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## Form factors and asymptotic freedom in the O(3) $\sigma$ -model

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Using the exact form factors we calculate the 2 and 4-particle contributions to the current-current 2-point function in the 2dimensional O(3)  $\sigma$ -model. The results are consistent with perturbative calculations (including the use of the exact value of the mass gap) and Monte Carlo measurements.

**1.** The O(N)  $\sigma$ -model in 1+1 dimensions is a toy model of QCD in 3+1 dimensions. It has received a lot of attention for nearly two decades and the following properties of the model have been established. It is asymptotically free at high energies according to perturbation theory [1], shows dimensional transmutation and requires non-perturbative treatment at low energies. It is integrable due to the presence of infinitely many conserved charges [2]. Its S-matrix and 2-particle form factors have been found using the bootstrap approach [3,4]. For the case of N=4 and 3 the S-matrix of the model has also been derived dynamically [5,6] and for N=3 all the many-particle form factors have been computed [7]. The use of the thermal Bethe ansatz integral equation for the description of the ground state of the model in the presence of a large chemical potential has allowed the exact determination of the  $m/\Lambda$  ratio [8].

Most of these results are based on unproven assumptions and none of them can be considered to be an unambiguous and satisfactory "solution" of the model. The basic unproven assumptions of the abstract bootstrap approach are: the absence of bound states and the absence of CDD factors. It remains to be proven that this description really corresponds to a relativistic field theory model and that this model is the  $O(N) \sigma$ -model. The dynamical calculations in refs. [5,6] are based on substantial changes of the

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original variables and dynamics and it is not clear whether they are really solutions of the original problem. In ref. [8] the ground state of the system (in the presence of a large chemical potential) is assumed to be consisting of one type of particles only. Even the validity of perturbation theory and hence asymptotic freedom has been questioned [9].

Despite of all the logical loopholes, as a result of a large number of studies of the model in the 1/N expansion and also in the lattice version of the model, a picture of overall consistency has emerged. The purpose of this paper is to substantiate this picture by computing the current-current 2-point function in the abstract bootstrap approach and comparing it to the perturbative 2-point function and Monte Carlo results.

Our approach is basically the same as that of ref. [10], where the 2-point function of the Ising model was studied. We insert a complete set of states into the current-current 2-point function and truncate the infinite sum by allowing only a finite number of physical particles in the intermediate states. The results of this preliminary study show that already a modest number of intermediate particles give a reasonable agreement up to relatively high energies.

2. We restrict ourselves to the case of N=3, because the many-particle form factors are not available for N>3. For N=3 they are given in principle in ref. [7], but unfortunately this paper contains so many misprints that forbid the use of its formulae for practical calculations. We can, however, compute the

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necessary form factors by solving the equations of the form factor bootstrap program. This approach was initiated in ref. [4], further developed by the authors of ref. [7] and it is summarized in ref. [10]. (See also ref. [11].)

We define the *n*-particle form factor functions  $f^{a}_{\mathcal{A}_{1}...\mathcal{A}_{n}}(\beta_{1},...,\beta_{n})$  by

$$\langle 0 | J_{\omega}^{a}(0) | A_{1}, \beta_{1}; ...; A_{n}, \beta_{n} \rangle^{\text{in}}$$
  
=  $-m\omega \left( \sum_{k=1}^{n} \exp(-\omega\beta_{k}) \right) f_{A_{1}..A_{n}}^{a}(\beta_{1}, ..., \beta_{n}) ,$   
(1)

where  $J_{\omega}^{a}$  ( $\omega = \pm$ , a = 1, 2, 3) are the O(3) current operators, *m* is the mass of the physical particles,  $A_{i}=1, 2, 3$  are the particle O(3) quantum numbers and  $\beta_{i}$  are the particle rapidities. (Note that current conservation is ensured by this representation.)

