QUANTIZATION OF SOLITONS

V. E. Korepin and L. D. Faddeev

Quantization of particlelike solutions is considered for the example of the Sine-Gordon equation. It is shown that the quasiclassical treatment is a good approximation for a small coupling constant. The quantum corrections are calculated by path integrals.

The Sine-Gordon equation is a completely integrable Hamilton system. The structure of the phase space and the basic observables – the Hamiltonian and momentum – are described in [1-3]. The results show that the equation at the quasiclassical level corresponds to a rich spectrum of particles. Apart from quanta corresponding to the considered field in the linear approximation, the spectrum contains particles corresponding to particlelike solutions – solitons. In this note we describe the quasiclassical characteristics of these particles and calculate the quantum corrections to them.

Usually, to calculate the Green's function of a particle one needs to know the wave functional of a particle with definite momentum. For a particlelike solution it is not easy to calculate even in the quasiclassical approximation since the quasiclassical solution gives simultaneously the momentum and coordinate of the particle, which contradicts Heisenberg's uncertainty principle.

In the present paper, we propose a way of circumventing this difficulty. Using the experience of nonrelativistic quantum mechanics, we consider only the asymptotic behavior of the Green's function at large times. In this limit, the coordinate dependence of the particlelike solution disappears.

In the first section, we describe the classical Hamilton system associated with the equation. The second section is devoted to constructing perturbation theory for the Green's function of the soliton. In the third section, we consider the renormalizations resulting from quantization of solitons. In the fourth section, we calculate the S matrix for the scattering of solitons.

The brief exposition of the results of the present paper in [19] contains a number of errors which are corrected in the present text.

1. Description of Classical System

In two-dimensional spacetime we consider the chiral field $\varkappa(x, t) = \exp\{iu(x, t)\}$. We impose the boundary condition $\varkappa(x, t) \to 1$ as $|x| \to \infty$. The field $\varkappa(x, t)$ varies on the unit circle, i.e., the fields $u(x, t)$ and $u(x, t) + 2\pi m$ are indistinguishable.

The system has a conserved charge $Q=\frac{1}{2\pi}\{u(\infty, t)-u(-\infty, t)\}$, which takes integral values. The corresponding current is $J^{\mu}=\varepsilon^{\mu\nu}\partial_{\nu}u$. Its divergence vanishes independently of the equations of motion.

The Lagrangian of this field is given by

$$\mathscr{L}=\frac{1}{\gamma}\int_{-\infty}^{\infty}dx\left[{}^{1}/_{2}u_{t}^{2}-{}^{1}/_{2}u_{x}^{2}-m^{2}(1-\cos u)\right], \tag{1a}$$

where m is the mass and γ the dimensionless coupling constant. We use units for which $\hbar = c = 1$.

The classical equation has the form

$$\Box u+m^{2}\sin u=0. \tag{1b}$$

The Hamilton system is determined by the Poisson brackets

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$$\left\{\pi(x), u(y)\right\} = \delta(x-y), \pi(x) = \frac{1}{\gamma} u_t(x).$$

In [1-3] the inverse problem method (see, for example, the review [14]) is used to describe a nonlinear canonical transformation from the field variables to action–angle variables. In these variables, the phase space is a product of three sets.

We give the list of variables that parametrize these sets and write out the nonvanishing Poisson brackets:

1) $0 \leqslant \rho(p) < \infty$, $0 \leqslant \varphi(p) < 2\pi$; $\{\rho(p_1), \varphi(p_2)\} = \delta(p_1 - p_2)$;

2) $-\infty < p_a < \infty$, $-\infty < q_a < \infty$; $\{p_{a_1}, q_{a_2}\} = \delta_{a_1 a_2}$, $a = 1, \ldots, A$;

3) $-\infty < \xi_b < \infty$, $-\infty < \eta_b < \infty$; $0 \leqslant \alpha_b < 2\pi$, $0 \leqslant \beta_b < 8\pi/\gamma$; $\{\eta_{b_1}, \xi_{b_2}\} = \delta_{b_1 b_2}$;

$\{\beta_{b_1}, \alpha_{b_2}\} = \delta_{b_1 b_2}$; b = 1, ..., B, where A and B are arbitrary integers.

The total energy and momentum can be expressed in terms of these variables:

$$P_0 = \int_{-\infty}^{\infty} \sqrt{p^2+m^2}\rho(p)\,dp + \sum_{a=1}^{A} \sqrt{p_a^2+M^2} + \sum_{b=1}^{B} \sqrt{\eta_b^2 + (2M\sin\theta_b)^2}, \quad P_1 = \int_{-\infty}^{\infty} p\rho(p)\,dp + \sum_{a=1}^{A} p_a + \sum_{b=1}^{B} \eta_b, \quad M = \frac{8m}{\gamma}, \quad \theta = \frac{\gamma}{16}\beta.$$

In the case of quasiclassical quantization, all the canonical variables are transformed into operators and the Poisson brackets replaced by commutators. An ordinary scalar particle of mass m corresponds to a variable of the first type in quasiclassical quantization. We shall call these the basic particles. They have zero charge. Only these particles can be obtained from (1) by perturbation theory.

