Universal amplitude ratios in the two-dimensional q-state Potts model and percolation from quantum field theory[1]

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Abstract

We consider the scaling limit of the two-dimensional q-state Potts model for $q \leq 4$. We use the exact scattering theory proposed by Chim and Zamolodchikov to determine the one and two-kink form factors of the energy, order and disorder operators in the model. Correlation functions and universal combinations of critical amplitudes are then computed within the two-kink approximation in the form factor approach. Very good agreement is found whenever comparison with exact results is possible. We finally consider the limit $q \to 1$ which is related to the isotropic percolation problem. Although this case presents a serious technical difficulty, we predict a value close to 74 for the ratio of the mean cluster size amplitudes above and below the percolation threshold. Previous estimates for this quantity range from 14 to 220.

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1 Introduction

Two-dimensional integrable models enjoy a very particular role in the framework of quantum field theory. As a matter of fact, they provide the only known examples of non-trivial, interacting relativistic quantum theories to be exactly solved. As is well known, this circumstance has to be traced back to the existence in these models of non-trivial integrals of motions resulting in a strong simplification of the scattering theory and in the possibility of determining the exact S-matrix through bootstrap techniques. A large number of integrable models has been discovered and solved in this way over the last two decades [1, 2, 3].

If the theoretical relevance of integrable field theories is obvious, surely their effectiveness for the accurate quantitative description of interesting physical systems is not less important. In this respect, two-dimensional statistical mechanics offers a natural testing ground. Many statistical mechanical models are known to be exactly solvable in two dimensions [4] and are then natural candidates for a description in terms of integrable field theory in the scaling limit in which the correlation length becomes much larger than the lattice spacing. Moreover, one of the remarkable results of the recent investigations of two-dimensional quantum field theory is that integrability can be a property of the scaling limit of statistical models even in cases where it has never been found on the lattice. The critical Ising model in a magnetic field provides the most known example of such a situation[1][2].

It is clear that, for the purpose of the quantitative study of statistical systems through the methods of quantum field theory, only universal quantities should be considered, i.e. those quantities which do not depend on the particular microscopic realisation of the system and are determined instead by its global features (essentially, internal symmetries and dimensionality). Universal properties emerge when the system approaches a second order phase transition point. For a magnetic system in zero external field, the usual characterisation of criticality is in terms of the classical thermodynamic quantities: in the vicinity of the critical point the specific heat, spontaneous magnetisation, susceptibility and correlation length behave as

$$\begin{aligned}
C &\simeq (A_\pm/\alpha)\,\tau^{-\alpha}\,,\\
M &\simeq B\,(-\tau)^{\beta}\,,\\
\chi &\simeq \Gamma_\pm\,\tau^{-\gamma}\,,\\
\xi &\simeq \xi_0^\pm\,\tau^{-\nu}\,,
\end{aligned} \tag{1.1}$$

[1]It is interesting to notice that for this specific case a lattice model has been identified which is in the same universality class than the Ising model in a magnetic field and is integrable [41].

where we denoted by τ the reduced temperature, $\tau = a(T - T_c)$, $a > 0$, and the labels $\pm$ on the critical amplitudes refer to the critical point being approached from above or from below. While the critical exponents are characteristic of the critical point and in $d = 2$ are determined by conformal field theory (CFT) [5, 6], the critical amplitudes carry information about the renormalisation group trajectory along which the critical point is approached and their determination requires a study of the system away from criticality. Critical amplitudes depend on metric factors and are not themselves universal, but universal combinations of them can by constructed which characterise the scaling region around the critical point. The following universal amplitude ratios are usually considered in the literature [7]

$$\begin{aligned} &A_+/A_- \,, \quad \Gamma_+/\Gamma_- \,, \quad \xi_0^+/\xi_0^- \,, \\ &R_C = A_+\Gamma_+/B^2 \,, \quad R_\xi^+ = A_+^{1/d}\xi_0^+ \,. \end{aligned} \tag{1.2}$$

The scale factor independence of R_C and R_ξ^+ is a direct consequence of the scaling and hyperscaling relations

$$\alpha = 2 - 2\beta - \gamma \,, \qquad 2 - \alpha = d\nu \,. \tag{1.3}$$

The critical amplitudes can be expressed as moments of correlation functions of the spin and energy operators in the off-critical theory. As a consequence, the computation of the amplitude ratios in the framework of integrable field theory requires bridging the gap between the scattering theory, in terms of which the solution of the model is given, and the off-shell dynamics. With respect to this problem, the so called form factor bootstrap is the method which proved so far as the most effective. In this approach, correlation functions are expressed as spectral sums over n-particle intermediate states. The operator matrix elements entering the decomposition (known as form factors) are subject to a set of monodromy and residue equations [8, 9] which have been solved exactly for many integrable models. These equations, however, are fixed by the S-matrix alone and do not distinguish between different operators. Although the selection rules coming from internal symmetries together with some minimality assumptions are sufficient in some cases to identify the form factor solutions corresponding to specific operators, it turns out that in general more information about the operator space has to be injected in order to handle this problem [10, 11].

The operators which appear in physical applications are the scaling operators. Since they are naturally defined in the scaling limit towards the ultraviolet fixed point, it is not surprising that the constraints for their identification in the form factor approach come from the high energy asymptotics of the matrix elements. Two such constraints have been identified which are crucial for the results obtained in this paper. The first one relates the asymptotic behaviour of form factors of the scaling operators to their scaling dimension [10]; the second constraint takes the form of a cluster decomposition of the matrix elements when the momenta of a subset of particles become much larger than the others. It was argued in Ref. [12] that the latter property is related to the decoupling of the theory into holomorphic and anti-holomorphic sectors in the ultraviolet limit.

This level of understanding proved so far sufficient for the determination of the two-particle form factors of physically relevant operators in integrable theories. From the general point of view, the computation of the two-particle matrix elements amounts to fixing the initial conditions of the bootstrap procedure. The determination of the form factors with more than two particles through the residue equations is a mathematical problem which can be straightforward or extremely difficult depending on the degree of complexity of the underlying scattering theory. Although the problem has been solved in several specific cases (see e.g. [28, 9, 13]), no general method of solution is known for the case when the scattering allows for the exchange of quantum numbers among particles with the same mass.

If in principle this circumstance severely restricts the range of applicability of the form factor approach for the exact computation of correlation functions, in practice very accurate results can be obtained anyway in a large number of cases. In fact, it has been by now verified for several integrable models that the spectral series over form factors is characterised by a remarkably fast rate of convergence (see, among others, Refs. [14, 13, 15, 10]). A theoretical justification of this property was proposed in [15] relying on phase space considerations and some peculiarities of integrable dynamics. As a matter of fact, it turns out that zeroth moments of two-point correlators can be computed within a typical accuracy of order 1%, including in the spectral series the one and two-particle contributions only. At fixed level of approximation, the accuracy rapidly increases if higher moments are considered in which the contribution coming from the short distances (more sensitive to the exclusion from the sum of many-particle states) is suppressed.

It is the purpose of this paper to use the form factor approach to compute the universal amplitude ratios for the q-state Potts model ($q \leq 4$) and the isotropic percolation problem. Although the latter does not fit immediately in the standard terminology of thermal phase transitions, it can be formally related to the $q \to 1$ limit of the Potts model. Through this mapping, the thermodynamic quantities listed in (1.1) become related to the mean number of clusters, the percolation probability, the mean cluster size and the pair connectivity, respectively.

Our starting point will be the exact scattering theory for the q-state Potts model proposed by Chim and Zamolodchikov in Ref. [16]. Due to the fact that the fundamental constraint of permutation symmetry which characterises the model is very naturally imposed on the interaction of the kinks interpolating between the q degenerate vacua of the spontaneously broken phase, the scattering theory is formulated at $T < T_c$. Form factors over kinks have not been previously considered in the literature. Although most of the formalism used to deal with ordinary particles goes through with minor adaptations, some interesting new features appear. This happens, in particular, when operators which are non-local with respect to the kinks (e.g. the magnetisation operator) are considered. It turns out that the way non-locality is implemented in the low-temperature formalism is markedly different from that characteristic of the unbroken phase (see [14] for a review of the latter).

The amplitude ratios (1.2) involve both high and low-temperature amplitudes. We will work all the time in the low-temperature phase and rely on duality to get the information about the unbroken phase. In particular, the correlators of the magnetisation at $T > T_c$ will be obtained computing those of the disorder operators at $T < T_c$. An interesting problem, however, arises at this point. As the form factors of the magnetisation and disorder operators are determined quite independently from each other, the relative normalisation between the two operators remains unfixed. This needs to be determined if the two operators are to describe the same physical quantity in the two different phases. The obvious solution to this problem would be to compute the two-point correlators of the two operators and to match the coefficients of their short distance asymptotics. In practice this cannot be done even in the cases in which all the multi-particle form factors are known, simply because nobody knows how to resum the spectral series. Once again the solution is provided by the asymptotic properties of form factors and in particular by the cluster factorisation mentioned above: a two-kink magnetisation matrix element has to factorise into the product of two one-kink disorder form factors, and this requirement fixes the relative normalisation.

We will compute correlation functions and their moments in the q-state Potts model within the two-kink approximation in the form factor approach. When comparison with exact results is possible, the fast convergence of the spectral series is confirmed and results with the typical accuracy mentioned above are obtained. A serious technical difficulty however arises in the computation of the two-kink form factor of the magnetisation operator. Here, the conspiracy of non-locality and non-diagonal scattering results in a new form of monodromy problem expressed by a functional equation that we are not able to solve for generic values of q. The problem can be overcome for $q = 2, 3, 4$ and the complete list of amplitude ratios (1.2) will be presented for this cases. For percolation, however, we can give accurate results only for some universal ratios. For the others we propose an extrapolation in q which, although naive, leads to results in good agreement with the available lattice estimates (series enumerations, Monte Carlo). Quite particular is the case of the ratio of the cluster size amplitudes Γ_+/Γ_-. Here the existing estimates range from 14 to 220, and our extrapolated value ≈ 74 represents a considerable improvement in accuracy.

The layout of the paper is the following. In the next section we briefly review the description of the scaling limit of the q-state Potts model as a perturbed CFT and the associated scattering theory. The one and two-particle form factors of the energy, order and disorder operators are determined in section 3. The results given by the form factor approach in the two-kink approximation for the central charge and scaling dimension sum rules, and for the universal amplitude ratios in the Potts model, are presented in section 4. We discuss percolation in section 5 before summarising our results and making some final remarks in section 6.

