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ASYMPTOTIC SCALE INVARIANCE IN A MASSIVE THIRRING MODEL *

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Abstract: The absence of Callan-Symanzik coupling-constant renormalization in a massive Thirring model is demonstrated to all orders using normal product methods. The derivation depends crucially on the mildness of the "anomaly" of the axial-vector Ward identity in this model, as well as on the special relationship between vector and axial-vector currents in two-dimensional field theory. Application of power-counting arguments establishes the asymptotic scale invariance of the vertex functions when the mass m tends to zero with the normalization point μ either fixed (Gell-Mann-Low limit) or vanishing with m (Callan-Symanzik limit).

1. INTRODUCTION

Wilson's conjecture [1] that the dominant short-distance singularities of products of fields are those of a scale-invariant zero-mass theory has stimulated considerable interest in the question of how quantum field theories behave in the limit of vanishing mass. Among the various models studied in this regard, the massive Thirring model [2] of a self-interacting spinor field in a space-time of two dimensions has the almost unique advantage of having for its associated zero-mass theory a well known, exactly soluble, scale-invariant model [3–5]. The important issue here is whether or not the Green's functions of the massive Thirring model, which one can calculate to arbitrary order in renormalized perturbation theory, pass over smoothly into the known Green's functions [4] of the massless theory. In other words, does the asymptotic short-distance behavior of the massive model coincide with that of the massless case?

Mueller and Trueman [2] have provided impressive evidence that the conjectured smooth zero-mass limit does indeed occur. They consider a model in which the Thirring field of mass $m > 0$ interacts via a trilinear coupling with a heavy vector boson of mass M. The zero-mass limit is defined to be that in which m tends to zero

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and M becomes infinite, in that order. The authors verify that, at least to fourth order in the four-fermion coupling, the renormalization of the coupling constant is finite in the limit, and hence the limiting theory, which should exist [6] provided the Green's functions are normalized away from the mass shell, is scale invariant. They then conclude that the N-point (proper) vertex functions of the zero-mass limiting theory satisfy the same "anomalous" scaling law (which determines the two-point function completely, up to normalization)

$$\Gamma^{(N)}(\lambda p_1 \ldots \lambda p_N) = \lambda^{2-N(1/2+\gamma)}\Gamma^{(N)}(p_1 \ldots p_N) \tag{1.1}$$

as those of the massless Thirring model, and moreover that the vertex functions of the original massive theory satisfy (1.1) asymptotically for sufficiently large λ.

In the present article we investigate a massive Thirring model which differs in certain essential respects from that of Mueller and Trueman. In particular we shall consider from the beginning a four-fermion contact interaction (this corresponds to letting $M \to \infty$ *before* $m \to 0$), and shall avoid ultra-violet divergences by applying Zimmermann's version [7] of the BPH subtraction scheme (without cutoffs). We apply powerful normal product techniques [8–10] to demonstrate the absence of Callan-Symanzik [11, 12] coupling constant renormalization to all orders without resorting to laborious graphical analysis or to the usual power-counting arguments (whose validity is yet to be established). Power counting is, of course, a necessary ingredient in showing the asymptotic scale invariance of the vertex functions, but is not needed to establish the vanishing of the Callan-Symanzik coefficient β. The latter is rather a direct consequence of the Ward identities of the massive theory, and in particular of the remarkably mild form of the "anomaly" [13] of the axial-vector Ward identity in this model.

2. THE MASSIVE THIRRING MODEL

2.1. *Specification of the Green's functions*

We consider a two-dimensional spinor field theory whose effective Lagrangian density (Zimmermann's terminology [8]) is given by

$$L_{\mathrm{EFF}} = \tfrac{1}{2} i(1+b)\,\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - (m-a)\bar{\psi}\psi - \tfrac{1}{2}(g-c)\,(\bar{\psi}\gamma^\mu\psi)\,(\bar{\psi}\gamma_\mu\psi) \equiv L_0 + L_1,$$

$$L_0 = \tfrac{1}{2} i\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m\bar{\psi}\psi, \tag{2.1}$$

where the finite renormalization constants a, b and c are power series in the coupling constant g to be determined by the normalization conditions of the theory. The Green's functions of the model are unambiguously defined by the modified Gell–Mann–Low formula [8],

$$\langle 0|\mathrm{T}\prod_{i=1}^{m}\psi(x_i)\prod_{j=1}^{m}\bar{\psi}(y_j)|0\rangle$$

$$= \text{finite part of } {}^{(0)}\langle 0|\mathrm{T}\prod_{i=1}^{m}\psi^{(0)}(x_i)\prod_{j=1}^{m}\bar{\psi}^{(0)}(y_j) \tag{2.2}$$

$$\times \exp\{i\int \mathrm{d}^2x : L_{\mathrm{I}}[\psi^{(0)},\bar{\psi}^{(0)}] :\}|0\rangle^{(0)} ,$$

where $\psi^{(0)}$ is the free field whose propagator is specified by L_0 and the finite part prescription is that of Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ) [8].

