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LETTER TO THE EDITOR

Diagonalisation of GL(N) invariant transfer matrices and quantum N-wave system (Lee model)

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Abstract. The algebraic Bethe ansatz is constructed for the GL(N) invariant transfer matrices with arbitrary GL(N) spin. For this purpose the notion of vacuum subspace is introduced. It is shown that the GL(N) magnet can be interpreted as an integrable discrete approximation of vector or matrix nonlinear Schrödinger models or of the quantum N-wave system.

The quantum inverse problem method (QIPM) connects exactly solvable field-theoretical models on a line or chain with solutions to the Yang–Baxter equation. We refer the reader to the review papers on QIPM of Faddeev (1979), Faddeev and Takhtajan (1979) and Kulish *et al* (1981) for details and references. As a result of the development of QIPM many interesting quantum models were solved and their lattice approximations were found, e.g. the sine-Gordon equation, the nonlinear Schrödinger equation and its multicomponent generalisation (see e.g. Izergin and Korepin 1982, Kulish *et al* 1981). The eigenstates of the corresponding Hamiltonian were constructed by means of the algebraic Bethe ansatz (ABA) for scalar equations.

The generalisation of the ABA to the multicomponent case incorporating the hierarchy of the Bethe ansatz was made by Kulish (1980) on an example of the quantum vector nonlinear Schrödinger (NS) model and by Kulish and Reshetikhin (1981) for the Sutherland (1975) magnet. In Babelon *et al* (1982) and Schultz (1983) this method was applied to the model with Z_N symmetrical R-matrices. All these models have a pseudovacuum with some special property (see below).

A class of GL(N) invariant solutions of the Yang–Baxter equation (YBE) was found by Kulish *et al* (1981). The transfer matrices corresponding to these solutions describe the chains of interacting GL(N) spins, which transform according to an irreducible representation with the highest weight ($m_1 \geqslant m_2 \geqslant \ldots \geqslant m_N$). The Sutherland model corresponds to the case of spins which transform according to the vector representation ($m_i = \delta_{i1}$). This case also corresponds to the Yang solution of YBE (Yang 1967).

In the present work we suggest a method of diagonalisation for transfer matrices of systems of interacting GL(N) spins with arbitrary highest weight. For this purpose, we had to introduce the notion of vacuum subspace instead of pseudovacuum in the traditional scheme of ABA.

It is shown that in correspondence with the choice of representation, acting in the quantum space of the system, the present model of the GL(N) magnet can be interpreted as an integrable discrete approximation of quantum vector or matrix NS models or as an integrable discrete approximation of the quantum N-wave model.

The model under consideration is the system of interacting GL(N) spins p_n^{ij} situated on a chain with M sites. The space of quantum states at each site $n=1,\ldots,M$ is the representation space V_n of GL(N), corresponding to the highest weight $(m_1^{(n)},\ldots,m_N^{(n)})$. The space $\mathscr{H}=V_1\otimes\ldots\otimes V_M$ is a (full) quantum space of the system.

In accordance with QIPM, let us define the monodromy matrix

$$T_a(u)=b_M L_{aM}(u-v_M)b_{M-1}L_{aM-1}(u-v_{M-1})\ldots b_1 L_{a1}(u-v_1). \tag{1}$$

This operator acts in the tensor product $W_a\otimes\mathscr{H}$ ($W_a\equiv C^N$ is an auxiliary space). The matrices b_k act non-trivially only in W_a and $b_k=\mathrm{diag}(b_k^1,\ldots,b_k^N)$. The operator $L_{an}(u)$ acts non-trivially only in $W_a\otimes V_n$ and has the form

$$L_{an}(u)=u+\sum_{i,j=1}^{N} e_a^{ij}p_n^{ji} \tag{2}$$

where $(e_a^{ij})_{kl}=\delta_k^i\delta_l^j$ are basic matrices in the auxiliary space and p_n^{ij} are generators of the Lie algebra gl(N)-representation V_n:

$$[p_n^{ij},p_m^{kl}]=\delta_{nm}(\delta_{jk}p_n^{il}-\delta_{li}p_n^{kj}), \qquad p_n^{ii}|0\rangle_n=m_i^{(n)}|0\rangle_n. \tag{3}$$