2 Scattering theory of the scaling Potts model

The q-state Potts model [17, 18] is the generalisation of the Ising model defined by the lattice Hamiltonian

$$H = -J \sum_{(x,y)} \delta_{s(x),s(y)}\,, \tag{2.1}$$

where the sum is over nearest neighbours and the site variable $s(x)$ can assume q possible values (colours). Clearly, the model is invariant under the group S_q of permutations of the colours. In the ferromagnetic case $J > 0$ we are interested in, the states in which all the sites have the same colour minimise the energy and the system exhibits spontaneous magnetisation at sufficiently low temperatures. There exists a critical temperature T_c above which the thermal fluctuations become dominant and the system is in a disordered phase. If we introduce the variables

$$\sigma_\alpha(x) = \delta_{s(x),\alpha} - \frac{1}{q}\,, \qquad \alpha = 1, 2, \ldots, q \tag{2.2}$$

constrained by the condition

$$\sum_{\alpha=1}^{q} \sigma_\alpha(x) = 0\,, \tag{2.3}$$

the expectation values $\langle\sigma_\alpha\rangle$ differ from zero only in the low-temperature phase and can be used as order parameters.

After defining $x \equiv e^{J/T} - 1$, the partition function of the model can be written in the form

$$Z = \mathrm{Tr}_s \prod_{(x,y)} \left(1 + x\delta_{s(x),s(y)}\right)\,. \tag{2.4}$$

A graph $\mathcal{G}$ on the lattice can be associated to each Potts configuration by drawing a bond between two sites with the same colour. In the above expression, a power of x is associated to each bond in the graph. Taking into account the summation over colours one arrives to the expansion [19]

$$Z = \sum_{\mathcal{G}} q^{N_c} x^{N_b} , \tag{2.5}$$

where N_b is the total number of bonds in the graph $\mathcal{G}$ and N_c is the number of connected components (clusters) in $\mathcal{G}$ (each isolated site is also counted as a cluster). In terms of the partition function (2.5) the q-state Potts model is well defined even for noninteger values of q.

In two dimensions, the Potts model is known to undergo a first order phase transition at $T = T_c$ for $q > 4$ [4]. For $q \leq 4$, however, the transition is continuous and the critical point can be described by a CFT. Relying on the knowledge of some critical exponents [20], Dotsenko and Fateev identified the central charge of this CFT to be [21]

$$c = 1 - \frac{6}{t(t+1)} , \tag{2.6}$$

where the parameter t is related to q by the formula

$$\sqrt{q} = 2 \sin \frac{\pi(t-1)}{2(t+1)} . \tag{2.7}$$

The scaling dimension x_σ of the magnetisation operators $\sigma_\alpha(x)$ (the continuous version of the quantities (2.2)) is identified with that of the primary operator $\phi_{(t-1)/2,(t+1)/2}$ in the CFT

$$x_\sigma = \frac{(t-1)(t+3)}{8t(t+1)} . \tag{2.8}$$

The energy density operator $\varepsilon(x)$ ($\sim \sum_y \delta_{s(x),s(y)}$ on the lattice) coincides with the primary operator $\phi_{2,1}$ with scaling dimension

$$x_\varepsilon = \frac{1}{2}\left(1 + \frac{3}{t}\right) . \tag{2.9}$$

In view of these identifications, the continuum limit of the q-state Potts model ($q \leq 4$) is described by the perturbed CFT

$$\mathcal{A} = \mathcal{A}_{CFT} + \tau \int d^2x \, \varepsilon(x) , \tag{2.10}$$

where $\mathcal{A}_{CFT}$ denotes the critical point action. Since a series of non-trivial integrals of motions is known to survive the deformation of a CFT by the operator $\phi_{2,1}$ [2], the off-critical theory (2.10) is integrable. This circumstance was exploited in Ref. [16] to propose an exact scattering theory which we now briefly review[2].

The low-temperature phase of the model is characterised by the presence of q degenerate vacua that we label by the index $\alpha = 1, 2, \ldots, q$. The elementary excitations are then provided by kinks[3] $K_{\alpha\beta}(\theta)$ interpolating between the two vacua α and β ($\alpha \neq \beta$). The space of physical asymptotic states consists of multi-kink configurations of the type $K_{\alpha_0\alpha_1}(\theta_1)K_{\alpha_1\alpha_2}(\theta_2)\ldots K_{\alpha_{n-1}\alpha_n}(\theta_n)$ ($\alpha_i \neq \alpha_{i+1}$) interpolating between the vacua α_0 and α_n. As a consequence of the invariance under permutations, all the n-kink states fall into two topological sectors: the "neutral" sector, corresponding to $\alpha_0 = \alpha_n$, and the "charged" sector, corresponding to $\alpha_0 \neq \alpha_n$.

Integrability implies that the scattering processes are completely elastic and factorised into the product of two-kink interactions. Since topological charge is conserved, an outgoing two-kink state can only differ from the ingoing one by the vacuum state between the kinks. Hence, the two-kink scattering can formally be described through the Faddeev-Zamolodchikov commutation relation

$$K_{\alpha\gamma}(\theta_1)K_{\gamma\beta}(\theta_2) = \sum_{\delta\neq\alpha,\beta} S^{\gamma\delta}_{\alpha\beta}(\theta_{12})K_{\alpha\delta}(\theta_2)K_{\delta\beta}(\theta_1)\,, \tag{2.11}$$

where $\theta_{12} \equiv \theta_1 - \theta_2$ and $S^{\gamma\delta}_{\alpha\beta}(\theta_{12})$ denotes the two-body scattering amplitude (Fig. 1a). S_q-invariance reduce to four the number of independent amplitudes, two for the charged and two for the neutral topological sector

$$\begin{aligned}
K_{\alpha\gamma}(\theta_1)K_{\gamma\beta}(\theta_2) &= S_0(\theta_{12})\sum_{\delta\neq\gamma} K_{\alpha\delta}(\theta_2)K_{\delta\beta}(\theta_1) + S_1(\theta_{12})K_{\alpha\gamma}(\theta_2)K_{\gamma\beta}(\theta_1) \qquad \alpha\neq\beta\\
K_{\alpha\gamma}(\theta_1)K_{\gamma\alpha}(\theta_2) &= S_2(\theta_{12})\sum_{\delta\neq\gamma} K_{\alpha\delta}(\theta_2)K_{\delta\alpha}(\theta_1) + S_3(\theta_{12})K_{\alpha\gamma}(\theta_2)K_{\gamma\alpha}(\theta_1)\,,
\end{aligned} \tag{2.12}$$

Using the commutation relation (2.11) twice one obtains the unitarity constraint

$$\sum_{\varepsilon} S^{\gamma\varepsilon}_{\alpha\beta}(\theta)S^{\varepsilon\delta}_{\alpha\beta}(-\theta) = \delta^{\gamma\delta}\,, \tag{2.13}$$

which amounts to the set of equations

$$(q-3)S_0(\theta)S_0(-\theta) + S_1(\theta)S_1(-\theta) = 1\,, \tag{2.14}$$

[2]Notice that a different S-matrix for the $\phi_{2,1}$ perturbation of minimal models was determined by Smirnov [22]. We use here the scattering description of Ref. [16] because it is more suitable for analytic continuation in q. Both descriptions must lead to the same results for the correlation functions. See [23] for a detailed discussion of this point in the case of the $\phi_{1,3}$ perturbation.

[3]We parameterise on-shell momenta as $p^\mu = (m\cosh\theta, m\sinh\theta)$, where m denotes the mass of the kinks.

$$(q-4)S_0(\theta)S_0(-\theta)+S_0(\theta)S_1(-\theta)+S_1(\theta)S_0(-\theta)=0\,, \tag{2.15}$$
$$(q-2)S_2(\theta)S_2(-\theta)+S_3(\theta)S_3(-\theta)=1\,, \tag{2.16}$$
$$(q-3)S_2(\theta)S_2(-\theta)+S_3(\theta)S_2(-\theta)+S_2(\theta)S_3(-\theta)=0\;. \tag{2.17}$$

Crossing symmetry provides the relations

$$S_0(\theta)=S_0(i\pi-\theta) \tag{2.18}$$
$$S_1(\theta)=S_2(i\pi-\theta) \tag{2.19}$$
$$S_3(\theta)=S_3(i\pi-\theta)\;. \tag{2.20}$$

Using these constraints together with the Yang-Baxter and bootstrap equations (that we do not need to reproduce here) the following expressions for the four elementary amplitudes were determined in Ref. [16]

$$S_0(\theta)=\frac{\sinh\lambda\theta\,\sinh\lambda(\theta-i\pi)}{\sinh\lambda\left(\theta-\frac{2\pi i}{3}\right)\,\sinh\lambda\left(\theta-\frac{i\pi}{3}\right)}\,\Pi\left(\frac{\lambda\theta}{i\pi}\right) \tag{2.21}$$
$$S_1(\theta)=\frac{\sin\frac{2\pi\lambda}{3}\,\sinh\lambda(\theta-i\pi)}{\sin\frac{\pi\lambda}{3}\,\sinh\lambda\left(\theta-\frac{2i\pi}{3}\right)}\,\Pi\left(\frac{\lambda\theta}{i\pi}\right) \tag{2.22}$$
$$S_2(\theta)=\frac{\sin\frac{2\pi\lambda}{3}\,\sinh\lambda\theta}{\sin\frac{\pi\lambda}{3}\,\sinh\lambda\left(\theta-\frac{i\pi}{3}\right)}\,\Pi\left(\frac{\lambda\theta}{i\pi}\right) \tag{2.23}$$
$$S_3(\theta)=\frac{\sin\lambda\pi}{\sin\frac{\pi\lambda}{3}}\,\Pi\left(\frac{\lambda\theta}{i\pi}\right)\,, \tag{2.24}$$

where λ is related to q as

$$\sqrt{q}=2\sin\frac{\pi\lambda}{3}\,, \tag{2.25}$$

and

$$\Pi(x)=-\frac{\Gamma(1-x)\Gamma(1-\lambda+x)\Gamma\left(\frac{7}{3}\lambda-x\right)\Gamma\left(\frac{4}{3}\lambda+x\right)}{\Gamma(1+x)\Gamma(1+\lambda-x)\Gamma\left(\frac{1}{3}\lambda+x\right)\Gamma\left(\frac{4}{3}\lambda-x\right)}\prod_{k=1}^{\infty}\Pi_k(x)\Pi_k(\lambda-x)\,,$$
$$\Pi_k(x)=\frac{\Gamma(1+2k\lambda-x)\Gamma(2k\lambda-x)\Gamma\left[1+\left(2k-\frac{1}{3}\right)\lambda-x\right]\Gamma\left[\left(2k+\frac{7}{3}\right)\lambda-x\right]}{\Gamma[1+(2k+1)\lambda-x]\Gamma[(2k+1)\lambda-x]\Gamma\left[1+\left(2k-\frac{4}{3}\right)\lambda-x\right]\Gamma\left[\left(2k+\frac{4}{3}\right)\lambda-x\right]}\;.$$

We quote here the following integral representation of the function $\Pi(x)$ which will be useful in the following

$$\Pi\left(\frac{\lambda\theta}{i\pi}\right)=\frac{\sinh\lambda\left(\theta+i\frac{\pi}{3}\right)}{\sinh\lambda(\theta-i\pi)}\,e^{\mathcal{A}(\theta)}\,, \tag{2.26}$$
$$\mathcal{A}(\theta)=\int_0^\infty\frac{dx}{x}\,\frac{\sinh\frac{x}{2}\left(1-\frac{1}{\lambda}\right)-\sinh\frac{x}{2}\left(\frac{1}{\lambda}-\frac{5}{3}\right)}{\sinh\frac{x}{2\lambda}\cosh\frac{x}{2}}\,\sinh\frac{x\theta}{i\pi}\;. \tag{2.27}$$

