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On the universal Representation of the Scattering Matrix of Affine Toda Field Theory

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Abstract

By exploiting the properties of q-deformed Coxeter elements, the scattering matrices of affine Toda field theories with real coupling constant related to any dual pair of simple Lie algebras may be expressed in a completely generic way. We discuss the governing equations for the existence of bound states, i.e. the fusing rules, in terms of q-deformed Coxeter elements, twisted q-deformed Coxeter elements and undeformed Coxeter elements. We establish the precise relation between these different formulations and study their solutions. The generalized S-matrix bootstrap equations are shown to be equivalent to the fusing rules. The relation between different versions of fusing rules and quantum conserved quantities, which result as nullvectors of a doubly q-deformed Cartan like matrix, is presented. The properties of this matrix together with the so-called combined bootstrap equations are utilised in order to derive generic integral representations for the scattering matrix in terms of quantities of either of the two dual algebras. We present extensive case-by-case data, in particular on the orbits generated by the various Coxeter elements.

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1 Introduction

The perturbation of 1+1 dimensional conformal field theories* [1] in a suitable way leads to massive quantum field theories which possess a rich underlying structure. Soon after the seminal paper by Zamolodchikov [3] a decade ago on the perturbation of the Ising model, it was realized [4] that most of these massive theories are closely related to affine Toda field theories [5], either in a "minimal" sense or with the coupling constant included. On the base of case-by-case studies for various algebras several explicit scattering matrices were constructed thereafter [6]. For the simply laced algebras (ADE) this series of investigations culminated with the formulation of universal formulae which encompass all these algebras at once [7, 8]. The universal nature of these representations for the scattering matrices allowed also to establish the equivalence between the bootstrap equations and a classical fusing rule [7] formulated with the orbits generated by Coxeter elements of the related algebra [9]. Furthermore the fusing rule is closely linked to the quantum conservation laws. The origin for the structural interrelation between the classical and the quantum field theory is the fact that for the simply laced theories all masses of the theory renormalise with an overall factor [6]. It is the breakdown of this property for theories related to a non-simply laced algebra which constituted the main obstacle in the construction of consistent scattering matrices on the base of the boostrap principle. Once again numerous candidates were proposed on the base of case-by-case studies [10, 11, 12, 13], but it remained a challenge to find a closed universal representation similar to the simply laced case for these theories, until Oota recently [14] succeeded.

The main conceptual breakthrough towards this goal was the proposal by Dorey [4], that one may regard these theories in a dual sense, mathematically in a Lie algebraic way and physically equivalent to this in the strong-weak duality sense in the coupling constant and the generalization of the bootstrap principle [11] by Corrigan, Dorey and Sasaki. From this point of view affine Toda theories constitute some concrete simple examples for the Olive-Montonen duality [16]. Technically it was also very important to express the scattering matrices in the adequate building blocks [12]. Chari and Pressley [17] succeeded thereafter to work out in detail the suggested [4] fusing rules in

*There exist earlier considerations of field theories in 1+1 dimensions which focus on the aspect of conformal invariance, e.g. [2]. However, the key feature, i.e. the role played by the Virasoro algebra, which lead to a more universal formulation and allowed to find their solution was first realised and exploited in [1].

terms of the two dual algebras which reproduced precisely the allowed fusing processes. Oota [14] suggested to re-formulate these fusing rules in terms of q-deformed Coxeter transformations of either of the two dual Lie algebras. Viewing matters in the latter fashion allows to link the fusing rules to the scattering matrices and find closed universal representations.

One of the purposes of this paper is to precisely establish and derive the interrelation between the different versions of the fusing rules. We further demonstrate that these fusing rules are equivalent to the S-matrix bootstrap equations. Numerous identities which were hitherto only claimed on the base of case-by-case analysis are rigorously derived. We manifest the relation between quantum conserved quantities and the various versions of the fusing rules. We derive a set of equations, which we refer to as combined bootstrap equations, and exploit them systematically to derive generic integral representations for the scattering matrix.

Our manuscript is organized as follows: We first develop the mathematics needed and apply it thereafter in the physical context. In section 2 we define two different q-deformed Coxeter elements related to two Lie algebras dual to each other. We derive some of their properties which we need later on in the physical context. In particular their action in the root space and inner product relations. In section 3 we formulate several equivalent versions of the fusing rule, study their different solutions and establish their relation to quantum conserved quantities. In section 4 we apply our results to a universal formula for the scattering matrices of affine Toda field theories in terms of basic building blocks consisting of specific combinations of hyperbolic functions, whose powers may be obtained from q-deformed quantities of either of the two dual algebras. An alternative formula for the scattering matrix in form of an integral representation is derived in section 5. We exploit the properties of matrices M and $\hat{N}$ related to the untwisted and twisted algebra, respectively, and establish their equality. In section 6 we reduce the expressions for the scattering matrix to the simply laced case. In section 7 we provide a case-by-case analysis for all non-simply laced algebras. Our conclusions are stated in section 8.

2 q-deformed Coxeter Elements of dual Pairs

Adopting the standard notation of [18], we let $X_n^{(1)}$ be a simple simply laced Lie algebra of rank n endowed with a Dynkin diagram automorphism ω of order l. Employing this automorphism to fix a subalgebra in $X_n^{(1)}$ we obtain the twisted Lie algebra $\hat{X}_n^{(l)}$ of rank r. Changing the orientation of the arrows of the Dynkin diagram related to this twisted Lie algebra $\hat{X}_n^{(l)}$, that is interchanging long and short roots, produces a Dynkin diagram related to a Lie algebra $X_r^{(1)}$. Two Lie algebras which are related by this map are referred to as dual pair $(X_r^{(1)}, \hat{X}_n^{(l)})$. Simply laced Lie algebras are self-dual in this sense.

Before we move on to the q-deformed case we shall collect a few well known facts in order to define our notations. To each simple root α_i of $X_r^{(1)}$ or $\hat{\alpha}_i$ of $\hat{X}_n^{(l)}$ a reflection on the hyperplane through the origin orthogonal to α_i or $\hat{\alpha}_i$ may be associated

$$\sigma_i(x) = x - 2\frac{x \cdot \alpha_i}{\alpha_i^2}\alpha_i \qquad \text{or} \qquad \hat{\sigma}_i(x) = x - 2\frac{x \cdot \hat{\alpha}_i}{\hat{\alpha}_i^2}\hat{\alpha}_i \, . \tag{1}$$

Note that there is no sum over i implied here on the r.h.s. These are the Weyl reflections constituting the Weyl group which are used to construct the so-called Coxeter- and twisted Coxeter element

$$\sigma = \prod_{i=1}^{r} \sigma_i \qquad \text{and} \qquad \hat{\sigma} = \prod_{i=1}^{r} \hat{\sigma}_i \omega \tag{2}$$

for $X_r^{(1)}$ and $\hat{X}_n^{(l)}$, respectively. The latter definition is originally due to Springer [19]. We also note here that these elements are not unique and only defined up to conjugation. There are several Coxeter numbers (see e.g. [18]), whose intimate relations we wish to exploit. Expressing the highest root of $X_r^{(1)}$ as $\psi = \sum_{i=1}^{r} n_i \alpha_i$, the corresponding Coxeter- and the dual Coxeter numbers are defined as

$$h = 1 + \sum_{i=1}^{r} n_i \qquad \text{and} \qquad h^\vee = 1 + \sum_{i=1}^{r} n_i^\vee \, . \tag{3}$$

The so-called marks n_i (or Kac labels) and co-marks $n_i^\vee$ are related by $n_i^\vee = n_i \alpha_i^2/2$. Since dual algebras are obtained from each other by the interchange of roots and co-roots, i.e. $\alpha_i \to 2\alpha_i/\alpha_i^2$, one deduces easily that

$$h = \hat{h}^\vee \qquad \text{and} \qquad h^\vee = \hat{h} \, , \tag{4}$$

where $\hat{h}^\vee$,$\hat{h}$ are the Coxeter numbers of $\hat{X}_n^{(l)}$. The order of the Coxeter elements read

$$\sigma^h = 1 \qquad \text{and} \qquad \hat{\sigma}^H = 1 \tag{5}$$

where H is the l-th Coxeter number of $\hat{X}_n^{(l)}$, i.e. $H = l\hat{h}$.

Following now essentially Oota [14] the definitions of the Coxeter elements (2) can be generalized by introducing a q-deformation.

2.1 q-deformed Coxeter Element of $X_r^{(1)}$

2.1.1 Definitions

Using the standard notation $[n]_q = (q^n - q^{-n})/(q^1 - q^{-1})$ for q-deformed integers, we define the action of the q-deformed Weyl reflection σ_i^q on a simple root α_i as

$$\sigma_i^q(\alpha_j) := \alpha_j - \left(2\delta_{ij} - [I_{ji}]_q\right)\alpha_i \ . \tag{6}$$

Here I denotes the incidence matrix, i.e. twice the unit matrix minus the Cartan matrix $K_{ij} = 2\alpha_i \cdot \alpha_j/\alpha_j^2$, related to the simply laced Lie algebra $X_r^{(1)}$. We easily verify the usual properties of a reflection $(\sigma_i^q)^2 = 1$. For the time being we assume the deformation parameter q to be completely generic, that is some complex number which is not a root of unity. In some later applications we will specify q to be a root of unity and also introduce a particular parameterization $q(\beta)$, where β is a coupling constant. In that situation the "classical" limit $q \to 1$ corresponds to the vanishing of the coupling constant.

Since in general Weyl reflections do not commute, Coxeter elements, i.e. the products of all Weyl reflections related to simple roots, only form a conjugacy class. However, by introducing a particular ordering amongst the simple roots, one is able to define the Coxeter element uniquely. For this purpose we partition the set of simple roots, denoted by Δ, into two disjoint sets of roots, say $\Delta_\pm$, by associating the values $c_i = \pm 1$ to the vertices i of the Dynkin diagram of $X_r^{(1)}$, in such a way that no two vertices related to the same set are linked together. Then it clearly holds by (6) that two reflections related to simple roots belonging to the same colour set commute,

$$\left[\sigma_i^q, \sigma_j^q\right] = 0 \qquad \text{for} \quad c_i = c_j \ . \tag{7}$$

Consequently the two special elements

$$\sigma_\pm^q := \prod_{\alpha_i \in \Delta_\pm} \sigma_i^q \ , \tag{8}$$

are uniquely defined, having obviously the property $(\sigma_\pm^q)^2 = 1$. For reasons which become more apparent below, it is convenient to introduce the simple root times its colour value as a separate quantity $\gamma_i := c_i \alpha_i$. Then, the action of the reflections on these elements is easily worked out. With the help of (6), (7) and (8) we obtain

$$\sigma^q_{c_i}(\gamma_i) = -\gamma_i \quad \text{and} \quad \sigma^q_{-c_i}(\gamma_i) = \gamma_i - \sum_{\alpha_j \in \Delta_{-c_i}} [I_{ij}]_q \, \gamma_j \ . \tag{9}$$

Here we introduced the notation $\sigma^q_{c_i}$, meaning that it takes the values σ^q_+ or σ^q_- when $c_i = 1$ or $c_i = -1$, respectively. Denoting now by $\alpha_s \in \Delta_s$ and $\alpha_l \in \Delta_l$ the short and the long roots, respectively, we define some integers

$$t_i = \begin{cases} 1 & \text{for } \alpha_i \in \Delta_s \\ \alpha_l^2/\alpha_s^2 & \text{for } \alpha_i \in \Delta_l \end{cases} \tag{10}$$

which symmetrize the incidence matrix

$$I_{ij} t_j = I_{ji} t_i \ . \tag{11}$$

The ratio α_l^2/α_s^2 is indeed an integer, which follows directly from the definition of the Cartan matrix. In fact it equals l $(1, 2$ or $3)$, the highest order of the Dynkin diagram automorphism of the algebra $X_n^{(1)}$. The occurrence of quantities of $X_n^{(1)}$, despite the fact that we are discussing $X_r^{(1)}$, is a feature we will encounter more frequently in the course of our discussion and indicates the close interrelation between the two dual algebras. We employ the symmetrizers (10) to introduce the map

$$\tau(\gamma_i) := q^{t_i} \gamma_i \ . \tag{12}$$

We have now assembled all the ingredients in order to define the q-deformed Coxeter element

$$\sigma_q := \sigma^q_- \, \tau \, \sigma^q_+ \, \tau \ . \tag{13}$$

Having eliminated the ambiguity in the ordering of the q-deformed Weyl reflections within $\sigma^q_\pm$, the only matter left to convention with regard to the q-deformed Coxeter element is the ordering of the four maps in (13) and the two possible choices for the colour values we attribute to the vertices of the Dynkin diagram. The former ambiguity is fixed by the choice in (13) and the latter by choosing the unique vertex of the short root which is connected to a long root as $c_i = -1$. Note also that $\lim_{q \to 1} \sigma_q = \sigma$, that is in the "classical" limit we recover the usual Coxeter element (2) from the q-deformed Coxeter element (13).

2.1.2 Action of σ_q in the Root Space

There are several properties of the q-deformed Coxeter element which we wish to exploit in the context of the scattering matrix of affine Toda field theories. First we state the identities

$$\sigma_{q^{-1}}^x = \tau^{\frac{1+c_i}{2}} \sigma_{c_i}^q \tau^{\frac{1+c_i}{2}} \; \sigma_q^{-x} \; \tau^{-\frac{1+c_i}{2}} \sigma_{c_i}^q \tau^{-\frac{1+c_i}{2}} \tag{14}$$

$$= \tau^{\frac{1-c_j}{2}} \sigma_{-c_j}^q \tau^{\frac{1-c_j}{2}} \; \sigma_q^{-x+\frac{c_i+c_j}{2}} \; \tau^{-\frac{1+c_i}{2}} \sigma_{c_i}^q \tau^{-\frac{1+c_i}{2}} \; , \tag{15}$$

which follow immediately by noting that under the interchange of q and q^{-1} the elements $\sigma_{\pm}^q$ remain invariant and $\tau \to \tau^{-1}$. In fact the r.h.s. of these equations correspond to several equations which are combined to one by including the colour values c_i and c_j in the way we need them. Obviously, $(\sigma_q)^{-1} = \tau^{-1}\sigma_+^q \, \tau^{-1}\sigma_-^q$ is the inverse q-deformed Coxeter element†.

We further need to know the action of σ_q on the simple roots. From (9), (12) and (13) we obtain

$$\sigma_q(\alpha_i) + q^{2t_i}\alpha_i = \sum_{\alpha_j \in \Delta_{-c_i}} q^{\frac{3+c_i}{2}t_i + \frac{1+c_j}{2}t_j} \left[I_{ij}\right]_q \gamma_j + \frac{c_i - 1}{2} \sum_{\substack{\alpha_j \in \Delta_+ \\ \alpha_l \in \Delta_-}} q^{t_i + t_j} \left[I_{ij}\right]_q \left[I_{jl}\right]_q \gamma_l \tag{16}$$

and also the crucial identity

$$\left(q^{-c_i t_i} (\sigma_q)^{c_i} + q^{c_i t_i}\right)(\gamma_i) = \sum_{\alpha_j \in \Delta_{-c_i}} q^{\frac{1+c_i}{2}t_i - \frac{1+c_j}{2}t_j} \left[I_{ij}\right]_q \gamma_j \; . \tag{17}$$

Acting now successively with σ_q on γ_i and the multiplication with powers of q will create an orbit which we denote by Ω_i^q, i.e. for x, y being arbitrary integers a typical element in Ω_i^q reads $q^x \sigma_q^y(\gamma_i)$. The periodicity of these orbits reads

$$q^{-2H} \left(\sigma_q\right)^h = 1 \; . \tag{18}$$

We do not have a general proof of (18), but it is confirmed on the base of a case-by-case analysis in section 7 as may be seen from the data presented. To each orbit Ω_i^q we associate a particle species. The anti-particle is identified with the orbit in which we find the element

$$- q^{-H + \frac{c_{\bar{\imath}} - c_i}{2} t_i} \sigma_q^{\frac{h}{2} + \frac{c_i - c_{\bar{\imath}}}{4}} \gamma_i \; = \gamma_{\bar{\imath}} \in \Omega_{\bar{\imath}}^q \; . \tag{19}$$

†We differ here from the definition of the inverse in [14].

The property $c_i c_{\bar{\imath}} = (-1)^h$ [8] ensures that the power of the Coxeter element is always an integer. Conjugating $\bar{\imath}$ once more in (19) leads to (18), when $t_i = t_{\bar{\imath}}$. For the non-simply laced algebras, the relation (19) reduces to

$$\sigma_q^{\frac{h}{2}} \gamma_i \;= -q^H \gamma_{\bar{\imath}} \;, \tag{20}$$

since in that case all particles are self-conjugate. The motivation of this definition is analogue to the one known from the simply laced case [9]. This means complex conjugating the field which creates the particle of type i in the classical theory corresponds to the creation of the anti-particle $\bar{\imath}$, suggesting to associate $-\gamma_i$ to the anti-particle. However, one should keep in mind that in this context the classical theory is only known in the extreme weak or extreme strong limit of the coupling constant. In the classical limit we recover the known identity [8] for the simply laced case $\sigma^{\frac{h}{2}+\frac{c_i - c_{\bar{\imath}}}{4}} \gamma_i \;= \gamma_{\bar{\imath}}$ which relates particles and anti-particles.

2.1.3 Inner Product Identities

We introduce now the co-fundamental weights $\tilde{\lambda}_i$, related to the fundamental weights λ_i as $\tilde{\lambda}_i := 2\lambda_i/\alpha_i^2$, such that they constitute a dual base to the simple roots, i.e. $\tilde{\lambda}_i \cdot \alpha_j = \delta_{ij}$. In comparison to the non-deformed (simply laced) case, it is important to note that σ_q does in general not preserve the inner product, i.e. $\tilde{\lambda}_j \cdot (\sigma_q)^x \gamma_i \neq (\sigma_q)^{-x} \tilde{\lambda}_j \cdot \gamma_i$.

In view of (9), (12) and the orthogonality of roots and co-fundamental weights we can write

$$\tilde{\lambda}_j \cdot \sigma_q^x \gamma_i = \tilde{\lambda}_j \cdot \sigma_{-c_j}^q \sigma_q^x \gamma_i = q^{-t_j} \, \tilde{\lambda}_j \cdot \tau \sigma_q^x \gamma_i \;. \tag{21}$$

Using now (9), (15) and exploiting (21) we derive

$$\begin{aligned} \tilde{\lambda}_j \cdot \sigma_q^x \gamma_i \;&=\; \tilde{\lambda}_j \cdot \tau^{\frac{c_j-1}{2}} \sigma_{-c_j}^q \tau^{\frac{c_j-1}{2}} \; \sigma_{q^{-1}}^{-x+\frac{c_i+c_j}{2}} \; \tau^{\frac{1+c_i}{2}} \sigma_{c_i}^q \tau^{\frac{1+c_i}{2}} \gamma_i & (22) \\ &=\; -q^{(c_j-1)t_j + (1+c_i)t_i} \; \tilde{\lambda}_j \cdot \sigma_{q^{-1}}^{-x+\frac{c_i+c_j}{2}} \; \gamma_i \;, & (23) \end{aligned}$$

which may also be re-written as

$$q^{\frac{(1-c_j)t_j - (1+c_i)t_i}{2}} \, \tilde{\lambda}_j \cdot \sigma_q^x \gamma_i + \; q^{2H + \frac{(c_j-1)t_j + (1+c_i)t_i}{2}} \, \tilde{\lambda}_j \cdot \sigma_{q^{-1}}^{h-x+(c_i+c_j)/2} \gamma_i = 0 \;, \tag{24}$$

with the help of (18).