Before making the BPHZ subtractions, the right-hand side of (2.2) is the usual sum over contributions from Feynman diagrams. If I_{G} is the Feynman integrand (in momentum space, before integrating over the independent internal momenta) corresponding to the diagram G, then the BPHZ subtracted integrand is given by

$$R_{\mathrm{G}} = \sum_{\mathrm{U}\in F_{\mathrm{G}}}\sum_{\gamma\in \mathrm{U}}(-t_\gamma)I_{\mathrm{G}} , \tag{2.3}$$

where F is the set of forests (families of non-overlapping, one particle irreducible subdiagrams) of G, and t_γ is the Taylor series to order $\delta(\gamma)$ in the independent external momenta of γ, taken about the point where all such momenta are zero. The degree function $\delta(\gamma)$ which determines the number of subtractions of the subdiagram γ is given in this model by

$$\delta(\gamma) = 2 - \tfrac{1}{2}N_\gamma , \tag{2.4}$$

where N_γ is the number of external lines of γ.

Normal products [8] are introduced by a slight modification of the above formulas. If $\mathcal{O}_a$, $a = 1, 2, \ldots$ are formal products of the basic fields and their derivatives we define

$$\langle 0|\mathrm{T}\prod_{a=1}^{l}N_{\delta_a}[\mathcal{O}_a](x_a)\prod_{i=1}^{m}\psi(y_i)\prod_{j=1}^{n}\bar{\psi}(z_j)|0\rangle = \text{finite part of}$$

$${}^{(0)}\langle 0|\mathrm{T}\prod_{a=1}^{l} :\mathcal{O}_a{}^{(0)}:(x_a)\prod_{i=1}^{m}\psi^{(0)}(y_i)\prod_{j=1}^{n}\bar{\psi}^{(0)}(z_j)$$

$$\times \exp\{i\int \mathrm{d}^2x : L_{\mathrm{I}}[\psi^{(0)},\bar{\psi}^{(0)}] :\}|0\rangle^{(0)} , \tag{2.5}$$

where again the finite part is that of BPHZ, but with

$$\delta(\gamma) = 2 - \tfrac{1}{2}N_\gamma - \sum_{V_a \in \gamma} (2 - \delta_a) . \tag{2.6}$$

Here V_a is the special vertex associated with $\mathcal{O}_a$ and δ_a is an integer or half-integer greater than or equal to the dimension of $\mathcal{O}_a$.

Defining the $2N$-point vertex functions $\Gamma^{(2N)}$ by

$$(2\pi)^2\delta\left(\sum_{i=1}^{N} p_i + \sum_{i=1}^{N} q_i\right)\Gamma^{(2N)}(p_1 \ldots p_N; q_1 \ldots q_N)$$

$$= \int \prod_{i=1}^{N} \mathrm{d}^2x_i\, \mathrm{d}^2y_i \exp\left\{i \sum_{i=1}^{N} (p_i \cdot x_i + q_i \cdot y_i)\right\} \langle 0|\mathrm{T} \prod_{i=1}^{N} \psi(x_i) \prod_{j=1}^{N} \bar{\psi}(y_j)|0\rangle^{\mathrm{PROP}}$$

$$\Gamma^{(2)}(p, -p) = -S'^{-1}_{\mathrm{F}}(p) ,$$

$$S'_{\mathrm{F}}(p) = \int \mathrm{d}x\, \mathrm{e}^{ip \cdot x} \langle 0|\mathrm{T}\psi(x)\bar{\psi}(0)|0\rangle , \tag{2.7}$$

where the superscript PROP indicates that only proper (amputated, one-particle irreducible) diagrams are included, we impose the normalization conditions

$$\Gamma^{(2)}(p, -p)\big|_{\not{p} = m} = 0 ,$$

$$\Gamma^{(2)}(p, -p)\big|_{\not{p} = \mu} = i(\mu - m) ,$$

$$\tfrac{1}{16}\delta_{\alpha_1\alpha_2\alpha_3\alpha_4}\, \Gamma^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1 p_2; p_3 p_4)\Big|_{p_i \cdot p_j = \frac{1}{3}\mu^2(4\delta_{ij} - 1)} = -ig ,$$

with

$$\delta_{\alpha_1\alpha_2\alpha_3\alpha_4} \equiv \gamma^\mu_{\alpha_1\alpha_4}\gamma_{\mu\alpha_2\alpha_3} - \gamma^\mu_{\alpha_1\alpha_3}\gamma_{\mu\alpha_2\alpha_4} = 2(\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} - \delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}) . \tag{2.8}$$

The finite renormalization constants appearing in (2.1) are determined implicitly by (2.8).