It is easily seen that the function $\Pi(\lambda\theta/i\pi)$ is free of poles in the physical strip $\mathrm{Im}\theta\in(0,\pi)$ for $q<3$ (i.e. $\lambda<1$). Hence, in this range of q the only poles of the scattering amplitudes in the physical strip are those located at $\theta = 2i\pi/3$ and $\theta = i\pi/3$ and correspond to the appearance of the elementary kink itself as a bound state in the direct and crossed channel, respectively. The residues

$$\begin{aligned}
&\mathrm{Res}_{\theta=2i\pi/3} S_0(\theta) = -\mathrm{Res}_{\theta=i\pi/3} S_0(\theta) = \\
&\mathrm{Res}_{\theta=2i\pi/3} S_1(\theta) = -\mathrm{Res}_{\theta=i\pi/3} S_2(\theta) = i(\Gamma_{KK}^K)^2 \,,
\end{aligned}$$

determine the coupling at the three-kink vertex (Fig. 2a)

$$\Gamma_{KK}^K = \left[\frac{1}{\lambda}\sin\frac{2\pi\lambda}{3}\, e^{\mathcal{A}(i\frac{\pi}{3})}\right]^{1/2} \quad . \tag{2.28}$$

For $q > 3$ ($\lambda > 1$) a direct channel (positive residue) pole located at $\theta = 2i\kappa$,

$$\kappa = \frac{\pi}{2}\left(1 - \frac{1}{\lambda}\right) , \tag{2.29}$$

enters the physical strip in the amplitudes $S_2(\theta)$ and $S_3(\theta)$. Such a pole must be accordingly associated to a (topologically neutral) kink-antikink bound state B with mass

$$m_B = 2m\cos\kappa \ . \tag{2.30}$$

Of course, the amplitudes $S_1(\theta)$ and $S_3(\theta)$ exhibit the corresponding crossed channel (negative residue) pole at $\theta = i\pi - 2i\kappa$. The coupling at the kink-kink-bound state vertex (Fig. 2b) is given by

$$\Gamma_{KK}^B = [-i\mathrm{Res}_{\theta=2i\kappa} S_3(\theta)]^{1/2} = \left[\frac{1}{\lambda}\sin\frac{4\pi\lambda}{3}\,\frac{\sin\lambda\pi}{\sin\frac{\lambda\pi}{3}}\, e^{\mathcal{A}(2i\kappa)}\right]^{1/2} \quad . \tag{2.31}$$

The amplitudes $S_{KB}(\theta)$ and $S_{BB}(\theta)$ (Fig. 1b,c) describing the kink-bound state scattering and the bound state self-interaction are determined by the bootstrap equations

$$\begin{aligned}
&S_{KB}(\theta) = (q-2)S_2(\theta - i\kappa)S_1(\theta + i\kappa) + S_3(\theta - i\kappa)S_3(\theta + i\kappa) \,, \\
&S_{BB}(\theta) = S_{BK}(\theta - i\kappa)S_{BK}(\theta + i\kappa) \,,
\end{aligned} \tag{2.32}$$

and read

$$\begin{aligned}
&S_{BK}(\theta) = t_{1-\kappa/\pi}(\theta)\, t_{2/3-\kappa/\pi}(\theta) \,, \\
&S_{BB}(\theta) = t_{2/3}(\theta)\, t_{1-2\kappa/\pi}(\theta)\, t_{2/3-2\kappa/\pi}(\theta) \,,
\end{aligned} \tag{2.33}$$

in terms of the building blocks

$$t_a(\theta) = \frac{\tanh\frac{1}{2}(\theta + i\pi a)}{\tanh\frac{1}{2}(\theta - i\pi a)} \ . \tag{2.34}$$

The poles located at $\theta = i(\pi - \kappa)$ in S_{BK} and at $\theta = 2\pi/3$ in S_{BB} are bound state poles corresponding to K and B, respectively. The coupling at the BBB vertex (Fig. 2c) is

$$\Gamma^B_{BB} = \left[-i\,\mathrm{Res}_{\theta=2i\pi/3} S_{BB}(\theta)\right]^{1/2} =$$
$$\left[2\sqrt{3}\cot\frac{\pi}{2}\left(1-\frac{1}{\lambda}\right)\cot\pi\left(\frac{1}{2\lambda}-\frac{1}{3}\right)\tan\frac{\pi}{6}\left(4-\frac{3}{\lambda}\right)\tan\frac{\pi}{6}\left(5-\frac{3}{\lambda}\right)\right]^{1/2} . \tag{2.35}$$

It can be shown [24] that, at least in the region $q \leq 4$ we are interested in, the remaining poles in the amplitudes S_{KB} and S_{BB} are associated to multi-scattering processes rather than to new particles[4]. Hence, the elementary kinks and their neutral bound state B are the only particles entering the spectrum of the theory in this range of the parameter q.

3 Form factors

The two-kink form factor of an operator $\Phi(x)$ (Fig. 3) is defined as the matrix element[5]

$$F^{\Phi}_{\alpha\gamma\beta}(\theta_{12}) \equiv \langle 0_\alpha|\Phi(0)|K_{\alpha\gamma}(\theta_1)K_{\gamma\beta}(\theta_2)\rangle\,, \tag{3.1}$$

where $|0_\alpha\rangle$ denotes the vacuum state in which all the sites have colour α. The fundamental equations constraining this matrix element come from the requirements of unitarity and crossing symmetry. The unitarity equation for form factors follows immediately from Eq. (2.11)

$$F^{\Phi}_{\alpha\gamma\beta}(\theta) = \sum_{\delta\neq\alpha,\beta} S^{\gamma\delta}_{\alpha\beta}(\theta) F^{\Phi}_{\alpha\delta\beta}(-\theta)\ . \tag{3.2}$$

The crossing equations read (see the Appendix)

$$F^{\Phi}_{\alpha\gamma\alpha}(\theta+2i\pi) = F^{\Phi}_{\gamma\alpha\gamma}(-\theta)\,, \tag{3.3}$$

for the neutral sector, and

$$F^{\Phi}_{\alpha\gamma\beta}(\theta+2i\pi) = F^{\Phi}_{\alpha\gamma\beta}(-\theta)\,, \qquad \alpha\neq\beta \tag{3.4}$$

for the charged sector. Let us consider how Eqs. (3.2)–(3.4) specialise for the different operators we are interested in.

[4] Multi-scattering singularities usually show up as higher order poles in two dimensions (they lead to anomalous thresholds in four dimensions) [25]. In the present case a simultaneous vanishing of the residues reduces the singularities to simple poles [24].

[5] We will consider only operators which are scalar under Lorentz transformations. Their matrix elements depend on rapidity differences only.

3.1 Energy operator

The energy operator $\varepsilon(x)$ is the operator which perturbs conformal invariance in the action (2.10). It is then proportional to the trace of the stress-energy tensor $\Theta(x)$

$$\Theta(x) = 2\pi(2 - x_\varepsilon)\tau\,\varepsilon(x)\ . \tag{3.5}$$

In view of this relation, we will mainly refer to $\Theta(x)$ in the following.

Being invariant under the S_q symmetry, $\Theta(x)$ couples to the neutral sector of the space of states. Moreover, its two-kink form factors

$$F^\Theta(\theta_{12}) \equiv \langle 0_\alpha|\Theta(0)|K_{\alpha\gamma}(\theta_1)K_{\gamma\alpha}(\theta_2)\rangle\,, \tag{3.6}$$

do not depend on the choice of α and γ. The unitarity and crossing equations can then be written in the form

$$\begin{aligned} F^\Theta(\theta) &= \Lambda(\theta)F^\Theta(-\theta)\,, \\ F^\Theta(\theta+2i\pi) &= F^\Theta(-\theta)\,, \end{aligned} \tag{3.7}$$

where

$$\Lambda(\theta) = (q-2)S_2(\theta) + S_3(\theta) = \frac{\sinh\lambda(\theta+i\pi)}{\sinh\lambda(\theta-i\pi)}\,\mathcal{E}(\theta)\,, \tag{3.8}$$

$$\mathcal{E}(\theta) = \exp\left[\int_0^\infty \frac{dx}{x}\, g_{\mathcal{E}}(x)\sinh\frac{x\theta}{i\pi}\right]\,, \tag{3.9}$$

$$g_{\mathcal{E}}(x) = 2\,\frac{\sinh\frac{x}{3}\cosh\left(\frac{1}{3}-\frac{1}{\lambda}\right)\frac{x}{2}}{\sinh\frac{x}{2\lambda}\cosh\frac{x}{2}}\ . \tag{3.10}$$

Since

$$\frac{\sinh\lambda(\theta+i\pi a)}{\sinh\lambda(\theta-i\pi a)} = -\exp\left[\int_0^\infty \frac{dx}{x}\, g_a(x)\sinh\frac{x\theta}{i\pi}\right]\,, \tag{3.11}$$

$$g_a(x) = 2\,\frac{\sinh\left(\frac{1}{2\lambda}-a\right)x}{\sinh\frac{x}{2\lambda}}\,, \tag{3.12}$$

it is easily checked that the function

$$F_\Lambda(\theta) = -i\sinh\frac{\theta}{2}\,\exp\left\{\int_0^\infty \frac{dx}{x}\,\frac{g_1(x)+g_{\mathcal{E}}(x)}{\sinh x}\,\sin^2\frac{(i\pi-\theta)x}{2\pi}\right\}\,,\qquad \lambda<1 \tag{3.13}$$

solves the system of functional equations (3.7). We note here for later convenience the asymptotic behaviour

$$F_\Lambda(\theta) \sim \exp\left(1-\frac{2\lambda}{3}\right)\theta\,,\qquad \theta\longrightarrow+\infty\ . \tag{3.14}$$

The integral in (3.13) is convergent on the real θ-axis for $\lambda < 1$. For $\lambda > 1$ one needs to use the analytically continued expression

$$F_\Lambda(\theta) = \frac{-\cos^2\kappa}{\sinh\frac{1}{2}(\theta - 2i\kappa)\sinh\frac{1}{2}(\theta + 2i\kappa)}\Upsilon(\theta)\,, \qquad \lambda > 1 \tag{3.15}$$

$$\Upsilon(\theta) = -i\sinh\frac{\theta}{2}\exp\left\{\int_0^\infty \frac{dx}{x}\,\frac{g_{1-1/\lambda}(x) + g_\mathcal{E}(x)}{\sinh x}\,\sin^2\frac{(i\pi - \theta)x}{2\pi}\right\}\,, \tag{3.16}$$

which explicitly exhibits the pole at $\theta = 2i\kappa$ in the physical strip corresponding to the kink-antikink bound state B (Fig. 4a). Taking into account the normalisation condition

$$F^\Theta(i\pi) = 2\pi m^2\,, \tag{3.17}$$

we can write

$$F^\Theta(\theta) = 2\pi m^2 F_\Lambda(\theta)\,, \tag{3.18}$$

$$F_B^\Theta \equiv \langle 0_\alpha|\Theta(0)|B\rangle = \frac{1}{i\Gamma_{KK}^B}\mathrm{Res}_{\theta=2i\kappa}F^\Theta(\theta) = \frac{\pi m_B^2}{\sin 2\kappa}\,\frac{\Upsilon(2i\kappa)}{\Gamma_{KK}^B}\,. \tag{3.19}$$