As the last inner product identity we show

$$(q^{2t_j}-1)(\,\tilde{\lambda}_j\cdot(\sigma_q)^x\gamma_i)=(q^{2t_i}-1)(\,\tilde{\lambda}_i\cdot(\sigma_q)^x\gamma_j)\ . \tag{25}$$

We prove (25) by induction and demonstrate therefore first that it holds for $x=1$. With the help of (16) we obtain

$$\begin{aligned}(q^{2t_j}-1)\sigma_q(\alpha_i)\cdot\,\tilde{\lambda}_j \;&=\; (q^{2t_j}-1)\left(-\frac{1-c_i}{2}\sum_{p\in\Delta_{-c_i}}q^{t_i+t_p}\,[I_{ip}]_q\,[I_{pj}]_q\right.\\ &\left.\qquad -q^{2t_i}\delta_{ij}+c_j q^{\frac{3+c_i}{2}t_i+\frac{1+c_j}{2}t_j}\,[I_{ij}]_q\,\delta_{c_i,-c_j}\right)\ . \end{aligned} \tag{26}$$

Noting that (11) also holds for the q-deformed quantities, i.e. $[I_{ij}]_q[t_j]_q=[I_{ji}]_q[t_i]_q$, it is easy to verify that the r.h.s. of equation (26) is symmetric in i and j. Assuming now relation (25) to be valid for x, one deduces by the similar reasoning as for the case $x=1$, that (25) also holds for $x+1$ and therefore for all integers x. This establishes (25).

It should be mentioned that once we have the matrix representation of section 5.1 the symmetry property (25) follows more easily.

2.2 q-deformed twisted Coxeter Element of $\hat{X}_n^{(l)}$

2.2.1 Definitions

Let us now consider a Lie algebra $X_n^{(1)}$, whose associated Dynkin diagram is endowed with an automorphism ω which acts on a simple root α_i with length l_i, i.e. $\omega^{l_i}\alpha_i=\alpha_i$. The largest value of l_i corresponds to l. Sometimes we will also use the common notation $\omega\alpha_i=\alpha_{\omega(i)}$. We may employ this automorphism to define the orbits Ω_i^ω by successive actions of ω on a simple root α_i. By selecting a representative of the orbit Ω_i^ω, we can build up a set of roots, which we denote by $\hat{\alpha}_i\in\hat{\Delta}$. The algebra related to these roots is the twisted Lie algebra $\hat{X}_n^{(l)}$. To each of the r elements in $\hat{\Delta}$ we associate a particular particle species. We choose the conventions in such a way that we may carry out a one-to-one correspondence between the two dual algebras without renaming the particles, see section 7. The Weyl reflections related to these representatives are now defined in the usual fashion as in (1)

$$\sigma_i(\alpha_j)=\alpha_j-K_{ji}\alpha_i\ \ , \tag{27}$$

where K denotes the Cartan matrix of $X_n^{(1)}$. Analogously to the non-twisted case, treated in the previous section, we can bi-colour the Dynkin diagram related to $X_n^{(1)}$ and divide the set of representatives into two sets $\hat{\Delta}_-$ and $\hat{\Delta}_+$. Note that roots related by the automorphism ω possess naturally the same colour value. Hence we may define uniquely the elements

$$\hat{\sigma}_\pm := \prod_{\hat{\alpha}_i \in \hat{\Delta}_\pm} \sigma_i \quad . \tag{28}$$

Besides the absence of the q-deformation, the difference between these special elements of the Weyl group in comparison with the non-twisted case is that the product runs only over the representatives. We define now the integers

$$\hat{t}_i = \begin{cases} 1 & \text{for } \alpha_i \in \hat{\Delta} \\ 0 & \text{for } \alpha_i \notin \hat{\Delta} \end{cases} \quad . \tag{29}$$

With the help of (27) we easily compute the action of $\hat{\sigma}_\pm$ on some $\gamma_i := c_i \alpha_i$, where we stress that α_i is not necessarily a representative

$$\hat{\sigma}_{c_i} \gamma_i = (-1)^{\hat{t}_i} \gamma_i \qquad \text{and} \qquad \hat{\sigma}_{-c_i} \gamma_i = \gamma_i - \sum_{\hat{\alpha}_j \in \hat{\Delta}_{-c_i}} I_{ij} \hat{\gamma}_j \quad . \tag{30}$$

The incidence matrix I is here related to $X_n^{(1)}$, but note that $1 \leq i \leq n$ and $1 \leq j \leq r$. In addition we introduce the map which will serve as a q-deformation

$$\hat{\tau}(\alpha_i) := q^{2\hat{t}_i} \alpha_i \quad . \tag{31}$$

At last we are in the position to define, analogously to [14], the q-deformed twisted Coxeter element as

$$\hat{\sigma}_q := \omega^{-1}\, \hat{\sigma}_-\, \hat{\tau}\, \hat{\sigma}_+ \quad . \tag{32}$$

Once again by means of the bi-colouration, we have achieved that $\hat{\sigma}_q$ is uniquely defined up to the ordering of the maps occurring in (32). For $q \to 1$ we obtain one of the standard twisted Coxeter elements in the conjugacy class as originally introduced by Springer (2). We will not elaborate here on the alternative characterization of the twisted Coxeter element, which may be obtained from the folding of an affine simply laced Dynkin diagram, see e.g. [18, 24, 25].

2.2.2 Action of $\hat{\sigma}_q$ in the Root Space

Introducing for convenience the quantities $\gamma_i^\pm := \omega^{\frac{\pm c_i - 1}{2}}\gamma_i$, the action of $\hat{\sigma}_q$ on the simple roots is computed to

$$\hat{\sigma}_q(\hat{\gamma}_i) + q^2\omega^{-1}\hat{\gamma}_i = -\sum_{\hat{\alpha}_j \in \hat{\Delta}_{-c_i}} q^2 I_{ij}\omega^{-1}\hat{\alpha}_j + \frac{1-c_i}{2} q^2 \sum_{\substack{\alpha_j \in \hat{\Delta}_+ \\ \alpha_l \in \hat{\Delta}_-}} I_{ij} I_{jl}\omega^{-1}\hat{\gamma}_l \tag{33}$$

and

$$\gamma_i^- = (-q^{-2c_i})^{\hat{t}_i}\hat{\sigma}_q^{c_i}\left(\gamma_i^+\right) + \sum_{\alpha_j \in \Delta_{-c_i}} I_{ij}\hat{\gamma}_j^+ \ , \tag{34}$$

with the help of (30), (31) and (32).

Acting successively with $\hat{\sigma}_q$ and q on the elements of $\hat{\Delta}$, we construct the orbits of the q-deformed twisted Coxeter element, which we denote by $\hat{\Omega}_i^q$. The order of the q-deformed twisted Coxeter element reads

$$q^{-2h}\hat{\sigma}_q^H = 1 \ . \tag{35}$$

Thus in comparison with (18) the roles of h and H are just interchanged. Like in the non-twisted case we do not have a generic proof of this periodicity property, but we have verified it case-by-case in section 7.

The anti-particle is identified with the orbit in which we find the element

$$-\, q^{-h+\frac{c_{\bar{\imath}}-c_i}{2}\hat{t}_i}\,\hat{\sigma}_q^{\frac{H}{2}+\frac{c_i-c_{\bar{\imath}}}{4}(2-l_i)}\hat{\gamma}_i^+ \ = \hat{\gamma}_{\bar{\imath}}^+ \in \hat{\Omega}_{\bar{\imath}}^q \ . \tag{36}$$

Conjugating $\bar{\imath}$ once more in (36) leads to (35), when $l_i = l_{\bar{\imath}}$. For the non-simply laced algebras, the relation (19) reduces to

$$\hat{\sigma}_q^{\frac{H}{2}}\hat{\gamma}_i^+ \ = -q^h\hat{\gamma}_{\bar{\imath}}^+ \ , \tag{37}$$

since in that case all particles are self-conjugate. In the limit $q \to 1$ we obtain $\hat{\sigma}^{\frac{H}{2}+\frac{c_i-c_{\bar{\imath}}}{4}(2-l_i)}\hat{\gamma}_i^+ \ = \hat{\gamma}_{\bar{\imath}}^+$, which relates particles and anti-particles in twisted algebras.

2.2.3 Inner Product Identities

To each orbit $\hat{\Omega}_i^q$ we associate now a fundamental weight $\hat{\lambda}_i$ which is dual to all elements inside the ω-orbit, i.e.

$$\hat{\lambda}_i \cdot \sum_{k=1}^{l_j}\omega^k(\alpha_j) = \delta_{ij} \ , \tag{38}$$

for α_i being a root of $X_n^{(1)}$. With the help of (31), (30) and the orthogonality relation (38) we derive easily

$$\hat{\lambda}_j \cdot \hat{\sigma}_q^x \hat{\gamma}_i = \hat{\lambda}_j \cdot \hat{\sigma}_{-c_j}^q \hat{\sigma}_q^x \hat{\gamma}_i = q^{-2\hat{t}_j}\, \hat{\lambda}_j \cdot \hat{\tau} \hat{\sigma}_q^x \gamma_i \ . \tag{39}$$

We also have the identities

$$\hat{\lambda}_j \cdot \hat{\sigma}_q^x \hat{\gamma}_i^+ = \hat{\lambda}_i \cdot \hat{\sigma}_q^{x+\frac{c_j-c_i}{2}+\frac{c_i-1}{2}l_i+\frac{1-c_j}{2}l_j} \hat{\gamma}_j^+ \tag{40}$$

and

$$\hat{\lambda}_j \cdot \hat{\sigma}_q^x \hat{\gamma}_i^+ = -q^{2h+c_i+c_j}\, \hat{\lambda}_j \cdot \hat{\sigma}_{q^{-1}}^{H-x+c_i+\frac{1-c_i}{2}l_i+\frac{c_j-1}{2}l_j} \hat{\gamma}_i^+ \tag{41}$$

To prove these identities directly is much more involved as for the equivalent relations in the untwisted case. We will therefore postpone the proof until section 5.2. where we can exploit properties of a different quantity which then implies the validity of (40) and (41).

3 The Fusing Rules

We are now in the position to formulate the universal fusing rules. This may be done either by exploiting the properties of the orbits of the q-deformed Coxeter Element of $X_r^{(1)}$ or the q-deformed twisted Coxeter Element of $\hat{X}_n^{(l)}$ similar to the approach of Oota [14] or alternatively in the spirit of Chari and Pressley [17] one may consider the orbits of the non-deformed Coxeter Element of $X_r^{(1)}$and simultaneously the non-deformed twisted Coxeter Element of $\hat{X}_n^{(l)}$. Additionally one may formulate the fusing rule in terms of the quantum conserved quantities. We will discuss the solutions to these different fusing rules and prove in general that they are in fact all equivalent. We derive the precise quantitative relation between the relevant quantities.

3.1 The Fusing Rule in Ω^q

The generalized[‡] *three-point-coupling related to three particles of the type i, j and k is non-vanishing, i.e. the process $i + j \to \bar{k}$ is possible, if and only*

[‡]Usually we really refer to the three-point-coupling in the common sense, i.e. related to the process $i + j \to \bar{k}$. The only exceptions are the processes $2 + 2 \to 2$ and $3 + 3 \to 3$ in $(F_4^{(1)}, E_6^{(2)})$, which are possible from the fusing rule point of view. However, on the S-matrix bootstrap side these processes correspond to third order poles.

if there exist representatives of the q-deformed orbits Ω_i^q, Ω_j^q and Ω_k^q whose sum is zero.

This means there should exist two triplets of integers (ξ_i, ξ_j, ξ_k) and $(\zeta_i, \zeta_j, \zeta_k)$ such that

$$\sum_{l=i,j,k} q^{\zeta_l} \sigma_q^{\xi_l} \gamma_l = 0 \ . \tag{42}$$

Multiplying (42) by q^n or σ_q^m corresponds naturally to the same process and we should therefore view the triplets as equivalence classes[§]. In this sense we regard two pairs of triplets as equivalent if they may be constructed from each other by the displacements $\zeta_l \to \zeta_l + m$ or $\xi_l \to \xi_l + n$. Similarly as in the simply laced case [8], it will turn out to be crucial that there exists a second solution to (42)

$$\sum_{l=i,j,k} q^{\zeta_l'} \sigma_q^{\xi_l'} \gamma_l = 0 \ . \tag{43}$$

The two solutions may not be obtained from each other by simple shifts, but they are related as

$$\xi_l' = -\xi_l + \frac{c_l - 1}{2} \quad \text{and} \quad \zeta_l' = -\zeta_l - (1 + c_l) t_l \ , \qquad l = i, j, k. \tag{44}$$

Nonetheless, as an existence criterion for the fusing process, the variant (42) is sufficient, since the second solution may always be constructed from the first as we now demonstrate. Changing q to q^{-1} in the fusing rule (42) and using (14) thereafter, we obtain

$$\sum_{l=i,j,k} q^{-\zeta_l} \tau^{\frac{1+c_l}{2}} \sigma_{c_l}^q \tau^{\frac{1+c_l}{2}} \ \sigma_q^{-\xi_l} \ \tau^{-\frac{1+c_l}{2}} \sigma_{c_l}^q \tau^{-\frac{1+c_l}{2}} \ \gamma_l = 0 \ . \tag{45}$$

Acting on this equation with $\tau^{-1}\sigma_+^q \tau^{-1}$ yields (43), with the help of (9), (12) and (13). What remains to be shown is that these two solutions are indeed non-equivalent in the sense defined above. For this purpose we may take the limit $q \to 1$ and note that the quantities ξ_l and ξ_l' are related to each other in the same way as in the simply laced case. We may now simply refer to [8] for the proof of the non-equivalence of this two triplets. This is sufficient to establish the non-equivalence between the two solutions. In addition we shall demonstrate below that there exists in fact no further non-equivalent solution.

[§] We shall see below that from a physical point of view this corresponds to a simple shift in the bootstrap functional equations which involve the scattering matrix.

3.2 The Fusing Rule in $\hat{\Omega}^q$

The generalized three-point-coupling related to three particles of the type i, j and k is non-vanishing, i.e. the process $i + j \to \bar{k}$ is possible, if and only if there exist representatives of the q-deformed orbits $\omega^{\frac{c_i-1}{2}}\hat{\Omega}_i^q$, $\omega^{\frac{c_j-1}{2}}\hat{\Omega}_j^q$ and $\omega^{\frac{c_k-1}{2}}\hat{\Omega}_k^q$ whose sum is zero.

This means there should exist two triplets of integers $(\hat{\xi}_i, \hat{\xi}_j, \hat{\xi}_k)$ and $(\hat{\zeta}_i, \hat{\zeta}_j, \hat{\zeta}_k)$ such that

$$\sum_{l=i,j,k} q^{\hat{\zeta}_l} \hat{\sigma}^{\hat{\xi}_l} \hat{\gamma}_l^+ = 0 \ . \tag{46}$$

Equivalence of two solutions is defined as in the previous section, i.e. two triplets which are obtained by simple shifts of the type $\hat{\xi}_l \to \hat{\xi}_l + m$ and $\hat{\zeta}_l \to \hat{\zeta}_l + n$ are considered equivalent to the original solution. However, as in the non-twisted case, also (46) always admits a second non-equivalent solution

$$\sum_{l=i,j,k} q^{\hat{\zeta}_l'} \ \hat{\sigma}_q^{\hat{\xi}_l'} \hat{\gamma}_l^+ = 0 \ . \tag{47}$$

The relations between the two solutions read

$$\hat{\zeta}_l' = -\hat{\zeta}_l + 1 - c_l \quad \text{and} \quad \hat{\xi}_l' = -\hat{\xi}_l + \frac{1 - c_l}{2} l_l + 1 + c_l \,, \quad l = i, j, k. \tag{48}$$

As in the previous section the second solution may be constructed from the first, and therefore the variant (46) is sufficient as an existence criterion.

3.3 The Fusing Rule in Ω and $\hat{\Omega}$

The generalized three-point-coupling related to three particles of the type i, j and k is non-vanishing, i.e. the process $i + j \to \bar{k}$ is possible, if and only if there exist representatives of the orbits Ω_i, Ω_j and Ω_k whose sum is zero and if in addition there exist representatives of the orbits $\omega^{\frac{c_i-1}{2}}\hat{\Omega}_i$, $\omega^{\frac{c_j-1}{2}}\hat{\Omega}_j$ and $\omega^{\frac{c_k-1}{2}}\hat{\Omega}_k$ which also sum up to zero.

Quantitatively this means there should exist two triplets of integers (ξ_i, ξ_j, ξ_k) and $(\hat{\xi}_i, \hat{\xi}_j, \hat{\xi}_k)$ such that

$$\sum_{l=i,j,k} \sigma^{\xi_l} \gamma_l = 0 \qquad \text{and} \qquad \sum_{l=i,j,k} \hat{\sigma}^{\hat{\xi}_l} \hat{\gamma}_l^+ = 0 \ . \tag{49}$$

The version (49) of the fusing rule was first stated by Chari and Pressley [17], with the only difference that our $\hat{\sigma}$ corresponds to the inverse twisted Coxeter element in [17] and also $\hat{\gamma}_l^+$ is defined differently in their formulation. The multiplication of the first equation in (49) by powers of the Coxeter element σ and the second by powers of the twisted Coxeter element $\hat{\sigma}$ will produce further solutions, which we regard as equivalent. Once again there exists a second non-equivalent solution

$$\sum_{l=i,j,k} \sigma^{\xi'_l}\gamma_l = 0 \qquad \text{and} \qquad \sum_{l=i,j,k} \hat{\sigma}^{\hat{\xi}'_l}\hat{\gamma}_l^+ = 0 \ , \tag{50}$$

which is related to the first by the relevant relations in (44) and (48). The equations (49) and (50) may be obtained in the limit $q \to 1$ from (42), (43) and (46), (47), respectively. Since we have already shown that neither the triplet (ξ'_i, ξ'_j, ξ'_k) may be obtained from (ξ_i, ξ_j, ξ_k) by simple shifts nor $(\hat{\xi}'_i, \hat{\xi}'_j, \hat{\xi}'_k)$ from $(\hat{\xi}_i, \hat{\xi}_j, \hat{\xi}_k)$ by the same means, we have established the nonequivalence between the two solutions. It is also clear from the preceding sections that we may construct the second solution always from the first.

3.4 The Fusing Rule and conserved Quantities

Let $y(n)$ $(1 \leq n \leq r)$ be a vector¶ whose components are labeled by particle types. In particular for $n = 1$ we identify $y_i(1)$ with the quantum mass m_i of the particle of species i. Then we may formulate a further variant of the fusing rule:

The generalized three-point-coupling related to three particles of the type i, j and k is non-vanishing, i.e. the process $i + j \to \bar{k}$ is possible, if there exist two triplets of integers (η_i, η_j, η_k) and $(\bar{\eta}_i, \bar{\eta}_j, \bar{\eta}_k)$ such that

$$\sum_{l=i,j,k} e^{s_n(\eta_l\theta_h + \bar{\eta}_l\theta_H)}\, y_l(n) = 0 \ . \tag{51}$$

The s_n $(1 \leq n \leq r)$ label the exponents of the algebra $X_r^{(1)}$ in increasing order. We further introduced the angles

$$\theta_h := \frac{i\pi(2-B)}{2h} \qquad \text{and} \qquad \theta_H := \frac{i\pi B}{2H} \, , \tag{52}$$

¶In fact we see below that this will be the nullvector of a particular matrix as specified in equation (100).

whose deeper origin becomes more apparent when we discuss the scattering matrix in section 4. The coupling constant β enters here the expressions through the function $B = 2H\beta^2/(H\beta^2 + 4\pi h)$ which takes values between 0 and 2. Obviously, multiplying equation (51) by $e^{m s_n \eta_l \theta_h}$ and $e^{k s_n \bar{\eta}_l \theta_H}$, with m, k being arbitrary integers, will also produce a solution, which we regard as equivalent in the same spirit as in the previous subsections. Likewise there exists a second non-equivalent solution

$$\sum_{l=i,j,k} e^{s_n(\eta'_l \theta_h + \bar{\eta}'_l \theta_H)} \, y_l(n) = 0 \qquad , \tag{53}$$

related to the first simply as

$$\eta'_l = -\eta_l \qquad \text{and} \qquad \bar{\eta}'_l = -\bar{\eta}_l \ . \tag{54}$$

Clearly we can not construct (51) from (53) by multiplication of $e^{s_n m \eta_l \theta_h}$ and $e^{s_n k \bar{\eta}_l \theta_H}$ unless $\eta_i = \eta_j = \eta_k$ and $\bar{\eta}_i = \bar{\eta}_j = \bar{\eta}_k$. The latter fact would mean that $\sum_{l=i,j,k} y_l(n) = 0$, which in particular for $n = 1$ is impossible since all quantities in the sum, the masses, are positive. We have therefore established that the two solutions are indeed non-equivalent. However, one solution may always be constructed from the other simply by replacing $s_n \to -s_n$ or complex conjugation of (51) or (53).