2.2. *Callan-Symanzik and renormalization group equations*

The Callan-Symanzik [11, 12] and renormalization group [6, 14] equations of the massive Thirring model are most easily derived using differential vertex operations (DVO) [10] defined by

$$\Delta^\delta_{\mathcal{O}} G^{(2N)}(x_1, \ldots x_N; y_1 \ldots y_N) = \int \mathrm{d}^2z \langle 0|\mathrm{T}N_\delta[\mathcal{O}](z) \prod_{i=1}^{N} \psi(x_i) \prod_{j=1}^{N} \bar{\psi}(y_j)|0\rangle \tag{2.9}$$

with corresponding formulas for the vertex functions. Clearly a DVO corresponds to the presence of a special *internal* vertex in each Feynman diagram. We shall need only four of the DVO's, namely the ones closely associated with the Lagrangian (2.1). They are, symbolically,

$$\Delta_0 = i\int \mathrm{d}^2x N_1[\bar{\psi}\psi](x)\,,$$

$$\Delta_1 = i\int \mathrm{d}^2x N_2\,[\bar{\psi}\psi](x)\,,$$

$$\Delta_2 = -\tfrac{1}{2}\int \mathrm{d}^2x N_2[\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi](x)\,,$$

$$\Delta_3 = \tfrac{1}{2}i\int \mathrm{d}^2x N_2[\bar{\psi}\gamma^\mu \psi)(\bar{\psi}\gamma_\mu\psi)](x)\,. \tag{2.10}$$

Note that $\bar{\psi}\psi$ has dimension one, so that Δ_1 is defined with one more subtraction than is necessary to avoid ultraviolet divergences. This will be very useful, however, for making mass insertions without disturbing the BPHZ subtraction scheme. Referring to ref. [10], we may write down the following identities for the N-point vertex functions (in momentum space):

$$\frac{\partial}{\partial m}\Gamma^{(N)} = \left[\left(\frac{\partial a}{\partial m} - 1\right)\Delta_1 + \frac{\partial b}{\partial m}\Delta_2 + \frac{\partial c}{\partial m}\Delta_3\right]\Gamma^{(N)}\,,$$

$$\frac{\partial}{\partial \mu}\Gamma^{(N)} = \left[\frac{\partial a}{\partial \mu}\Delta_1 + \frac{\partial b}{\partial \mu}\Delta_2 + \frac{\partial c}{\partial \mu}\Delta_3\right]\Gamma^{(N)}\,,$$

$$\frac{\partial}{\partial g}\Gamma^{(N)} = \left[\frac{\partial a}{\partial g}\Delta_1 + \frac{\partial b}{\partial g}\Delta_2 + \left(\frac{\partial c}{\partial g} - 1\right)\Delta_3\right]\Gamma^{(N)}\,,$$

$$N\Gamma^{(N)} = [2(a-m)\Delta_1 + 2(1+b)\Delta_2 + 4(c-g)\Delta_3]\,\Gamma^{(N)}\,,$$

$$\Delta_0\Gamma^{(N)} = [\Delta_1 + r\Delta_2 + s\Delta_3]\,\Gamma^{(N)}\,, \tag{2.11}$$

where

$$r = -\tfrac{1}{4}i\mathrm{T}r\gamma^\mu \frac{\partial}{\partial p^\mu}\Delta_0\Gamma^{(2)}(p,-p)\Big|_{p=0}\,,$$

$$s = -\tfrac{1}{16}i\delta_{\alpha_1\alpha_2\alpha_3\alpha_4}\Delta_0\Gamma^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(00,00)\,.$$

The first of these are simple consequences of the structure of Feynman diagrams and the nature of the BPHZ subtraction scheme. The last is an identity relating normal products of different degree of the type derived by Zimmermann [8].

Eqs. (2.11) express five quantities as linear combinations of the three (linearly independent) $\Delta_i\Gamma^{(N)}$, $i = 1, 2, 3$. Thus there must be two independent linear relations among the quantities on the lefthand side. These are conveniently taken to be [10] the Callan-Symanzik and renormalization group equations,

$$[m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} + \beta\left(\frac{m}{\mu}, g\right)\frac{\partial}{\partial g} - N\gamma\left(\frac{m}{\mu}, g\right)]\,\Gamma^{(N)}\,, \tag{2.12}$$

$$[\mu\frac{\partial}{\partial \mu} + \sigma\left(\frac{m}{\mu}, g\right)\frac{\partial}{\partial g} - N\tau\left(\frac{m}{\mu}, g\right)]\,\Gamma^{(N)} = 0\,. \tag{2.13}$$

Here α, β, γ, σ and τ may be determined from (2.11) by equating to zero the coefficient of each $\Delta_i\Gamma^{(N)}$:

$$[m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} + \beta\frac{\partial}{\partial g} - 2\gamma](a - m) = \alpha m\,,$$

$$[m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} + \beta\frac{\partial}{\partial g} - 2\gamma](1 + b) = \alpha m r\,,$$

$$[m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} + \beta\frac{\partial}{\partial g} - 4\gamma](c - g) = \alpha m s\,, \tag{2.14}$$

$$(\mu\frac{\partial}{\partial \mu} + \sigma\frac{\partial}{\partial g} - 2\tau)(a - m) = 0\,,$$

$$(\mu\frac{\partial}{\partial \mu} + \sigma\frac{\partial}{\partial g} - 2\tau)(1 + b) = 0\,,$$

$$(\mu\frac{\partial}{\partial \mu} + \sigma\frac{\partial}{\partial g} - 4\tau)(c - g) = 0\,. \tag{2.15}$$