Let us finally determine the matrix element $F_{BB}^\Theta(\theta_{12}) \equiv \langle 0_\alpha|\Theta(0)|B(\theta_1)B(\theta_2)\rangle$. It has to satisfy the monodromy equations

$$\begin{aligned} F_{BB}^\Theta(\theta) &= S_{BB}(\theta)F_{BB}^\Theta(-\theta)\,,\\ F_{BB}^\Theta(\theta + 2i\pi) &= F_{BB}^\Theta(-\theta)\,. \end{aligned} \tag{3.20}$$

The solution to these equations with the expected pole structure is

$$F_{BB}^\Theta(\theta) = [a\cosh\theta + b]\sinh\frac{\theta}{2}\,\mathcal{R}_{2/3}(\theta)\mathcal{R}_{1-2\kappa/\pi}(\theta)\mathcal{R}_{2/3-2\kappa/\pi}(\theta)\,, \tag{3.21}$$

where

$$\mathcal{R}_a(\theta) = \frac{1}{\cosh\theta - \cos\pi a}\exp\left\{2\int_0^\infty \frac{dx}{x}\,\frac{\cosh\left(a - \frac{1}{2}\right)x}{\cosh\frac{x}{2}\sinh x}\,\sin^2\frac{(i\pi - \theta)x}{2\pi}\right\}\,. \tag{3.22}$$

The coefficients a and b in (3.21) are uniquely determined by the residue equation

$$-i\,\mathrm{Res}_{\theta=2i\pi/3}F_{BB}^\Theta(\theta) = \Gamma_{BB}^B F_B^\Theta\,, \tag{3.23}$$

and the normalisation condition

$$F_{BB}^\Theta(i\pi) = 2\pi m_B^2\,. \tag{3.24}$$

Having computed F_B^Θ and $F_{BB}^\Theta(\theta)$, we can determine the vacuum expectation value $F_0^\Theta \equiv \langle 0_\alpha|\Theta|0_\alpha\rangle$ through the asymptotic factorisation relation [12]

$$\lim_{\theta\longrightarrow\infty} F_{BB}^\Theta(\theta) = \frac{(F_B^\Theta)^2}{F_0^\Theta}\,. \tag{3.25}$$

We find

$$F_0^\Theta = -\frac{\pi \sin\frac{\pi}{2\lambda}}{\sqrt{3}\sin\pi\left(\frac{1}{3}+\frac{1}{2\lambda}\right)}\, m^2\,, \tag{3.26}$$

in perfect agreement with the result obtained in [26] using the thermodynamic Bethe ansatz (TBA).

3.2 Disorder operators

The theory contains disorder operators which are dual to the $q-1$ independent order operators $\sigma_\alpha(x)$ and have the same scaling dimension (2.8). Following the usual interpretation of the disorder operator as the kink creation operator, we denote by $\mu_\beta(x)$ the disorder operator which, acting on the vacuum state $|0_\alpha\rangle$, creates states interpolating between the vacua α and β ($\beta \neq \alpha$). The two-kink form factor

$$\langle 0_\alpha|\mu_\beta(0)|K_{\alpha\gamma}(\theta_1)K_{\gamma\delta}(\theta_2)\rangle = \delta_{\beta\delta}F^\mu(\theta_{12})\,, \quad \beta\neq\alpha \tag{3.27}$$

does not depend on the intermediate index γ and satisfies the unitarity and crossing equations

$$\begin{aligned} F^\mu(\theta) &= \Sigma(\theta)F^\mu(-\theta)\,,\\ F^\mu(\theta+2i\pi) &= F^\mu(-\theta)\,, \end{aligned} \tag{3.28}$$

with

$$\Sigma(\theta) = (q-3)S_0(\theta)+S_1(\theta) = \frac{\sinh\lambda\left(\theta+\frac{2i\pi}{3}\right)}{\sinh\lambda\left(\theta-\frac{2i\pi}{3}\right)}\,\mathcal{E}(\theta)\ . \tag{3.29}$$

Hence, it can be written as

$$F^\mu(\theta) = Z_\mu\,\frac{F_\Sigma(\theta)}{\sinh\frac{1}{2}\left(\theta+\frac{2i\pi}{3}\right)\sinh\frac{1}{2}\left(\theta-\frac{2i\pi}{3}\right)}\,, \tag{3.30}$$

where

$$F_\Sigma(\theta) = -i\sinh\frac{\theta}{2}\,\exp\left\{\int_0^\infty \frac{dx}{x}\,\frac{g_{2/3}(x)+g_{\mathcal{E}}(x)}{\sinh x}\,\sin^2\frac{(i\pi-\theta)x}{2\pi}\right\}\,, \tag{3.31}$$

is the solution of the system (3.28) without poles in the physical strip; the pole at $\theta = 2i\pi/3$ corresponding to the φ^3-property of the kinks has been inserted explicitly. The function $F_\Sigma(\theta)$ behaves asymptotically as

$$F_\Sigma(\theta) \sim \exp\left(1-\frac{\lambda}{3}\right)\theta\,, \qquad \theta \longrightarrow +\infty\ . \tag{3.32}$$

The one-kink form factor

$$\langle 0_\alpha|\mu_\beta(0)|K_{\alpha\gamma}\rangle = \delta_{\beta\gamma}F_K^\mu\,, \quad \beta\neq\alpha \tag{3.33}$$

is given by (Fig. 4b)

$$F_K^\mu = \frac{1}{i\Gamma_{KK}^K}\mathrm{Res}_{\theta=2i\pi/3}F^\mu(\theta) = -\frac{4F_\Sigma(2i\pi/3)}{\sqrt{3}\,\Gamma_{KK}^K}\,Z_\mu\ . \tag{3.34}$$

3.3 Magnetisation operators

The magnetisation operators $\sigma_\alpha(x)$ couple to neutral states. Since σ_α carries an index, its matrix elements are not in general invariant under permutations. In the two-kink form factor we distinguish the three components

$$\langle 0_\alpha|\sigma_\beta(0)|K_{\alpha\gamma}(\theta_1)K_{\gamma\alpha}(\theta_2)\rangle = \delta_{\beta\alpha}F_1^\sigma(\theta_{12}) + \delta_{\beta\gamma}F_2^\sigma(\theta_{12}) + (1-\delta_{\beta\alpha}-\delta_{\beta\gamma})F_3^\sigma(\theta_{12})\,, \quad (3.35)$$

which eq. (2.3) relates as

$$F_1^\sigma(\theta) + F_2^\sigma(\theta) + (q-2)F_3^\sigma(\theta) = 0\ . \quad (3.36)$$

Eq. (3.2) provides the relations

$$\begin{aligned} F_1^\sigma(\theta) &= \Lambda(\theta)F_1^\sigma(-\theta)\,, \\ F_2^\sigma(\theta) &= S_3(\theta)F_2(-\theta) + (q-2)S_2(\theta)F_3^\sigma(-\theta)\,, \\ F_3^\sigma(\theta) &= S_2(\theta)F_2^\sigma(-\theta) + [(q-3)S_2(\theta)+S_3(\theta)]F_3^\sigma(-\theta)\ . \end{aligned} \quad (3.37)$$

The crossing equations

$$\begin{aligned} F_1^\sigma(\theta+2i\pi) &= F_2^\sigma(-\theta)\,, \\ F_3^\sigma(\theta+2i\pi) &= F_3^\sigma(-\theta)\,, \end{aligned} \quad (3.38)$$

are an immediate consequence of (3.3). Recalling that the function $F_\Lambda(\theta)$ defined in (3.13) satisfies Eqs. (3.7), we parameterise $F_1^\sigma(\theta)$ as

$$F_1^\sigma(\theta) = \Omega(\theta)F_\Lambda(\theta)\,; \quad (3.39)$$

then, the first of (3.37) is automatically fulfilled provided the function $\Omega(\theta)$ satisfies

$$\Omega(\theta) = \Omega(-\theta)\ . \quad (3.40)$$

Simple manipulations involving Eqs. (3.36)-(3.38) provide the basic equation that $\Omega(\theta)$ should satisfy together with the previous one:

$$\Omega(\theta) = \left[\frac{S_3(\theta)}{S_2(\theta)} - 1\right]\Omega(2i\pi+\theta) - \frac{\Lambda(\theta)}{S_2(\theta)}\,\Omega(2i\pi-\theta)\,, \quad (3.41)$$

or, more explicitly,

$$\Omega(\theta) = \frac{\sinh\lambda(\theta-i\pi)\,\Omega(2i\pi+\theta) + \sinh\lambda(\theta+i\pi)\,\Omega(2i\pi-\theta)}{2\cos\frac{\pi\lambda}{3}\,\sinh\lambda\theta}\ . \quad (3.42)$$

Unfortunately, we do not know how to solve this equation for generic values of λ. Before turning to the solution for specific values of this parameter, we collect some additional physical information about the magnetisation matrix elements.

The order operators $\sigma_\alpha(x)$ are non-local with respect to the disorder operators $\mu_\beta(x)$ which interpolate the kinks. This kind of mutual non-locality is known to lead to the presence of an "annihilation pole" at $\theta = i\pi$ in the two-particle form factor [9, 14, 27]. We show in the Appendix that in our case the residue on such a pole takes the form

$$- i \, \text{Res}_{\theta_1-\theta_2=i\pi} \langle 0_\alpha | \sigma_\gamma(0) | K_{\alpha\beta}(\theta_1) K_{\beta\alpha}(\theta_2) \rangle = \langle 0_\alpha | \sigma_\gamma | 0_\alpha \rangle - \langle 0_\beta | \sigma_\gamma | 0_\beta \rangle \, . \tag{3.43}$$

Denoting

$$F_0^\sigma \equiv \langle 0_\alpha | \sigma_\alpha | 0_\alpha \rangle \, , \tag{3.44}$$

and taking into account the constraint (2.3), the order parameter can be written as

$$\langle 0_\alpha | \sigma_\gamma | 0_\alpha \rangle = \frac{F_0^\sigma}{q-1} \left(q \delta_{\gamma\alpha} - 1 \right) \, . \tag{3.45}$$

Hence, for the different components of the two-kink form factor we conclude

$$\begin{aligned} -i \, \text{Res}_{\theta=i\pi} F_1^\sigma(\theta) &= -i \, \text{Res}_{\theta=i\pi} F_2^\sigma(\theta) = \frac{q}{q-1} F_0^\sigma \\ -i \, \text{Res}_{\theta=i\pi} F_3^\sigma(\theta) &= 0 \end{aligned} \tag{3.46}$$

For $\lambda > 1$ the magnetisation has a one-particle form factor on the bound state B. Denoting

$$F_B^\sigma \equiv \langle 0_\alpha | \sigma_\alpha(0) | B \rangle \, , \tag{3.47}$$

we can write

$$\langle 0_\alpha | \sigma_\gamma(0) | B \rangle = \frac{F_B^\sigma}{q-1} \left(q \delta_{\gamma\alpha} - 1 \right) \, . \tag{3.48}$$

These one-particle form factors can be obtained from the two-kink matrix elements through the residue equation

$$- i \, \text{Res}_{\theta_1-\theta_2=2i\kappa} \langle 0_\alpha | \sigma_\gamma(0) | K_{\alpha\beta}(\theta_1) K_{\beta\alpha}(\theta_2) \rangle = \Gamma_{KK}^B \, \langle 0_\alpha | \sigma_\gamma(0) | B \rangle \, . \tag{3.49}$$

The last two equations, together with the crossing relation (3.38), immediately lead to the following identity for the function $\Omega(\theta)$ introduced in (3.39)

$$\Omega(2i\pi - 2i\kappa) = -\frac{1}{q-1} \, \Omega(2i\kappa) \, . \tag{3.50}$$

Although obtained for $\lambda > 1$, this relation is expected to hold for generic values of the parameter.