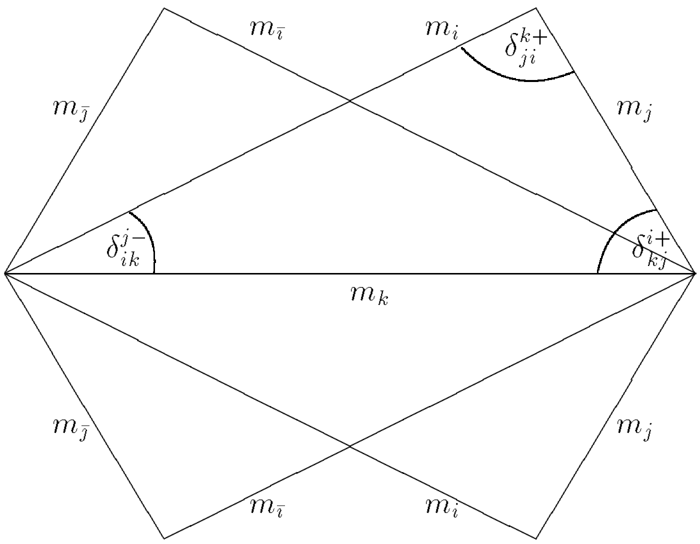


Figure 1: Mass triangles in the complex velocity plane. The angles are defined as $i\delta_{jk}^{i\pm} = (\eta_j - \eta_k)\theta_h + (\bar{\eta}_j - \bar{\eta}_k)\theta_H \pm i\pi$.

Having obtained the fusing angles η we may immediately compute relations among the quantum conserved quantities. Combining (51) and (53) we derive

$$\frac{y_i(n)}{y_j(n)} = \frac{\sinh\left(s_n(\eta_k - \eta_j)\theta_h + s_n(\bar{\eta}_k - \bar{\eta}_j)\theta_H\right)}{\sinh\left(s_n(\eta_i - \eta_k)\theta_h + s_n(\bar{\eta}_i - \bar{\eta}_k)\theta_H\right)} . \tag{55}$$

We may interpret these relations in the complex velocity plane as explained in [8]. In particular for $s_1 = 1$ we obtain the important ratios of the quantum masses

$$\frac{m_i}{m_j} = \frac{\sinh\left((\eta_k - \eta_j)\theta_h + (\bar{\eta}_k - \bar{\eta}_j)\theta_H\right)}{\sinh\left((\eta_i - \eta_k)\theta_h + (\bar{\eta}_i - \bar{\eta}_k)\theta_H\right)} . \tag{56}$$

As the main difference to the simply laced case we note that the masses now depend on the coupling constant. The relevant triangles are depicted in figure 1. We will now be more specific on how to calculate the fusing angles from Lie algebraic properties.

3.5 Relations between the Fusing Rules

The four versions of the fusing rules are all related to each other, meaning that having one solution of one particular formulation of the fusing rule we are able to construct all the other solutions. The precise relations read

$$\eta_l = \xi'_l - \xi_l = \frac{\hat{\zeta}_l - \hat{\zeta}'_l}{2} \quad \text{and} \quad \bar{\eta}_l = \frac{\zeta_l - \zeta'_l}{2} = \hat{\xi}'_l - \hat{\xi}_l \qquad \text{for } l = i, j, k. \tag{57}$$

We see that the interchange of the two solutions of one version of the fusing rule immediately demands that the two solutions of the other rules should also be exchanged. In particular it follows that

$$-2\xi_l = \hat{\zeta}_l, \qquad \zeta_l = -2\hat{\xi}_l + \frac{1-c_l}{2} l_l - \frac{1+c_l}{2} t_l + 1 + c_l \quad \text{for } l = i, j, k. \tag{58}$$

These relations do not only relate the fusing rule in Ω^q and $\hat{\Omega}^q$ to each other, but they also provide the precise link between the q-deformed and non-deformed versions of the fusing rule. It will take until subsection 5.1. to have assembled all the ingredients for the proof of (57).

There is one last question which we should answer with regard to possible solutions of the fusing rules: Are there any further non-equivalent solutions to these equations? The answer is no. For the proof of this statement we assume at this point that the rules are indeed equivalent, such that it suffices to discuss only one version. We adopt the argumentation of [8] for this purpose. The only four triangles which we may construct in the complex velocity plane from three sides with fixed modulus are the ones depicted in figure 1. Hence there are no further possible angles, meaning no additional non-equivalent solution to (51) exist. By (57) this fact is also established for all other versions of the fusing rule we have stated.

Treating the fusing rule as a pure existence criterion for the possibility of certain fusing processes, one version is as good as the other. We observed however that the relevant data from the "classical" fusing rules, which correspond to two equations in section 3.3., may be merged together into one single equation by the q-deformation. This is the key feature which can be exploited in the quantum field theory and which appears to be absolutely necessary for the construction of generic expressions for the scattering matrices.

4 Block Representation

The scattering matrices for affine Toda field theories have been the subject of numerous investigations [6, 8, 11, 12, 13, 14]. Restricting the attention to the case when the coupling constant is real, the two-particle scattering matrix for all simple Lie algebras, involving particles of the species i and j as a function of the relative rapidity θ, may be cast into the universal expression

$$S_{ij}(\theta) = \prod_{x=1}^{h} \prod_{y=1}^{H} \{x,y\}_{\theta}^{2\mu_{ij}(x,y)} \quad . \tag{59}$$

Here $\{x,y\}_\theta$ are certain combinations of hyperbolic functions and the $\mu_{ij}(x,y)$ are positive semi-integers for the given range in (59).

4.1 The Building Blocks

Before explaining how the powers $\mu_{ij}(x,y)$ may be computed, we present several representations of the general building blocks, which will serve different purposes. As a crucial step in the process of formulating generic expressions for scattering matrices one should view the observation of P. Dorey [12] who noticed that the building blocks may all be expressed in a very elegant form.

We slightly modify them to simplify certain computations and define[||]

$$\{x,y\}_\theta := \frac{[x,y]_\theta}{[x,y]_{-\theta}} \tag{60}$$

and

$$[x,y]_\theta := \frac{\langle x-1,y-1\rangle_\theta \langle x+1,y+1\rangle_\theta}{\langle x-1,y+1\rangle_\theta \langle x+1,y-1\rangle_\theta} \quad \langle x,y\rangle_\theta := \sinh\frac{1}{2}(\theta + x\theta_h + y\theta_H)\ . \tag{61}$$

We used the angles θ_h and θ_H as introduced in section 3.4. Notice that the strong-weak duality transformation $\beta \to 4\pi/\beta$ $(B \to 2-B)$, $h \leftrightarrow H$ leaves the scattering matrix invariant. One should stress that besides the strong-weak interchange the invariance also demands the interchange of the Coxeter numbers.

Alternatively, each block (60) admits an integral representation in the form

$$\{x,y\}_\theta = \exp \int_0^\infty \frac{dt}{t\sinh t}\ f_{x,y}^{h,H}(t)\sinh\left(\frac{\theta t}{i\pi}\right) \tag{62}$$

with

$$f_{x,y}^{h,H}(t) = 8\sinh(\vartheta_h t)\sinh(\vartheta_H t)\sinh(t - x\vartheta_h t - y\vartheta_H t)\ . \tag{63}$$

This may be verified for instance by the explicit computation of the integral in (62). We abbreviated here $\vartheta_h := (2-B)/2h$ and $\vartheta_H := B/2H$. Particular attention has to be paid to the convergence of the integral representation (62), especially when we analytically continue. Shifting $\theta \to \theta + x'\theta_h + y'\theta_H$, convergence requires that

$$0 \le (x - x' - 1)\vartheta_h + (y - y' - 1)\vartheta_H \le 2(1 - (1+x')\vartheta_h - (1+y')\vartheta_H)\ . \tag{64}$$

In particular for real rapidity θ the convergence is guaranteed if $0 < x < h$ and $0 < y < H$.

With regard to several applications, the values of the scattering matrices at $\theta = 0$ are of special interest and we therefore comment on it for definiteness. In general we have $\{x,y\}_{\theta=0} = 1$, apart from the case $\{1,1\}_{\theta=0} = -1$.

[||] In [11, 22] a different type of blocks was used. They may be translated into each other by simple replacements, e.g. for G_2 and F_4 one sets $H^{-1} = \theta_h + \theta_H$.

This means we have to pay attention to the ordering of certain limits. When writing the blocks in form of hyperbolic functions (60), we have to set first $x = y = 1$ and then take the limit $\theta \to 0$, whereas in the integral representation (62) we have to set $x = y = 1$, integrate thereafter and finally take the limit $\theta \to 0$.

The following obvious identities will turn out to be useful in the course of our argumentation

$$\{x,y\}_\theta = \{x+2h, y+2H\}_\theta = \{-x,-y\}_\theta^{-1} \quad (65)$$

$$\{x,y\}_{\theta+x'\theta_h+y'\theta_H} = \frac{[x+x', y+y']_\theta}{[x-x', y-y']_{-\theta}} \quad (66)$$

$$\{x,y\}_{\theta+p\theta_h+q\theta_H}\,\{x,y\}_{\theta-p\theta_h-q\theta_H} = \{x+p, y+q\}_\theta\,\{x-p, y-q\}_\theta\ . \quad (67)$$

Furthermore, it will be convenient to adopt the slightly more compact notation for the product of several blocks

$$\{x_1,y_1\}_\theta^{\mu_1}\,\{x_2,y_2\}_\theta^{\mu_2}\ldots\{x_n,y_n\}_\theta^{\mu_n} =: \{x_1,y_1^{\mu_1};x_2,y_2^{\mu_2};\cdots;x_n,y_n^{\mu_n}\}_\theta \quad (68)$$

from time to time.

We shall now come to the characterization of the powers $\mu_{ij}(x,y)$ of particular blocks $\{x,y\}_\theta$, which may be computed either by using the properties of $X_r^{(1)}$ or $\hat{X}_n^{(l)}$.

4.2 The Powers from $X_r^{(1)}$

The powers in (59) can be evaluated from the matrix-valued generating function

$$\sum_y \mu_{ij}\left(2x - \frac{c_i+c_j}{2}, y\right) q^y = -\frac{[t_j]_q}{2} q^{\frac{(1-c_j)t_j-(1+c_i)t_i}{2}}(\,\tilde{\lambda}_j\cdot\sigma_q^x\gamma_i)\ , \quad (69)$$

for fixed x. Taking x in the range $(3-c_i)/2 \le x \le h + (1-c_i)/2$ ensures that the first argument of μ is between 1 and $2h$. This formula is a natural generalization of the one for the simply laced case (128), where now the q-deformation incorporates the information of both dual algebras. At this point we have only stated (69) and we shall now convince ourselves that it is indeed satisfying all the requirements we need.

When applying formula (59), we have to guarantee that the properties of the combinations of hyperbolic functions in the building blocks $\{x,y\}_\theta$ are reflected in the correct way by the Lie algebraic quantities. This means, that according to the identities in (65) we should have

$$\mu_{ij}(x,y) = \mu_{ij}(x+2h, y+2H) \quad \text{and} \quad \mu_{ij}(x,y) = -\mu_{ij}(2h-x, 2H-y) \ . \quad (70)$$

Considering (69), the first relation in (70) follows trivially from (18). Together with the r.h.s. of (69) the second relation in (70) may be proven directly with the help of (24). The second relation is important, since it ensures that we can always find two blocks which combine in such a way that the total power of each building block becomes an integer. Therefore it guarantees that the scattering matrix is a meromorphic function, even if we choose (this is sometimes very convenient) the ranges in (59) to be $1 \leq x \leq h$ and $1 \leq y \leq H$.

Having established the formal legitimacy of (59), it is clear that properties of the μ's may be carried over into properties of the scattering matrix. We will therefore prove several identities which we exploit below when discussing the scattering matrix.

First we note that

$$\mu_{ij}(x,y) = \mu_{ji}(x,y) = -\mu_{\bar{\imath}j}(x+h, y+H) \ . \quad (71)$$

The symmetry in the subscripts follows directly from the defining relation for the μ's (69) and the symmetry property of the inner product (25). The second equation follows in view of the definition of the anti-particle (19) and (69). The latter identity relates the powers involving the particle on one hand and the anti-particle on the other and will therefore turn out to be useful to show the crossing relation.

From the fusing rule in Ω^q follows by similar manipulations as we have just performed

$$\sum_{l=i,j,k} \mu_{lp}\left(x \pm \eta_l, y \pm \bar{\eta}_l\right) = 0 \ , \quad (72)$$

where the lower sign relates to the first (42) and the upper sign to the second solution (43). The integers η_l and $\bar{\eta}_l$ are related to the two solutions of the fusing rules by (57). It still needs to be established that they are indeed the same as the ones occurring in the equations involving the conserved quantities, (51) and (53). It will turn out that both relations in (72) will be crucial to prove the bootstrap equations for the scattering matrices.

The final relation in this section follows from (17) and (69)

$$\mu_{ij}(x+1, y+t_i) + \mu_{ij}(x-1, y-t_i) = \sum_{n=1}^{I_{il}} \sum_{l\in\Delta} \mu_{lj}(x, y+2n-1-I_{il}) \ , \quad (73)$$

where we understand that the sum $\sum_{n=1}^{I_{il}}$ yields zero when $I_{il}=0$. We can view (73) as a particular solution of the recursive equations (2.4) quoted in [14]. One may take these equations as a starting point and use them to construct the powers μ_{ij} recursively. However, it remains unclear how to obtain the equations (73) from first principles. In fact (73) should be regarded as a consequence of (72) and we therefore view the latter equations as more fundamental. We demonstrate this fact only for the equivalent equations of the scattering matrices, since in that setting they correspond to a simple physical property, see section 7.

4.3 The Powers from $\hat{X}_n^{(l)}$

Alternatively we can use the data of the twisted algebra $\hat{X}_n^{(l)}$ in order to compute the powers of the building blocks. In this case the role of two arguments x and y in the generating function is reversed, that is now we fix a particular y and read off the possible values for x from the generating functions

$$\sum_x \nu_{ij}\left(x, 2y - c_i + \frac{c_i-1}{2}l_i - \frac{c_j-1}{2}l_j\right) q^x = -\frac{q^{-\frac{c_i+c_j}{2}}}{2}(\,\hat{\lambda}_j\cdot\hat{\sigma}_q^y\hat{\gamma}_i^+)\ . \quad (74)$$

Since the two descriptions, i.e. in terms of the data of $X_r^{(1)}$ or in terms of the data of $\hat{X}_n^{(l)}$ are supposed to be the same, we expect similar relations as we obtained in the previous section for the μ's also to hold for the ν's. Now property (65) of the blocks demands that

$$\nu_{ij}(x,y)=\nu_{ij}(x+2h,y+2H) \quad \text{and} \quad \nu_{ij}(x,y)=-\nu_{ij}(2h-x,2H-y)\ . \quad (75)$$

The first relation in (75) follows trivially from (35). Once again we may guarantee that the scattering matrix is a meromorphic function by means of the second relation in (75), which follows from (41). We also have the identities which imply parity and crossing

$$\nu_{ij}(x,y)=\nu_{ji}(x,y)=-\nu_{\bar{\imath}j}(x+h,y+H)\ . \quad (76)$$

The first equation follows now from (40) and the second from (36)**. The relation which implies the bootstrap identity

**We should keep in mind here that we did not yet prove (41) and (40). In fact we reverse the logic and prove first the properties for the ν's in section 5.2. and deduce from them the inner product identities in section 2.2.3.

$$\sum_{l=i,j,k} \nu_{lp}\left(x \pm \eta_l, y \pm \bar{\eta}_l\right) = 0 \ , \tag{77}$$

follows from the version of the fusing rules related to the q-deformed twisted Coxeter element in $\hat{\Omega}^q$ (section 3.2). As the counterpart of (73) we derive from the defining relations of the ν's and (34)

$$\nu_{ij}(x+c_i, y)+\nu_{\omega^{-c_i}(i)j}(x-c_i, y-2c_i) = \sum_{\alpha_l \in \hat{\Delta}_{-c_i}} I_{il}\nu_{lj}\left(x, y+\frac{1-c_i}{2}l_i - \frac{1+c_i}{2}l_l\right) \tag{78}$$

Having finally assembled the main properties of all the ingredients from which we construct the scattering matrices, we are now in the position to utilize them in order to study the properties of S.

4.4 Bootstrap Properties

The exact expressions for two-particle scattering matrices of integrable quantum field theories may be obtained by solving certain consistency equations, the so-called bootstrap equations. We will now demonstrate that (59) fulfills indeed all the requirements and take this as a proof for the conjectured formulae stated in the previous subsection.

4.4.1 Unitarity, Crossing and Parity Invariance

The unitarity-analyticity equation $S_{ij}(\theta)S_{ij}(-\theta) = 1$ follows trivially from the property $\{x,y\}_\theta \{x,y\}_{-\theta} = 1$ of each individual building blocks. The crossing relation $S_{ij}(\theta) = S_{\bar{\imath}j}(i\pi - \theta)$ requires in general a little bit more effort, e.g. [25]. Using (65) and (66) we obtain

$$S_{ij}(i\pi-\theta) = S_{ij}(h\theta_h+H\theta_H-\theta) = \prod_{x=1}^{h}\prod_{y=1}^{H} \{x+h, y+H\}_\theta^{-\mu_{ij}(x+h,y+H)} \ . \tag{79}$$

Employing now the second identity in (71), the r.h.s. of (79) equals $S_{\bar{\imath}j}(\theta)$, which establishes the crossing relation. The parity invariance of the scattering matrix, i.e. $S_{ij}(\theta) = S_{ji}(\theta)$, is guaranteed by the symmetry property of the μ's in the lower indices, i.e. the first equation in (71).

Alternatively we can use the data of the q-deformed twisted Coxeter element and repeat the argumentation once more, using now the relations (76) instead of (71).

4.4.2 Bootstrap Identities

We will now come to the key equations, whose names are sometimes associated with this whole approach, the bootstrap equations. The claim is that once the fusing rules in section 3 hold, the following identity is true for the scattering matrices

$$\prod_{l=i,j,k} S_{pl}(\theta + \eta_l \theta_h + \bar{\eta}_l \theta_H) = 1\,. \tag{80}$$

The integers η_l and $\bar{\eta}_l$ may be expressed by using the data from the various versions of the fusing rules (57). The proofs of the relations (80) are straightforward. We obtain with the help of (66)

$$\prod_{\substack{x,y\\ l=i,j,k}} \{x,y\}^{\mu_{pl}(x,y)}_{\theta+\eta_l\theta_h+\bar{\eta}_l\theta_H} = \prod_{\substack{x,y\\ l=i,j,k}} \frac{[x+\eta_l, y+\bar{\eta}_l]^{\mu_{pl}(x,y)}_{\theta}}{[x-\eta_l, y-\bar{\eta}_l]^{\mu_{pl}(x,y)}_{-\theta}} = 1\ . \tag{81}$$

The last step follows by shifting $x \to x - \eta_l$ and $x \to x + \bar{\eta}_l$ in the numerator and denominator, respectively, such that we can employ the two equations in (72). We note that it is crucial to have both solutions at hand. Alternatively we can derive the bootstrap equation (80) by exploiting the property (77) of the ν's and repeating the arguments once more.