To arrive at (2.13) one chooses σ and τ to satisfy the last two equations of (2.15) and then verifies the remaining equation using (2.11) and the relation

$$[(\mu\frac{\partial}{\partial \mu} + \sigma\frac{\partial}{\partial g} - 2\tau)\Gamma^{(2)}(p, -p)]\Big|_{p^2 = m} = 0\,. \tag{2.16}$$

Subtracting (2.13) from (2.12) yields

$$[m\frac{\partial}{\partial m} + \eta\frac{\partial}{\partial g} - N\xi]\,\Gamma^{(N)} = \alpha m\Delta_0\Gamma^{(N)}\,, \tag{2.17}$$

where $\eta = \beta - \sigma$ and $\xi = \gamma - \tau$ may be determined from (2.14) and (2.15) or, alternatively, from the normalization conditions at μ applied to (2.17):

$$\xi = \frac{im}{\mu - m}\,(\alpha\Delta_0\Gamma^{(2)}(p, -p)\Big|_{p^2=\mu} + i)\,, \tag{2.18}$$

$$\eta = 4g\xi + \tfrac{1}{16} im\delta_{\alpha_1\alpha_2\alpha_3\alpha_4}\Delta_0\Gamma^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1p_2, p_3p_4)\Big|_{p_i\cdot p_j=\frac{1}{3}\mu^2(4\delta_{ij}-1)} \tag{2.19}$$

Analogues of (2.12) and (2.13) may be derived for the proper functions of the current operator,

$$(2\pi)^2\delta(p + \sum_{i=1}^{N} p_i + \sum_{j=1}^{N} q_i)\Gamma^{(2N)}_\lambda(p, p_1 \ldots p_N, q_1 \ldots q_N)$$

$$= \int d^2x \prod_{i=1}^{N} d^2x_i\, d^2y_i \exp\{ipx + i\sum_j (p_j\cdot x_j + q_j\cdot y_j)\}$$

$$\times \langle 0|T\, N_1[\bar\psi\gamma_\lambda\psi](x) \prod_{i=1}^{N} \psi(x_i) \prod_{j=1}^{N} \bar\psi(y_j)|0\rangle^{\mathrm{PROP}}$$

in a similar way. $\Gamma^{(N)}_\lambda$ satisfies the first three of the eqs. (2.11), and in addition

$$(N-2)\Gamma^{(N)}_\lambda = [2(a-m)\Delta_1 + 2(1+b)\Delta_2 + 4(c-g)\Delta_3]\,\Gamma^{(N)}_\lambda\,,$$

$$(\Delta_0 - t)\Gamma^{(N)}_\lambda = [\Delta_1 + r\Delta_2 + s\Delta_3]\,\Gamma^{(N)}_\lambda\,, \tag{2.20}$$

where

$$t = \tfrac{1}{2}\,\mathrm{Tr}\gamma^\lambda\Delta_0\Gamma^{(2)}_\lambda(0,0,0)\,.$$

The presence of the term $t\Gamma^{(N)}_\lambda$ in the second equation is due to the BPHZ subtractions for proper subdiagrams which contain both the $\bar\psi\psi$ and $\bar\psi\gamma_\lambda\psi$ normal product vertices. By the same reasoning as before, we obtain

$$[m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - (N-2)\gamma]\,\Gamma^{(N)}_\lambda = \alpha m(\Delta_0 - t)\Gamma^{(N)}_\lambda\,, \tag{2.21}$$

$$[\mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial g} - (N-2)\tau]\,\Gamma^{(N)}_\lambda = 0\,, \tag{2.22}$$

$$[m\frac{\partial}{\partial m} + \eta\frac{\partial}{\partial g} - (N-2)\xi]\,\Gamma^{(N)}_\lambda = \alpha m(\Delta_0 - t)\Gamma^{(N)}_\lambda\,. \tag{2.23}$$

2.3. *Equations of motion and Ward identities*

Equations of motion for the massive Thirring model may be derived in a straightforward manner using the methods of refs. [8] and [9]. In particular

$$(i\gamma\cdot\partial_{x_1}-m)\langle 0|\mathrm{T}\psi(x_1)\ldots\psi(x_N)\bar{\psi}(y_1)\ldots\bar{\psi}(y_N)|0\rangle$$

$$=\langle 0|\mathrm{T}\{(g-c)N_{3/2}[(\bar{\psi}\gamma^\mu\psi)\gamma_\mu\psi](x_1)-a\psi(x_1)-ib\partial\!\!\!/\psi(x_1)\}\psi(x_2)\ldots\bar{\psi}(y_N)|0\rangle$$

$$+i\sum_{k=1}^{N}(-1)^{k+N}\delta(x_1-y_k)\langle 0|\mathrm{T}\psi(x_2)\ldots\bar{\psi}(y_{k-1})\bar{\psi}(y_{k+1})\ldots\bar{\psi}(y_N)|0\rangle,$$