The matrix elements of scaling operators in unitary theories must satisfy the asymptotic bound obtained in Ref. [10]. For the kink-kink form factor of the magnetisation operators this reads

$$\lim_{\theta \longrightarrow +\infty} F_1^\sigma(\theta) \leq constant \, e^{x_\sigma \theta/2} \, . \tag{3.51}$$

The functional equation (3.42) becomes trivial at the points $q = 2, 3, 4$. Let us investigate these cases in more detail.

q=2

For this value of q (corresponding to $\lambda = 3/4$) the solution to Eqs. (3.40) and (3.42) satisfying the asymptotic bound (3.51) is simply

$$\Omega(\theta) = \frac{iF_0^\sigma}{\cosh\frac{\theta}{2}}, \tag{3.52}$$

where the normalisation has been fixed using Eq. (3.46). The identity (3.50) is automatically satisfied. Since $F_\Lambda(\theta) = -i\sinh\theta/2$, one obtains

$$F_1^\sigma(\theta) = -F_2^\sigma(\theta) = iF_0^\sigma \tanh\frac{\theta}{2}, \tag{3.53}$$

which is the well known result usually obtained in the high temperature formalism [28, 14].

q=3

For $\lambda = 1$ Eq. (3.42) reduces to

$$\Omega(\theta) = -[\Omega(\theta + 2i\pi) + \Omega(\theta - 2i\pi)]\ . \tag{3.54}$$

Together with Eqs. (3.40), (3.51) and (3.46), it fixes[6]

$$\Omega(\theta) = -\frac{\sqrt{3}}{2}\, F_0^\sigma\, \frac{\cosh\frac{\theta}{6}}{\cosh\frac{\theta}{2}}\ . \tag{3.55}$$

Again Eq. (3.50) is automatically satisfied.

q=4

For $\lambda = 3/2$ Eq. (3.42) becomes

$$\Omega(\theta + 2i\pi) = \Omega(\theta - 2i\pi)\ . \tag{3.56}$$

This time Eq. (3.50) has to be enforced together with the other constraints in order to fix the solution

$$\Omega(\theta) = -\frac{2F_0^\sigma}{3\sqrt{3}}\, \frac{\cosh\frac{\theta}{2} + \sqrt{3}}{\cosh\frac{\theta}{2}}\ . \tag{3.57}$$

We conclude this section considering the problem of the relative normalisation of the order and disorder operators we discussed in the introduction. Solving Eq. (3.42) for $\theta \longrightarrow \infty$ one easily finds that $\Omega(\theta)$ has to behave in this asymptotic limit as $e^{\eta\theta}$, with

[6] An equivalent result can be obtained working with the high temperature scattering theory which is defined in terms of particles rather than kinks [29, 30].

$\eta = \lambda/3 + k$ *or* $\eta = 2\lambda/3 + k$ (k integer). The solutions obtained at $q = 2, 3, 4$ indicate that the correct choice is $\eta = 2\lambda/3 - 1$. Combining this result with the asymptotic behaviour (3.14) for $F_\Lambda(\theta)$, we conclude that $F_1^\sigma(\theta)$ goes to a constant in the limit of large rapidity difference. Hence, according to the discussion of Ref. [12], we expect the following asymptotic factorisation equation to hold

$$\lim_{\theta \longrightarrow \infty} |F_1^\sigma(\theta)| = \frac{(F_K^\mu)^2}{F_0^\sigma} \,. \tag{3.58}$$

In the next section, this relation will enable us to express the critical amplitudes in terms of a single arbitrary normalisation constant (say F_0^σ) which in turn cancels out when the universal amplitude ratios are considered.

4 Correlation functions and amplitude ratios

The knowledge of the form factors allows to express correlation functions as spectral sums over intermediate asymptotic states. For the two-point euclidean correlator of two scalar operators $\Phi_1(x)$ and $\Phi_2(x)$ we have

$$\langle \Phi_1(x)\Phi_2(0)\rangle = \sum_{n=0}^{\infty} \int_{\theta_1 > \ldots > \theta_n} \frac{d\theta_1}{2\pi} \ldots \frac{d\theta_n}{2\pi} \langle 0|\Phi_1(0)|n\rangle\langle n|\Phi_2(0)|0\rangle e^{-|x|E_n} \,, \tag{4.1}$$

where E_n denotes the total energy of the n-particle state $|n\rangle$.

Clearly, the above expression is a large distance expansion: while the intermediate states with the lowest total mass provide the dominant contribution for large separations, in principle the whole series should be resummed in order to reproduce the correct ultraviolet behaviour. Relying on the properties of fast convergence of the spectral series we mentioned in the introduction, we consider partial sums of the series truncated at the level of the two-kink intermediate state (the relevant form factors have been computed in the previous section). For example, the expansions for the two-point correlators of the order and disorder parameters read

$$\begin{aligned} \langle 0_\alpha|\sigma_\beta(x)\sigma_\gamma(0)|0_\alpha\rangle &= \frac{(q\delta_{\beta\alpha}-1)(q\delta_{\gamma\alpha}-1)}{(q-1)^2}\left[|F_0^\sigma|^2 + H(q-3)\frac{|F_B^\sigma|^2}{\pi}K_0(m_B|x|)\right] + \\ &+ \frac{1}{\pi^2}\int_0^\infty d\theta\, f_{\beta\gamma}^\alpha(2\theta)\, K_0(2m|x|\cosh\theta) + \ldots \qquad (4.2) \\ \langle 0_\alpha|\mu_\beta(x)\mu_\gamma(0)|0_\alpha\rangle &= \delta_{\beta\gamma}\left[\frac{|F_k^\mu|^2}{\pi}K_0(m|x|) + \right. \\ &+ \left. \frac{q-2}{\pi^2}\int_0^\infty d\theta\, |F^\mu(2\theta)|^2\, K_0(2m|x|\cosh\theta)\right] + \ldots \qquad \beta,\gamma \neq \alpha (4.3) \end{aligned}$$

where we introduced the function

$$
\begin{aligned}
f^{\alpha}_{\beta\gamma}(\theta) &= [\delta_{\beta\alpha}\delta_{\gamma\alpha}(q-1) - \delta_{\beta\alpha}(1-\delta_{\gamma\alpha}) - (1-\delta_{\beta\alpha})\delta_{\gamma\alpha}]|F^{\sigma}_{1}(\theta)|^2 + \\
&+ \delta_{\beta\gamma}(1-\delta_{\beta\alpha})[|F^{\sigma}_{2}(\theta)|^2 + (q-2)|F^{\sigma}_{3}(\theta)|^2] + \\
&+ (1-\delta_{\beta\gamma})(1-\delta_{\beta\alpha})[F^{\sigma}_{2}(\theta)F^{\sigma}_{3}(-\theta) + F^{\sigma}_{3}(\theta)F^{\sigma}_{2}(-\theta) + (q-3)|F^{\sigma}_{3}(\theta)|^2] ,
\end{aligned}
\tag{4.4}
$$

and the step function $H(y)$ which equals 1 for $y > 0$ and is 0 otherwise. The terms we omitted in the r.h.s. are of order $e^{-3m|x|}$ for $|x|$ large (at least for $q \leq 3$). We show immediately that this level of approximation is sufficient to obtain remarkably accurate numerical results for the quantities we need to evaluate in this paper, namely moments of two-point correlators of the type $\int d^2x|x|^p\langle\Phi_1(x)\Phi_2(0)\rangle_c$. An interesting check is provided by the following sum rules for the central charge of the ultraviolet CFT [31] and the scaling dimension of the magnetisation operator [12]

$$
\begin{aligned}
c &= \frac{3}{4\pi}\int d^2x\,|x|^2\langle 0_\alpha|\Theta(x)\Theta(0)|0_\alpha\rangle_c\,, \\
x_\sigma &= -\frac{1}{2\pi\langle 0_\alpha|\sigma_\gamma|0_\alpha\rangle}\int d^2x\,\langle 0_\alpha|\Theta(x)\sigma_\gamma(0)|0_\alpha\rangle_c\,,
\end{aligned}
\tag{4.5}
$$

where $\langle\cdots\rangle_c$ denotes connected correlators. The result obtained for c as a function of q using the truncated spectral expansion of the trace-trace correlator is shown in Fig. 5 and compared with the exact value (2.6)[7]. The numerical results obtained for c and Δ_σ for $q = 2, 3, 4$ are listed in Table 1. Since the energy operator in the two-dimensional Ising model couples to the two-kink state only, the results obtained for $q = 2$ are exact. In the other cases the observed deviation from the exact result is of few percent at most.

The critical amplitudes (1.1) are linked to the off-critical correlators by the relations

$$
C = \int d^2x\,\langle 0_\alpha|\varepsilon(x)\varepsilon(0)|0_\alpha\rangle_c\ , \tag{4.6}
$$

$$
M = \langle 0_\alpha|\sigma_\alpha|0_\alpha\rangle\,, \tag{4.7}
$$

$$
\chi = \int d^2x\,\langle 0_\alpha|\sigma_\alpha(x)\sigma_\alpha(0)|0_\alpha\rangle_c\ , \tag{4.8}
$$

$$
\xi^2 = \frac{1}{4}\,\frac{\int d^2x\,|x|^2\langle 0_\alpha|\sigma_\alpha(x)\sigma_\alpha(0)|0_\alpha\rangle_c}{\int d^2x\,\langle 0_\alpha|\sigma_\alpha(x)\sigma_\alpha(0)|0_\alpha\rangle_c}\ . \tag{4.9}
$$

[7] It can be appreciated from the figure that the values of c obtained from this first contribution to the spectral sum are slightly larger than the exact result in the range $1 < q < 2$. At first sight this may seem strange since each intermediate state which remains to be included into the sum will give a positive contribution (an integral over the modulus square of the corresponding trace form factor). Notice however that the number of three-kink intermediate states in the neutral topological sector is $(q-1)(q-2)$ and becomes negative when $1 < q < 2$!