Note that the variables $\rho(p)$ and $\varphi(p)$ are modulus–phase variables, and $\rho(p)$ is the number density of the basic particles. After quantization, the operator $\rho(p)$ has eigenvalues of the form $\sum_i \delta(p-p_i)$. The eigenvalues of the energy and the momentum take the form $\sum_i \sqrt{p_i^2+m^2}$, $\sum_i p_i$, respectively.

The variables of the second and third types correspond to localized solutions of Eq. (1). The energy of these particlelike solutions is concentrated in a finite region of the configuration space. Particles of mass $M = 8m/\gamma$ and charge ± 1 correspond to variables of the second type in quasiclassical quantization.

Following the established terminology, we shall call them solitons, i.e., the same name as the classical solutions corresponding to them. The eigenvalues of the operators P_0 and P_1 on these states are

$$\sum_{a=1}^{A} \sqrt{p_a^2+M^2} \quad \text{and} \quad \sum_{a=1}^{A} p_a.$$

We write down the explicit solution of Eq. (1) for A = 1, B = ρ = 0:

$$u_t(x, t|p, q_0\pm) = 4\,\mathrm{arctg}\exp\left\{\pm m\frac{x-vt-q_0}{\sqrt{1-v^2}}\right\}, \quad p = \frac{Mv}{\sqrt{1-v^2}}. \tag{2}$$

The upper sign corresponds to solitons of positive charge; the lower, to ones of negative charge. The variables p_a and q_a are the momenta and coordinates of the solitons.

Double solitons correspond to variables of the third type in the quasiclassical treatment. These are particles of mass $2M\sin\theta_b$ and zero charge. On these states, the operators P_0 and P_1 have the eigenvalues

$$\sum_{b=1}^{B} \sqrt{\eta_b^2 + (2M\sin\theta_b)^2}, \qquad \sum_{b=1}^{B} \eta_b.$$

These particles have an internal degree of freedom, which is described by the variables β_b and α_b. The variable η_b is the momentum of the soliton and ξ_b is the coordinate of the center of mass. The solution corresponding to ρ = A = 0 and B = 1 has the form

$$4\,\mathrm{arctg}\left\{\mathrm{tg}\,\theta\frac{\cos[m\cos\theta(t\,\mathrm{ch}\,\psi - x\,\mathrm{sh}\,\psi) - \alpha_0]}{\mathrm{ch}[m\sin\theta((x-\xi_0)\,\mathrm{ch}\,\psi - t\,\mathrm{sh}\,\psi)]}\right\}, \tag{3}$$

where ψ is introduced by the expression $\eta = 2M\sin\theta\,\sinh\psi$.

Note that in the case of a double soliton we have an unusual phase space: The pair of variables α and β describing the internal state of the double soliton varies in a finite region. The total area of this

phase space, which is equal to $16\pi^2/\gamma$ in the quasiclassical treatment in units of 2π, is the number N of states. The condition $16\pi^2/\gamma=2\pi N$, $N=8\pi/\gamma$ is valid approximately for large N or small γ. The quantities α and β have a finite number of values, and in the first approximation the eigenvalues of β have the form $\beta_k=k$, $k=1,\ldots, N$; $k<8\pi/\gamma$. In other words, in the quasiclassical approximation the masses of the double solitons have the finite number of values $M_k=\frac{16m}{\gamma}\sin\frac{\gamma}{16}k$. This last expression is valid, generally speaking, for small γ and large k. However, because of the complete integrability of the Sine-Gordon equation it is to be expected that it has a larger region of application and, moreover, may be in fact exact. In particular, the limit of M_1 as $\gamma \to 0$ is m, i.e., the mass of the basic particle. In the recent preprint [20], Dashen, Hasslacher, and Neveu suggested that the first bound state of solitons and the basic particle are identical in all physical manifestations. A similar result is well known in the case of a nonlinear Schrödinger equation [21, 22]. Below, we shall give some more results that confirm this conjecture.

We also require a solution that describes the scattering of two solitons. Let us write it out explicitly. In the case of equal charges of the solitons,

$$u_2(x,t|p_1p_2q_1^0q_2^0{+}{+}) = 4\operatorname{arctg}\left[\operatorname{cth}\frac{\psi_1-\psi_2}{2}\frac{\operatorname{ch}\frac{d_1-d_2}{2}}{\operatorname{sh}\left(\frac{d_1+d_2}{2}-\ln\operatorname{th}\frac{\psi_1-\psi_2}{2}\right)}\right], \tag{4}$$

$$d_a=m\operatorname{ch}\psi_a(x-q_a^0)-m\operatorname{sh}\psi_a t,\quad p_a=M\operatorname{sh}\psi_a.$$