$$\langle 0|\mathrm{T}\,N_2[\bar{\psi}(i\partial\!\!\!/-m)\psi](x)X|0\rangle$$

$$=\langle 0|\mathrm{T}\{(g-c)N_2[\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)](x)-aN_2[\bar{\psi}\psi](x)-bN_2[\bar{\psi}(i\partial\!\!\!/)\psi](x)\}X|0\rangle$$

$$+i\sum_{k=1}^{N}\delta(x-y_k)\langle 0|\mathrm{T}X|0\rangle. \tag{2.24}$$

where X is an arbitrary product of the basic fields,

$$X=\prod_{j=1}^{N}\psi(x_j)\prod_{k=1}^{N}\bar{\psi}(y_k).$$

As in ref. [9]. eq. (2.24) may be used to derive Ward identities for the vector and axial-vector currents:

$$(1+b)\partial_x^\mu\langle 0|\mathrm{T}\,N_1[\bar{\psi}\gamma_\mu\psi](x)X|0\rangle=(1+b)\langle 0|\mathrm{T}\,N_2[\partial^\mu(\bar{\psi}\gamma_\mu\psi)](x)X|0\rangle$$

$$=(1+b)i\langle 0|\mathrm{T}\{N_2[\bar{\psi}(-i\overleftarrow{\partial\!\!\!/}-m)\psi](x)-N_2[\bar{\psi}(i\overrightarrow{\partial\!\!\!/}-m)\psi](x)\}X|0\rangle$$

$$=\sum_{k=1}^{N}[\delta(x-y_k)-\delta(x-x_k)]\langle 0|\mathrm{T}X|0\rangle; \tag{2.25}$$

$$(1+b)\partial_x^\mu\langle 0|\mathrm{T}\,N_1[\bar{\psi}\gamma_\mu\gamma^5\psi](x)X|0\rangle=(1+b)\langle 0|\mathrm{T}\,N_2[\partial^\mu(\bar{\psi}\gamma_\mu\gamma^5\psi)](x)X|0\rangle$$

$$=(1+b)i\langle 0|\mathrm{T}\{N_2[\bar{\psi}(-i\overleftarrow{\partial\!\!\!/}-m)\gamma^5\psi](x)+N_2[\bar{\psi}\gamma^5(i\overrightarrow{\partial\!\!\!/}-m)\psi](x)+2mN_2[\bar{\psi}\gamma^5\psi](x)\}X|0\rangle$$

$$=-\sum_{k=1}^{N}(\delta(x-x_k)\gamma_{x_k}^5+\delta(x-y_k)\gamma_{y_k}^{5\mathrm{T}})\langle 0|\mathrm{T}X|0\rangle+2i(m-a)\langle 0|\mathrm{T}N_2[\bar{\psi}\gamma^5\psi](x)X|0\rangle. \tag{2.26}$$

For a justification of our bringing the derivative inside the time-ordering and normal product symbols, the reader is referred to the appendix of ref. [9].

The fact that the right-hand side of (2.26) involves $N_2[\bar{\psi}\gamma^5\psi]$ rather than $N_1[\bar{\psi}\gamma^5\psi]$ is the two-dimensional analogue of the well-known "anomaly" of the axial-vector Ward identity in quantum electrodynamics and other four-dimensional models [13]. In our case, however, the "anomaly" is less drastic in its consequences,

since it merely produces a change of normalization of the axial-vector current and hence does not prevent it from being conserved in the zero-mass limit (see sect.3). This is because Zimmermann's identity relating normal products of different degree is severely restricted by the requirements of P-, C- and Lorentz invariance, as well as Fermi statistics, and assumes the particularly simple form,

$$N_2[\bar{\psi}\gamma^5\psi] = N_1[\bar{\psi}\gamma^5\psi] + eN_2[\partial^\mu(\bar{\psi}\gamma_\mu\gamma^5\psi)] \ , \tag{2.27}$$

where

$$e\gamma_\mu\gamma^5 = i\frac{\partial}{\partial q_\mu}\langle 0|\mathrm{T}N_1[\bar{\psi}\gamma^5\psi](0)\tilde{\psi}(\tfrac{1}{2}q)\tilde{\psi}(\tfrac{1}{2}q)|0\rangle^{\mathrm{PROP}}\Big|_{q=0} \ .$$

Thus we may rewrite (2.26) as

$$\begin{aligned}&(1+b)(1-h)\partial_x^\mu\langle 0|\mathrm{T}N_1[\psi\gamma_\mu\gamma^5\psi](x)X|0\rangle\\ &= -\sum_{k=1}^{N}[\delta(x-x_k)\gamma^5{}_{x_k} + \delta(x-y_k)\gamma^{5\mathrm{T}}_{y_k}]\langle 0|\mathrm{T}X|0\rangle\\ &+ 2i(m-a)\langle 0|\mathrm{T}N_1[\bar{\psi}\gamma^5\psi](x)X|0\rangle \ ,\end{aligned} \tag{2.28}$$

where

$$h = 2i\frac{m-a}{1+b}e \ .$$

2.4. *Vanishing of* $\beta(m/\mu, g)$

The Ward identities of the massive Thirring model may be used to demonstrate that the coefficient of the coupling-constant derivative in the Callan-Symanzik equation vanishes to all orders. This property, which we shall prove without reference to the asymptotic behavior of vertex functions, will be employed in sect. 3 to show the scale invariance of the theory in the limit of vanishing mass.