The dimensional parameter entering our S-matrix approach is the mass m of the kink. It is related to the reduced temperature τ appearing in (1.1) as[8]

$$m = m_0\,\tau^{\nu}\ . \tag{4.10}$$

Equation (4.9) defines the "second moment" correlation length. In the literature the so-called "true" correlation length ξ_t is often considered which is defined through the large distance asymptotic decay of the spin-spin correlators

$$\langle\sigma(x)\sigma(0)\rangle \sim \exp(-|x|/\xi_t)\,, \qquad |x|\longrightarrow\infty\ . \tag{4.11}$$

It follows from (4.3) together with duality that $\xi_t = 1/m$ at $T > T_c$. At $T < T_c$, Eq. (4.2) implies instead $\xi_t = 1/2m$ for $q \leq 3$, and $\xi_t = 1/m_B$ for $3 < q \leq 4$.

Since the zeroth moment of the energy-energy correlator appears in the scaling dimension sum rule

$$x_\varepsilon = -\frac{1}{2\pi\langle 0_\alpha|\varepsilon|0_\alpha\rangle}\int d^2x\,\langle 0_\alpha|\Theta(x)\varepsilon(0)|0_\alpha\rangle_c\,, \tag{4.12}$$

the specific heat amplitudes $A_\pm$ can be computed exactly as[9]

$$A_\pm = -\alpha(1-\alpha)(2-\alpha)\left(\frac{m_0}{m}\right)^2\frac{F_0^\Theta}{4\pi}\,, \tag{4.13}$$

where F_0^Θ is given in (3.26). The equality of A_+ and A_- follows from the fact that the energy operator $\varepsilon(x)$ simply changes sign under the duality transformation exchanging the low and high temperature phases.

Remembering the definition (3.45), the magnetisation amplitude B can be written as

$$B = \left(\frac{m_0}{m}\right)^{x_\sigma} F_\sigma^0\ . \tag{4.14}$$

The susceptibility and correlation length amplitudes can be evaluated using the expansions (4.2) and (4.3) for the correlators and the corresponding form factors. By duality, the high temperature amplitudes are obtained substituting σ_α by μ_β ($\beta \neq \alpha$) in Eqs. (4.8) and (4.9). Some of the values obtained in this way for integer q are listed in Table 2. In particular, the results for $\xi_0^+ m_0$ show that the "second moment" and "true" correlation lengths differ very slightly from each other at $T > T_c$.

Table 3 collects the results we find for the amplitude ratios at $q = 2, 3, 4$. Excepting A_+/A_-, we are not aware of previous reliable estimates of these ratios for the cases

[8]The dimensionless number m_0 is exactly computable through the TBA [26]; we will not need it here.

[9]The specific heat diverges logarithmically in the Ising model ($\alpha = 0$) and the definition of the amplitudes is accordingly modified to $C \simeq -A_\pm \ln\tau$. In the limit $q \longrightarrow 2$, Eq. (4.13) gives the correct result $A_\pm = m_0^2/2\pi$.

$q = 3, 4$. At $q = 2$, however, comparison with the known exact results [32, 33] (see also [34])

$$\begin{aligned} \Gamma_+/\Gamma_- &= 37.69365.. , \\ R_C &= 0.318569.. , \\ \xi_0^+/\xi_0^- &= 3.16.. , \end{aligned} \tag{4.15}$$

further confirms the remarkable accuracy of the results yielded by our two-kink approximation.

5 Percolation

Percolation is the purely geometrical problem (no temperature involved) in which bonds[10] are randomly distributed on a lattice with occupation probability p [35]. A set of bonds forming a connected path on the lattice is called a *cluster*. There exist a critical value p_c of the occupation probability above which an infinite cluster appears in the system; p_c is called the *percolation threshold*. If N is the total number of bonds in the lattice, the probability of a configuration with N_b occupied bonds is $p^{N_b}(1-p)^{N-N_b}$. Hence, the average of a quantity X over all configurations $\mathcal{G}$ is

$$\langle X \rangle = \sum_{\mathcal{G}} X \, p^{N_b}(1-p)^{N-N_b} \, . \tag{5.1}$$

Let x and y denote the positions of two bonds on the lattice and consider the function $C(x, y)$ which takes value 1 if x and y belong to the same cluster, and 0 otherwise. Then, the function $g(x, y) = \langle C(x, y) \rangle$ is the probability that x and y belong to the same cluster and is called *pair connectivity*. If we denote by P the probability that a bond belongs to the infinite cluster ($P = 0$ at $p < p_c$), clearly we have

$$\lim_{|x-y| \longrightarrow \infty} g(x, y) = P^2 \, . \tag{5.2}$$

Since $\sum_y C(x, y)$ counts the total number of bonds in the cluster x belongs to, the *mean cluster size* can be obtained as

$$S = \sum_y g_c(0, y) \, , \tag{5.3}$$

where the subscript c means that the connected part of the pair connectivity is taken in order to get rid of the contribution coming from the infinite cluster at $p > p_c$. The second

[10] We refer here to *bond* percolation; in *site* percolation, sites rather than bonds are occupied with probability p. Of course, all universal results are independent of this distinction.

moment correlation length is also naturally defined in terms of the pair connectivity as

$$\tilde{\xi}^2 = \frac{1}{2d} \frac{\sum_x |x|^2 g_c(x,0)}{\sum_x g_c(x,0)} \ . \tag{5.4}$$

The following relation with the q-state Potts model is particularly important for the theoretical study of percolation processes [36]. Remembering (2.5), the average of a quantity X in the q-state Potts model can be written as

$$\langle X \rangle_q = Z^{-1} \sum_{\mathcal{G}} X \, q^{N_c} x^{N_b} \ ; \tag{5.5}$$

hence, it is sufficient to make the formal identification $x = p/(1-p)$ to see that $\langle X \rangle_1$ coincides with the percolation average (5.1). For example, the *mean cluster number* in percolation can be expressed as

$$\langle N_c \rangle = \lim_{q \longrightarrow 1} \frac{\partial \ln Z}{\partial q} = \lim_{q \longrightarrow 1} \frac{\ln Z}{q-1} \ . \tag{5.6}$$

To proceed further with this mapping we need a representation of pair connectivity in the Potts model formalism. For this purpose observe that the insertion of a delta function $\delta_{s(x)\alpha}$ in a Potts configuration fixes to α the colour of the cluster x belongs to. Hence, at $T > T_c$,

$$\begin{aligned} \langle \delta_{s(x)\alpha} \rangle_q &= \frac{1}{q} \, , \\ \langle \delta_{s(x)\alpha} \delta_{s(y)\alpha} \rangle_q &= \frac{1}{q} \, g_q(x,y) + \frac{1}{q^2} \left[1 - g_q(x,y) \right] , \end{aligned} \tag{5.7}$$

where $g_q(x,y)$ is the probability that x and y belong to the same cluster. Using the definition (2.2) we immediately find

$$\langle \sigma_\alpha(x) \sigma_\alpha(y) \rangle_q = \frac{q-1}{q^2} \, g_q(x,y) \, , \tag{5.8}$$

from which we see that the pair connectivity in percolation can be obtained as

$$g(x,y) = \lim_{q \longrightarrow 1} \frac{1}{q-1} \langle \sigma_\alpha(x) \sigma_\alpha(y) \rangle_q \ . \tag{5.9}$$

Comparison with (5.2), (5.3) and (5.4) provides the following relations with the magnetisation, susceptibility and correlation length in the Potts model

$$\begin{aligned} P &= \lim_{q \longrightarrow 1} \frac{M}{\sqrt{q-1}} \, , \\ S &= \lim_{q \longrightarrow 1} \frac{\chi}{q-1} \, , \\ \tilde{\xi} &= \xi|_{q=1} \ . \end{aligned} \tag{5.10}$$

Together with (5.6), they imply the following critical behaviour near the percolation threshold

$$
\begin{aligned}
&\frac{\langle N_c\rangle}{N} \simeq \tilde{A}_\pm |p_c - p|^{2-\alpha} \,, \\
&P \simeq \tilde{B}(p - p_c)^\beta \,, \\
&S \simeq \tilde{\Gamma}_\pm |p_c - p|^{-\gamma} \,, \\
&\tilde{\xi} \simeq \tilde{\xi}_0^\pm |p_c - p|^{-\nu} \,,
\end{aligned}
\tag{5.11}
$$

where the critical exponents are those of the Potts model evaluated at $q = 1$, and the amplitudes are related to the Potts amplitudes as[11].

$$
\begin{aligned}
\tilde{A}_\pm &= \lim_{q \longrightarrow 1} \frac{A_\pm \tau_0^{2-\alpha}}{(q-1)\alpha(1-\alpha)(2-\alpha)} \,, \\
\tilde{B} &= \lim_{q \longrightarrow 1} \frac{B\tau_0^\beta}{\sqrt{q-1}} \,, \\
\tilde{\Gamma}_\pm &= \lim_{q \longrightarrow 1} \frac{\Gamma_\pm \tau_0^{-\gamma}}{q-1} \,, \\
\tilde{\xi}_0^\pm &= \xi_0^\pm \tau_0^{-\nu}\Big|_{q=1} \;;
\end{aligned}
\tag{5.12}
$$

here we introduced the non-universal positive constant τ_0 entering the relation $\tau \simeq \tau_0(p_c - p)$. It follows from these relations that the following combinations of critical amplitudes in percolation

$$
\begin{aligned}
&\tilde{A}_+/\tilde{A}_- = \lim_{q \longrightarrow 1} A_+/A_- \,, \\
&\tilde{\Gamma}_+/\tilde{\Gamma}_- = \lim_{q \longrightarrow 1} \Gamma_+/\Gamma_- \,, \\
&\tilde{\xi}_0^+/\tilde{\xi}_0^- = \xi_0^+/\xi_0^- |_{q=1} \,, \\
&\tilde{R}_C \equiv \alpha(1-\alpha)(2-\alpha)\tilde{A}_+\tilde{\Gamma}_+/\tilde{B}^2 = \lim_{q \longrightarrow 1} \frac{R_C}{q-1} \,, \\
&\tilde{R}_\xi^+ \equiv [\alpha(1-\alpha)(2-\alpha)\tilde{A}_+]^{1/d}\tilde{\xi}_0^+ = \lim_{q \longrightarrow 1} \frac{R_\xi^+}{(q-1)^{1/d}} \,,
\end{aligned}
\tag{5.13}
$$

are universal and can be computed from the $q \longrightarrow 1$ limit of the Potts amplitude ratios.