With the help of (67) we translate (73) into what we refer to as the "combined bootstrap" identity for the scattering matrix

$$S_{ij}\left(\theta + \theta_h + t_i \theta_H\right) S_{ij}\left(\theta - \theta_h - t_i\theta_H\right) = \prod_{l=1}^{r}\prod_{n=1}^{I_{il}} S_{jl}\left(\theta + (2n-1-I_{il})\theta_H\right) \quad . \tag{82}$$

Here we understand that the product $\prod_{n=1}^{I_{il}}$ contributes 1 when $I_{il} = 0$. Sometimes this identity is identical to some bootstrap equation, but in general it has to be constructed by combining several identities of the type (80) in a very particular way. Its significance is, that it may be employed in order to derive the matrix representation for the scattering matrix (see section 4.3). Reducing (82) to the simply laced case, i.e. $[I_{il}]_q \to I_{il}$, $H \to h, t_i \to 1$, we recover an identity quoted in [23], see section 7.

4.4.3 Occurrence of certain special Blocks

For various purposes it is important to exhibit explicitly the occurrence of particular blocks $\{x,y\}$ in the general formula (59). It is possible to extract the blocks of the form $\{1,y\}_\theta\ \ \{2,y'\}_\theta$ from the general product and re-write the scattering matrix as

$$S_{ij}(\theta) = \{1,1_{t_i}\}_\theta^{\delta_{ij}}\ \left\{2,2_{I_{ij}\,t_j}\right\}_\theta \prod_{x\neq 1,2}\prod_y \{x,y\}_\theta \quad . \tag{83}$$

For the proof of (83) we exploit the properties of the q-deformed Coxeter element σ_q. Considering the identity (69), we notice that for $i=j$ a block of the form $\{1,y\}$ may only occur for $x=0, c_i=-1$ or $x=1, c_i=1$. From (17) and the orthogonality of simple roots and co-fundamental weights, we obtain $\tilde{\lambda}_i\cdot\sigma_q^{\frac{c_i+1}{2}}\gamma_i = -q^{(1+c_i)t_i}$ and therefore we get

$$\sum_y \mu_{ii}\,(1,y)\,q^y = \frac{[t_i]_q}{2}q^{t_i} = \frac{1}{2}\sum_{n=1}^{t_i} q^{2n-1}\ , \tag{84}$$

which establishes the first factor in (83). In order to prove the occurrence of the second factor, we observe that a block of the form $\{2,y\}$ may only be generated if $c_i\neq c_j$. Due to the parity property of the μ's (70), we may choose $c_i=1$ and $c_j=-1$ w.l.g., such that we obtain from (17) $\tilde{\lambda}_j\cdot\sigma_q\gamma_i = -q^{2t_i}[I_{ij}]_q$. Hence we obtain

$$\sum_y \mu_{ij}\,(2,y)\,q^y = \frac{q^{t_i+t_j}}{2}[I_{ij}]_q[t_j]_q = \frac{1}{2}\sum_{n=1}^{t_j} q^{2n-1}\sum_{n=1}^{I_{ij}} q^{2n-1} = \frac{1}{2}\sum_{n=1}^{I_{ij}t_j} q^{2n}\,. \tag{85}$$

In the last equality we have used the fact that either t_j or I_{ij} has to be one. This establishes (83).

There are several consequences we may draw from (83). An immediate conclusion concerns the value of the scattering matrix at vanishing rapidities. With the remark made in section 4.1 we deduce from (83) that

$$S_{ij}(0) = (-1)^{\delta_{ij}}\ . \tag{86}$$

The knowledge of this value is for instance important in the context of the thermodynamic Bethe ansatz [21].

4.4.4 Singularities and the generalized Bootstrap

As we have seen the blocks of the form (60) are extremely useful to exhibit the Lie algebraic structure of the scattering matrix. However, they are quite misleading with regard to the singularity structure due to the possible cancellation of zeros and poles. This may happen whenever we have a product of two blocks $\{x,y\}\{x',y'\}$ and x,x' or y,y' differ by 2. It suffices to consider the latter case, since it will cover all examples we shall be constructing. Motivated by this observation we introduce the quantity

$$\begin{aligned} \{x,y_n\}_\theta &= \prod_{l=0}^{n-1}\{x,y+2l\}_\theta && (87)\\ &= \frac{\langle x-1,y-1\rangle_\theta\,\langle x+1,y-1+2n\rangle_\theta}{\langle x+1,y-1\rangle_\theta\,\langle x-1,y-1+2n\rangle_\theta}\times(\theta\to-\theta)^{-1}\ , && (88)\end{aligned}$$

and also define the angles

$$\theta^{\pm}_{x,y,n}=(x\pm 1)\theta_h+(2n+y-1)\theta_H \quad . \tag{89}$$

which serve to characterize the precise location of the singularities of the blocks $\{x,y_n\}_\theta$. Obviously the four zeros are situated at $\pm\theta^{\pm}_{x,y,0},\mp\theta^{\pm}_{x,y,n}$ and the four poles at $\pm\theta^{\pm}_{x,y,n},\pm\theta^{\mp}_{x,y,0}$ respectively. In order to interpret these singularities from the physical point of view we should know when they are situated on the physical sheet, i.e. $0\le \operatorname{Im}\theta\le i\pi$. Recalling that the range for the possible arguments of the blocks $0<x<h$, $0<y<H$ and the range in which the effective coupling takes its value, i.e. $0\le B\le 2$ we evaluate

$$0\le \operatorname{Im}(\theta^{\pm}_{x,y,n})\le\pi \qquad \text{for } B\le\frac{2H(h-x\mp 1)}{|h(2n+y-1)-H(x\pm 1)|}\,. \tag{90}$$

The relevant residues are computed to

$$\operatorname*{Res}_{\theta=\theta^{-}_{x,y,0}}\{x,y_n\}=\frac{-2\sinh\theta_h\sinh(n\theta_H)\sinh(x\theta_h+(n+y-1)\theta_H)\sinh\left(\theta^{-}_{x,y,0}\right)}{\sinh(\theta_h+n\theta_H)\sinh(x\theta_h+(y-1)\theta_H)\sinh((x-1)\theta_h+(y+n-1)\theta_H)} \tag{91}$$

$$\operatorname*{Res}_{\theta=\theta^{+}_{x,y,n}}\{x,y_n\}=\frac{2\sinh\theta_h\sinh(n\theta_H)\sinh(x\theta_h+(n-1+y)\theta_H)\sinh\left(\theta^{+}_{x,y,n}\right)}{\sinh(\theta_h+n\theta_H)\sinh((1+x)\theta_h+(n+y-1)\theta_H)\sinh(x\theta_h+(2n+y-1)\theta_H)}\ . \tag{92}$$

It is easy to convince oneself that with the stated range for x,y,B,n together with (90) we have

$$\operatorname{Im}\left(\operatorname*{Res}_{\theta=\theta^{-}_{x,y,0}}\{x,y_n\}_\theta\right)<0 \qquad\text{and}\qquad \operatorname{Im}\left(\operatorname*{Res}_{\theta=\theta^{+}_{x,y,n}}\{x,y_n\}_\theta\right)>0\ , \tag{93}$$

such that the $\theta^+_{x,y,n}$ could correspond to the direct channel poles. In the simply laced case this knowledge is enough to judge about the sign of the residue of the whole S-matrix, e.g. [8]. For the case at hand matters are more involved since the remaining blocks in the scattering matrix do in general not possess a definite sign. It is this feature which lead the authors of [11] to the formulation of the generalized bootstrap. According to this prescription only odd order poles, whose imaginary part of the residue is positive in the *whole* range of the effective coupling B, participate in the bootstrap.

So let us have a closer look at the behaviour of a block $\{x', y'_{n'}\}_{\theta^+_{x,y,n}}$. We obtain a first criterion for a possible sign change by considering the extreme limits in the coupling constant. In general we have $\lim_{\beta\to 0,\infty} \{x', y'_{n'}\}_{\theta^+_{x,y,n}} = 1$. However, if $x' = x$ we have

$$\lim_{\beta\to 0} \{x', y'_{n'}\}_{\theta^\pm_{x,y,n}} = \left(\frac{y' - y - 2n}{y' - y + 2n' - 2n}\right)^{\pm 1} \tag{94}$$

$$\lim_{\beta\to\infty} \{x', y'_{n'}\}_{\theta^\pm_{x,y,n}} = 1 \; . \tag{95}$$

This means if the block responsible for the pole is $\{x, y_n\}_\theta$ and the right hand side of (94) is negative the imaginary parts of the possible additional blocks

$$\{x, y'_{n'}\}_{\theta^+_{x,y,n}} \qquad \text{and} \qquad \{x+2, y''_{n''}\}_{\theta^+_{x,y,n}} \tag{96}$$

both change their sign while β runs from zero to infinity. This means the pole $\theta^+_{x,y,n}$ does not participate in the bootstrap if in the scattering matrix also the blocks (96) occur to an odd power and if they do not cross the real axis at the same position. This means having the scattering matrix given explicitly in blockform the condition on y, y', n, n' by which the l.h.s. of (94) becomes negative, together with the occurrence of blocks like (96) provides a simple criterion which allows to judge whether a pole resulting from a certain block should be excluded from the generalized bootstrap or not.

Exploiting the fusing rules and reading off the relative rapidities from (51) we obtain the precise location, say ϕ, of a pole in the scattering matrix which participates in the generalized bootstrap

$$\phi = \pm(\eta_i - \eta_j)\theta_h \pm (\bar{\eta}_i - \bar{\eta}_j)\theta_H \; . \tag{97}$$

The two signs result from the two non-equivalent solutions of the fusing rule.

5 Matrix-Integral Representation

Alternatively to the universal form for the scattering matrix in form of blocks there exists a remarkable integral representation. This version of the scattering matrix is particularly useful when applied in the context of the thermodynamic Bethe ansatz [21, 22] or off-shell when computing form factors [20]. We can express the scattering matrix as[††]

$$S_{ij}(\theta) = \exp \int_0^\infty \frac{dt}{t} \; \Phi_{ij}(t) \sinh\left(\frac{\theta t}{i\pi}\right) \;, \tag{98}$$

with

$$\Phi_{ij}(t) = 8 \sinh(\vartheta_h t) \sinh(t_j \vartheta_H t) \left([K]_{q(t)\bar{q}(t)} \right)^{-1}_{ij} \quad . \tag{99}$$

We introduced here the particular deformation parameters $q(t) = \exp(\vartheta_h t)$ and $\bar{q}(t) = \exp(\vartheta_H t)$ and the matrix

$$[K_{ij}]_{q\bar{q}} = (q\bar{q}^{t_i} + q^{-1}\bar{q}^{-t_i})\delta_{ij} - [I_{ij}]_{\bar{q}} \; . \tag{100}$$

In the limit $q \to 1$ and $\bar{q} \to 1$ the matrix $[K_{ij}]_{q\bar{q}}$ obviously reduces to the ordinary Cartan matrix K, such that one is tempted to view this matrix as a doubly q-deformed Cartan matrix. However, this viewpoint is slightly misleading as we now argue. For the simply laced cases it was proven [9], that the conserved quantities may be organized as right eigenvectors of the Cartan matrix $\sum_j K_{ij} y_j(n) = 4\sin^2(s_n \pi/h) \; y_i(n)$ with s_n labeling the exponents of the algebra as already introduced. In particular we have that $y_i(1) \sim m_i$. It is then easy to see that this may also be re-written as

$$\sum_{j=1}^{r} [K_{ij}]_{q(i\pi s_n)\bar{q}(i\pi s_n)} y_j(n) = 0 \; . \tag{101}$$

Hence, we can alternatively organize the conserved charges as nullvectors of the matrix $[K_{ij}]_{q(i\pi t)\bar{q}(i\pi t)}$ evaluated at exponents of the Lie algebra, i.e. $t = is_n$. Based on a case-by-case investigation, Oota pointed out [14] that equation (101) also holds for the non-simply laced case. A general proof of

[††]A very similar formula was first obtained by Oota ((5.2) in [14]) on the base of a case-by-case study. In comparision, the formula (98) differs only by a factor $(-1)^{\delta_{ij}}$ $\exp\left(2\delta_{ij}\int_0^\infty \frac{dt}{t}\sinh\left(\frac{\theta t}{i\pi}\right)\right)$, which is 1 for θ real, but different from one if the rapidity becomes complex. Similar expressions also appear in [26].

this statement is still outstanding. There is, however, one important difference in comparison with the simply laced case. In general we can not reverse the interpretation anymore, such that we are not able to recover a genuine eigenvalue equation. In particular for $s_1 = 1$ this leads to

$$\sum_j [I_{ij}]_{\bar{q}(i\pi)} m_j = 2\cosh\left(\theta_h + t_i\theta_H\right) m_i \ . \tag{102}$$

We observe that the eigenvalue depends now through the symmetrizer t_i on the component of the "eigenvector". In the limit $\beta \to 0$ we restore the old picture and recover the equation $\sum_j I_{ij} y_j(n) = 2\cos(\pi s_n/h) y_i(n)$ valid for all simple Lie algebras. With the help of (11) we also obtain the equation for the left nullvector $x_i(n)$ related to the right as $y_i(n) = [t_i]_{\bar{q}}\, x_i(n)$.

The determinant of the matrix (100) may be computed [14] to

$$\det[K]_{q(i\pi t)\bar{q}(i\pi t)} = \prod_{n=1}^{r} 4\sin((t+s_n)\pi/h)\sin((s_n - t)\pi/h) \ . \tag{103}$$

We do not have a general proof of this formula, but we have verified it case-by-case. Two important features which we exploit below should be noticed here, first the determinant becomes independent of the coupling constant β and second it vanishes for t being an exponent.

Before we provide the proof for the representation (98), we will introduce two further auxiliary matrices.

5.1 The M-Matrix

We restrict now the sum of the generating function for the powers of the building blocks (69) to a finite range and also include an additional deformation parameter $\bar{q}$ into our consideration. We define the matrix

$$M_{ij}(q,\bar{q}) = \sum_{x=1}^{2h}\sum_{y=1}^{2H} \mu_{ij}(x,y) q^x \bar{q}^y \quad , \tag{104}$$

where initially we keep both deformation parameters completely generic. From the properties for the μ's, which we deduced in section 4.2., we can immediately derive several features for the matrix M

$$M_{ij}(q,\bar{q}) = -q^{2h}\bar{q}^{2H} M_{ij}(q^{-1},\bar{q}^{-1}) = M_{ji}(q,\bar{q}) \, . \tag{105}$$

The first identity in (105) is a consequence of the two relations in (70) together with the fact that $\mu_{ij}(0,y) = \mu_{ij}(2h,y) = 0$ for all y. The second follows trivially from the symmetry properties of the μ's from the first relation in (71).

Most crucial is once more the combined bootstrap equation, which on the Lie algebraic side corresponds to the property (73). In fact, this identity will enable us to compute the matrix M explicitly. By some straightforward manipulations of this relation we deduce with (73) that $M(q,\bar{q})$ has to satisfy

$$(q^{-1}\bar{q}^{-t_i} + q\bar{q}^{t_i})M_{ij}(q,\bar{q}) - \sum_{k=1}^{r}[I_{ik}]_{\bar{q}}M_{kj}(q,\bar{q}) = \frac{1-q^{2h}\bar{q}^{2H}}{2}\,[t_i]_{\bar{q}}\delta_{ij}\,. \quad (106)$$

Solving this equation for $M(q,\bar{q})$ yields

$$M_{ij}(q,\bar{q}) = \frac{1-q^{2h}\bar{q}^{2H}}{2}\,([K]_{q\bar{q}})^{-1}_{ij}\ [t_j]_{\bar{q}}\ . \quad (107)$$

At first sight (107) does not seem to be a finite polynomial of the form (104). However, the doubly q-deformed Cartan matrix becomes singular at certain values and the pre-factor $(1-q^{2h}\bar{q}^{2H})$ ensures the whole expression to remain finite. In other words this term may always be factorized into the determinant of $[K_{ij}]_{q\bar{q}}$ and some rest, such that the r.h.s. of (107) will indeed be a polynomial as defined in (104). In [26] a similar matrix as (107) also occurs. However, apart from the ordering of $[t]$, $[K]$, the pre-factor $(1-q^{2h}\bar{q}^{2H})/2$, which is crucial for the polynomial aspect we discuss below, is not mentioned in there.

We will deviate now from our generic consideration and specify the deformation parameters to be $q(t)$ and $\bar{q}(t)$ as introduced after equation (99). Noting first of all that $q(t)^{2h}\bar{q}(t)^{2H} = e^{2t}$, we observe that for $t = i\pi m$ the r.h.s. of (106) always vanishes. Furthermore it follows from (107) that $M(q(i\pi m),\bar{q}(i\pi m))$ is also always zero unless m is an exponent by (103). From this we deduce that $M(q(i\pi s_n),\bar{q}(i\pi s_n))$ is proportional to the right nullvector $y(n)$ as specified in (101). In view of the symmetry property (105), we conclude that

$$M_{ij}(q(i\pi s_n),\bar{q}(i\pi s_n)) \sim y_i(n)y_j(n) \quad (108)$$

where the factor of proportionality does neither depend on the particle index i nor on j. Most importantly we derive from (72) a matrix version of the fusing rule (51) and (53)

$$\sum_{l=i,j,k} q(i\pi s_n)^{\eta_l}\, \bar{q}(i\pi s_n)^{\bar{\eta}_l}\, M_{lp}(q(i\pi s_n), \bar{q}(i\pi s_n)) = 0 \qquad \text{for } 1 \leq p \leq r \ . \tag{109}$$

By means of (108) we may divide out $y_p(n)$ and the factor of proportionality from (109), such that we have at last established the relation (51) involving the conserved quantities.

We may specify the deformation parameters further and take q and $\bar{q}$ to be roots of unity of order $2h$ and $2H$, respectively. This may be done safely after we have cancelled the determinant against the pre-factor. As a consequence this means in particular that together with the periodicity property of the μ's (the first property in (70)), we may simultaneously shift the upper and lower limit in the sum (104) arbitrarily. The properties of the blocks are now also reflected by the polynomial (104), such that we can not only carry out a one-to-one identification between $\{x,y\}_\theta$ and $q^x\bar{q}^y$, but in addition we can also manipulate them in an identical way. If in analogy to $\{-x,-y\}_\theta = \{x,y\}_\theta^{-1}$, we further define $q^{-x}\bar{q}^{-y} = -q^x\bar{q}^y$ we can even guarantee that the range of x and y is $1 \leq x \leq h$, $1 \leq y \leq H$. With these assumptions in mind we derive

$$M_{ij}(q,\bar{q}) = q^h\bar{q}^H M_{\bar{\imath}j}(q^{-1},\bar{q}^{-1}) = -q^h\bar{q}^H M_{\bar{\imath}j}(q,\bar{q}) \tag{110}$$

from the last relation in (71).

As a final remark of this section, we note that at roots of unity the defining relation for the M-matrix (104) may be viewed as the discrete Fourier transformation of $\mu_{ij}(x,y)$, the inverse of which reads

$$\mu_{ij}(x,y) = \frac{1}{4\,h\,H} \sum_{m=1}^{2h} \sum_{n=1}^{2H} M_{ij}\left(\omega^m, \hat{\omega}^n\right) \omega^{-mx} \hat{\omega}^{-ny}, \tag{111}$$

with ω and $\hat{\omega}$ being the $2h$'th and $2H$'th primitive roots of unity, respectively. This allows us to compute the powers of the blocks, i.e. the μ's, in an alternative way from the explicit expression of $M(q,\bar{q})$ in matrix form (107). We may also utilize (111) to verify the properties of the μ's by exploiting now explicitly matrix representation of $M(q,\bar{q})$, instead of the orbits of the q-deformed Coxeter element as in of section 4.2. In addition the computing rules, which we stated in the previous paragraph for generic q and $\bar{q}$ are automatically satisfied for q and $\bar{q}$ being roots of unity.