We shall need the following Ward identities, which may be proved using methods analogous to those which led to (2.25) and (2.28),

$$\begin{aligned}&(1+b)p^\lambda\Gamma_\lambda^{(2N)}(p, p_1\ldots p_N, q_1\ldots q_N)\\ &= i\sum_{k=1}^{N}\{\Gamma^{(2N)}(p_1\ldots p_N, q_1\ldots q_k+p\ldots q_N)\\ &- \Gamma^{(2N)}(p_1\ldots p_k+p\ldots p_N, q_1\ldots q_N)\} \ ,\end{aligned} \tag{2.29}$$

$$(1+b)(1-h)p_\nu \epsilon^{\nu\lambda} \Gamma_\lambda^{(2N)} (p, p_1 \ldots p_N, q_1 \ldots q_N)$$
$$= i \sum_{k=1}^{N} \{\Gamma^{(2N)}(p_1 \ldots p_N, q_1 \ldots q_k + p \ldots q_N) \gamma_{q_k}^5$$
$$+ \gamma_{p_k}^5 \Gamma^{(2N)}(p_1 \ldots p_k + p \ldots p_N, q_1 \ldots q_N)\}$$
$$- 2(m-a)\Gamma_5^{(2N)} (p, p_1 \ldots p_N, q_1 \ldots q_N) , \tag{2.30}$$

$$(1+b)p^\lambda \Delta_0 \Gamma_\lambda^{(2N)}(p, p_1 \ldots p_N, q_1 \ldots q_N)$$
$$= i \sum_{k=1}^{N} \{\Delta_0 \Gamma^{(2N)}(p_1 \ldots p_N, q_1 \ldots q_k + p \ldots q_N)$$
$$- \Delta_0 \Gamma^{(2N)}(p_1 \ldots p_k + p \ldots p_N, q_i \ldots q_N)\} , \tag{2.31}$$

$$(1+b)(1-h)p_\nu \epsilon^{\nu\lambda} \Delta_0 \Gamma_\lambda^{(2N)}(p, p_1 \ldots p_N, q_1 \ldots q_N)$$
$$= i \sum_{k=1}^{N} \{\Delta_0 \Gamma^{(2N)}(p_1 \ldots p_N, q_1 \ldots q_k + p \ldots q_N)\gamma_{q_k}^5$$
$$+ \gamma_{p_k}^5 \Delta_0 \Gamma^{(2N)}(p_1 \ldots p_k + p \ldots p_N, q_1 \ldots q_N)\}$$
$$- 2[1 + (m-a)(\Delta_0 - u)] \Gamma_5^{(2N)}(p, p_1 \ldots p_N, q_1 \ldots q_N) , \tag{2.32}$$

where

$$(\epsilon^{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad \epsilon^{\mu\nu}\gamma_\nu = \gamma^\mu \gamma^5 ,$$

$$u\gamma^5 = \Delta_0 \Gamma_5(0, 0, 0) ,$$

and $\Gamma_5^{(2N)}$ is defined in the same way as $\Gamma_\lambda^{(2N)}$, with γ_λ is replaced by γ^5.

The extra term in the last equation is due to the fact that the Zimmermann expansion relating $N_1[\psi\gamma^5\psi]$ and $N_2[\bar{\psi}\gamma^5\psi]$ contains a term arising from subtractions for which the γ^5 vertex and the Δ_0 vertex are in the same proper subdiagram with two external lines.

We now apply the Callan-Symanzik differential operator

$$m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} + \beta\frac{\partial}{\partial g}$$

to both sides of (2.29) and (2.30), simplifying the results with the aid of (2.31), (2.32), (2.12), (2.14), (2.21) and the analogue of (2.21) with γ^λ and t replaced by γ^5 and u, respectively. Thus

$$[\alpha mr + \alpha mt\,(1 + b)]\, p^\lambda \Gamma_\lambda^{(2N)} = 0\,,$$

$$\left[\alpha mr + \alpha mt\,(1 + b) - (1 + b)\beta \frac{\partial h}{\partial g}\right] p_\nu \epsilon^{\nu\lambda} \Gamma_\lambda^{(2N)} = 0\,, \tag{2.33}$$

and hence,

$$r + (1 + b)\, t = 0\,,$$

$$\beta \frac{\partial h}{\partial g} = 0\,. \tag{2.34}$$

But a simple computation shows that the first-order contribution to h is nonzero. Hence $\beta(m/\mu, g) = 0$.

Referring back to (2.14) we may now write a convenient expression for the "anomalous dimension", $\gamma(m/\mu, g)$:

$$2\gamma(m/\mu, g) = \frac{m'r}{1 + m'r} \neq 0\,, \tag{2.35}$$

where $m' = (m - a)/(1 + b)$.