The vector $|0\rangle_n$ is the highest vector in the space V_n: $p_n^{ij}|0\rangle_n=0$, $i<j$. The operator $L_{an}(u)$ satisfies the Yang–Baxter equation

$$R_{ab}(u)L_{an}(u+v)L_{bn}(v)=L_{bn}(v)L_{an}(u+v)R_{ab}(u) \tag{4}$$

which is written in the tensor product $W_a\otimes W_b\otimes V_n$, $W_a\equiv W_b\equiv C^N$. The $N^2\times N^2$ matrix $R_{ab}(u)$ is given by

$$R_{ab}(u)=u+\mathscr{P}_{ab}^{(N)}, \tag{5}$$

where $\mathscr{P}_{ab}^{(N)}$ is the permutation operator in $C^N\otimes C^N$: $\mathscr{P}_{ab}^{(N)}f\otimes g=g\otimes f$.

From (4) we obtain the Yang–Baxter relation for monodromy matrices

$$R_{ab}(u)T_a(u+v)T_b(v)=T_b(v)T_a(u+v)R_{ab}(u). \tag{6}$$

It means in particular that the trace of $T_a(u)$ (transfer matrix) over W_a forms a family of commuting operators

$$[t(u),t(v)]=0, \qquad t(u)=\mathrm{Tr}_a\,T_a(u)\equiv\sum_{i=1}^{N}T_{ii}(u). \tag{7}$$

We shall see later that entries of $T(u)$ can be used to construct the eigenstates of $t(u)$. To obtain the integrals of motion for our spin system with local densities it is necessary to consider L-operators in auxiliary spaces W_a corresponding to the representation with signature $(m_1,\ldots,m_N)$ and transfer matrices generated by these L-operators. For the fundamental representation $[1^k]$ the corresponding transfer matrices can be written in the form ($k=1,\ldots,N$):

$$t_k(u)=\mathrm{Tr}_{(a_1\ldots a_k)}\{P^-_{a_1\ldots a_k}T_{a_1}(u)\ldots T_{a_k}(u+k-1)\} \tag{8}$$

where the trace is taken over the antisymmetric subspace in the product $W_{a_1}\otimes\ldots\otimes W_{a_k}$, $W_{a_i}\equiv C^N$ and $P^-_{a_1\ldots a_k}$ is a projector on this subspace. The operators (8) commute with each other,

$$[t_k(u),t_l(v)]=0, \tag{9}$$

the joint spectrum of these operators is simple modulo the global GL(N) degeneracy, and transfer matrices corresponding to other representations of GL(N) in auxiliary space are some algebraic function of $t_k(u)$ (Kulish and Reshetikhin 1982).

To construct the eigenstates of $t(u)$ we arrange the $N\times N$ matrix (5) in the block form

$$T(u)=\begin{pmatrix} A(u) & B(u)\\ C(u) & D(u)\end{pmatrix}, \tag{10}$$

where $D(u)$ is an $(N-1)\times(N-1)$ block, $B(u)$ is a row with $N-1$ components, $C(u)$ is a column. The R matrix (4) from (3), (6) in this basis has the form

$$R(u)=\begin{pmatrix} u+1 & 0 & 0 & 0\\ 0 & uI & I & 0\\ 0 & I & uI & 0\\ 0 & 0 & 0 & S(u)\end{pmatrix} \tag{11}$$

where I is the unit matrix in C^{N-1}, $S(u)$ acts in $C^{N-1}\otimes C^{N-1}$ and has the same structure as the R matrix: $S(u)=u+\mathscr{P}^{(N-1)}$. The relation (6) contains all the commutation relations between entries of $T(u)$. We shall only use the following ones of them:

$$(u-v)A(v)B(u)+B(v)A(u)=(u-v+1)B(u)A(v), \tag{12}$$

$$(u-v)D_a(u)B_b(v)+B_a(u)D_b(v)=B_a(v)D_b(u)S_{ab}(u-v), \tag{13}$$

the subscripts indicating the auxiliary spaces W_a, W_b.