In the case of opposite charges, the solution can be written in the form

$$u_2(x,t|p_1p_2q_1^0q_2^0{+}{-}) = 4\operatorname{arctg}\left[\operatorname{cth}\frac{\psi_1-\psi_2}{2}\frac{\operatorname{sh}\frac{d_1-d_2}{2}}{\operatorname{ch}\left(\frac{d_1+d_2}{2}-\ln\operatorname{th}\frac{\psi_1-\psi_2}{2}\right)}\right]. \tag{5}$$

A characteristic feature of this system is the infinite number of conservation laws [4]. They can be conveniently described on the basis of the conservation laws for the free system $\Box u + m^2u = 0$ which (see [9]) has an infinite number of conservation laws. We write them in the explicit form

$$P_{2n}=\int dx[{}^1/_2(\partial_t u_x^{(n)})^2+{}^1/_2(u_x^{(n+1)})^2+{}^1/_2m^2(u_x^{(n)})^2],\quad P_{2n+1}=\int dx[u_t u_x^{(2n+1)}],\quad n=0,1,\ldots .$$

The conservation laws for Eq. (1) can be obtained from these by adding to the densities terms that contain higher powers of the field u(x, t). We do not require the explicit form of these terms. The presence of conservation laws imposes rigorous restrictions on the dynamics. A variant of the argument given below was pointed out to us by A. M. Polyakov.

We express the conservation laws of Eq. (1) in terms of in- and out-variables. The limit as $t \to -\infty(\infty)$ will coincide with the free laws. This means that after the interaction the following sums over all soliton particles, double solitons, and basic particles remain unchanged:

$$\sum_a p_{a\,\mathrm{in}}^{2n+1}=\sum_a p_{a\,\mathrm{out}}^{2n+1},\quad \sum_a (p_a^0 p_a^{2n})_{\mathrm{in}}=\sum_a (p_a^0 p_a^{2n})_{\mathrm{out}},\quad p_a^0=\sqrt{p_a^2+m_a^2},\quad n=0,1,\ldots .$$

Solving this system of equations, we arrive at the conclusion that the number of particles of each type and their individual momenta are conserved after interaction. Therefore, the S matrix is proportional to the identity operator:

$$\hat{S}=IS,\quad I=\operatorname{sym}\prod_{\alpha_i}\delta(p_{\mathrm{in}}^{\alpha_i}-p_{\mathrm{out}}^{\alpha_i}). \tag{6}$$

The symmetrization is performed with respect to each type of particle separately in accordance with their statistics. The factor S of I has unit modulus.

2. Perturbation Theory

We now consider a refinement of quasiclassical quantization. To investigate the quantum corrections, it is convenient to use a path integral. First, in this formalism we directly obtain conservation of the charge; for there does not exist a path which is continuous with respect to the time and joins field configurations with different charge Q (see [10] for the topological meaning of this assertion). But on discontinuous paths the action becomes infinite, and the contribution of these paths to the functional integral is

zero. Second, a path integral enables one to develop a perturbation theory in the coupling constant γ if the method of stationary phase is used.

In this and the following section, we shall illustrate perturbation theory by calculating the corrections to the soliton mass; to calculate them, we use the Green's function

$$G(p_1, t_1 | p_2, t_2) = \int \Psi_{p_2}^*(u, t_2) \exp\left\{ i \int_{t_1}^{t_2} \mathscr{L} dt \right\} \Psi_{p_1}(u, t_1) \prod_{xt} du,$$

where $\Psi_p(u, t)$ is the wave functional of a soliton with momentum p at time t. However, we do not know the explicit form of $\Psi_p(u, t)$. One cannot assume that the functional is related simply to the one-soliton classical solution (2); for the latter contains both the momentum and the coordinate of the soliton, and information about one of these variables must disappear on quantization because of the uncertainty principle. The way out of this dilemma is indicated by the nonrelativistic quantum mechanics of one particle, in which the function $G(p_1, p_2)$ of a transition from a state with definite momentum at $t = t_1$ to a state with definite momentum as $t = t_2$ in the principal order as $T = t_2 - t_1 \to \infty$ is obtained from the corresponding transition function $G(x_1, x_2)$ in the coordinate representation as follows. One must set $x_1 = \frac{p_1}{m} t_1 + x_1^0$, $x_2 = \frac{p_2}{m} t_2 + x_2^0$ and then go to the limit $t_1 \to -\infty$, $t_2 \to +\infty$. The limit does not depend on x_1^0 or x_2^0.