5.2 The N-Matrix

As to be expected, we may also express the scattering matrix in terms of the data of the twisted algebra $\hat{X}_n^{(l)}$. In analogy to the M-matrix (104) we define the $n \times n$-matrix

$$N_{ij}(q,\bar{q}) = \sum_{x=1}^{2h} \sum_{y=1}^{2H} \nu_{ij}(x,y) q^x \bar{q}^y \quad , \tag{112}$$

where once again we keep both deformation parameters completely generic for the time being. It should be clear that our notation in (74) is slighly abused here at the cost of avoiding the introduction of new symbols. From the Lie algebraic analogue to the combined bootstrap equation (78) we derive

$$\begin{aligned} & (-1)^{\hat{t}_i+1}(q^{\hat{t}_i}\bar{q})^{-2c_i} N_{ij} + N_{\omega^{-c_i}(i)j} - \sum_{\alpha_l \in \hat{\Delta}_{-c_i}} q^{-c_i} \bar{q}^{-2c_i + \frac{c_i-1}{2} l_i + \frac{c_i+1}{2} l_l} I_{il} N_{lj} \\ = \; & (q\bar{q})^{-c_i} \frac{(1-q^{2h}\bar{q}^{2H})}{2} \delta_{i\omega^{\frac{1+c_i}{2}}(j)} \quad . \end{aligned} \tag{113}$$

Unlike the corresponding equation for the non-twisted case (106), we can not solve (113) directly due to the occurrence of indices transformed by the automorphism ω. However, we may consider equation (113) for $i \to \omega^{-c_i}(i)$ and iterate the resulting equations as long as we obtain $N_{\omega^{-lc_i}(i)j} = N_{ij}$. Thereafter we can safely solve the equation for the $r \times r$-submatrix, say $\hat{N}$, and obtain

$$\hat{N}_{ij}(q,\bar{q}) = \frac{1-q^{2h}\bar{q}^{2H}}{2} \left([\hat{K}]_{q\bar{q}}\right)^{-1}_{ij} [l_j]_{\bar{q}} \; . \tag{114}$$

Here we have introduced the doubly q-deformed twisted Cartan matrix

$$[\hat{K}]_{q\bar{q}} = (q\bar{q}^{l_i} + q^{-1}\bar{q}^{-l_i})\delta_{ij} - \left[\sum_{k=1}^{l_i} \hat{I}_{\omega^k(i)j}\right]_{\bar{q}} \quad . \tag{115}$$

Note that in the classical limit $q, \bar{q} \to 1$ we recover the transpose of the usual twisted Cartan matrix. The transposition results from our convention that particles in both dual algebras are denoted by the same particle index. Similarly as in the nontwisted case the determinant of the matrix (115) acquires a very neat form

$$\det[\hat{K}]_{q(i\pi t)\bar{q}(i\pi t)} = \prod_{k=1}^{n} 4\sin((t+\hat{s}_k)\pi/H)\sin((\hat{s}_k - t)\pi/H) \; , \tag{116}$$

where the $\hat{s}_k$ denote the l-th exponents of $\hat{X}_n^{(l)}$ [18]. We also do not have a general proof of this formula, but we have verified it once again case-by-case.

By direct computation, we may now derive several identities for the matrix $\hat{N}$, namely

$$\hat{N}_{ij}(q,\bar{q}) = \hat{N}_{ji}(q,\bar{q}) = -q^{2h}\bar{q}^{2H}\hat{N}_{ij}(q^{-1},\bar{q}^{-1}) \ . \tag{117}$$

The first and second relation in (117) imply the first property for the ν's in (76) and the second relation in (75), respectively, which on the other hand finally prove the inner product identities of section 2.2.3. Comparing (107) and (114) we see immediately that $M = \hat{N}$ and therefore $\nu(x,y) = \mu(x,y)$. A direct Lie algebraic proof of the latter equality would be desirable since it allows to express quantities of the twisted algebra in terms of the non-twisted algebra and vice versa. Having established several features of the matrix $M(q,\bar{q})$ and $\hat{N}(q,\bar{q})$ we will now supply the context in which they naturally originate.

5.3 From Block- to Integral Representation

Concerning the representation of the scattering matrix in blockform (59), an obvious question which arises is, whether it is possible to compute explicitly the product over x and y. Taking the explicit integral representations of the blocks (62) into account , this problem amounts to the evaluation of

$$\begin{aligned}
\Phi_{ij}(t) &= \frac{1}{\sinh t}\sum_{x=1}^{2h}\sum_{y=1}^{2H}\mu_{ij}(x,y)\, f_{x,y}^{h,H}(t,B), && (118)\\
&= -\frac{8\sinh(\vartheta_h t)\sinh(\vartheta_H t)}{\sinh t}e^{-t}M(q(t),\bar{q}(t)) && (119)\\
&= -\frac{8\sinh(\vartheta_h t)\sinh(\vartheta_H t)}{\sinh t}e^{-t}\hat{N}(q(t),\bar{q}(t)) && (120)
\end{aligned}$$

if we want to transform the scattering matrix into the form (98). From the first identity in (105), noting that $q(t)^{2h}\bar{q}(t)^{2H} = e^{2t}$, together with the explicit form of the M-matrix (107), we deduce the integral representation (98) with (99).

Some comments are due, since it appears that the convergence condition (64) is violated by the range we chose for x and y in the defining relation for M. However, for each individual block $\{x,y\}_\theta$ we can exploit the properties

(65) and bring the arguments x and y into a range for which the integral representation (62) is convergent. These features are reflected in the M-matrix if it is taken at roots of unity together with the already mentioned rule $q^{-x}\bar{q}^{-y} = -q^{x}\bar{q}^{y}$.

As an alternative proof we may proceed similar as in [22] for the simply laced case. This method turns out to be instructive with regard to particular applications as the thermodynamic Bethe ansatz and it will illustrate the origin of the slight difference between (99) and the formula in [14]. First we notice that the scattering matrix may also be written as

$$S_{ij}(\theta) = \mathcal{N} \exp \left(\int_0^\infty \frac{dt}{\pi t} \widetilde{\varphi}_{ij}(t/\pi) \sinh \left(\frac{it\theta}{\pi} \right) \right) \quad , \tag{121}$$

when we introduce the quantities

$$\varphi_{ij}(\theta) = -i \frac{d}{d\theta} \ln S_{ij}(\theta) \qquad \text{and} \qquad \widetilde{\varphi}_{ij}(k) = \int_{-\infty}^{\infty} d\theta \varphi_{ij}(\theta) e^{ik\theta} \, . \tag{122}$$

Due to the differentiation in (122), we have the freedom of a normalization constant $\mathcal{N}$ in (121), which may be fixed by some asymptotic condition. Acting now with $-i$ times the logarithmic derivative on the combined bootstrap identity (we concentrate here on the case $I_{il} = 1$) (82), multiplying with $\exp(ik\theta)$ and integrating thereafter with respect to θ we obtain

$$\mathcal{P} \int d\theta \left(\varphi_{ij} \left(\theta + \theta_h + t_i \theta_H \right) + \varphi_{ij} \left(\theta - \theta_h - t_i \theta_H \right) \right) e^{ik\theta} = \sum_{l=1}^{r} I_{il} \; \widetilde{\varphi}_{lj}(k) \; . \tag{123}$$

Here $\mathcal{P}$ denotes the Cauchy principal value. Alternatively we may compute $\widetilde{\varphi}_{ij}(k)$ directly. For this purpose we shift the Fourier integral into the complex plane and integrate along the contours $\mathcal{C}_\varrho^\pm$ as depicted in figure 2.
Due to (83) we know explicitly the occurrence of the relevant blocks which will give a contribution when we integrate along $\mathcal{C}_\varrho^\pm$.

$$\lim_{\varrho \to \infty} \oint_{\mathcal{C}_\varrho^\pm} d\theta \varphi_{ij}(\theta) e \;=\; \widetilde{\varphi}_{ij}(k) - \mathcal{P} \int d\theta \varphi_{ij} \left(\theta \pm \theta_h \pm t_i \theta_H \right) e^{ik(\theta \pm \theta_h \pm t_i \theta_H)} \tag{124}$$

$$= \; 2\pi \delta_{ij} e^{\mp 2\pi \vartheta_{h/H}} - \pi I_{ij} e^{\mp k\pi(\vartheta_h + \vartheta_H)} \, . \tag{125}$$

On the other hand the l.h.s. of (123) may be computed alternatively from the right hand sides of (124) and (125), such that we obtain

$$\widetilde{\varphi}_{ij}(k/\pi) = 2\pi \left(\delta_{ij} - 4 \sinh k\vartheta_h \sinh k\vartheta_H \left(2 \cosh t(\vartheta_h + \vartheta_H) - I\right)^{-1}_{ij} \right) \ , \tag{126}$$

and therefore (99) by means of (121). The other cases when $I_{il} = 2, 3$ may be obtained similarly with the singularity structure as indicated in figure 2.

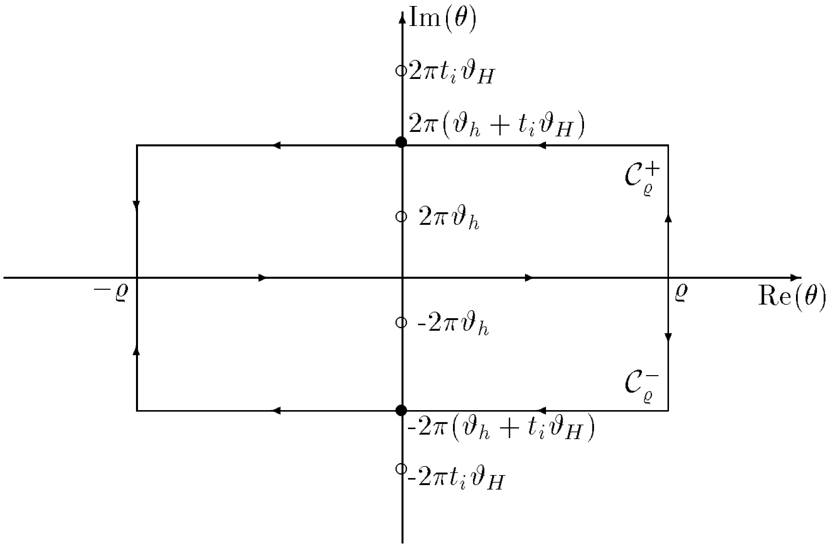


Figure 2: The contours $\mathcal{C}_\varrho^\pm$ in the complex θ-plane. The bullets $\bullet$ belong to poles resulting from $-id/d\theta \ln\{1,1\}_\theta$ and the open circles $\circ$ to poles of $-id/d\theta \ln\{2,2\}_\theta$, for the situation $B > 2H/(H+t_i h)$. When $B < 2H/(H+t_i h)$ the poles at $\pm 2\pi t_i \vartheta_H$ and $\pm 2\pi\vartheta_h$ reverse their roles.

6 Reduction to the simply laced Case

It is instructive to investigate how the general formulae valid for all simple Lie algebras behave when we specialize to simply laced Lie algebras. Considering the data of $X_r^{(1)}$, we notice first of all that there is no distinction anymore between H and h. The length of all roots is the same in the simply laced case, such that $t_i = 1$ for all i and the incidence matrix becomes therefore symmetric. The q-deformed incidence matrix reduces now to the usual incidence matrix, i.e. $[I_{ij}]_q \to I_{ij}$, since it does not have entries different from 1. As a consequence the q-deformed Weyl reflections in (6) become the ordinary Weyl reflections, such that $\sigma^q_{c(i)} \to \sigma_{c(i)}$. The map τ commutes now with the $\sigma_{c(i)}$ and therefore the q-deformed Coxeter element becomes

$$\sigma_q \to q^2\sigma_-\sigma_+ \ = q^2\sigma \ , \tag{127}$$

with σ being the ordinary non-deformed Coxeter element of $X_n^{(1)}$. Noting further that co-weights become identical to weights, i.e. $\hat{\lambda}_i \to \lambda_i$, the generating function (69) acquires the form

$$\sum_y \mu_{ij}\left(2x - \frac{c_i + c_j}{2}, y\right) q^y = -\frac{1}{2} q^{2x - \frac{c_i+c_j}{2}} (\,\lambda_j \cdot \sigma^x \gamma_i) \ . \tag{128}$$

Hence we always have $y = 2x - \frac{c_i+c_j}{2}$ and the only type of blocks which emerges is $\{x, x\}_\theta$ [‡‡]. Therefore the block form of the scattering matrix reads

$$S_{ij}(\theta) = \prod_{q=1}^{h} \left\{ 2q - \frac{c_i + c_j}{2}, 2q - \frac{c_i + c_j}{2} \right\}_\theta^{-\frac{1}{2}\lambda_i \cdot \sigma^q \gamma_j} , \quad X_r^{(1)} \equiv ADE. \tag{129}$$

This means, that also conceptually the simply laced case admits a slightly different formulation. In the generic case we compute the powers of the building blocks indirectly via a generating function, whilst in the simply laced case we may compute them directly.

We can also consider the data of $\hat{X}_n^{(l)}$ and undo the twist, which means that $\omega \to 1$, $l_i \to 1$ and $\hat{t}_i \to 1$ for all i, such that the twisted q-deformed Coxeter element becomes

$$\hat{\sigma}_q \to q^2\sigma_-\sigma_+ \ = q^2\sigma \ . \tag{130}$$

Therefore the generating function (74) becomes

$$\sum_x \nu_{ij}\left(x, 2y - \frac{c_i + c_j}{2}\right) q^x = -\frac{q^{2y - \frac{c_i+c_j}{2}}}{2} (\,\lambda_j \cdot \sigma^y \gamma_i) \ \ , \tag{131}$$

which means that $x = 2y - \frac{c_i+c_j}{2}$ and the only type of blocks which emerge are once again $\{x, x\}_\theta$. Hence, the scattering matrix reduces also in this analysis to the form (129).

The matrix inside the integral representation (99) for the simply laced case follows likewise and acquires the form

$$\Phi_{ij}(t) = 8 \sinh\left(\frac{Bt}{2h}\right) \sinh\left(\frac{(2-B)t}{2h}\right) (2\cosh t/h - I)^{-1}_{ij} \ . \tag{132}$$

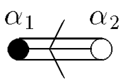
Hence we have recovered the formulae of [8] or [22].

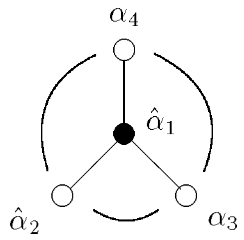
‡‡The block $\{x, x\}_\theta$ corresponds to the block $\{x\}_\theta$ as defined in [8] or [22].

7 Case-by-Case

In order to illustrate the working of our general formulae it is useful to work them out explicitly for some concrete examples. We concentrate here on the non-simply laced case, since the simply laced case is covered extensively in the literature [6]. We will be most detailed for the $(G_2^{(1)}, D_4^{(3)})$-case. Our conventions with regard to numbering and colouring may be read off from the Dynkin diagrams. As usual the arrow points towards the short roots. A black and white vertex corresponds to the colour value $c_i = -1$ and $c_i = 1$, respectively.

7.1 $(G_2^{(1)}, D_4^{(3)})$





The S-matrices of the theory read [10]

$$
\begin{aligned}
S_{11}(\theta) &= \{\overbrace{1,1}^{2};\overbrace{3,5_2}^{1};5,11\}_\theta &(133)\\
S_{12}(\theta) &= \{2,2_3;\overbrace{4,6_3}^{1}\}_\theta &(134)\\
S_{22}(\theta) &= \{1,1_3;\overbrace{3,3_3}^{2};3,5_3;5,7_3\}_\theta\ . &(135)
\end{aligned}
$$

Here we indicated which block is responsible for which type of fusing process. We have $h = 6$ and $H = 12$ for the Coxeter numbers. With the help of (67), we easily verify that for (133), (134) and (135) the following bootstrap identities hold

$$
\begin{aligned}
S_{1l}(\theta+\theta_h+\theta_H)\,S_{1l}(\theta-\theta_h-\theta_H) &= S_{2l}(\theta) \qquad l=1,2 &(136)\\
S_{1l}(\theta+2\theta_h+4\theta_H)\,S_{1l}(\theta-2\theta_h-4\theta_H) &= S_{1l}(\theta) \qquad l=1,2 &(137)\\
S_{2l}(\theta+2\theta_h+4\theta_H)\,S_{2l}(\theta-2\theta_h-4\theta_H) &= S_{2l}(\theta) \qquad l=1,2\ . &(138)
\end{aligned}
$$

As an example for the working of the generalized bootstrap and our criterion provided in section 4.4.4., we plotted the imaginary part of the residues of $S_{22}(\theta)$ in figure 3 for several poles. We observe that the sign changes throughout the range for poles resulting from $\{1, 1_3\}$ and $\{3, 5_3\}$. Only the poles responsible for the self-coupling of particle 2 has a positive imaginary part of the residue throughout the range of the coupling constant β. Except at $B = 4/3$ where it is zero, such that this fusing process decouples.

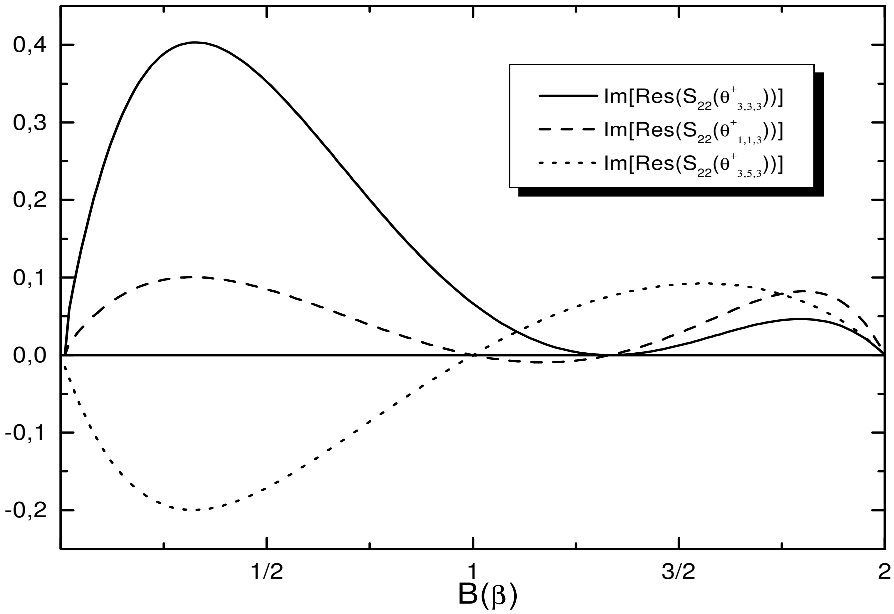


Figure 3: The imaginary part of several residues of $S_{22}(\theta)$ as a function of the effective coupling constant.

Besides (136) the combined bootstrap identities (82) also yield

$$S_{l2}\left(\theta+\theta_h+3\theta_H\right) S_{l2}\left(\theta-\theta_h-3\theta_H\right)=S_{l1}\left(\theta\right) S_{l1}\left(\theta+2\theta_H\right) S_{l1}\left(\theta-2\theta_H\right), \tag{139}$$

for $l = 1, 2$. These equations may be derived from (136) and (137) or verified directly for (133), (134) and (135), with the help of (67). The process corresponding to the combined bootstrap identity (139) is depicted in figure 4.

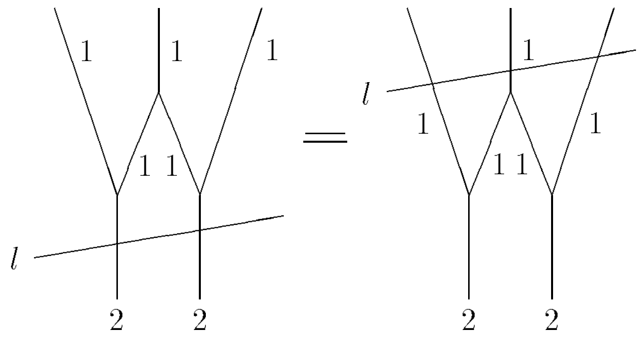


Figure 4: $(G_2^{(1)}, D_4^{(3)})$-combined bootstrap identities (139).