In contrast to the traditional approach let us consider the vacuum subspace $\mathscr{H}_0=V_1^{(0)}\otimes\ldots\otimes V_M^{(0)}$ in $\mathscr{H}$ instead of the pseudovacuum. The subspace $V_k^{(0)}\subset V_k$ consists of vectors $f_k\in V_k$ satisfying

$$p_k^{11}f_k=m_1^{(k)}f_k, \qquad p_k^{ij}f_k=0, \qquad j=2,3,\ldots,N. \tag{14}$$

The subspace $V_k^{(0)}$ corresponds to the natural embedding $\mathrm{GL}(N-1)\subset\mathrm{GL}(N)$ and is an irreducible $\mathrm{GL}(N-1)$-module of the highest weight $(m_2,\ldots,m_N)$. Vectors $f\in\mathscr{H}_0$ satisfy

$$A(u)f=\prod_{n=1}^{M}(u-v_n+m_1^{(n)})b_n^1 f, \qquad C^i(u)f=0, \qquad i=2,\ldots,N, \tag{15}$$

$$D_a(u)f=b_M^{(1)}L_{aM}^{(1)}(u-v_M)\ldots b_1^{(1)}L_{a1}^{(1)}(u-v_1)f\equiv T_a^{(1)}(u)f, \tag{16}$$

$$L_{an}^{(1)}(u)=u+\sum_{i,j=2}^{N}e_a^{ij}p_n^{ji}, \qquad b_k^{(1)}=\mathrm{diag}(b_k^2,\ldots,b_k^N). \tag{17}$$

Here p_n^{ij}, $i,j=2,\ldots,N$ are generators of the $\mathrm{GL}(N-1)$ acting in $V_n^{(0)}$ and auxiliary space $W_a^{(1)}\equiv C^{N-1}$. We construct the eigenvectors of (7) by the formula

$$F=B^{i_1}(u_1^{(1)})\ldots B^{i_{n_1}}(u_{n_1}^{(1)})F^{(1)}_{i_1\ldots i_{n_1}}, \qquad i=2,\ldots,N. \tag{18}$$

The set of vectors $F^{(1)}_{i_1\ldots i_{n_1}}$ from $\mathscr{H}_0$ will be found later. Acting on (18) by $t(u)=A(u)+\mathrm{Tr}_a D_a(u)$ according to Kulish (1980) and Kulish and Reshetikhin (1981), we get

$$A(u)F=\prod_{n=1}^{M}(u-v_n+m_1^{(n)})b_n^1\prod_{k=1}^{n_1}\frac{u-u_k^{(1)}-1}{u-u_k^{(1)}}+\text{'unwanted terms'}, \tag{19}$$

$$\mathrm{Tr}_a D_a(u)F=\prod_{k=1}^{n_1}(u-u_k^{(1)})^{-1}B^{i_1}(u_1^{(1)})\ldots B^{i_{n_1}}(u_{n_1}^{(1)})t^{(1)}(u\,|\{u_k^{(1)}\})^{j_1\ldots j_{n_1}}_{i_1\ldots i_{n_1}}F^{(1)}_{j_1\ldots j_{n_1}}$$
$$+\text{'unwanted terms'}. \tag{20}$$

The 'unwanted terms' do not contain the vector (18). The condition of their vanishing in the sum of (19), (20) gives rise to the equations on quasimomenta $\{u_k^{(1)}\}$. The operator $t^{(1)}(u|\{u_k^{(1)}\})$ introduced in (20) acts in the space

$$\mathscr{H}^{(1)} = \mathscr{H}_0 \otimes W_1^{(1)} \otimes W_2^{(1)} \otimes \ldots \otimes W_{n_1}^{(1)}, \qquad W_l^{(1)} \equiv C^{N-1}, \tag{21}$$

$$t^{(1)}(u|\{u_k^{(1)}\}) = \mathrm{Tr}_a \tilde{T}_a^{(1)}(u), \tag{22}$$

$$\tilde{T}_a^{(1)}(u) = T_a^{(1)}(u) S_{an_1}(u - u_{n_1}^{(1)}) \ldots S_{a1}(u - u_1^{(1)}). \tag{23}$$