We proceed similarly here. The Green's function $G(p_1, t_1 | p_2, t_2)$ describing the transition from the state that is a soliton with momentum p_1 at $t = t_1$ to the state which is a soliton with momentum p_2 at $t = t_2$ is given in the limit $T \to \infty$ by the integral

$$\int \exp\left\{ i \int_{t_1}^{t_2} d^2x [\pi u_t - H] \right\} \prod_{xt} du\, d\pi, \qquad H = \frac{\gamma}{2} \pi^2 + \frac{1}{2\gamma} u_x^2 + \frac{m^2}{\gamma} V(u). \tag{7}$$

In (7), the integration is with respect to the field $u(x, t)$ which is such that $u(x, t) = u_1(x, t | p_1, q_1^0)$ for $t = t_1$ and $u(x, t) = u_1(x, t | p_2, q_2^0)$ for $t = t_2$. Bearing in mind the general nature of the arguments, we write $V(u)$ intead of $1 - \cos u$. The limit as $T \to \infty$ does not depend on q_1^0 or q_2^0. Because of momentum conservation, G must be proportional to $\delta(p_1 - p_2)$:

$$G(p_1, t_1 | p_2, t_2) = \delta(p_1 - p_2) F(p_1) \tag{8}$$

as $T \to \infty$. Let us prove this. If $p_1 \neq p_2$, then no classical path exists which has these asymptotic behaviors and $G = 0$. If $p_1 = p_2$, there are infinitely many such paths. Each one-soliton solution $u_1(x, t | p_1, q^0)$ for given q^0 is such a path. The action on these paths does not depend on q^0, and (7) is proportional to $\int dq^0 = 2\pi\delta(0)$. The degeneracy can be lifted by going over to a subspace with fixed total momentum. In the language of the functional integral, this can be done as follows (see [5]). Consider an arbitrary additional condition $\chi(u, \pi)$ such that $\{P_1, \chi\} \neq 0$, where P_1 is the total momentum functional: $P_1 = \int \pi u_x dx$. The transition function between states with momentum p_1 is given by

$$\int \exp\left\{ i \int_{t_1}^{t_2} [\pi u_t - H] \right\} \prod_t \delta(P_1 - p_1) \delta(\chi) \{P_1, \chi\} \prod_{x,t} du\, d\pi. \tag{9}$$

The coefficient F is given by the integral (9) for $u|_{t_1} = u_1(x, t | p_1, q_1^0)$, $u|_{t_2} = u_1(x, t | p_1, q_2^0)$ and as $T \to \infty$ does not depend on q_1^0 or q_2^0, so that in what follows we set $q_1^0 = q_2^0 = 0$. The function defined by the integral (9) is Lorentz covariant and must depend on T and p through the combination $M_q \sqrt{1 - v^2} T = M_q^2 T / \sqrt{p^2 + M_q^2}$, where M_q is the exact quantum mass of the soliton. We have verified this assertion in perturbation theory. The calculations made below are in the rest system $v = 0$, in which they simplify appreciably. In (9), we set $p = 0$ and choose as additional condition

$$\chi = \int xH\, dx \Big/ \int H\, dx. \tag{10}$$

This condition is convenient in that $\{P_1, \chi\} = 1$. In (9), we make the change of variables

$$u = u_1(x) + \sqrt{\gamma}\, z(x, t), \quad \pi = \frac{1}{\sqrt{\gamma}} w(x, t), \quad u_1(x) = u_1(x, t | 00), \tag{11}$$

where the functions z and w are the deviations from the classical path. Restricting ourselves to the first two orders in γ, we obtain $F = F_{-1} \cdot F_0$, where

$$F_{-1}=\exp\{-iMT\},$$

$$F_0=\int\exp\left\{i\int_{t_1}^{t_2}d^2x\,(wz_t-{}^1/_2w^2-{}^1/_2zKz)\right\}\prod_t\delta\left(\int_{-\infty}^{\infty}wu_{1x}(x)\,dx\right)\delta\left(\int_{-\infty}^{\infty}x\,(z_xu_{1x}+m^2V'(u_1)z)\,dx\right)\prod_x dz\,dw, \tag{12a}$$

$$K=-\frac{d^2}{dx^2}+m^2V''(u_1(x)).$$

We transform the integral in the second δ function:

$$\int x\,(z_xu_{1x}+m^2V'(u_1)z)\,dx=-\int zu_{1x}\,dx-\quad xz\,(u_{1xx}-m^2V'(u_1))\,dx.$$

The second term on the right-hand side vanishes because of the classical equation. Finally, for F_0 we obtain the expression

$$\int\exp\left\{i\int_{t_1}^{t_2}d^2x\,(z_tw-{}^1/_2w^2-{}^1/_2zKz)\right\}\prod_t\delta\left(\int wu_{1x}dx\right)\delta\left(\int zu_{1x}\,dx\right)\prod_x dz\,dw. \tag{12b}$$

We now note that $u_{1x}(x)$ is an eigenfunction of K with zero eigenvalue. This is a general result; see [8, 17]. The last integral is Gaussian and equal to

$$F_0=\exp\left\{-{}^1/_2\mathrm{Tr}'\ln\left(\frac{d^2}{dt^2}+K\right)\right\}. \tag{13}$$

