle\,0\,|\,e^{\vec{a}\vec{\varphi}}\,|\,B_n\,\rangle}{\langle\,0\,|\,e^{\vec{a}\vec{\varphi}}\,|\,0\,\rangle}\ , \quad (4.8)$$

as $|\theta_1-\theta_2|\to\infty$ while

$$\begin{aligned}\langle\,0\,|\,\vec{\varphi}\,|\,B_m(\theta_2)B_n(\theta_1)\,\rangle\to-\langle\,0\,|\,\vec{\varphi}\,|\,B_{m+n}\,\rangle\ \sqrt{\frac{Z_nZ_m}{Z_{n+m}}}\times\\ 2\,Q\sqrt{\frac{2N}{\pi}}\ \sin\big(\frac{\pi b}{NQ}\big)\sin\big(\frac{\pi}{NQb}\big)\ [m+n]\ e^{-|\theta_1-\theta_2|}\ .\end{aligned} \quad (4.9)$$

The last relation was established in the work [30] by an analysis of the Feynman diagrams.

The form-factors (4.3), (4.6) satisfy Watson's and crossing symmetry equations [32] and have required analytical structure [30]. Specifically, they develop simple poles at the points

$$\theta_2 - \theta_1 = i\frac{\pi}{N}\,(m+n-2q) \qquad (q = 1, 2, ..., m-1)\ .$$

The latter do not correspond to bound states and are directly related to the double pole structure of the two-body S-matrix (1.13). At the same time, the form-factors singularity at $\theta_2 - \theta_1 = i\frac{\pi}{N}(m+n)$,

$$\langle\, 0\,|\, e^{\vec{a}\vec{\varphi}}\,|\, B_m(\theta_2)B_n(\theta_1)\,\rangle \to \frac{i\ \Gamma_{nm}}{\theta_2 - \theta_1 - i\frac{\pi}{N}(m+n)}\ \ \langle\, 0\,|\, e^{\vec{a}\vec{\varphi}}\,|\, B_{m+n}\,\rangle\ , \tag{4.10}$$

is in agreement with the bound state pole of (1.8). To find the residue it is useful to take advantage of the identity (4.1),

$$\Gamma_{nm} = \sqrt{\frac{Z_n Z_m}{Z_{n+m}}}\ Q\,\sqrt{\frac{2N}{\pi}}\ \frac{\sin\left(\frac{\pi b}{NQ}\right)\sin\left(\frac{\pi}{NQb}\right)}{\sin\left(\frac{\pi}{N}\right)}\ \mathcal{R}_{nm}\big(-\frac{i\pi}{N}(m+n)\big)\ \prod_{q=1}^{m-1} F\big(i\frac{\pi}{N}(2q+1)\big)\ . \tag{4.11}$$

According to the common requirement [32], Γ_{nm} should also define the residue of the two-body S-matrix in the following manner,

$$\mathcal{S}_{nm}(\theta) \to \frac{i\ \big(\Gamma_{nm}\big)^2}{\theta - i\frac{\pi}{N}(m+n)}\,, \quad \text{as} \quad \theta \to i\frac{\pi}{N}(m+n)\ . \tag{4.12}$$

Using (1.11), we conclude that

$$\big(\Gamma_{nm}\big)^2 = 2\,\frac{\sin\left(\frac{\pi b}{NQ}\right)\sin\left(\frac{\pi}{NQb}\right)}{\sin\left(\frac{\pi}{N}\right)}\ \frac{\prod_{q=1}^{m-1} F\big(i\frac{\pi}{N}(2q+1)\big)}{\prod_{q=1}^{m} F\big(i\frac{\pi}{N}(2n+2q-1)\big)}\ . \tag{4.13}$$

Relations (4.11) and (4.13) restrict Z_n up to a n-independent constant c,

$$Z_n = \frac{\pi\ c^n\ \sin\left(\frac{\pi}{N}\right)}{N\,Q^2\ \sin\left(\frac{\pi b}{NQ}\right)\sin\left(\frac{\pi}{NQb}\right)}\ \prod_{p=1}^{n-1} \frac{\Big[R\big(-\frac{2\pi i}{N}p\big)\,F\big(\frac{i\pi}{N}(2p+1)\big)\Big]^{2n-2p}}{F\big(\frac{i\pi}{N}(2p+1)\big)}\ . \tag{4.14}$$

If we enclose also the condition

$$Z_n = Z_{N-n}\ , \tag{4.15}$$

which follows from the invariance with respect to the charge conjugation $\hat{\mathcal{C}}$ (1.3), (1.4), the constant c is specified uniquely. Finally we have

$$
\begin{aligned}
Z_n =& \frac{\pi}{Q^2\sin(\frac{\pi b}{Q})}\times \\
& \exp\left\{2\int_0^\infty \frac{dt}{t}\ \frac{\sinh\big(\frac{tb}{NQ}\big)\ \sinh\big(\frac{t}{NQb}\big)}{\sinh^2(t)\sinh(\frac{t}{N})}\ \Big(\sinh^2\big(\frac{tn}{N}\big)+\sinh^2\big(t(1-\frac{n}{N})\big)\Big)\right\}.
\end{aligned}
\tag{4.16}
$$

The expression (4.16) for the wave-function renormalization constants was previously obtained in the work [30]. It matches all necessary field-theoretic bounds, including the one loop perturbative check. In such a manner we conclude that the explicit formulas for the one and two particle form-factors support the proposed free field representation.

In closing we note that the free field representation admits straightforward generalization for any untwisted affine ADE Toda QFT [33].

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