Let us see which results we can obtain for percolation in $d = 2$ from our study of the q-state Potts model of the previous sections. From (4.13) we find

$$
\tilde{A}_\pm = -\left.\frac{m_0^2 \tau_0^{2-\alpha}}{2\sqrt{3}}\right|_{q=1} \,. \tag{5.14}
$$

The negative sign of this amplitude agrees with the series result of Domb and Pearce [37] which is listed in Table 4 together with other series and Monte Carlo estimates of

[11] Notice that the labels + and − refer to $p < p_c$ and $p > p_c$, respectively.

percolation amplitudes[12]. The equality of $\tilde{A}_+$ and $\tilde{A}_-$ follows from duality which is also a crucial ingredient of the Domb and Pearce lattice calculation[13]. When combined with the value of ξ_0^+ quoted in Table 2, (5.14) gives the result ($\alpha = -2/3$ at $q = 1$)

$$\tilde{R}_\xi^+ \simeq 0.926\,, \tag{5.15}$$

which we think substantially improves the value around 1.1 one can extract from the lattice amplitudes in Table 4.

We cannot compute directly the ratios involving the amplitudes $\tilde{B}$, $\tilde{\Gamma}_-$ and $\tilde{\xi}_0^-$ simply because we are not able to solve the functional equation (3.42) for $q = 1$. What we can do is to attempt a naive quadratic extrapolation at $q = 1$ of the results obtained for $q = 2, 3, 4$. Since all the results following from the scattering theory are analytic in λ, we perform a quadratic extrapolation in this variable[14]. From the values of Table 2 one extrapolates $\Gamma_+ m_0^2/B^2 \approx 3.5$ at $q = 1$, which in turn leads to $\tilde{R}_C = 40\Gamma_+ m_0^2/27\sqrt{3}B^2 \approx 3.0$. Similarly, the results of Table 3 lead to $\tilde{\xi}_0^+/\tilde{\xi}_0^- \approx 3.76$ and $\tilde{\Gamma}_+/\tilde{\Gamma}_- \approx 74.2$. The extrapolated values for $\tilde{\xi}_0^+/\tilde{\xi}_0^-$ and $\tilde{R}_C$ compare quite well with the Monte Carlo result $\tilde{\xi}_0^+/\tilde{\xi}_0^- = 4.0 \pm 0.5$ of Ref. [39] and with the estimate $\tilde{R}_C \approx 2.7 - 2.8$ we deduce from Table 4[15].

The status of the lattice estimates of the ratio $\tilde{\Gamma}_+/\tilde{\Gamma}_-$ is extremely controversial. While there exists a substantial agreement (within 20–30%) on the value of $\tilde{\Gamma}_+$, the amplitude $\tilde{\Gamma}_-$ turns out to be very small and difficult to determine. This resulted in a series of estimates for the ratio ranging from 14 to 220 (see [7])! When a value around 200 seemed to be accepted, the authors of Ref. [39] obtained $\tilde{\Gamma}_+/\tilde{\Gamma}_- = 75(\frac{+40}{-26})$ as a result of their Monte Carlo analysis (this is the most recent result known to us).

Since we expect the accuracy of our results at $q = 3, 4$ to be comparable with that found at $q = 2$, the main source of uncertainty for the extrapolated results is in the extrapolation itself. Comparison between the extrapolated and the Monte Carlo values for ξ_0^+/ξ_0^- shows that our error does not exceed 10% in this case. We find quite reasonable to assume the same level of accuracy for our prediction on Γ_+/Γ_-. We summarise in Table 5 the situation about the amplitude ratios in two-dimensional percolation we considered in this paper.

[12]The existing ε-expansion results are unreliable in two-dimensions since the upper critical dimension in the problem is $d_c = 6$.

[13]The value $\tilde{A}_+/\tilde{A}_- = -1$ is quoted in Refs. [38] and [7]. We do not understand the origin of this discrepancy.

[14]It is immediately checked that extrapolating in q gives essentially the same results for $\tilde{\xi}_0^+/\tilde{\xi}_0^-$ and $\tilde{\Gamma}_+/\tilde{\Gamma}_-$ while the value of $\tilde{R}_C$ gets modified by less than 10%.

[15]Starting apparently from the same lattice amplitudes, Aharony [38] finds $\tilde{R}_C \approx 4.1 - 4.2$ and this result is quoted also in [7].

6 Conclusion

In this paper we used the form factor bootstrap approach to compute several universal quantities for the q-state Potts model and isotropic percolation in two dimensions. The results have been obtained truncating the spectral series for the two-point correlators at the level of the two-kink contribution. We showed by comparison with exact results that this approximation is sufficient to provide remarkably accurate values for physical quantities like central charge, scaling dimensions and critical amplitudes. In particular, we gave the first theoretical prediction of the universal amplitude ratios in the q-state Potts model for $q = 3, 4$.

Our ability to provide precise predictions for percolation is limited by the difficulty to solve Eq. (3.42) around $q = 1$. We gave accurate results for some of the amplitude ratios and relied for the others on an extrapolation based on the values obtained at $q = 2, 3, 4$. Our predictions (including those from extrapolation) are found to be in good agreement with the existing lattice estimates when the latter are able to provide reasonably accurate results. This is not the case for the ratio of the mean cluster size amplitudes above and below the percolation threshold, for which the different lattice determinations span more than one order of magnitude. In light of this, our extrapolated value for this quantity represents substantial progress.

All the results of this paper have been obtained within the form factor framework, using the S-matrix as the only input and without relying on data coming from other approaches (e.g. TBA). This was possible also because we computed universal combinations of amplitudes concerning two renormalisation group trajectories related by dual symmetry, and then describable through the same scattering theory. Other universal ratios can be defined which involve also amplitudes computed at the critical temperature but in presence of an external magnetic field. When trying to determine them in integrable field theory, one faces the problem of fixing the relative normalisations of amplitudes computed through different integrable scattering theories. It is remarkable that also this problem can be solved using the results of the TBA [42, 26] and some more recent developments [43, 44]. It was shown in Ref. [34] through the basic example of the Ising model how the universal ratios at nonzero magnetic field can be computed along these lines.

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Appendix A

Consider the matrix element

$$\langle K_{\alpha\beta}(\theta')|\Phi(0)|K_{\beta\alpha}(\theta)\rangle \ . \tag{A.1}$$

It can be related by analytic continuation to a matrix element between the vacuum and a two-kink state. There are, however, two different ways to perform the analytic continuation, corresponding to the fact that the final two-kink state can be an "In" or "Out" asymptotic state. This leads to the two equations

$$\langle K_{\alpha\beta}(\theta')|\Phi(0)|K_{\beta\alpha}(\theta)\rangle = \langle 0_\alpha|\Phi(0)|K_{\alpha\beta}(\theta'+i\pi)K_{\beta\alpha}(\theta)\rangle + 2\pi\delta(\theta-\theta')\langle 0_\alpha|\Phi|0_\alpha\rangle\,, \tag{A.2}$$

$$\langle K_{\alpha\beta}(\theta')|\Phi(0)|K_{\beta\alpha}(\theta)\rangle = \langle 0_\beta|\Phi(0)|K_{\beta\alpha}(\theta)K_{\alpha\beta}(\theta'-i\pi)\rangle + 2\pi\delta(\theta-\theta')\langle 0_\beta|\Phi|0_\beta\rangle\,, \tag{A.3}$$

where the delta function terms are disconnected parts which take into account kink-antikink annihilation. While these equations are kinematically identical to those one obtains when dealing with ordinary particles, the important difference to be noticed is the appearance of two different vacuum states. Subtracting the second equation from the first, one gets

$$\begin{aligned} \langle 0_\alpha|\Phi(0)|K_{\alpha\beta}(\theta'+i\pi)K_{\beta\alpha}(\theta)\rangle &= \langle 0_\beta|\Phi(0)|K_{\beta\alpha}(\theta)K_{\alpha\beta}(\theta'-i\pi)\rangle \\ &+ 2\pi\delta(\theta-\theta')(\langle 0_\beta|\Phi|0_\beta\rangle - \langle 0_\alpha|\Phi|0_\alpha\rangle)\ . \end{aligned} \tag{A.4}$$

As long as $\theta \neq \theta'$, this is exactly the crossing relation (3.3). For $\theta = \theta'$, the above equation implies the presence of a simple pole in the two-kink form factor with residue

$$\begin{aligned} &\mathrm{Res}_{\theta=\theta'}\langle 0_\alpha|\Phi(0)|K_{\alpha\beta}(\theta'+i\pi)K_{\beta\alpha}(\theta)\rangle = -\mathrm{Res}_{\theta=\theta'}\langle 0_\beta|\Phi(0)|K_{\beta\alpha}(\theta)K_{\alpha\beta}(\theta'-i\pi)\rangle \\ &= i(\langle 0_\alpha|\Phi|0_\alpha\rangle - \langle 0_\beta|\Phi|0_\beta\rangle)\ . \end{aligned} \tag{A.5}$$

The presence of a pole in the two-particle form factors when the rapidity difference equals $i\pi$ is known to reflect the non-locality of the operator with respect to the fields which interpolate the asymptotic particles (see e.g. [14]). We see that in the low temperature formalism this amounts to the fact that the operator has different expectation values on different vacua. Among the operators considered in this paper, this is the case of the magnetisation $\sigma_\alpha(x)$ which is non-local with respect to the disorder operators which create the kinks.

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Table Caption

Table 1 . Central charge and scaling dimension of the magnetisation operator in the q-state Potts model. The results obtained through the form factor approach in the two-kink approximation are shown below the exact values.

Table 2 . Values of some amplitude combinations in the q-state Potts model as obtained through the form factor approach in the two-kink approximation.

Table 3 . Universal amplitude ratios in the q-state Potts model. The exact result for A_+/A_- follows from duality; the other values are computed through the form factor approach in the two-kink approximation.

Table 4 . Series and Monte Carlo estimates of amplitudes for bond percolation on the square lattice and site percolation on the triangular lattice. The results marked by the superscript a, b and c are taken from Refs. [37], [40] and [39], respectively.

Table 5 . Universal amplitude ratios in two-dimensional percolation. The dagger signals extrapolated values. The results marked by the superscript a, d and c are taken from Refs. [37], [7] and [39], respectively. The series/MC estimates for $\tilde{R}_C$ and $\tilde{R}_\xi^+$ are obtained using the results of Table 4.

Figure Caption

Figure 1 . Schematic representation of the two-body scattering amplitudes in the Potts model scattering theory. The continuous lines represent a kink, the dotted lines the bound state B.

Figure 2 . The three-particle vertices in the Potts model scattering theory.

Figure 3 . Schematic representation of a two-kink form factor.

Figure 4 . (a) Two-kink form factor of the energy operator at the resonant rapidity difference $\theta = 2i\kappa$; (b) two-kink form factor of the disorder operator at the resonant rapidity difference $\theta = 2i\pi/3$.