The prime means that the contribution of the zero eigenvalue to the spectrum is not taken into account. We transform $\mathrm{Tr}'\ln\left(\frac{d^2}{dt^2}+K\right)$. For this, we consider

$$L_\varepsilon=\mathrm{Tr}'\ln\left(\frac{d^2}{dt^2}+\varepsilon K\right),\quad F_0=\exp\{-{}^1/_2L_1\}.$$

We differentiate L_ε and obtain

$$\frac{dL_\varepsilon}{d\varepsilon}=\mathrm{Tr}'\left(\frac{d^2}{dt^2}+\varepsilon K\right)^{-1}K=\mathrm{Tr}'\frac{e^{i\sqrt{\varepsilon K}(t-t')}}{2i\sqrt{\varepsilon K}}\cdot K.$$

Recall that K does not depend on t. We write the expression on the right-hand side in the form

$$\int_{t_1}^{t_2}dt\int_{-\infty}^{\infty}dx\frac{e^{i\sqrt{\varepsilon K}\,(t-t')}}{2i\sqrt{\varepsilon}}\sqrt{K}\,|_{t=t'}=-\frac{i}{2\sqrt{\varepsilon}}\cdot T\cdot\mathrm{tr}'\sqrt{K}.$$

We integrate the expression on the right-hand side with respect to ε, set $\varepsilon = 1$, and obtain

$$L_1=\mathrm{Tr}'\ln\left(\frac{d^2}{dt^2}+K\right)=-i\,\mathrm{tr}'\sqrt{K}\cdot T, \tag{14}$$

where tr′ means that the trace is taken only in the x space and the zero in the K spectrum is omitted. We transform the expression on the right-hand side of (14) by means of the trace identities for the Schrödinger operator [6]:

$$\mathrm{tr}[W(K)-W(K_0)]=\frac{1}{2\pi i}\int d\lambda W'(\lambda)\ln\det S_\lambda+\mathrm{tr}_1, \tag{15}$$

where $K_0 = -d^2/dx^2 + m^2V''(0)$, W is an arbitrary function, S_λ is the scattering matrix of the operator K, and tr_1 is the contribution of the discrete spectrum. We do not write out tr_1 explicitly since for $V(u) = 1 - \cos u$ the discrete spectrum does not contribute to tr′. Indeed, for this potential

$$K=-\frac{d^2}{dx^2}+m^2-\frac{2m^2}{\mathrm{ch}^2mx}. \tag{16}$$

This operator has one eigenvalue $\lambda = 0$, whose contribution is omitted. The continuous spectrum of K is on the interval $m^2 < \lambda < \infty$. The potential in K is reflectionless and the S matrix has the form

$$S=\begin{pmatrix}S_\lambda & 0\\ 0 & S_\lambda\end{pmatrix},\quad S_\lambda=\frac{\sqrt{\lambda-m^2}+im}{\sqrt{\lambda-m^2}-im}.$$

We calculate F_0 using (13), (14), and (15). After elementary calculations, we find

$$F_0=\exp\left\{\frac{i}{2\pi^2}mD(0)\,T+i\frac{m}{\pi}T\right\},\quad D(0)=\int\frac{d^2K}{k^2+m^2}. \tag{17}$$

We have obtained a logarithmically divergent expression, but the infinity can in fact be eliminated. We recall that in the case of quantization of the basic field in the one-loop approximation expressions proportional to D(0) also arise. These infinities are eliminated by a renormalization of m^2. In Sec. 3 we show that the same renormalization removes the divergence in (17); after the renormalization, it is found that

$$F_{-1}\cdot F_0=\exp\left\{-i\left(\frac{8m_r}{\gamma}-\frac{m_r}{\pi}+0(\gamma)\right)T\right\}.$$

Thus, the soliton mass in the one-loop approximation is $M_q^0=8m_r/\gamma-m_r/\pi$. For the $\lambda\varphi^4$ model a similar correction was calculated in [15]. Note that M_q^0 vanishes for $\gamma = 8\pi$. It is known that this value of γ is critical for the Lagrangian (1); see [18, 13]. Dashen, Hasslacher, and Neveu assume that this expression is exact. To prove this conjecture, one must show that the higher corrections reduce to only a renormalization. Unfortunately, the calculations are very cumbersome, and we do not give them.

3. Renormalization

In the calculation of the quantum correction to the various quantities that characterize solitons ultraviolet divergences arise [see, for example, (17)]. We must introduce counter terms into the Lagrangian to eliminate them. In the same order of perturbation theory, we are forced to introduce other counter terms. They cancel the divergences of the ordinary Feynman graphs which arise when the basic field is quantized.

The two lots of counter terms must be identical since otherwise we have a contradiction. We now show that this contradiction does not arise. For this, we consider the generating functional of the S matrix of the basic field [11]:

$$\int\exp\left\{\frac{i}{\gamma}\int d^2x[{}^1/_2(\partial_\mu u)^2-m^2V(u)]\right\}\prod_{xt}du, \tag{18}$$

where in the limit $|t|\to\infty$ we have $u\to u_{as}$ where u_{as} is a fixed asymptotic behavior, $V''(0) = 1$.