The indices (i_l, j_l) in (20) refer to the spaces $W_l^{(1)}$, $l = 1, \ldots, n_{n_1}$. From (21)–(23) we conclude that $t^{(1)}(u|\{u_k^{(1)}\})$ is the transfer matrix of the GL$(N-1)$-invariant magnet on a chain with $M + n_1$ sites. As for initial $T(u)$ for $T^{(1)}(u)$ the relations

$$S_{ab}(u)\tilde{T}_a^{(1)}(u+v)\tilde{T}_b^{(1)}(v) = \tilde{T}_b^{(1)}(v)\tilde{T}_a^{(1)}(u+v)S_{ab}(u), \tag{24}$$

$$[t^{(1)}(u|\{u_k^{(1)}\}), t^{(1)}(v|\{u_k^{(1)}\})] = 0, \tag{25}$$

are valid. These relations imply the following theorem.

Theorem. The vector (18) is an eigenvector of $t(u)$ iff $F^{(1)} \in \mathscr{H}^{(1)}$ is an eigenvector of $t^{(1)}(u|\{u_k^{(1)}\})$, so that

$$t(u)F = \left\{ \prod_{n=1}^{M} (u - v_n + m_1^{(n)}) b_n^1 \prod_{i=1}^{n_1} \frac{u - u_i^{(1)} - 1}{u - u_i^{(1)}} + \Lambda(u) \prod_{i=1}^{n_1} (u - u_i^{(1)})^{-1} \right\} F. \tag{26}$$

Here $\Lambda(u)$ is the eigenvalue of $t^{(1)}(u|\{u_k^{(1)}\})$

$$t^{(1)}(u|\{u_k^{(1)}\}) F^{(1)} = \Lambda(u) F^{(1)}. \tag{27}$$

The quasimomenta $\{u_k^{(1)}\}_{k=1}^{n_1}$ satisfy the equations

$$\prod_{n=1}^{M} (u_j^{(1)} - v_n + m_1^{(n)}) b_n^1 \prod_{\substack{i=1 \\ i \neq j}}^{n_1} (u_j^{(1)} - u_i^{(1)} - 1) = -\Lambda(u_j^{(1)}). \tag{28}$$

The theorem is proved by a straightforward calculation following Kulish and Reshetikhin (1981).

Thus we have reduced the diagonalisation of the GL(N)-invariant transfer matrix (7) to the same problem for the transfer matrix with GL$(N-1)$ symmetry. Using the embeddings GL$(N) \supset$ GL$(N-1) \supset \cdots \supset$ GL(2) and repeating the procedure $N-2$ times we get the problem of diagonalisation of the GL(2) invariant transfer matrix (inhomogeneous *XXX*-model). The solution of the last problem is well known (see e.g. Faddeev and Takhtajan 1979, or Baxter 1971).

As a result the eigenvectors of $t(u)$ are parametrised by $N-1$ sets of quasimomenta $\{u_k^{(1)}\}_{k=1}^{n_1}, \ldots, \{u_k^{(N-1)}\}_{k=1}^{n_{N-1}}$. The corresponding eigenvalue is

$$\begin{aligned} \lambda(u) = & \prod_{n=1}^{M} (u - v_n + m_1^{(n)}) b_n^1 \prod_{k=1}^{n_1} \frac{u - u_k^{(1)} - 1}{u - u_k^{(1)}} \\ & + \sum_{s=2}^{N-1} \prod_{n=1}^{M} (u - v_n + m_s^{(n)}) b_n^s \prod_{k=1}^{n_{s-1}} \frac{u - u_k^{(s-1)} + 1}{u - u_k^{(s-1)}} \prod_{k=1}^{n_s} \frac{u - u_k^{(s)} - 1}{u - u_k^{(s)}} \\ & + \prod_{n=1}^{M} (u - v_n + m_1^{(n)}) b_n^N \prod_{k=1}^{n_{N-1}} \frac{u - u_k^{(N-1)} + 1}{u - u_k^{(N-1)}}. \end{aligned} \tag{29}$$