Reading off the fusing angles from the bootstrap equations we obtain the mass ratios according to (56)

$$\frac{m_1}{m_2} = \frac{\sinh(\theta_h + \theta_H)}{\sinh(2\theta_h + 2\theta_H)} \,. \tag{140}$$

We may construct all these formulae from the Lie algebraic data in two alternative ways.

7.1.1 $S_{ij}(\theta)$ from $G_2^{(1)}$

We start by exploiting the properties of $G_2^{(1)}$. The non-vanishing entries of the incidence matrix are $I_{12} = 1$ and $I_{21} = 3$. Consequently equation (11) yields $t_1 = 1$ and $t_2 = 3$. As indicated in the Dynkin diagram we choose $c_1 = -1$ and $c_2 = 1$, such that the q-deformed Coxeter element reads $\sigma_q = \sigma_1^q \tau \sigma_2^q \tau$. The result of successive actions of this element on the simple roots is reported in table 1. Here and in all further tables we choose the following conventions: To each γ_i we associate a column in which we report the powers of the q of the coefficients of the simple roots. We abbreviate

$$\pm(q^{\mu_1^1} + \ldots + q^{\mu_1^{l_1}})\alpha_1 \pm \ldots \pm (q^{\mu_r^1} + \ldots + q^{\mu_r^{l_r}})\alpha_r \to \pm\mu_1^1, \ldots, \mu_{l_1}^1; \ldots; \mu_r^1, \ldots, \mu_{l_r}^1 \,, \tag{141}$$

with $r = \text{rank}\,\mathbf{g}$. When q^μ occurs x-times we denote this by μ^x. Like in the undeformed case the overall sign of any element in Ω_q is definite. Therefore it suffices to report the sign only once as stated in (141). In the complete orbit we always have an equal number of plus and minus signs. When we do not report any signs in the column at all, the signs of the column to the left are adopted. In case the coefficient of the root is zero, we indicate this by a $*$. For instance from table 1 we read off : $\sigma_q \gamma_1 = -(q^4 + q^6)\alpha_1 - q^4 \alpha_2$.

σ_q^x	$\alpha_1 = -\gamma_1$	$\alpha_2 = \gamma_2$
1	$4,6;4$	$-4,6,8;6$
2	$10;8$	$-8,10,12;8,10$
3	$-12;*$	$-*;12$
4	$-16,18;16$	$16,18,20;18$
5	$-22;20$	$20,22,24;20,22$
6	$24;*$	$*;24$

Table 1: The orbits Ω_i^q created by the action of σ_q^x on γ_i

For the conventions chosen the generating functions (69) for the powers of the building blocks are obtainable from the generating functions

$$\sum_y \mu_{11}(2x+1,y)\,q^y = -q^1(\tilde{\lambda}_1 \cdot (\sigma_q)^x \gamma_1)/2 \tag{142}$$

$$\sum_y \mu_{21}(2x,y)\,q^y = -q^{-2}(\tilde{\lambda}_1 \cdot (\sigma_q)^x \gamma_2)/2 \tag{143}$$

$$\sum_y \mu_{22}(2x-1,y)\,q^y = -q^{-3}\,[3]_q\,(\tilde{\lambda}_2 \cdot (\sigma_q)^x \gamma_2)/2\ . \tag{144}$$

We may now read off the Lie algebraic data from the table 1 and we can construct the scattering matrices (133), (134) and (135) according to formula (59).

The two non-equivalent solutions to (42) corresponding to the S-matrix bootstrap equations (136), (137) and (138) read

$$q\sigma_q^{-1}\gamma_1 + q^{-1}\gamma_1 = q^{-3}\gamma_2, \qquad q^{-1}\gamma_1 + q\sigma_q^{-1}\gamma_1 = q^{-3}\gamma_2\,, \tag{145}$$

$$q^3\sigma_q^{-1}\gamma_1 + q^{-5}\sigma_q\gamma_1 = q^{-1}\gamma_1, \qquad q^{-3}\gamma_1 + q^5\sigma_q^{-2}\gamma_1 = q\sigma_q^{-1}\gamma_1\ , \tag{146}$$

$$q^{16}\sigma_q\gamma_2 + \sigma_q^5\gamma_2 = q^{20}\gamma_2, \qquad q^4\sigma_q^4\gamma_2 + q^{20}\gamma_2 = \sigma_q^5\gamma_2\ , \tag{147}$$

respectively. These relations may be obtained either from (136), (137) and (138) together with the formulae which relate the fusing angles to the solution of the fusing rules in terms of the q-deformed Coxeter element (57) or alternatively they may be read off directly from table 1. For a direct comparison with (57) one should cross all term to one side of the equation by means of (19).

It is also instructive to consider explicitly the matrix representation and verify the general formulae of section 5. The doubly q–deformed Cartan matrix for generic q and $\bar{q}$ reads

$$[K]_{q\bar{q}} = \begin{pmatrix} q\bar{q} + q^{-1}\bar{q}^{-1} & -1 \\ -(1+\bar{q}^2+\bar{q}^{-2}) & q\bar{q}^3 + q^{-1}\bar{q}^{-3} \end{pmatrix} \tag{148}$$

with determinant $\det[K]_{q\bar{q}} = q^2\bar{q}^4 + q^{-2}\bar{q}^{-4} - 1$. The right nullvectors are evaluated to

$$\begin{aligned} y(1) &= (\sinh(\theta_h+\theta_H), \sinh(2\theta_h+2\theta_H)) && (149)\\ y(2) &= (\sinh(5\theta_h+5\theta_H), \sinh(10\theta_h+10\theta_H)) \ . && (150) \end{aligned}$$

From (148) we compute the M-matrix according to (107)

$$M(q,\bar{q}) = \frac{1-q^{12}\bar{q}^{24}}{2} \begin{pmatrix} \frac{q\bar{q}+q^3\bar{q}^7}{1-q^2\bar{q}^4+q^4\bar{q}^8} & \frac{1+\bar{q}^2+\bar{q}^{-2}}{q^2\bar{q}^4+q^{-2}\bar{q}^{-4}-1} \\ \frac{1+\bar{q}^2+\bar{q}^{-2}}{q^2\bar{q}^4+q^{-2}\bar{q}^{-4}-1} & \frac{(q\bar{q}+q^3\bar{q}^3)(1+\bar{q}^2+\bar{q}^4)}{1+q^2\bar{q}^4+q^4\bar{q}^8} \end{pmatrix} \tag{151}$$

$$= \begin{pmatrix} \frac{(1+q^2\bar{q}^4-q^6\bar{q}^{12}-q^8\bar{q}^{16})(q\bar{q}+q^3\bar{q}^7)}{2} & \frac{(1+\bar{q}^2+\bar{q}^{-2})(q^2\bar{q}^4+q^4\bar{q}^8-q^{10}\bar{q}^{20}-q^8\bar{q}^{16})}{2} \\ \frac{(1+\bar{q}^2+\bar{q}^{-2})(q^2\bar{q}^4+q^4\bar{q}^8-q^{10}\bar{q}^{20}-q^8\bar{q}^{16})}{2} & \frac{(1+\bar{q}^2+\bar{q}^4)(q\bar{q}+q^3\bar{q}^3)(1+q^6\bar{q}^{12}-q^2\bar{q}^4-q^8\bar{q}^{16})}{2} \end{pmatrix} .$$

Evaluating the M-matrix at $M(q(i\pi s_n), \bar{q}(i\pi s_n))$ leads to

$$\begin{aligned} M_{ij}(q(i\pi),\bar{q}(i\pi)) &= \frac{2i\sqrt{3}(1+2\cosh\theta_H)}{\sinh(\theta_h+\theta_H)\sinh(2\theta_h+2\theta_H)}\ y_i(1)y_j(1) && (152)\\ M_{ij}(q(5i\pi),\bar{q}(5i\pi)) &= \frac{-2i\sqrt{3}(1+2\cosh(5\theta_H))}{\sinh(5\theta_h+5\theta_H)\sinh(10\theta_h+10\theta_H)}\ y_i(2)y_j(2)\ , && (153) \end{aligned}$$

which confirms equation (108) including also the precise factor of proportionality.

7.1.2 $S_{ij}(\theta)$ from $D_4^{(3)}$

Instead of using the data from $G_2^{(1)}$, we can also employ the properties of $D_4^{(3)}$. As indicated in the Dynkin diagram, we choose the values of the bi-colouration to be $c_1 = -1$ and $c_2 = c_3 = c_4 = 1$. Our conventions for the incidence matrix I, the action of $\hat{\tau}$ on the simple roots and the action of the automorphism ω on the simple roots are

$$I = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{\tau}(\vec{\alpha}) = \begin{pmatrix} q^2\alpha_1 \\ q^2\alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}, \qquad \omega(\vec{\alpha}) = \begin{pmatrix} \alpha_1 \\ \alpha_4 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} . \tag{154}$$

The lengths of the orbits are $l_1 = 1$, $l_2 = l_3 = l_4 = 3$ and the q-deformed twisted Coxeter element reads therefore $\hat{\sigma}_q = \omega^{-1}\hat{\sigma}_1^q \hat{\tau} \hat{\sigma}_2^q$. Successive actions of this element on the representatives of Ω_i^ω are reported in table 2.

$\hat{\sigma}_q^x$	$\hat{\alpha}_1 = -\hat{\gamma}_1^+$	$\hat{\alpha}_2 = \hat{\gamma}_2^+$
1	$*;*;2;*$	$-2;*;2;*$
2	$2;*;*;2$	$-2;*;4;2$
3	$2;2;4;*$	$-2,4;2;4;4$
4	$*;*;*;4$	$-4;4;6;4$
5	$4;4;*;*$	$-4;4;*;6$
6	$-6;*;*;*$	$-*;6;*;*$
7	$-*;*;8;*$	$8;*;8;*$
8	$-8;*;*;8$	$8;*;10;8$
9	$-8;8;10;*$	$8,10;8;10;10$
10	$-*;*;*;10$	$10;10;12;10$
11	$-10;10;*;*$	$10;10;*;12$
12	$12;*;*;*$	$*;12;*;*$

Table 2: The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on γ_i

For the generating functions (74) we obtain

$$\sum_x \nu_{11}(x,2y+1)\, q^x = -q(\hat{\lambda}_1 \cdot (\hat{\sigma}_q)^y \hat{\gamma}_1)/2 \tag{155}$$

$$\sum_x \nu_{12}(x,2y)\, q^x = -(\hat{\lambda}_2 \cdot (\hat{\sigma}_q)^y \hat{\gamma}_1)/2 \tag{156}$$

$$\sum_x \nu_{22}(x,2y-1)\, q^x = -q^{-1}(\hat{\lambda}_2 \cdot (\hat{\sigma}_q)^y \hat{\gamma}_2)/2 \tag{157}$$

which yield the scattering matrices (133), (134) and (135) with the help of table 2.

The two non-equivalent solutions to (46) corresponding to (136), (137) and (138) read

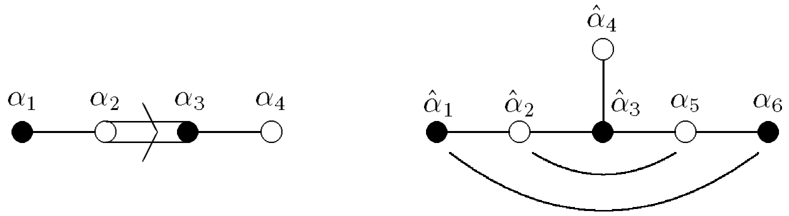
$$q^2\hat{\gamma}_1^+ + \hat{\sigma}_q\hat{\gamma}_1^+ = \hat{\sigma}_q\hat{\gamma}_2^+, \qquad \hat{\sigma}_q\hat{\gamma}_1^+ + q^2\hat{\gamma}_1^+ = \hat{\sigma}_q\hat{\gamma}_2^+, \tag{158}$$

$$q\hat{\sigma}_q^{-1}\hat{\gamma}_1^+ + q^{-3}\hat{\sigma}_q^3\hat{\gamma}_1^+ = q^{-1}\hat{\sigma}_q\hat{\gamma}_1^+, \qquad q^{-3}\hat{\sigma}_q^3\hat{\gamma}_1^+ + q\hat{\sigma}_q^{-1}\hat{\gamma}_1^+ = q^{-1}\hat{\sigma}_q\hat{\gamma}_1^+, \tag{159}$$

$$q^{-2}\hat{\sigma}_q^6\hat{\gamma}_2^+ + q^2\hat{\sigma}_q^2\hat{\gamma}_2^+ = \hat{\sigma}_q^4\hat{\gamma}_2^+, \qquad q^2\hat{\sigma}_q^2\hat{\gamma}_2^+ + q^{-2}\hat{\sigma}_q^6\hat{\gamma}_2^+ = \hat{\sigma}_q^4\hat{\gamma}_2^+, \tag{160}$$

respectively. These relations may be obtained either from (136), (137) and (138) together with the relation which relates the fusing angles to the solution of the fusing rules in terms of the q-deformed twisted Coxeter element (57) or alternatively they may be read off directly from table 2. Exploiting the relationship between the different versions of the fusing rules (57), we may also obtain (158), (159) and (160) from (145), (146) and (147).

7.2 $(F_4^{(1)}, E_6^{(2)})$



The S-matrices of the theory read [7]

$$
\begin{aligned}
S_{11}(\theta) &= \{1,1_2;5,7_2;7,9_2;11,15_2\}_\theta \\
S_{12}(\theta) &= \{2,3_2;4,5_2;6,7_2;6,9_2;8,11_2;10,13_2\}_\theta \\
S_{13}(\theta) &= \{3,4_2;5,6_2;7,10_2;9,12_2\}_\theta \\
S_{14}(\theta) &= \{4,5_2;8,11_2\}_\theta \\
S_{22}(\theta) &= \{1,1_2;3,3_2;3,5_2;5,5_2;5,7_2^2;7,9_2^2;7,11_2;9,11_2;9,13_2;11,15_2\}_\theta \\
S_{23}(\theta) &= \{2,2_2;4,4_2;4,6_2;6,8_2^2;8,10_2;8,12_2;10,14_2\}_\theta \\
S_{24}(\theta) &= \{3,3_2;5,7_2;7,9_2;9,13_2\}_\theta \\
S_{33}(\theta) &= \{1,1;3,3_2;5,7;5,7_2;7,9_2;7,11;9,13_2;11,17\}_\theta \\
S_{34}(\theta) &= \{2,2;4,6;6,8_2;8,12;10,16\}_\theta \\
S_{44}(\theta) &= \{1,1;5,7;7,11;11,17\}_\theta \ .
\end{aligned}
$$

We have $h = 12$ and $H = 18$ for the Coxeter numbers. We will not report here all boostrap identities, but we state the combined bootstrap identities (82)

$$
\begin{aligned}
S_{1l}(\theta+\theta_h+2\theta_H)S_{1l}(\theta-\theta_h-2\theta_H) &= S_{l2}(\theta) \qquad (161) \\
S_{2l}(\theta+\theta_h+2\theta_H)S_{2l}(\theta-\theta_h-2\theta_H) &= S_{l1}(\theta)S_{l3}(\theta-\theta_H)S_{l3}(\theta+\theta_H) \qquad (162) \\
S_{3l}(\theta+\theta_h+\theta_H)S_{3l}(\theta-\theta_h-\theta_H) &= S_{l2}(\theta)S_{l4}(\theta) \qquad (163) \\
S_{4l}(\theta+\theta_h+\theta_H)S_{4l}(\theta-\theta_h-\theta_H) &= S_{l3}(\theta) \qquad (164)
\end{aligned}
$$

for $l=1,2,3,4$. Once again there occurs one equation which is more involved than the usual bootstrap which we depict in figure 5.

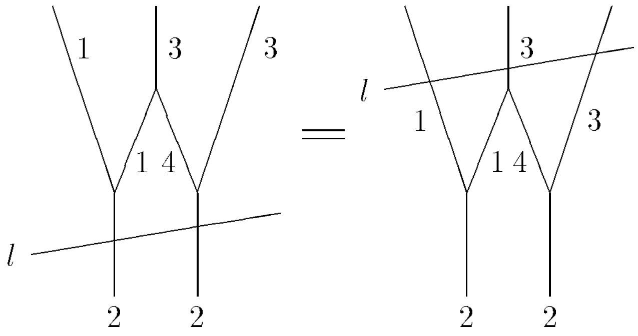


Figure 5: $(F_4^{(1)}, E_6^{(2)})$-combined bootstrap identities (162).

Reading off the fusing angles from the bootstrap equations we obtain the mass ratios from (56)

$$\frac{m_1}{m_2} = \frac{\sinh(\theta_h + 2\theta_H)}{\sinh(10\theta_h + 14\theta_H)} \qquad \frac{m_1}{m_3} = \frac{\sinh(3\theta_h + 5\theta_H)}{\sinh(7\theta_h + 10\theta_H)} \tag{165}$$

$$\frac{m_1}{m_4} = \frac{\sinh(3\theta_h + 5\theta_H)}{\sinh(2\theta_h + 3\theta_H)} \qquad \frac{m_2}{m_3} = \frac{\sinh(9\theta_h + 15\theta_H)}{\sinh(2\theta_h + 2\theta_H)} \tag{166}$$

$$\frac{m_2}{m_4} = \frac{\sinh(9\theta_h + 15\theta_H)}{\sinh(\theta_h + \theta_H)} \qquad \frac{m_3}{m_4} = \frac{\sinh(2\theta_h + 2\theta_H)}{\sinh(\theta_h + \theta_H)}. \tag{167}$$

As in the previous case these formulae can be re-constructed from the twisted as well as the untwisted Lie algebra.

7.2.1 $S_{ij}(\theta)$ from $F_4^{(1)}$

According to our conventions the q-deformed Coxeter element reads $\sigma_q = \sigma_1^q \sigma_3^q \tau \sigma_2^q \sigma_4^q \tau$. The result of successive actions of this element on the simple roots is reported in table 3.

σ_q^x	$\alpha_1 = -\gamma_1$	$\alpha_3 = -\gamma_3$	$\alpha_2 = \gamma_2$	$\alpha_4 = \gamma_4$
1	$*; 4; 3, 5; *$	$3; 3; 2, 4; 2$	$-4; 4; 3, 5; *$	$*; *; 2; 2$
2	$6; 6; 5, 7; 5, 7$	$5; 5, 7; 6^2, 8; 6$	$-6; 6, 8; 5, 7^2, 9; 5, 7$	$5; 5; 6; *$
3	$8; 8, 10; 9, 11; *$	$9; 9^2; 8, 10^2; 8, 10$	$-8, 10; 8, 10^2; 9^2, 11^2; 9, 11$	$*; 9; 8, 10; 8$
4	$*; 12; 11, 13; 11, 13$	$11; 11, 13; 12, 14; 12$	$-12; 12^2, 14; 11, 13^2, 15; 11, 13$	$11; 11; 12; 12$
5	$14; 14; *; *$	$*; 15; 16; 16$	$-14; 14, 16; 15, 17; 15, 17$	$*; 15; 16; *$
6	$-18; *; *; *$	$*; *; 18; *$	$-*; 18; *; *$	$*; *; *; 18$
7	$-*; 22; 21, 23; *$	$21; 21; 20, 22; 20$	$22; 22; 21, 23; *$	$*; *; 20; 20$
8	$-24; 24; 23, 25; 23, 25$	$23; 23, 25; 24^2, 26; 24$	$24; 24, 26; 23, 25^2, 27; 23, 25$	$23; 23; 24; *$
9	$-26; 26, 28; 27, 29; *$	$27; 27^2; 26, 28^2; 26, 28$	$26, 28; 26, 28^2; 27^2, 29^2; 27, 29$	$*; 27; 26, 28; 26$
10	$-*; 30; 29, 31; 29, 31$	$29; 29, 31; 30, 32; 30$	$30; 30^2, 32; 29, 31^2, 33; 29, 31$	$29; 29; 30; 30$
11	$-32; 32; *; *$	$*; 33; 34; 34$	$32; 32, 34; 33, 35; 33, 35$	$*; 33; 34; *$
12	$36; *; *; *$	$*; *; 36; *$	$*; 36; *; *$	$*; *; *; 36$

Table 3: The orbits Ω_i^q created by the action of σ_q^x on γ_i.