The form factor functions are originally defined by (1) for the ordered set of real rapidities  $\beta_1 > ... > \beta_n$ , corresponding to the physical "in" state, but they can be analytically extended to the complex  $\theta$  plane in all rapidity variables. They are meromorphic functions with first order poles only, the residues of which are explicitly given. (See below.) They satisfy the following requirements:

$$f_{A_{1}\dots A_{n}}^{a}(\theta_{1}+\Lambda,...,\theta_{n}+\Lambda) = f_{A_{1}\dots A_{n}}^{a}(\theta_{1},...,\theta_{n}), \quad (2)$$

$$f_{\dots XY\dots}^{a}(...\theta,\theta'...) = S_{XY;VU}(\theta-\theta')f_{\dots UV\dots}^{a}(...\theta',\theta...), \quad (3)$$

$$f^{a}_{A_{1}A_{2}...A_{n}}(\theta_{1} + 2\pi i, \theta_{2}, ..., \theta_{n})$$
  
=  $f^{a}_{A_{2}...A_{n}A_{1}}(\theta_{2}, ..., \theta_{n}, \theta_{1}),$  (4)

 $f^{a}_{ABU_{1...}U_{n}}(\alpha,\beta,\theta_{1},...,\theta_{n})$ 

$$= \frac{\mathrm{i}}{2\pi(\alpha - \beta - \mathrm{i}\pi)} \left[ \delta_{AB} f^{a}_{U_{1}...U_{n}}(\theta_{1}, ..., \theta_{n}) - S_{BU_{1}...U_{n}; V_{1}...V_{n}A}(\theta_{1}, ..., \theta_{n} | \beta) f^{a}_{V_{1}...V_{n}}(\theta_{1}, ..., \theta_{n}) \right]$$
  
+ terms regular at  $\alpha = \beta + \mathrm{i}\pi$ . (5)

The O(3) S-matrix entering (3) is given by [3]

$$S_{AB;CD}(\theta) = S_1(\theta) \delta_{AB} \delta_{CD} + S_2(\theta) \delta_{AC} \delta_{BD} + S_3(\theta) \delta_{AD} \delta_{BC}, \qquad (6)$$

where

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$$S_1(\theta) = \frac{2q\theta}{(\theta+q)(\theta-2q)}, \quad S_2(\theta) = \frac{\theta(\theta-q)}{(\theta+q)(\theta-2q)}$$

$$S_3(\theta) = \frac{2q(q-\theta)}{(\theta+q)(\theta-2q)},\tag{7}$$

and  $q = i\pi$ .

The coefficient  $S_{BU_1...U_n; V_1...V_nA}(\theta_1, ..., \theta_n | \beta)$  in (5) is a product of 2-particle S-matrices corresponding to the scattering of particle  $(B, \beta)$  through the set of particles  $(U_1, \theta_1; ...; U_n, \theta_n)$  into the set of  $(V_1, \theta_1; ...; V_n, \theta_n)$  and the particle  $(A, \beta)$ . (For details, see ref. [11].) Note that (5) determines the residue of the pole of the (n+2)-particle form factor function in terms of the *n*-particle form factor functions.

The lowest form factor function corresponds to 2 particles and is given by [4]

$$f^{a}_{AB}(\theta_1, \theta_2) = -\frac{1}{8} i\pi \,\epsilon^{aAB} \psi(\theta_1 - \theta_2) , \qquad (8)$$

where

$$\psi(\theta) = \frac{\theta - q}{\theta(2q - \theta)} \operatorname{th}^2 \frac{\theta}{2}.$$
(9)

Using the properties

$$\psi(-\theta) = \sigma(\theta)\psi(\theta)$$
 with  $\sigma(\theta) = \frac{\theta+q}{\theta-q}\frac{2q-\theta}{2q+\theta}$ , (10)

and

$$\psi(2q+\theta) = -\psi(-\theta) , \qquad (11)$$

it is easy to see that (8) indeed satisfies (2)-(4). Instead of (5), which applies only for  $n \ge 2$ , the overall normalization in (8) is determined by requiring that the integrated O(3) charges satisfy the O(3) algebra

$$[Q^{a}, Q^{b}] = i\epsilon^{abc}Q^{c}, \quad Q^{a} = \int_{-\infty}^{\infty} dx^{1} J_{0}^{a}(x^{0}, x^{1}).$$
(12)