The sets of quasimomenta $\{u_j^{(k)}\}_{j=1}^{n_k}$, $k=1,2,\ldots,N-1$ satisfy the equations

$$\prod_{n=1}^{M}\frac{(u_j^{(1)}-v_n+m_1^{(n)})b_n^1}{(u_j^{(1)}-v_n+m_2^{(n)})b_n^2}=\prod_{\substack{k\neq j\\k=1}}^{n_1}\frac{u_j^{(1)}-u_k^{(1)}+1}{u_j^{(1)}-u_k^{(1)}-1}\prod_{k=1}^{n_2}\frac{u_j^{(1)}-u_k^{(2)}-1}{u_j^{(1)}-u_k^{(2)}},\tag{30}$$

$$\prod_{n=1}^{M}\frac{(u_j^{(s)}-v_n+m_s^{(n)})b_n^s}{(u_j^{(s)}-v_n+m_{s+1}^{(n)})b_n^{s+1}}$$
$$=\prod_{\substack{k\neq j\\k=1}}^{n_s}\frac{u_j^{(s)}-u_k^{(s)}+1}{u_j^{(s)}-u_j^{(s)}-1}\prod_{k=1}^{n_{s+1}}\frac{u_j^{(s)}-u_k^{(s+1)}-1}{u_j^{(s)}-u_k^{(s+1)}}\prod_{k=1}^{n_{s-1}}\frac{u_j^{(s)}-u_k^{(s-1)}}{u_j^{(s)}-u_k^{(s+1)}+1},$$

$$\prod_{n=1}^{M}\frac{(u_j^{(N-1)}-v_n+m_{N-1}^{(n)})b_n^{N-1}}{(u_j^{(N-1)}-v_n+m_N^{(n)})b_n^N}$$
$$=\prod_{\substack{k\neq j\\k=1}}^{n_{N-1}}\frac{u_j^{(N-1)}-u_k^{(N-1)}+1}{u_j^{(N-1)}-u_k^{(N-1)}-1}\prod_{k=1}^{n_{N-2}}\frac{u_j^{(N-1)}-u_k^{(N-2)}}{u_j^{(N-1)}-u_k^{(N-2)}+1}.$$

The corresponding eigenvector is the highest vector of an irreducible representation of $SL(N)$ with the highest weight $(M+n_2-2n_1,\ldots,n_{k+1}+n_{k-1}-2n_k,\ldots,n_{N-2}-2n_{N-1})$.

In the cases solved by Kulish and Reshetikhin (1982) the spaces $V_k^{(0)}$ were one-dimensional, the highest weights were $(m_1^{(n)}, m_2^{(n)}, m_2^{(n)},\ldots,m_2^{(n)})$ and hence

$$D_a(u)\mathcal{H}_0=I_a\prod_{k=1}^{M}(u-v_k+m_2^{(k)})\mathcal{H}_0.$$

This situation corresponds to the usual pseudovacuum.

Depending on the choice of representation acting in $\mathcal{H}$, the $GL(N)$ magnet can be considered as a discrete version of the following field-theoretic models.

(i) The magnet with $m_i^{(n)}=m\delta_{i1}$, $b_n=\operatorname{diag}(-1,1,\ldots,1)$ is a discrete approximation of the quantum vector NS model with $N-1$ components. In the continuum limit $m=\varkappa/\Delta$, $M=\Delta^{-1}L$, $\Delta\to0$, $\varkappa$, L are fixed. For $N=3$, this was thoroughly discussed by Kulish and Reshetikhin (1982).