We also consider the propagator of solitons (in a naive formulation):

$$\int\exp\left\{\frac{i}{\gamma}\int_{t_1}^{t_2}d^2x[{}^1/_2(\partial_\mu u)^2-m^2V(u)]\right\}\prod_{xt}du, \tag{19}$$

where $u=\sum_{a=1}^{A}u_1(x,t|p_a,q_a^0)$ for $t = t_1,\ t_2$.

We shall calculate both integrals by the method of stationary phase. We expand the argument of the integrand in (18) in a series in the neighborhood of u^{cl}, where

$$\Box u^{cl}+m^2V'(u^{cl})=0;\quad u^{cl}\to u^{as},\quad |t|\to\infty.$$

In (19), we expand the action in the neighborhood of the purely soliton solution u^{sol}. It satisfies the equation

$$\Box u^{sol}+m^2V'(u^{sol})=0;\quad u^{sol}\to\sum_{a=1}^{A}u_1(x,t|p_a,q_a^0).$$

We write both expressions in the form

$$\exp\sum_{k=-1}^{\infty}\gamma^k g_k, \tag{20}$$

where g_k for $k \geq 0$ is the sum of connected (k + 1)-loop vacuum graphs, in which the propagators and vertices are given, respectively, by

$$[\Box+m^2V''(u^{cl})]\Delta_1=\delta^2(x-y),\quad V_k=\gamma^{k/2-1}\frac{\delta^k\int d^2xV(u)}{\delta u^k}\Big|_{u=u^{cl}} \tag{21}$$

for (18) and

$$[\Box+m^2V''(u^{cl})]\Delta_2=\delta^2(x-y),\quad V_k=\gamma^{k/2-1}\frac{\delta^k\int d^2xV(u)}{\delta u^k}\Big|_{u=u^{sol}} \tag{22}$$

for (19).

From the analogy it is clear that if in the l-th perturbation order in (18) there arises the divergence $i\gamma^l\ln\Lambda\int C(u^{cl})d^2x$, then in (19) in the same order there arises the infinity $i\gamma^l\ln\Lambda\int C(u^{sol})d^2x$. Both infinities can be eliminated by adding the single counter term $-\gamma^{l+1}\ln\Lambda\int d^2xC(u)$ to the original Lagrangian.

Let us make this more precise in the example of the one-loop approximation. The integral (18) in this approximation reduces to

$$\exp\{-1/2\,\mathrm{Tr}\ln(\Box+m^2V''(u^{cl}))\}, \tag{23}$$

and (19) to

$$\exp\{-1/2\,\mathrm{Tr}\ln(\Box+m^2V''(u^{sol}))\}. \tag{24}$$

The ultraviolet divergence in (23) has the form

$$\exp\left\{-\frac{im^2}{8\pi^2}D(0)\int d^2x[V''(u^{cl})-1]\right\},$$

and in (24) it is

$$\exp\left\{-\frac{im^2}{8\pi^2}D(0)\int d^2x[V''(u^{sol})-1]\right\}.$$

The infinities can be eliminated by adding to the original Lagrangian the counter term

$$\frac{\gamma m^2}{8\pi^2}D(0)\int d^2x[V''(u)-1].$$

It is easy to show that with allowance for this counter term in (17) F_0 becomes $\exp\left\{i\frac{m}{\pi}T\right\}$.

Our arguments have a general nature and can be applied to an arbitrary model.

Note finally that for the Sine-Gordon equation all the divergences can be eliminated by the multiplicative renormalization

$$m_r^2=m^2\exp\left\{\frac{\gamma}{8\pi^2}D(0)\right\},$$

and in this connection see also [7].

4. Scattering Matrix of Solitons

In this section we discuss the quasiclassical approximation for the S matrix of solitons and ways of obtaining the quantum corrections to it (see [19]). We consider in detail the case of two simple solitons, saying a few words about more general processes at the end of the section. The quasiclassical approximation can be calculated by proceeding from the known description of the scattering in the classical treatment. The latter is based on knowledge of exact solutions of classical equations describing the interaction of an arbitrary number of solitons [4]. The result in the case of two solitons is as follows.

1. Solitons of different charges. The solution of (5) as $|t|\to\infty$ has the asymptotic behaviors

$$u_2(x,t|p_1p_2q_1^0q_2^0+-)=\begin{cases}u_1(x,t|p_1^-q_1^-+)+u_1(x,t|p_2^-q_2^--), & t\to-\infty;\\ u_1(x,t|p_1^+q_1^++)+u_1(x,t|p_2^+q_2^+-), & t\to\infty;\end{cases}$$

$$p_1^-=p_1^+,\ p_2^-=p_2^+,\ q_1^+=q_1^-+\frac{\partial}{\partial p_1^-}K(p_1^-,p_2^-),\ q_2^+=q_2^-+\frac{\partial}{\partial p_2^-}K(p_1^-,p_2^-);$$

$$K(p_1,p_2)=K(\xi)=i\frac{8}{\gamma}\int_0^{\pi}d\theta\ln\frac{\xi e^{-i\theta}+1}{\xi+e^{-i\theta}},\ p_1^-=p_1,\ p_2^-=p_2; \tag{25}$$

$$\xi=\frac{s-2M^2+\sqrt{s(s-4M^2)}}{2M^2},\quad s=(p_1^0+p_2^0)^2-(p_1+p_2)^2;\quad s>4M^2.$$