By using table 3 we may recover the $(F_4^{(1)}, E_6^{(2)})$–S-matrices with the help of generating functions (69). The two non-equivalent solutions of the fusing rule in Ω_q are

$$
\begin{aligned}
\gamma_l + q^{-12}\sigma_q^4\gamma_l &= q^{-6}\sigma_q^2\gamma_l, \quad \sigma_q^{-1}\gamma_l + q^{12}\sigma_q^{-5}\gamma_l = q^6\sigma_q^{-3}\gamma_l, \quad l = 1,2,3,4 \\
\gamma_1 + q^{-4}\sigma_q\gamma_1 &= q^{-4}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_1 + q^4\sigma_q^{-2}\gamma_1 = \sigma_q^{-1}\gamma_2, \\
\gamma_2 + q^{-14}\sigma_q^5\gamma_1 &= \gamma_1, \quad q^{-4}\gamma_2 + q^{14}\sigma_q^{-6}\gamma_1 = \sigma_q^{-1}\gamma_1, \\
\gamma_4 + q^{-2}\sigma_q\gamma_4 &= \gamma_3, \quad q^{-2}\gamma_4 + \sigma_q^{-1}\gamma_4 = \sigma_q^{-1}\gamma_3, \\
\gamma_4 + q^{-16}\sigma_q^5\gamma_3 &= q^{-16}\sigma_q^5\gamma_4, \quad q^{-2}\gamma_4 + q^{16}\sigma_q^{-6}\gamma_3 = q^{14}\sigma_q^{-5}\gamma_4, \\
\gamma_1 + q^{-15}\sigma_q^5\gamma_3 &= q^{-11}\sigma_q^4\gamma_4, \quad \sigma_q^{-1}\gamma_1 + q^{15}\sigma_q^{-6}\gamma_3 = q^9\sigma_q^{-4}\gamma_4, \\
\gamma_1 + q^{-9}\sigma_q^3\gamma_4 &= q^{-3}\sigma_q\gamma_3, \quad \sigma_q^{-1}\gamma_1 + q^7\sigma_q^{-3}\gamma_4 = q^3\sigma_q^{-2}\gamma_3, \\
\gamma_3 + q^{-14}\sigma_q^5\gamma_4 &= q^{-3}\sigma_q\gamma_1, \quad \sigma_q^{-1}\gamma_3 + q^{12}\sigma_q^{-5}\gamma_4 = q^3\sigma_q^{-2}\gamma_1, \\
\gamma_2 + q^{-15}\sigma_q^5\gamma_3 &= q^{-1}\sigma_q\gamma_4, \quad q^{-4}\gamma_2 + q^{15}\sigma_q^{-6}\gamma_3 = q^{-1}\sigma_q^{-1}\gamma_4, \\
\gamma_2 + q^{-15}\sigma_q^5\gamma_4 &= q\gamma_3, \quad q^{-4}\gamma_2 + q^{13}\sigma_q^{-5}\gamma_4 = q^{-1}\sigma_q^{-1}\gamma_3, \\
\gamma_3 + q^{-4}\sigma_q^2\gamma_4 &= q^{-3}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_3 + q^2\sigma_q^{-2}\gamma_4 = q^{-1}\sigma_q^{-1}\gamma_2, \\
\gamma_4 + q^{-8}\sigma_q^3\gamma_4 &= q^{-3}\sigma_q\gamma_1, \quad q^{-2}\gamma_4 + q^6\sigma_q^{-3}\gamma_4 = q^3\sigma_q^{-2}\gamma_1, \\
\gamma_4 + q^{-13}\sigma_q^4\gamma_1 &= q^{-10}\sigma_q^3\gamma_4, \quad q^{-2}\gamma_4 + q^{13}\sigma_q^{-5}\gamma_1 = q^8\sigma_q^{-3}\gamma_4.
\end{aligned}
$$

Once again we can confirm from these solution the equivalence of the bootstrap equations and the fusing rules by means of (57) and also verify the relation for the mass ratios (56).

7.2.2 $S_{ij}(\theta)$ from $\hat{E}_6^{(2)}$

The q-deformed twisted Coxeter element in our conventions reads $\hat{\sigma}_q = \omega^{-1}\hat{\sigma}_1^q\hat{\sigma}_3^q\hat{\tau}\hat{\sigma}_2^q\hat{\sigma}_4^q$. We report successive actions of this element on the representatives of Ω_i^ω in table 4.

$\hat{\sigma}_q^x$	$\alpha_6 = -\hat{\gamma}_1^+$	$\hat{\alpha}_3 = -\hat{\gamma}_3^+$	$\hat{\alpha}_2 = \hat{\gamma}_2^+$	$\hat{\alpha}_4 = \hat{\gamma}_4^+$
1	$0;*;*;*;*;*$	$*;*;2;2;2;2$	$-*;*;2;*;2;2$	$*;*;2;2;*;*$
2	$*;*;2;*;2;*$	$2;2;2;*;4;4$	$-2;2;2,4;4;4;4$	$*;*;*;*;4;4$
3	$*;2;2,4;4;4;4$	$4;4;4^2;4;4;*$	$-4;4;4^2;4;4,6;6$	$4;4;4;*;*;*$
4	$4;4;4;4;6;6$	$*;4;4,6;6;6^2;6$	$-6;4,6;4,6^2;6;6^2;6$	$*;*;6;6;6;*$
5	$6;6;6;*;6;*$	$6;6^2;6^2;6;8;8$	$-6;6^2;6^2,8;6,8;8^2;8$	$*;6;6;*;8;8$
6	$*;6;6,8;8;8;*$	$8;8;8;8;8;*$	$-8;8^2;8^2;8;8,10;10$	$8;8;8;8;*;*$
7	$*;8;8;8;10;10$	$*;8;8;*;10;*$	$-10;8,10;8,10;10;10;*$	$*;*;*;*;10;*$
8	$10;10;*;*;*;*$	$*;10;10;10;*;*$	$-*;10;10;10;12;*$	$*;10;10;*;*;*$
9	$-*;*;*;*;*;12$	$*;*;12;*;*;*$	$-*;12;*;*;*;*$	$*;*;*;12;*;*$
10	$-12;*;*;*;*;*$	$*;*;14;14;14;14$	$*;*;14;*;14;14$	$*;*;14;14*;*$
11	$-*;*;14;*;14;*$	$14;14;14;*;16;16$	$14;14;14,16;16;16;16$	$*;*;*;*;16;16$
12	$-*;14;14,16;16;16;16$	$16;16;16^2;16;16;*$	$16;16;16^2;16;16,18;18$	$16;16;16;*;*;*$
13	$-16;16;16;16;18;18$	$*;16;16,18;18;18^2;18$	$18;16,18;16,18^2;18;18^2;18$	$*;*;18;18;18;*$
14	$-18;18;18;*;18;*$	$18;18^2;18^2;18;20;20$	$18;18^2;18^2,20;18,20;20^2;20$	$*;18;18;*;20;20$
15	$-*;18;18,20;20;20;*$	$20;20;20;20;20;*$	$20;20^2;20^2;20;20,22;22$	$20;20;20;20;*;*$
16	$-*;20;20;20;22;22$	$*;20;20;*;22;*$	$22;20,22;20,22;22;22;*$	$*;*;*;*;22;*$
17	$-22;22;*;*;*;*$	$*;22;22;22;*;*$	$*;22;22;22;24;*$	$*;22;22;*;*;*$
18	$*;*;*;*;*;24$	$*;*;24;*;*;*$	$*;24;*;*;*;*$	$*;*;*;24;*;*$

Table 4: The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on γ_i.

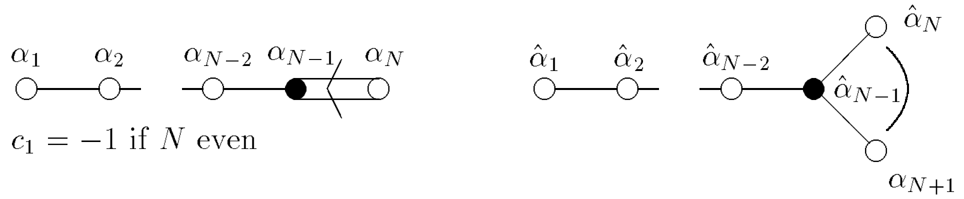
Using the orbits $\hat{\Omega}_i^q$ listed in table 4 we recover with help of the generating functions (74) the $(F_4^{(1)}, E_6^{(2)})$-S-matrices. The two non-equivalent solutions to the fusing rule in $\hat{\Omega}_q$ read

$$\begin{aligned}
\hat{\gamma}_l^+ + q^{-8}\hat{\sigma}_q^6\hat{\gamma}_l^+ &= q^{-4}\hat{\sigma}_q^3\hat{\gamma}_l^+, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_l^+ + q^{10}\hat{\sigma}_q^{-4}\hat{\gamma}_l^+ = q^6\hat{\sigma}_q^{-1}\hat{\gamma}_l^+, \quad l = 1,2,3,4 \\
\hat{\gamma}_1^+ + q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^+, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^+ + q^4\hat{\gamma}_1^+ = q^2\hat{\sigma}_q\hat{\gamma}_2^+, \\
\hat{\gamma}_2^+ + q^{-10}\hat{\sigma}_q^8\hat{\gamma}_1^+ &= \hat{\sigma}_q\hat{\gamma}_1^+, \quad \hat{\sigma}_q^2\hat{\gamma}_2^+ + q^{12}\hat{\sigma}_q^{-6}\hat{\gamma}_1^+ = q^2\hat{\sigma}_q\hat{\gamma}_1^+, \\
\hat{\gamma}_4^+ + q^{-2}\hat{\sigma}_q\hat{\gamma}_4^+ &= \hat{\gamma}_3^+, \quad \hat{\sigma}_q^2\hat{\gamma}_4^+ + q^2\hat{\sigma}_q\hat{\gamma}_4^+ = q^2\hat{\sigma}_q\hat{\gamma}_3^+, \\
\hat{\gamma}_4^+ + q^{-10}\hat{\sigma}_q^8\hat{\gamma}_3^+ &= q^{-10}\hat{\sigma}_q^8\hat{\gamma}_4^+, \quad \hat{\sigma}_q^2\hat{\gamma}_4^+ + q^{12}\hat{\sigma}_q^{-7}\hat{\gamma}_3^+ = q^{10}\hat{\sigma}_q^{-6}\hat{\gamma}_4^+, \\
\hat{\gamma}_1^+ + q^{-10}\hat{\sigma}_q^7\hat{\gamma}_3^+ &= q^{-8}\hat{\sigma}_q^5\hat{\gamma}_4^+, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^+ + q^{12}\hat{\sigma}_q^{-6}\hat{\gamma}_3^+ = q^8\hat{\sigma}_q^{-3}\hat{\gamma}_4^+, \\
\hat{\gamma}_1^+ + q^{-6}\hat{\sigma}_q^4\hat{\gamma}_4^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_3^+, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^+ + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_4^+ = q^4\hat{\gamma}_3^+, \\
\hat{\gamma}_3^+ + q^{-10}\hat{\sigma}_q^7\hat{\gamma}_4^+ &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^+, \quad q^2\hat{\sigma}_q\hat{\gamma}_3^+ + q^{10}\hat{\sigma}_q^{-5}\hat{\gamma}_4^+ = q^4\hat{\gamma}_1^+,
\end{aligned}$$

$$
\begin{aligned}
\hat{\gamma}_2^+ + q^{-10}\hat{\sigma}_q^8\hat{\gamma}_3^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_4^+, \quad \hat{\sigma}_q^2\hat{\gamma}_2^+ + q^{12}\hat{\sigma}_q^{-7}\hat{\gamma}_3^+ = q^2\hat{\sigma}_q\hat{\gamma}_4^+, \\
\hat{\gamma}_2^+ + q^{-10}\hat{\sigma}_q^8\hat{\gamma}_4^+ &= \hat{\gamma}_3^+, \quad \hat{\sigma}_q^2\hat{\gamma}_2^+ + q^{10}\hat{\sigma}_q^{-6}\hat{\gamma}_4^+ = q^2\hat{\sigma}_q\hat{\gamma}_3^+, \\
\hat{\gamma}_3^+ + q^{-4}\hat{\sigma}_q^2\hat{\gamma}_4^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^+, \quad q^2\hat{\sigma}_q\hat{\gamma}_3^+ + q^4\hat{\gamma}_4^+ = q^2\hat{\sigma}_q\hat{\gamma}_2^+, \\
\hat{\gamma}_4^+ + q^{-6}\hat{\sigma}_q^4\hat{\gamma}_4^+ &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^+, \quad \hat{\sigma}_q^2\hat{\gamma}_4^+ + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_4^+ = q^4\hat{\gamma}_1^+, \\
\hat{\gamma}_4^+ + q^{-8}\hat{\sigma}_q^7\hat{\gamma}_1^+ &= q^{-6}\hat{\sigma}_q^5\hat{\gamma}_4^+, \quad \hat{\sigma}_q^2\hat{\gamma}_4^+ + q^{10}\hat{\sigma}_q^{-5}\hat{\gamma}_1^+ = q^6\hat{\sigma}_q^{-3}\hat{\gamma}_4^+.
\end{aligned}
$$

Again we confirm from these solution the equivalence between the bootstrap equations and the fusing rules by means of (57) and also verify the relation for the mass ratios (56).

7.3 $(C_2^{(1)}, D_3^{(2)})$



The S-matrices are given as

$$
S_{11}(\theta) = \{1,1;3,5\}_\theta \quad S_{12}(\theta) = \{2,2_2\}_\theta \quad S_{22}(\theta) = \{1,1_2;3,3_2\}_\theta .
$$

We have $h = 4$ and $H = 6$ for the Coxeter numbers. The combined bootstrap equations (82) yield

$$
\begin{aligned}
S_{1l}(\theta+\theta_h+\theta_H)S_{1l}(\theta-\theta_h-\theta_H) &= S_{l2}(\theta) && (168) \\
S_{2l}(\theta+\theta_h+2\theta_H)S_{2l}(\theta-\theta_h-2\theta_H) &= S_{l1}(\theta-\theta_H)S_{l1}(\theta+\theta_H) && (169)
\end{aligned}
$$

for $l = 1, 2$.

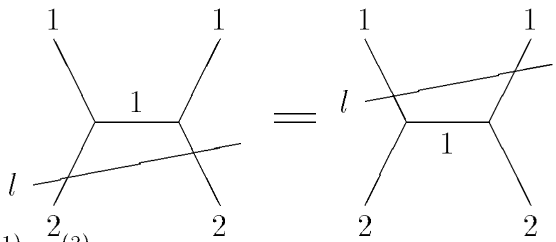


Figure 6: $(C_2^{(1)}, D_3^{(2)})$-combined bootstrap identities (169).

The mass ratio according to (56) are

$$\frac{m_1}{m_2} = \frac{\sinh(\theta_h + \theta_H)}{\sinh(2\theta_h + 4\theta_H)}. \tag{170}$$

7.3.1 $S_{ij}(\theta)$ from $C_2^{(1)}$:

The result of successive actions of the q-deformed Coxeter element on the simple roots is reported in table 5.

σ_q^x	$\alpha_1 = -\gamma_1$	$\alpha_2 = \gamma_2$
1	$4; 3$	$-3, 5; 4$
2	$-6; *$	$-*; 6$
3	$-10; 9$	$9, 11; 10$
4	$12; *$	$*; 12$

Table 5: The orbits Ω_i^q created by the action of σ_q^x on γ_i

The two non-equivalent solutions to the fusing rule in Ω_q read

$$\begin{aligned}
\gamma_1 + q^{-2}\sigma_q\gamma_1 &= q^{-3}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_1 + q^2\sigma_q^{-2}\gamma_1 = q^{-1}\sigma_q^{-1}\gamma_2, \\
\gamma_1 + q^{-7}\sigma_q^2\gamma_2 &= q^{-4}\sigma_q\gamma_1, \quad \sigma_q^{-1}\gamma_1 + q^3\sigma_q^{-2}\gamma_2 = q^4\sigma_q^{-2}\gamma_1.
\end{aligned}$$

7.3.2 $S_{ij}(\theta)$ from $\hat{D}_3^{(2)}$:

The result of successive actions of the q-deformed twisted Coxeter element on the simple roots is reported in table 6.

$\hat{\sigma}_q^x$	$\hat{\alpha}_1 = -\hat{\gamma}_1^+$	$\hat{\alpha}_2 = \hat{\gamma}_2^+$
1	$*; *; 2$	$-2; *; 2$
2	$2; 2; *$	$-2; 2; 4$
3	$-4; *; *$	$-*; 4; *$
4	$-*; *; 6$	$6; *; 6$
5	$-6; 6; *$	$6; 6; 8$
6	$8; *; *$	$*; 8; *$

Table 6: The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on $\hat{\gamma}_i^+$

The two non-equivalent solutions to the fusing rule in $\hat{\Omega}_q$ read

$$\begin{aligned}
\hat{\gamma}_1^+ + q^{-2}\hat{\sigma}_q\hat{\gamma}_1^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^+, \quad q^2\hat{\sigma}_q\hat{\gamma}_1^+ + q^4\hat{\gamma}_1^+ = q^2\hat{\sigma}_q\hat{\gamma}_2^+, \\
\hat{\gamma}_1^+ + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_2^+ &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1, \quad q^2\hat{\sigma}_q\hat{\gamma}_1^+ + q^4\hat{\sigma}_q^{-1}\hat{\gamma}_2^+ = q^4\hat{\sigma}_q^{-1}\hat{\gamma}_1^+.
\end{aligned}$$

7.4 $(C_3^{(1)}, D_4^{(2)})$

The S-matrices are

$$\begin{aligned} S_{11}(\theta) &= \{1,1;5,7\}_\theta \quad S_{12}(\theta) = \{2,2;4,6\}_\theta \quad S_{33}(\theta) = \{1,1_2;3,3_2;5,5_2\}_\theta \\ S_{22}(\theta) &= \{1,1;3,3_2;5,7\}_\theta \quad S_{23}(\theta) = \{2,2_2;4,4_2\}_\theta \quad S_{13}(\theta) = \{3,3_2\}_\theta. \end{aligned}$$

We have $h = 6$ and $H = 8$ for the Coxeter numbers. The combined bootstrap identities read

$$S_{1l}(\theta+\theta_h+\theta_H)S_{1l}(\theta-\theta_h-\theta_H) = S_{l2}(\theta) \tag{171}$$

$$S_{2l}(\theta+\theta_h+\theta_H)S_{2l}(\theta-\theta_h-\theta_H) = S_{l1}(\theta)S_{l3}(\theta) \tag{172}$$

$$S_{3l}(\theta+\theta_h+2\theta_H)S_{3l}(\theta-\theta_h-2\theta_H) = S_{l2}(\theta-\theta_H)S_{l2}(\theta+\theta_H). \tag{173}$$

The mass ratios turn out to be

$$\frac{m_1}{m_2} = \frac{\sinh(\theta_h+\theta_H)}{\sinh(4\theta_h+6\theta_H)} \quad \frac{m_1}{m_3} = \frac{\sinh(\theta_h+\theta_H)}{\sinh(3\theta_h+5\theta_H)} \quad \frac{m_2}{m_3} = \frac{\sinh(2\theta_h+2\theta_H)}{\sinh(3\theta_h+5\theta_H)}. \tag{174}$$

7.4.1 $S_{ij}(\theta)$ from $C_3^{(1)}$

The result of successive actions of the q-deformed Coxeter element on the simple roots is reported in table 7.