Figure 5 . Central charge in the q-state Potts model. The continuous line is the exact formula (2.6). For $q \leq 3$ the dotted line gives the result of the c-theorem sum rule computed through the form factor approach in the two-kink approximation. For $q > 3$ the bound state B enters the physical spectrum. The upper dotted branch represents the sum of the kink-kink contribution (lower dotted branch) and the single particle bound state contribution.

q	2	3	4
c	1/2	4/5	1
	1/2	0.792	0.985
x_σ	1/8	2/15	1/8
	1/8	0.128	0.117

Table 1

q	1	2	3	4
$\Gamma_+ m_0^2/B^2$	–	2	0.973	0.476
$\xi_0^+ m_0$	1.001	1	0.998	0.992

Table 2

q	2	3	4
A_+/A_-	1	1	1
Γ_+/Γ_-	37.699	13.848	4.013
R_C	0.3183	0.1041	0.0204
ξ_0^+/ξ_0^-	3.162	2.657	1.935
R_ξ^+	0.3989	0.3262	0.2052

Table 3

	SQ bond	TR site
$\tilde{A}_+$	-4.24^a	-4.37^a
$\tilde{B}$	0.77^b	0.78^b
$\tilde{\Gamma}_+$	0.134^b	0.128^b
$\tilde{\xi}_0^+$	–	0.313^c

Table 4

	This work	Series/MC
$\tilde{A}_+/\tilde{A}_-$	1	1^a
$\tilde{\Gamma}_+/\tilde{\Gamma}_-$	$74.2^\dagger$	$14 - 220^d$
$\tilde{\xi}_0^+/\tilde{\xi}_0^-$	$3.76^\dagger$	4.0 ± 0.5^c
$\tilde{R}_C$	$3.0^\dagger$	2.7–2.8
$\tilde{R}_\xi^+$	0.926	1.1

Table 5

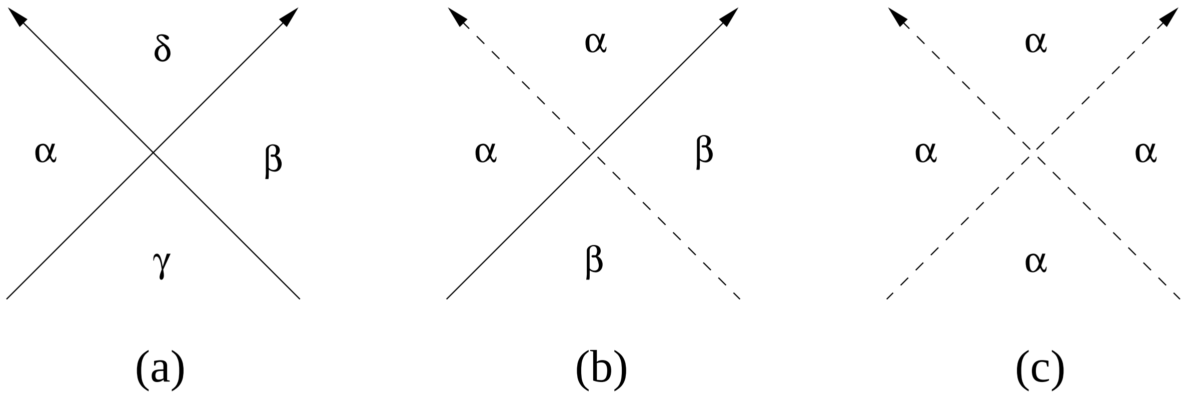

Figure 1

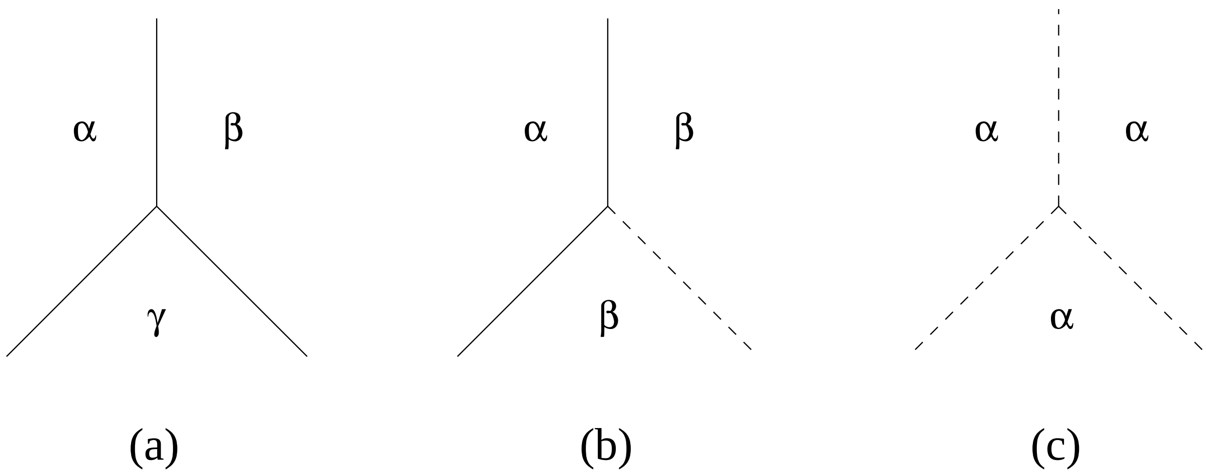

Figure 2

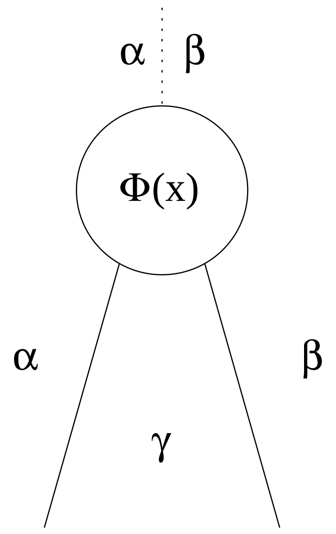

Figure 3

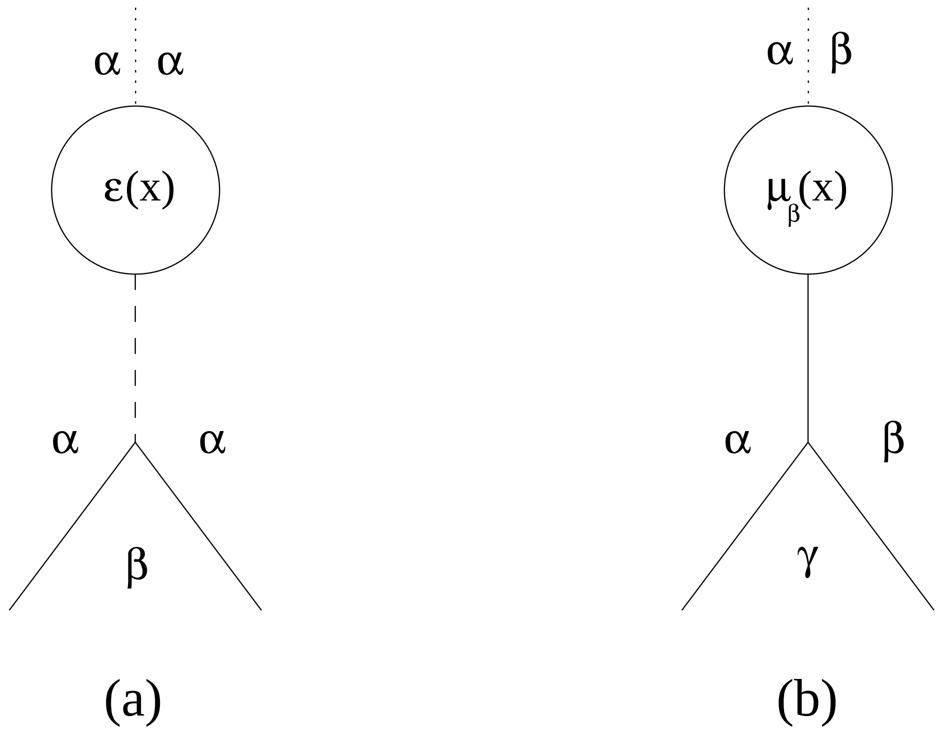

Figure 4

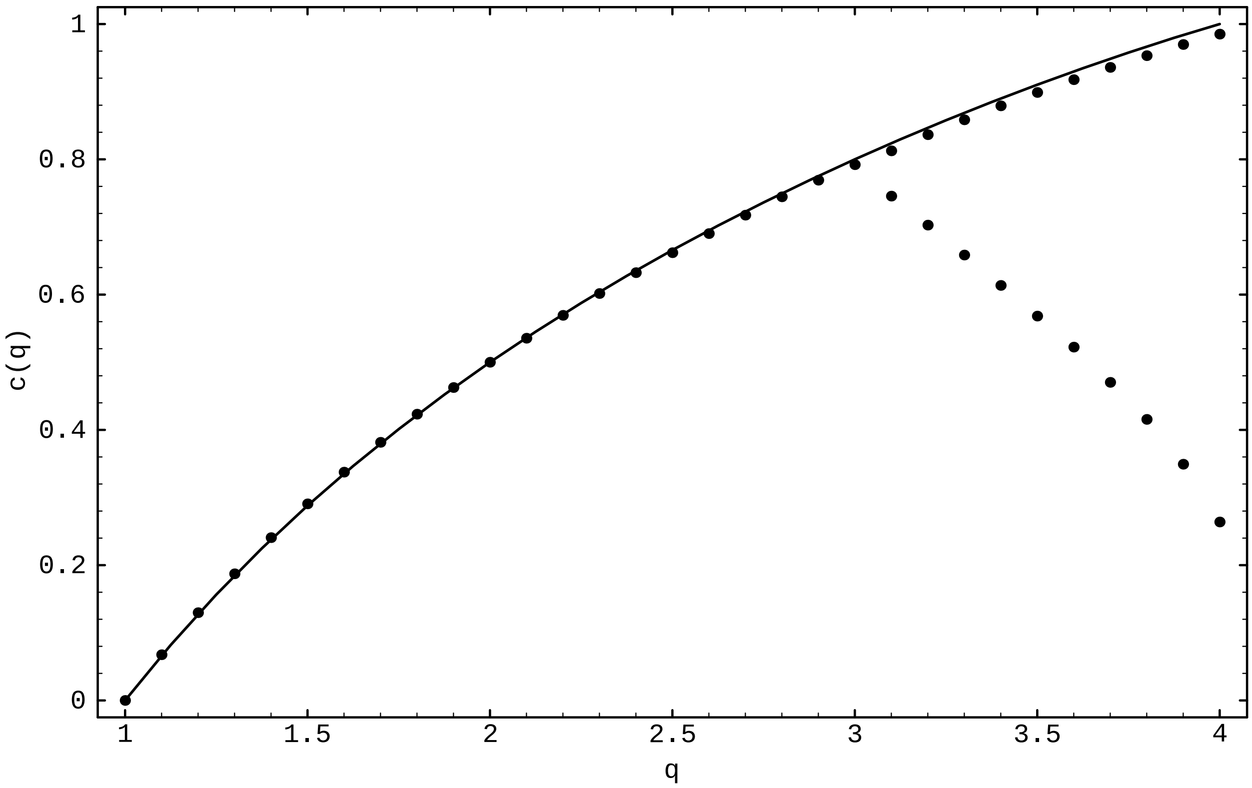

Figure 5