Now we turn to the determination of the 4-particle form factor function. (This is the next non-trivial case since the currents have non-vanishing matrix elements only between the vacuum and an even number of particles.) Using O(3) symmetry, we can parametrize it as

$$f^{a}_{ABCD} = F_{1} \epsilon^{aAB} \delta_{CD} + F_{2} \epsilon^{aAD} \delta_{BC} + F_{3} \epsilon^{aCD} \delta_{AB}$$
$$+ F_{4} \epsilon^{aBC} \delta_{AD} + F_{5} \epsilon^{aAC} \delta_{BD} + F_{6} \epsilon^{aBD} \delta_{AC} .$$
(13)

It is determined by the equations

 $f(1234) = M_0(34)f(1243), \qquad (3')$ 

$$f(\bar{1}234) = Q_0 f(2341) , \qquad (4')$$

$$\operatorname{Res} f(2234) = \Gamma_0(234) . \tag{5'}$$

Here we introduced a compact notation to make the formulae more transparent. The particle symbols 1, 2, 3, 4 represent both internal symmetry quantum numbers and rapidity variables. f in (3')-(5') is a 6-vector corresponding to the basis (13), whose components, because of (2), are functions of the rapidity differences.  $M_0$  and  $Q_0$  are  $6 \times 6$  matrices, the entries of which can be computed by applying (3) and (4) to the special case of 4 particles. ( $Q_0$  is a constant matrix, whereas the entries of  $M_0$  are S-matrix elements.) Moreover, the bar and dot over the particle symbol represent a shift of the rapidity variable by  $2\pi i$  and  $i\pi$ , respectively. Finally,  $\Gamma_0$  on the right-hand side of (5') stands for a known 6-vector, constructed from S-matrix elements and 2-particle form factors according to the right-hand side of (5).

To find a solution of (3')-(5'), we take the following ansatz:

$$f(1234) = -\frac{1}{32} i\pi^{3} \psi(12) \psi(13) \psi(14)$$
  
  $\times \psi(23) \psi(24) \psi(34) g(1234), \qquad (14)$ 

where g(1234) is a new function. (Note that the 6-vectors f and g only differ by an overall scalar factor.) Using (10) and (11), which are satisfied by the 2-particle function  $\psi$ , we can rewrite (3') and (4') as

$$g(1234) = M(34)g(1243), \quad M(\theta) = \sigma(\theta)M_0(\theta),$$
  
(3")

$$g(\bar{1}234) = Qg(2341), \quad Q = -Q_0.$$
 (4")

Now we make use of the following additional properties of the 2-particle function  $\psi$ 

$$\psi(\theta) = -\frac{4}{\pi^2} \frac{1}{\theta - q} + \text{terms regular at } \theta = q$$
, (15)

$$\psi(\theta)\psi(\theta+q) = \frac{1}{E(\theta)}, \quad E(\theta) = (\theta+q)(\theta-2q),$$
(16)

and rewrite (5') as

$$g(2234) = \Gamma(234) . \tag{5"}$$

Note that while (3'') and (4'') are of the same form as (3') and (4') (with modified coefficient matrices), there is a difference between (5') and (5''): (5') prescribes the residue of f at  $\theta_1 = \theta_2 + q$ , whereas (5'') gives the value of g at this point. More importantly, the right-hand side of (5'') turns out to be a (cubic) polynomial in the rapidity differences. This is a consequence of the fact that the function  $E(\theta)$ defined by (16) is not only a polynomial in  $\theta$ , but it is the same expression as the denominator of the Smatrix elements (7). We think that this wonderful coincidence (which happens only for N=3) explains why all the form factor functions of the O(3)  $\sigma$ -model can be explicitly given in terms of elementary functions.