σ_q^x	$\alpha_1=\gamma_1$	$\alpha_3=\gamma_3$	$\alpha_2=-\gamma_2$
1	$-2;2;*$	$*;3,5;4$	$2;2,4;3$
2	$-*;6;5$	$5,7;5,7;6$	$6;6;5$
3	$-8;*;*$	$*;*;8$	$-*;8;*$
4	$10;10;*$	$*;11,13;12$	$-10;10,12;11$
5	$*;14;13$	$13,15;13,15;14$	$-14;14;13$
6	$16;*;*$	$*;*;16$	$*;16;*$

Table 7 : The orbits Ω_i^q created by the action of σ_q^x on γ_i

The solutions of the fusing rule in Ω^q are

$$\begin{aligned} \gamma_1 + q^{-2}\sigma_q\gamma_1 &= \gamma_2, \quad q^{-2}\gamma_1 + \sigma_q^{-1}\gamma_1 = \sigma_q^{-1}\gamma_2, \\ \gamma_1 + q^{-6}\sigma_q^2\gamma_2 &= q^{-6}\sigma_q^2\gamma_1, \quad q^{-2}\gamma_1 + q^6\sigma_q^{-3}\gamma_2 = q^4\sigma_q^{-2}\gamma_1, \\ \gamma_1 + q^{-2}\sigma_q\gamma_2 &= q^{-3}\sigma_q\gamma_3, \quad q^{-2}\gamma_1 + q^2\sigma_q^{-2}\gamma_2 = q^{-1}\sigma_q^{-1}\gamma_3, \\ \gamma_1 + q^{-7}\sigma_q^2\gamma_3 &= q^{-4}\sigma_q\gamma_2, \quad q^{-2}\gamma_1 + q^3\sigma_q^{-2}\gamma_3 = q^4\sigma_q^{-2}\gamma_2, \\ \gamma_2 + q^{-9}\sigma_q^3\gamma_3 &= q^{-6}\sigma_q^2\gamma_1, \quad \sigma_q^{-1}\gamma_2 + q^5\sigma_q^{-3}\gamma_3 = q^4\sigma_q^{-2}\gamma_1. \end{aligned}$$

7.4.2 $S_{ij}(\theta)$ from $\hat{D}_4^{(2)}$

The result of successive actions of the q-deformed twisted Coxeter element on the simple roots is reported in table 8.

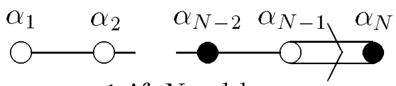
$\hat{\sigma}_q^x$	$\hat{\alpha}_1 = \hat{\gamma}_1^+$	$\hat{\alpha}_3 = \hat{\gamma}_3^+$	$\hat{\alpha}_2 = -\hat{\gamma}_2^+$
1	$-2; 2; *; *$	$*; 2; *; 2$	$2; 2; *; 2$
2	$-*; *; *; 4$	$4; 2, 4; 2; 4$	$*; 2; 2; 4$
3	$-*; 4; 4; *$	$4; 4; 4; 6$	$4; 4; 4; *$
4	$-6; *; *; *$	$*; *; 6; *$	$-*; 6; *; *$
5	$8; 8; *; *$	$*; 8; *; 8$	$-8; 8; *; 8$
6	$*; *; *; 10$	$10; 8, 10; 8; 10$	$-*; 8; 8; 10$
7	$*; 10; 10; *$	$10; 10; 10; 12$	$-10; 10; 10; *$
8	$12; *; *; *$	$*; *; 12; *$	$*; 12; *; *$

Table 8 : The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on $\hat{\gamma}_i^+$

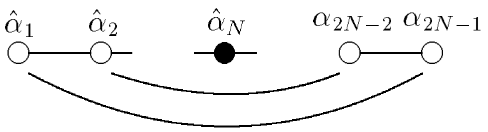
The solutions of the fusing rule in $\hat{\Omega}^q$ are

$$
\begin{aligned}
\hat{\gamma}_1^+ + q^{-2}\hat{\sigma}_q\hat{\gamma}_1^+ &= \hat{\gamma}_2^+, \quad \hat{\sigma}_q^2\hat{\gamma}_1^+ + q^2\hat{\sigma}_q\hat{\gamma}_1^+ = q^2\hat{\sigma}_q\hat{\gamma}_2^+, \\
\hat{\gamma}_1^+ + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_2^+ &= q^{-4}\hat{\sigma}_q^3\hat{\gamma}_1^+, \quad \hat{\sigma}_q^2\hat{\gamma}_1^+ + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_2^+ = q^4\hat{\sigma}_q^{-1}\hat{\gamma}_1^+, \\
\hat{\gamma}_1^+ + q^{-2}\hat{\sigma}_q\hat{\gamma}_2^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_3^+, \quad \hat{\sigma}_q^2\hat{\gamma}_1^+ + q^4\hat{\gamma}_2^+ = q^2\hat{\sigma}_q\hat{\gamma}_3^+, \\
\hat{\gamma}_1^+ + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_3^+ &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_2^+, \quad \hat{\sigma}_q^2\hat{\gamma}_1^+ + q^4\hat{\sigma}_q^{-1}\hat{\gamma}_3^+ = q^4\hat{\sigma}_q^{-1}\hat{\gamma}_2^+, \\
\hat{\gamma}_2^+ + q^{-6}\hat{\sigma}_q^4\hat{\gamma}_3^+ &= q^{-4}\hat{\sigma}_q^3\hat{\gamma}_1^+, \quad q^2\hat{\sigma}_q\hat{\gamma}_2^+ + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_3^+ = q^4\hat{\sigma}_q^{-1}\hat{\gamma}_1^+.
\end{aligned}
$$

7.5 $(B_2^{(1)}, A_3^{(2)})$



$c_1 = -1$ if N odd



$c_1 = -1$ if N odd

The S-matrices read

$$S_{11}(\theta) = \{1, 1_2; 3, 3_2\}_\theta \quad S_{12}(\theta) = \{2, 2_2\}_\theta \quad S_{22}(\theta) = \{1, 1; 3, 5\}_\theta.$$

We have $h = 4$ and $H = 6$ for the Coxeter numbers. The combined bootstrap identities are

$$S_{1l}(\theta+\theta_h+2\theta_H)S_{1l}(\theta-\theta_h-2\theta_H) = S_{l2}(\theta-\theta_H)S_{l2}(\theta+\theta_H) \quad (175)$$
$$S_{2l}(\theta+\theta_h+\theta_H)S_{2l}(\theta-\theta_h-\theta_H) = S_{l1}(\theta). \quad (176)$$

The mass ratio is
$$\frac{m_1}{m_2}=\frac{\sinh(2\theta_h+4\theta_H)}{\sinh(\theta_h+\theta_H)}. \quad (177)$$

7.5.1 $S_{ij}(\theta)$ from $B_2^{(1)}$

The result of successive actions of the q-deformed Coxeter element on the simple roots is reported in table 9.

σ_q^x	$\alpha_1=\gamma_1$	$\alpha_2=-\gamma_2$
1	$-4;3,5$	$3;4$
2	$-6;*$	$-*;6$
3	$10;9,11$	$-9;10$
4	$12;*$	$*;12$

Table 9: The orbits Ω_i^q created by the action of σ_q^x on γ_i
Solutions of the fusing rule in Ω^q

$$\gamma_1+q^{-3}\sigma_q\gamma_2 = q\gamma_2, \quad q^{-4}\gamma_1+q^3\sigma_q^{-2}\gamma_2=q^{-1}\sigma_q^{-1}\gamma_2,$$
$$\gamma_2+q^{-2}\sigma_q\gamma_2 = q^{-3}\sigma_q\gamma_1, \quad \sigma_q^{-1}\gamma_2+q^2\sigma_q^{-2}\gamma_2=q^{-1}\sigma_q^{-1}\gamma_1.$$

7.5.2 $S_{ij}(\theta)$ from $\hat{A}_3^{(2)}$

The result of successive actions of the q-deformed twisted Coxeter element on the simple roots is reported in table 10.

$\hat{\sigma}_q^x$	$\hat{\alpha}_1=\hat{\gamma}_1^+$	$\hat{\alpha}_2=-\hat{\gamma}_2^+$
1	$-*;2;2$	$*;*;2$
2	$-2;2;4$	$2;2;*$
3	$-4;*;*$	$-*;4;*$
4	$*;6;6$	$-*;*;6$
5	$6;6;8$	$-6;6;*$
6	$8;*;*$	$*;8;*$

Table 10 : The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on $\hat{\gamma}_i^+$
The solutions to the fusing rule in $\hat{\Omega}^q$

$$\begin{aligned}\hat{\gamma}_1^+ + q^{-2}\hat{\sigma}_q^2\hat{\gamma}_2^+ &= \hat{\gamma}_2^+, \quad \hat{\sigma}_q^2\hat{\gamma}_1^+ + q^4\hat{\sigma}_q^{-1}\hat{\gamma}_2^+ = q^2\hat{\sigma}_q\hat{\gamma}_2^+, \\ \hat{\gamma}_2^+ + q^{-2}\hat{\sigma}_q\hat{\gamma}_2^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_1, \quad q^2\hat{\sigma}_q\hat{\gamma}_2^+ + q^4\hat{\gamma}_2^+ = q^2\hat{\sigma}_q\hat{\gamma}_1^+.\end{aligned}$$

7.6 $(B_3^{(1)}, A_5^{(2)})$

The S-matrices read

$$\begin{aligned}S_{11}(\theta) &= \{1,1_2;5,7_2\}_\theta \quad S_{12}(\theta) = \{2,3_2;4,5_2\}_\theta \quad S_{33}(\theta) = \{1,1;3,5;5,9\}_\theta \\ S_{22}(\theta) &= \{1,1_2;3,3_2;3,5_2;5,7_2\}_\theta \quad S_{23}(\theta) = \{2,2_2;4,6_2\}_\theta \quad S_{13}(\theta) = \{3,4_2\}_\theta.\end{aligned}$$

We have $h = 6$ and $H = 10$ for the Coxeter numbers. The combined bootstrap identities read

$$\begin{aligned}S_{1l}(\theta+\theta_h+2\theta_H)S_{1l}(\theta-\theta_h-2\theta_H) &= S_{l2}(\theta) && (178)\\ S_{2l}(\theta+\theta_h+2\theta_H)S_{2l}(\theta-\theta_h-2\theta_H) &= S_{l1}(\theta)S_{l3}(\theta-\theta_H)S_{l3}(\theta+\theta_H) && (179)\\ S_{3l}(\theta+\theta_h+\theta_H)S_{3l}(\theta-\theta_h-\theta_H) &= S_{l2}(\theta). && (180)\end{aligned}$$

The mass ratios are

$$\frac{m_1}{m_2} = \frac{\sinh(\theta_h+2\theta_H)}{\sinh(4\theta_h+6\theta_H)} \quad \frac{m_1}{m_3} = \frac{\sinh(2\theta_h+4\theta_H)}{\sinh(2\theta_h+3\theta_H)} \quad \frac{m_2}{m_3} = \frac{\sinh(4\theta_h+8\theta_H)}{\sinh(\theta_h+\theta_H)}. \tag{181}$$

7.6.1 $S_{ij}(\theta)$ from $B_3^{(1)}$

The result of successive actions of the q-deformed Coxeter element on the simple roots is reported in table 11.

σ_q^x	$\alpha_1 = -\gamma_1$	$\alpha_3 = -\gamma_3$	$\alpha_2 = \gamma_2$
1	$*;4;3,5$	$3;3;4$	$-4;4;3,5$
2	$6;6;*$	$*;7;8$	$-6;6,8;7,9$
3	$-10;*;*$	$*;*;10$	$-*;10;*$
4	$-*;14;13,15$	$13;13;14$	$14;14;13,15$
5	$-16;16;*$	$*;17;18$	$16;16,18;17,19$
6	$20;*;*$	$*;*;20$	$*;20;*$

Table 11: The orbits Ω_i^q created by the action of σ_q^x on γ_i
The solutions of the fusing rule in Ω^q are

$$\begin{aligned}
\gamma_1 + q^{-4}\sigma_q\gamma_1 &= q^{-4}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_1 + q^4\sigma_q^{-2}\gamma_1 = \sigma_q^{-1}\gamma_2, \\
\gamma_1 + q^{-10}\sigma_q^3\gamma_2 &= q^{-6}\sigma_q^2\gamma_1, \quad \sigma_q^{-1}\gamma_1 + q^6\sigma_q^{-3}\gamma_2 = q^6\sigma_q^{-3}\gamma_1, \\
\gamma_1 + q^{-7}\sigma_q^2\gamma_3 &= q^{-3}\sigma_q\gamma_3, \quad \sigma_q^{-1}\gamma_1 + q^7\sigma_q^{-3}\gamma_3 = q^3\sigma_q^{-2}\gamma_3, \\
\gamma_2 + q^{-7}\sigma_q^2\gamma_3 &= q\gamma_3, \quad q^{-4}\gamma_2 + q^7\sigma_q^{-3}\gamma_3 = q^{-1}\sigma_q^{-1}\gamma_3, \\
\gamma_3 + q^{-6}\sigma_q^2\gamma_3 &= q^{-3}\sigma_q\gamma_1, \quad \sigma_q^{-1}\gamma_3 + q^6\sigma_q^{-3}\gamma_3 = q^3\sigma_q^{-2}\gamma_1, \\
\gamma_3 + q^{-2}\sigma_q\gamma_3 &= q^{-3}\sigma_q\gamma_2, \quad \sigma_q^{-1}\gamma_3 + q^2\sigma_q^{-2}\gamma_3 = q^{-1}\sigma_q^{-1}\gamma_2.
\end{aligned}$$

7.6.2 $S_{ij}(\theta)$ from $\hat{A}_5^{(2)}$:

The result of successive actions of the q-deformed twisted Coxeter element on the simple roots is reported in table 12.

$\hat{\sigma}_q^x$	$\alpha_5 = -\hat{\gamma}_1^+$	$\hat{\alpha}_3 = -\hat{\gamma}_3^+$	$\hat{\alpha}_2 = \hat{\gamma}_2^+$
1	$0;*;*;*;*$	$*;*;*;2;2$	$-*;*;2;2;2$
2	$*;*;2;2;*$	$2;2;2;*;*$	$-2;2;2;4;4$
3	$*;2;2;4;4$	$*;*;*;4;*$	$-4;4;4;4;*$
4	$4;4;*;*;*$	$*;4;4;*;*$	$-*;4;4;6;*$
5	$-*;*;*;*;6$	$*;*;6;*;*$	$-*;6;*;*;*$
6	$-6;*;*;*;*$	$*;*;*;8;8$	$*;*;8;8;8$
7	$-*;*;8;8;*$	$8;8;8;*;*$	$8;8;8;10;10$
8	$-*;8;8;10;10$	$*;*;*;10;*$	$10;10;10;10;*$
9	$-10;10;*;*;*$	$*;10;10;*;*$	$*;10;10;12;*$
10	$*;*;*;*;12$	$*;*;12;*;*$	$*;12;*;*,*$

Table 12 : The orbits $\hat{\Omega}_i^q$ created by the action of $\hat{\sigma}_q^x$ on $\hat{\gamma}_i^+$
The solutions of the fusing rule in $\hat{\Omega}^q$

$$\begin{aligned}
\hat{\gamma}_1^+ + q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^+, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^+ + q^4\hat{\gamma}_1^+ = q^2\hat{\sigma}_q\hat{\gamma}_2^+, \\
\hat{\gamma}_1^+ + q^{-6}\hat{\sigma}_q^4\hat{\gamma}_2^+ &= q^{-4}\hat{\sigma}_q^3\hat{\gamma}_1^+, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^+ + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_2^+ = q^6\hat{\sigma}_q^{-1}\hat{\gamma}_1^+, \\
\hat{\gamma}_1^+ + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_3^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_3^+, \quad q^2\hat{\sigma}_q^2\hat{\gamma}_1^+ + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_3^+ = q^4\hat{\gamma}_3^+, \\
\hat{\gamma}_2^+ + q^{-4}\hat{\sigma}_q^4\hat{\gamma}_3^+ &= \hat{\gamma}_3^+, \quad \hat{\sigma}_q^2\hat{\gamma}_2^+ + q^6\hat{\sigma}_q^{-3}\hat{\gamma}_3^+ = q^2\hat{\sigma}_q\hat{\gamma}_3^+, \\
\hat{\gamma}_3^+ + q^{-4}\hat{\sigma}_q^3\hat{\gamma}_3^+ &= q^{-2}\hat{\sigma}_q^2\hat{\gamma}_1^+, \quad q^2\hat{\sigma}_q\hat{\gamma}_3^+ + q^6\hat{\sigma}_q^{-2}\hat{\gamma}_3^+ = q^4\hat{\gamma}_1^+, \\
\hat{\gamma}_3^+ + q^{-2}\hat{\sigma}_q\hat{\gamma}_3^+ &= q^{-2}\hat{\sigma}_q\hat{\gamma}_2^+, \quad q^2\hat{\sigma}_q\hat{\gamma}_3^+ + q^4\hat{\gamma}_3^+ = q^2\hat{\sigma}_q\hat{\gamma}_2^+.
\end{aligned}$$

8 Conclusion

We have systematically developed the properties of the q-deformed Coxeter element and its twisted counterpart. The vanishing of the three-point-coupling is governed by the so-called fusing rules. They rules may be formulated either in the orbits Ω_q, $\hat{\Omega}_q$ or Ω and $\hat{\Omega}$. The precise relation between these alternative rules is worked out (57). All of these identities may be proven by appealing to physical arguments. The scattering matrices of affine Toda field theories with real coupling constant related to any dual pair of simple Lie algebras may be expressed in a completely generic way in terms of combinations of hyperbolic functions whose powers are computed from generating functions involving either q-deformed Coxeter elements (69) or alternatively twisted q-deformed Coxeter elements (74). The q-deformation appears to be vital in the construction since it achieves that the properties of the two dual algebras are merged together. It would be interesting to investigate whether it is possible at all to construct generic formulae solely from non-deformed quantities as it is possible in the simply laced case. However, it appears to us that the q-deformation is vital to describe non-simply laced theories. Closely related to this is the question of how to derive the q-deformed versions of the fusing rules directly from the non-deformed versions. We have demonstrated that the proposed scattering matrices fulfill all the requirements of the generalized bootstrap equations. In particular, we established the equivalence of the fusing rules and the generalized S-matrix boostrap equations. Furthermore, we provide a simple criterion which allows to exclude poles from the participation in the bootstrap.

It is intriguing that the combined bootstrap equation (82) incorporates the information of *all* individual fusing processes. These equations do in fact not constitute anything new since they may always be obtained from the individual fusing processes. They correspond to particular graphs (see figure 5 and 6) of higher order.

The matrix $[K]_{q\bar{q}}$ plays a central role in several ways. The components of its nullvectors constitute conserved quantities, e.g. the particle masses. We show how these quantities are related to the fusing rules. The properties of the matrix $[K]_{q\bar{q}}$ are further utilized in order to formulate a matrix M which serves to derive and prove a generic integral representation for the scattering matrix. The same goal may be achieved by exploiting the properties of the matrix $[\hat{K}]_{q\bar{q}}$ which is related to the twisted algebra and allows to define the matrix $\hat{N}$. We established the equality between these two matrices.

It is interesting to note that the properties of the blocks are reflected by the polynomial (104), such that we can carry out a one-to-one identification between $\{x,y\}_\theta$ and $q^x\bar{q}^y$. In addition we can also manipulate them in an identical way if we further define $q^{-x}\bar{q}^{-y} = -q^x\bar{q}^y$ in analogy to $\{-x,-y\}_\theta = \{x,y\}_\theta^{-1}$ or choose q and $\bar{q}$ to be roots of unity. This means we can treat the whole bootstrap properties in an entirely polynomial fashion.

From the matrix relation $\hat{N} = M$ one deduces immediately the equality $\mu(x,y) = \nu(x,y)$. However, it remains a challenge to develop a more direct Lie algebraic understanding of the equation.

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