The roots in the last equation are arithmetic. We see that when solitons collide they pass through each other and the entire scattering reduces to a change of their positions; the faster is moved forward along its momentum. This corresponds to attraction. Equation (25) is an expression of the canonical transformation generated by Hamiltonian $K(p_1, p_2)$.

2. Solitons with the same charges. The solution (4) describes the collision of identical solitons, after which they are elastically scattered. This can be seen by considering the evolution of the solution. If the momenta of the particles are of opposite signs, the solution (4) as $t\to-\infty$ has the asymptotic behavior

$$u_2(x,t|p_1p_2q_1^0q_2^0++)\to u_1(x,t|p_1^-q_1^-+)+u_1(x,t|p_2^-q_2^-+);\ p_1^-=p_1,\ p_2^-=p_2;$$

so that the energy density has two maxima: the first as $x\to-\infty$ and the second as $x\to\infty$. With

increasing time, they approach, come to a halt, and move apart in the opposite direction. The asymptotic behavior as $t \to \infty$ is

$$u_2(x, t|p_1p_2q_1^0q_2^0++) \to u_1(x, t|p_1^+q_1^+++) + u_1(x, t|p_2^+q_2^+++), \tag{26}$$

where $p_1^+ = p_2^-$, $p_2^+ = p_1^-$, $q_1^+ = q_2^- + \partial K/\partial p_2^-$, $q_2^+ = q_1^- + \partial K/\partial p_1^-$. $K(p_1^-, p_2^-)$ is the same function as in Case 1. We have appended numbers to the solitons, remembering that after the collision they are reflected, and do not pass through one another.

A quasiclassical S matrix arises when the canonical transformations are quantized. The corresponding formulas are

$$\begin{aligned} &S_{+-}^{\text{cl}}(p_1^-, p_2^-, p_1^+, p_2^+) = \delta(p_1^- - p_1^+)\delta(p_2^- - p_2^+)S_-^{\text{cl}}(s), \\ &S_{++}^{\text{cl}}(p_1^-, p_2^-, p_1^+, p_2^+) = \delta(p_1^- - p_2^+)\delta(p_2^- - p_1^+)S_+^{\text{cl}}(s), \\ &S_-^{\text{cl}}(s) = \exp\{-iK(\xi) + ic_1\}, \quad S_+^{\text{cl}}(s) = \exp\{-iK(\xi) + ic_2\}, \end{aligned} \tag{27a}$$

where we have added the constants of integration c_1 and c_2 to the phase $K(\xi)$ normalized by the condition $K(1) = 0$.

An alternative way of deriving (27a) is based on the general definition of the S matrix [11] in terms of a functional integral over paths with classical asymptotic behaviors. In the case of two solitons, the method of stationary phase reduces to calculating the asymptotic behavior of the truncated action A between t_1 and t_2 on the solutions (5) and (4) as $t_2 - t_1 \to \infty$. Calculations made in the center of mass system show that

$$A[u_2(++)] = -K(\xi) + 2p\Delta q, \; A[u_2(+-)] = -K(\xi) + 8\pi^2/\gamma + 2p\Delta q, \; \Delta q = v(t_2 - t_1) + \frac{16}{\gamma}\frac{\sqrt{1-v^2}}{M}\ln\frac{1}{v}, \quad v = \frac{p}{\sqrt{p^2+M^2}} \tag{27b}$$

where Δq is the change in the soliton coordinate in time $t_2 - t_1$. It follows from the experience of nonrelativistic quantum mechanics that the last term in (27b) must be subtracted, and the remainder is then the phase shift. Comparing (27a) and (27b), we see that $c_1 = 8\pi^2/\gamma$, $c_2 = 0$. This result and its interpretation were obtained for the first time by Jackiw and Woo [23].

Equation (25) gives one the possibility of continuing analytically the $S_\pm(s)$'s from the physical region $s > 4M^2$ into the complex plane with cut along the real axis for $s > 4M^2$, $s < 0$. Then $S_\pm(s)$ is not real in the gap $0 < s < 4M^2$. We interpret this as a manifestation of the accumulation of poles corresponding to bound states of solitons. Indeed, the number N of bound states tends to infinity as $\gamma \to 0$. The higher corrections must lead to replacement of the false cut in the gap by a finite number of poles.

This circumstance prevents a direct verification of crossing. One can however use Hermitian analyticity and write down the crossing condition in the form

$$S_+(4M^2 - s + i0) = S_-^*(s + i0). \tag{28}$$

It is not difficult to see that this equation is indeed satisfied in our approximation. The argument was first published by Coleman in the Appendix to [23].