Since we know that g is a (cubic) polynomial of the remaining rapidity differences if  $\theta_1 - \theta_2 = q$ , it is natural to take the following ansatz for g:

$$g(\theta_1, \theta_2, \theta_3, \theta_4) = \Gamma(\theta_2 - \theta_3, \theta_2 - \theta_4)$$
  
+  $(\theta_1 - \theta_2 - q)P(\theta_2 - \theta_3, \theta_2 - \theta_4)$   
+  $(\theta_1 - \theta_2 - q)^2 R(\theta_2 - \theta_3, \theta_2 - \theta_4)$ . (17)

Written in a basis analogous to (13),  $\Gamma$  is given as

$$\Gamma(x, y) = \begin{pmatrix} 2q(q-x)y \\ xy(3q-y) \\ -q(x^2-qx+y^2-qy-2q^2) \\ (x-q)(x+2q)(q-y) \\ x(x-q)y \\ (x-q)y(y-q) \end{pmatrix}.$$
 (18)

After some algebra, we find that (17) is really a solution of (3") and (4") if we take

$$P(x, y) = \begin{pmatrix} 2q^2 + q(x - 3y) \\ -2q^2 + q(x + 2y) - 2xy \\ 4q^2 - q(x + y) - (x - y)^2 \\ 4q^2 + q(x - 2y) - x^2 \\ -q(2x + y) + 2xy \\ 2q^2 - 3qy + y^2 \end{pmatrix},$$
 (19)

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$$R(x, y) = \begin{pmatrix} x - y \\ -x \\ 0 \\ 0 \\ y - 2q \\ 0 \end{pmatrix}.$$
 (19 cont'd)

In the case of the higher *n*-particle form factor functions, we take

$$f(12...n) = \left(\prod_{i < j} \psi(ij)\right) g(12...n) .$$
 (20)

It is easy to see that the analogue of (5") for the general case is always of the form

$$g(22...n) = \Gamma(2...n)$$
, (21)

where  $\Gamma$  is a polynomial, provided the g function for the (n-2) particle case was a polynomial.

We conjecture that the analogues of (3'') and (4'') have polynomial solutions for all n.

**3.** Having computed the form factor functions, we can construct the (euclidean) time-ordered 2-point function:

$$\sum_{n} \langle 0 | J_{\mu}^{a}(0) | n \rangle \langle n | J_{\nu}^{b}(0) | 0 \rangle \exp(-iP_{n}\xi_{1})$$

$$\times \left[ \theta(\xi_{2}) \exp(-E_{n}\xi_{2}) + (-1)^{\mu+\nu}\theta(-\xi_{2}) \exp(E_{n}\xi_{2}) \right]. \qquad (22)$$

Here  $P_n$  and  $E_n$  are the total momentum and energy of the intermediate state n, and the summation over n indicates a summation over the number of particles as well as summation over the internal indices and integration over the particle rapidities.

Because of short distance singularities, the Fourier transform of (22) does not exist. We can nevertheless represent it as

$$\langle J^{a}_{\mu}(\xi) J^{b}_{\nu}(0) \rangle = \int \frac{\mathrm{d}^{2}p}{(2\pi)^{2}} \exp(-\mathrm{i}p\xi) \\ \times \left[ C\delta_{\mu\nu} + \frac{I(p)}{3\pi} \left( \frac{p_{\mu}p_{\nu}}{p^{2}} - \delta_{\mu\nu} \right) \right] \delta^{ab}, \qquad (23)$$

where C is an undetermined (divergent) constant, but the physically interesting transversal part is unambiguously defined and is given by

$$I(p) = \sum_{n=2}^{\infty} I_n(Q) ,$$
 (24)

$$I_{n}(Q) = 12\pi^{2} \int_{0}^{\infty} du_{1} \int_{0}^{\infty} du_{2} \dots$$

$$\times \int_{0}^{\infty} du_{n-1} \rho_{n}(u) \frac{Q^{2}}{Q^{2} + M_{n}^{2}(u)}, \qquad (25)$$