3. ZERO–MASS LIMITS

3.1. *Preliminaries*

In this section we investigate the asymptotic behavior of the vertex functions in two different zero-mass limits: (i) the Gell-Mann-Low limit [6], in which m tends to zero with g and μ fixed, and (ii) the Callan-Symanzik limit [11, 12] in which m and μ tend to zero with g and m/μ fixed. Both limits define scale invariant zero-mass theories in which the Ward identities of the massless Thirring model are satisfied.

Our methods follow closely those of Symanzik [15]. With him we assume the validity of certain power counting arguments which are generally accepted as true but which have not yet been placed on a completely rigorous footing. Thus we shall suppose that for non-exceptional [15] momenta $p_1, \ldots p_N$ (no nontrivial sub-sum $\Sigma_k p_{i_k}$ is lightlike), a Feynman integral of the type

$$J(p, m, g) = \lim_{\epsilon \to 0} \int \mathrm{d}k R_{\mathrm{G}}\,(p, k, m, g, \epsilon)\,, \qquad \{p = p_1 \ldots p_N\}\,,\ \{k = k_1 \ldots k_N\}\,, \tag{3.1}$$

where R_G is given by (2.3) and the ϵ-dependence of the integrand enters through Zimmermann's form [7] for the propagator,

$$S'_F(p,\epsilon) = i(\not{p} + m)[p^2 - m^2 + i\epsilon(p^2 + m^2)]^{-1}$$

has an asymptotic expansion, for $\lambda \to \infty$ [15],

$$J(p,\lambda^{-1}m,g) = \lambda^{\alpha} \sum_{k,l=0}^{\infty} c_{kl}(p, m, g)\lambda^{-k}\log^l \lambda \,, \tag{3.2}$$

where the leading power α is an integer which may be determined by counting powers in $R_G(p, k, m, g, \epsilon)$. The latter leads to the following assignment of leading asymptotic powers in the massive Thirring model:

$$\begin{aligned} &\alpha = 1 : r, s, t, u, e. \\ &\alpha = 0 : \Gamma^{(N)}, \Delta_0\Gamma^{(N)}, \Gamma^{(N)}_{\lambda}, \Delta_0\Gamma^{(N)}_{\lambda}, b, c, h, \alpha, \beta, \gamma \\ &\alpha = -1 : a, \eta, \xi \,. \end{aligned} \tag{3.3}$$

The same power-counting is applicable for both Gell-Mann-Low and Callan-Symanzik zero-mass limits, since the asymptotic forms differ only by powers of $\log {}^{m}/\mu$.

3.2. *Gell-Mann-Low zero-mass limit*

From (3.3) we see that in the limit of vanishing m with μ fixed and positive, the vertex functions $\Gamma^{(N)}$, $\Delta_0\Gamma^{(N)}$, etc., are at worst logarithmically divergent, whereas η and ξ tend to zero like $m \log^x m$ ($x \equiv$ unknown logarithmic power). Thus, from (2.17), we have for non-exceptional momenta

$$m \frac{\partial}{\partial m} \Gamma^{(N)} = \mathrm{O}(m \log^x m) \,, \tag{3.4}$$

and so $\Gamma^{(N)}$ must be *constant* in the limit. Similarly, from (2.14), (2.15), (2.34) and (3.3),

$$m \frac{\partial b}{\partial m} = -\alpha m(1 + b)t + \mathrm{O}(m \log^x m) \,, \tag{3.5}$$

so that (2.23) yields

$$m \frac{\partial}{\partial m} [(1 + b)\Gamma^{(N)}_{\lambda}] = \mathrm{O}(m \log^x m) \,, \tag{3.6}$$

and the current $j_\lambda = (1+b)N_1[\bar{\psi}\gamma_\lambda\psi]$ has finite vertex functions in the zero-mass limit.

Eq. (3.4) allows us to define, at least for non-exceptional momenta, the *Gell-Mann-Low zero-mass theory* associated with the massive Thirring model as follows:

$$\Gamma^{(N)}_{\mathrm{GML}}(p_1 \ldots p_N, \mu, g) = \lim_{m\to 0} \Gamma^{(N)}(p_1 \ldots p_N, m, \mu, g) . \tag{3.7}$$

The limiting vertex functions are scale-invariant, satisfying

$$\mu \frac{\partial}{\partial \mu} \Gamma^{(N)}_{\mathrm{GML}} = N\tau(0,g)\Gamma^{(N)}_{\mathrm{GML}} = N\gamma(0,g)\Gamma^{(N)}_{\mathrm{GML}} \tag{3.8}$$

since

$$\sigma\left(\frac{m}{\mu}, g\right) = \beta\left(\frac{m}{\mu}, g\right) - \eta\left(\frac{m}{\mu}, g\right) = -\eta\left(\frac{m}{\mu}, g\right) = \mathrm{O}(m \log^x m) ,$$

$$\tau\left(\frac{m}{\mu}, g\right) = \gamma\left(\frac{m}{\mu}, g\right) - \xi\left(\frac{m}{\mu}, g\right) = \gamma\left(\frac{m}{\mu}, g\right) + \mathrm{O}(m \log^x m) . \tag{3.9}$$