(ii) If $m_i^{(n)}=m$, $b_n^i=-1$, $i\leqslant k$; $m_i^{(n)}=0$, $b_n^i=1$, $i>k$, the model is the lattice approximation of the matrix NS model where the field $\psi(x)$ is the $k\times(N-k)$ matrix. In the continuum limit $m=\varkappa/\Delta$, $M=\Delta^{-1}L$, $\Delta\to0$, $\varkappa$, L are fixed.

(iii) The case $m_i^{(n)}=m_i$, $b_n=\operatorname{diag}(b^1,\ldots,b^N)$ corresponds to the discrete approximation of the quantum N-wave system (Manakov 1976). In continuum limit $m_i=1/\Delta a_i$, $M=\Delta^{-1}L$, $b^i=1/m_i$, $\Delta\to0$, a_i, L are fixed. In this limit the monodromy matrix (1) tends to the monodromy matrix $\tilde{T}(u)$:

$$(\partial/\partial x)f(x,u)=-:(uA+Q(x))f(x,u):,\qquad A=\operatorname{diag}(a_1,\ldots,a_N),\tag{31}$$

$$Q(x)=\sum_{1\leqslant i<j\leqslant N}(q_{ij}(x)\,e^{ij}+q_{ij}^{+}(x)\,e^{ji})\sqrt{a_j-a_i},\tag{32}$$

$$[q_{ij}(x),q_{kl}^{+}(y)]=\delta_{ik}\delta_{jl}\delta(x-y),\tag{33}$$

$$f(0,u)=I,\qquad f(L,u)=A^{-1/2}\tilde{T}(u)A^{1/2}.\tag{34}$$

Here $:\ldots:$ means the normal ordering of the operators q^+, q, which act in Fock space $\mathcal{H}_F=\otimes_{1\leqslant i<j\leqslant N}\mathcal{H}_F(q_{ij})$. The vacuum space $\mathcal{H}_0$, which was used previously, when $\Delta\to0$ has the form $\mathcal{H}_0=\otimes_{2\leqslant i<j\leqslant N}\mathcal{H}_F(q_{ij})$. The eigenvalues of the trace of $\tilde{T}(u)$ are obtained

from (29), (30) in this limit. For example, for $N=3$, we have

$$\Lambda(u)=\exp(ua_1L)\prod_{k=1}^{n}\frac{u-u_k-1}{u-u_k}+\exp(ua_3L)\prod_{k=1}^{m}\frac{u-v_k+1}{u-v_k}$$
$$+\exp(ua_2L)\prod_{k=1}^{n}\frac{u-u_k+1}{u-u_k}\prod_{k=1}^{m}\frac{u-v_k-1}{u-v_k}, \tag{35}$$

$$\exp[u_k(a_1-a_2)L]=\prod_{\substack{l\neq k\\ l=1}}^{n}\frac{u_k-u_l+1}{u_k-u_l-1}\prod_{l=1}^{m}\frac{u_k-v_l-1}{u_k-v_l}, \tag{36}$$

$$\exp[v_k(a_2-a_3)L]=\prod_{\substack{l\neq k\\ l=1}}^{m}\frac{v_k-v_l+1}{v_k-v_l-1}\prod_{l=1}^{n}\frac{v_k-u_l}{v_k-u_l+1}. \tag{37}$$

The Hamiltonian of the N-wave system is expressed as a linear combination of coefficients in the expansion of $t(u)$ over u^{-1}. The Heisenberg equations for $N=3$ are

$$\mathrm{i}(\partial/\partial t+b_1(\partial/\partial x))q_{13}=\varkappa q_{12}\overset{+}{q}_{23},$$
$$\mathrm{i}(\partial/\partial t+b_2(\partial/\partial x))q_{12}=\varkappa q_{23}q_{13},$$
$$\mathrm{i}(\partial/\partial t+b_3(\partial/\partial x))q_{23}=\varkappa \overset{+}{q}_{13}q_{12},$$

where $\varkappa$ is defined by the Hamiltonian.

The application of the generalised Bethe ansatz to the multicomponent models with Z_N symmetry and with pseudovacuum was made by Babelon *et al* (1982) and Schultz (1983). The construction given above can be performed in an analogous fashion for the trigonometric case.

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