where Q = p/m,

$$u_{1} = \beta_{1} - \beta_{2} ,$$

$$u_{2} = \beta_{2} - \beta_{3} ,$$

$$\vdots$$

$$u_{n-1} = \beta_{n-1} - \beta_{n} ,$$

$$\rho_{n}(u) = \sum_{A_{1}...A_{n}} |f_{A_{1}...A_{n}}^{3}(\beta_{1}, ..., \beta_{n})|^{2} ,$$

$$M_{n}^{2}(u) = \left(\sum_{k=1}^{n} \operatorname{ch} \beta_{k}\right)^{2} - \left(\sum_{k=1}^{n} \operatorname{sh} \beta_{k}\right)^{2} .$$
(26)

4. Now we turn to the perturbative computation of the current-current 2-point function. We will work in the dimensional regularization scheme and consider the case of general N. The calculation is rather standard, the only difficulty is caused by infrared divergences. We will use the method of Lüscher [12] to solve this problem.

We start from the infrared regularized action

$$S = \int d^{D}x \left( \frac{1}{2g_{0}^{2}} \partial_{\mu} S^{a} \partial_{\nu} S^{a} - \frac{m_{0}^{2}}{g_{0}^{2}} S^{N} \right),$$
  

$$S^{a}S^{a} = 1, \quad D = 2 - \varepsilon$$
(27)

and define the currents as

$$J^{ab}_{\mu} = \frac{1}{g_0^2} \left( S^a \,\partial_{\mu} S^b - S^b \,\partial_{\mu} S^a \right) \,. \tag{28}$$

The mass term is introduced to make the perturbative expansion around the classical ground state  $S^i=0$  (i=1,...,N-1);  $S^N=1$  well-defined. It breaks the O(N) symmetry, but if we compute O(N) invariant quantities like

$$\langle J^{ab}_{\mu}(\xi) J^{ab}_{\nu}(0) \rangle = (N-1) \int \frac{\mathrm{d}^{D} p}{(2\pi)^{D}} \exp(-\mathrm{i} p\xi) \\ \times \left[ C \delta_{\mu\nu} + \frac{I(p)}{\pi} \left( \frac{p_{\mu} p_{\nu}}{p^{2}} - \delta_{\mu\nu} \right) \right], \tag{29}$$

they are expected to remain finite in the  $m_0^2 \rightarrow 0$  limit, in accordance with Elitzur's theorem.

Indeed, in an explicit 2-loop calculation we find (in the  $m_0^2 \rightarrow 0$  limit):

$$C = \frac{2}{g_0^2} + \mathcal{O}(g_0^4) , \qquad (30)$$

$$I(p) = 2\pi \left\{ \frac{1}{g_0^2} + p^{-\varepsilon} \left[ -\beta_0 \left( \frac{2}{\varepsilon} + \gamma + 2 \right) \right] + p^{-2\varepsilon} g_0^2 \left[ -\beta_1 \left( \frac{1}{\varepsilon} + \gamma + \frac{3}{2} \right) \right] + O(g_0^4) \right\}, \quad (31)$$

where

$$\beta_0 = \frac{N-2}{4\pi}, \quad \beta_1 = \frac{N-2}{8\pi^2}, \quad \gamma = \Gamma'(1) + \ln 4\pi.$$
(32)

((30) is actually true to all orders, due to a Ward identity.)

Using the renormalization group, we can translate the bare results (31) into an expansion in terms of the running coupling  $\alpha(p)$  defined by

$$\frac{1}{\alpha} + \kappa \ln \alpha = \ln \frac{p}{\Lambda}, \qquad (33)$$

where  $\kappa = \beta_1/2\beta_0^2 = 1/(N-2)$  and  $\Lambda$  is the  $\Lambda$ -parameter in the  $\overline{\text{MS}}$  scheme. We obtain

$$I(p) = (N-2) \left( \frac{1}{\alpha} - 1 - \kappa \alpha + \mathcal{O}(\alpha^2) \right).$$
(34)

(To obtain (34), the 2-loop RG improved perturbative result, we needed in addition to the bare 2-loop expression (31) also the 3-loop  $\beta$ -function coefficient  $\beta_2$ . This is a scheme dependent quantity, but fortunately it is known in the minimal dimensional regularization scheme [13]:  $\beta_2 = (N^2 - 4)/64\pi^3$ .)