From (3.8), the boundary conditions for time-ordered Green's functions and the normalization condition at μ, the two-point function of the zero-mass theory is completely determined:

$$\Gamma^{(2)}_{\mathrm{GML}}(p, -p) = i\not{p}\mu^{2\tau}(p^2 + i0)^{-\tau}, \qquad \tau = \tau(0,g) . \tag{3.10}$$

The zero-mass vector-current vertex functions (for non-exceptional momenta) are defined with the aid of (3.6):

$$\Gamma^{(N)\lambda}_{\mathrm{GML}}(p, p_1 \ldots p_N, \mu, g) \equiv \lim_{m\to 0} (1+b)\Gamma^{(N)\lambda}(p, p_1 \ldots p_N, m, \mu, g) \tag{3.11}$$

From (2.29), (2.30) and (3.3) these are easily seen to satisfy the Ward identities, characteristic of the massless Thirring model [4]

$$\begin{aligned} &p_\lambda \Gamma^{(2N)\lambda}_{\mathrm{GML}}(p, p_1 \ldots p_N, q_1 \ldots q_N) \\ &= \sum_{k=1}^{N} \{\Gamma^{(2N)}_{\mathrm{GML}}(p_1 \ldots p_N, q_1 \ldots q_k + p \ldots q_N) \\ &- \Gamma^{(2N)}_{\mathrm{GML}}(p_1 \ldots p_k + p \ldots p_N, q_1 \ldots q_N)\} , \end{aligned} \tag{3.12}$$

$$(1-h_0)p^\nu \epsilon_{\nu\lambda} \Gamma^{(2N)\,\lambda}_{\mathrm{GML}}(p, p_1 \ldots p_N, q_1 \ldots q_N)$$

$$= i \sum_{k=1}^{N} \{\Gamma^{(2N)}_{\mathrm{GML}}(p_1 \ldots p_N, q_1 \ldots q_k + p \ldots q_N)\gamma^5_{q_k}$$

$$+ \gamma^5_{p_k} \Gamma^{(2N)}_{\mathrm{GML}}(p_1 \ldots p_k + p \ldots p_N, q_1 \ldots q_N)\} , \tag{3.13}$$

where

$$h_0 = \lim_{m \to 0} h < \infty .$$

3.3. *Callan-Symanzik zero-mass limit*

The asymptotic behavior of the massive Thirring model in the limit $m \to 0$, $\mu \to 0$ and $\rho = \mu/m$ fixed may be studied in a manner which parallels closely the discussion of the preceding paragraphs. The controlling differential equation is now (2.12) which we rewrite, considering $\mu = \rho m$ as a function of m

$$[m \frac{\partial}{\partial m} - N\gamma] \Gamma^{(N)} = \alpha m \Delta_0 \Gamma^{(N)} . \tag{3.14}$$

From (3.3) the righthand side may be ignored in the limit $m \to 0$, and we may define the vertex functions of the *Callan-Symanzik zero-mass theory* for non-exceptional momenta as

$$\Gamma^{(N)}_{\mathrm{CS}}(p_1 \ldots p_N, m_0, \rho, g) = \lim_{m \to 0} \left(\frac{m}{m_0}\right)^{-N\gamma} \Gamma^{(N)}(p_1 \ldots p_N, m, \rho m, g) , \tag{3.15}$$

where m_0 is an arbitrary mass parameter which fixes the normalization. Similarly, the current vertex function may be defined as

$$\Gamma^{(N)\lambda}_{\mathrm{CS}}(p, p_1 \ldots p_N, m_0, \rho, g)$$

$$= \lim_{m \to 0} \left(\frac{m}{m_0}\right)^{-(N-2)\gamma + \alpha m t} \Gamma^{(N)\lambda}(p, p_1 \ldots p_N, m, \rho m, g) . \tag{3.16}$$

Using methods analogous to those of subsect. 3.2, one may derive in straightforward fashion the Ward identities characteristic of the massless Thirring model. This time the anomalous dimension, non-trivially dependent on the choice of ρ, is given by

$$\gamma(\rho, g) . \tag{3.17}$$

4. CONCLUSIONS

Why is the massive Thirring model asymptotically scale invariant? According to Mueller and Trueman [2], the crucial fact is that the interaction is renormalizable and of the form $\lambda j_\mu j^\mu$, with j_μ conserved. Our analysis reveals a quite different picture: the absence of Callan-Symanzik coupling-constant renormalization, which leads to asymptotic scaling in either of the two zero-mass limits of sect. 3, is a consequence of the asymptotic conservation of both vector and axial-vector currents (j_μ, resp. $j_{5\mu}$), with the two related by the relation (peculiar to two dimensions) $j_{5\mu} = \epsilon_{\mu\nu} j^\nu$.

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