Using the results of ref. [8], we can define a new, scheme independent (inverse) running coupling by

$$x - \kappa \ln x = \ln \frac{p}{m} \tag{35}$$

and rewrite (34) as a scheme independent expansion:

$$I(p) = (N-2) \left[ x + (\xi - 1) + \frac{\kappa(\xi - 1)}{x} + O\left(\frac{1}{x^2}\right) \right],$$
(36)

where [8]

$$\xi = \ln \frac{m}{\Lambda} = \kappa \left( \ln 8 - 1 \right) - \ln \Gamma(1 + \kappa) . \tag{37}$$

If we now specialize (36) to the N=3 case, we find that the 1 and 2-loop corrections to the leading term are extremely small:

$$I(p) = x + (\ln 8 - 2) + \frac{\ln 8 - 2}{x} + O\left(\frac{1}{x^2}\right)$$
$$= x + 0.08 + \frac{0.08}{x} + O\left(\frac{1}{x^2}\right).$$
(38)

Finally we note that the normalization of the current operators (28) is consistent with (12). More importantly, due to current conservation, no integration constant enters the solution of the RG equations. This made it possible to obtain the absolute magnitude of the current-current 2-point function in the asymptotic expansion (38).

5. Using the exact form factors, we computed the 2 and 4-particle contributions to (24) numerically. Our results are shown in figs. 1, 2. Fig. 1 shows that the 2point function is strongly dominated by the 2-particle contribution for low and medium energies. Also shown in fig. 1 are the 2-loop perturbative curve (38) and Monte Carlo measurements of the 2-point function. The MC data were obtained [14] on a  $262 \times 262$ lattice at (inverse) bare coupling  $\beta = 1.7$ . This corresponds to a correlation length  $\xi = 34.5$  in lattice units. Finite size effects are expected to be correspondingly small. Cutoff effects are also expected to be small, since the wavelength of the lattice Fourier mode, even for the largest energies considered here, is larger than 13 lattice units. The good agreement between the MC results and the analytic curve based on the exact form factors seems to indicate that the bootstrap method really represents a description of the same lagrangian theory. It remains to be understood, however, why the agreement here is much better than in previous studies [15], where the 2-particle form factor was directly measured on the lattice.



Fig. 1. The transversal part of the current-current 2-point function for low energy. The dotted line is the 2-particle contribution, the solid line is the sum of the 2- and 4-particle contributions and the dashed line is the 2-loop perturbative result. The data points are the Monte Carlo measurements of ref. [14].



Fig. 2. The transversal part of the current-current 2-point function for high energy. The notation is the same as for fig. 1.

Fig. 1 also contains a warning. If we did not have the exact result of ref. [8] and tried to find the value of the mass gap  $m/\Lambda$  by fitting the perturbative curve to the MC data in this low and medium energy range, we would miss the exact value by about 20% (from below). This illustrates the fact that it is difficult to obtain a value for the  $\Lambda$  parameter which is correct within a few percent. There is no real discrepancy between our results and perturbation theory (including the use of the exact mass gap). If we go to higher energies (fig. 2), where perturbation theory is supposed to be more reliable, we find that the agreement is better than 2% (between Q=50 and Q=500). Note that the deviation at even higher energies is positive so we can assume that it can be accounted for by the contributions of 6 and more particle intermediate states.

It is clear that we need a better understanding of the general structure of the *n*-particle form factors before we can estimate their contribution and draw quantitative conclusions. The results of these preliminary investigations are, however, encouraging.

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