Let us attempt to establish the complete quantum S matrix on the basis of its quasiclassical approximation. By virtue of what we have said in Sec. 3, allowance for quantum corrections reduces to replacing $S_-^{\text{cl}}(s)$ by a function $S_-(s)$ of the form

$$S_-(s) = S_-^{\text{cl}}(s)\exp\sum_{k=0}^{\infty}\left(\frac{1}{N}\right)^k g_k(s), \tag{29}$$

where $g_k(s)$ is a sum of $(k+1)$-loop connected vacuum graphs of the renormalized perturbation theory, which is constructed like the perturbation theory (20)-(22). Note that $\text{Re}\, g_k(s) = 0$ both for $s < 0$ and for $s > 4M^2$.

We have not been able to obtain any explicit expressions for the quantum corrections by means of this diagrammatic technique apart from the investigation of some asymptotic behaviors (a promising approach to this problem is contained in [20]). We therefore consider whether something could not be said about the quantum corrections on the basis of the general requirements of analyticity and unitarity. The quasiclassical S matrix has one unsatisfactory property – it is not real in the gap; the quantum corrections must evidently put right this shortcoming. Let us consider more carefully the expression for $S_-^{\text{cl}}(s)$ in the gap. After a change of variables, we obtain ($s = 4M^2\cos^2\nu/2$, $0 < \nu < \pi$)

$$S_-^{cl}=\exp\left\{\frac{N}{\pi}\int_0^{\pi}d\theta\ln\frac{\zeta e^{-i\theta}+1}{\zeta+e^{-i\theta}}+iN\pi\right\}=\exp\left\{iN\nu+iN\pi+\frac{N}{\pi}\int_0^{\pi}d\theta\ln\left|\frac{\cos\frac{\theta-\nu}{2}}{\cos\frac{\theta+\nu}{2}}\right|\right\},\quad N=\frac{8\pi}{\gamma}. \tag{30}$$

The first term in the argument of the exponential gives the undesired complex value. Quantum corrections cannot eliminate it since it has the order N (29). Thus, the phase of the quantum scattering matrix S_(s) is of order N and Im S_(s) = 0 (in the gap). This evidently means that the exponent of the power of S_(s) contains a function h(ν) that is piecewise constant and has discontinuities $i\pi$ at the points θ_k, whose number is N. An example of such a function h(ν) is the following: points of discontinuity distributed uniformly, $\theta_k = \pi k/N$, h(0) = 0.

Since in the physical region S_(s) is unitary in any order and has the correct asymptotic properties, from a given function h(ν) one can recover ln S_(s) uniquely to within a known polynomial arbitrariness. For the h(ν) given above, S_(s) has the form

$$S_-(s)=\exp\left\{\sum_{k=1}^{N}\ln\frac{\zeta e^{-i\theta_k}+1}{\zeta+e^{-i\theta_k}}+i\pi N\right\},\ \theta_k=\pi k/N,\ k=1,\ldots;\ k<8\pi/\gamma.$$

Note that the sum in the argument is an integral sum for the integral in Eq. (30). We rewrite S_(s) in the form

$$S_-(s)=e^{iN\pi}\prod_{k=1}^{N}\frac{\zeta e^{-i\theta_k}+1}{\zeta+e^{-i\theta_k}}. \tag{31}$$

It has first-order poles at the points $s_k=4M^2\sin^2\frac{\theta_k}{2}$, i.e., at the points corresponding to the masses of the double solitons in the quasiclassical approximation. Note that if the function h(ν) had discontinuity $il\pi$ (l integral) then S_(s) would have a pole of l-th order.

We cannot, of course, guarantee that S_(s) reduces to such a modification of the function $S_-^{cl}(s)$, though the possible candidates for the function h(ν) cannot differ too strongly from the example we have chosen. Thus, the points of division θ_k must be arranged fairly uniformly in the interval [0, π]. We assume that Eq. (31) is exact for $\gamma = 8\pi/N$, N = 2, 3, ...

The case of the scattering of A solitons is treated similarly. The classical scattering of A solitons reduces to the process of successive scattering of solitons on one another [16, 4] and the quasiclassical expression for the S matrix in the case of solitons of the same charge has the form

$$\exp\left\{-\frac{i}{2}\int K(p_1,p_2)\frac{dp_1\,dp_2}{p_1^0p_2^0}:\rho(p_1)\rho(p_2):\right\},\quad \rho(p)=a^+(p)a(p),\ [a(p_1,a^+(p_2)]=p_1^0\delta(p_1-p_2).$$

A similar expression holds for solitons of arbitrary charges, the S matrix being equal to the product of the pairwise S matrices.

We now turn to the case of the scattering of double solitons. The corresponding classical solution in the system $\eta = 0$ is periodice in time with period $2\pi/m\cos\theta$ [4]. We first calculate the propagation function of a double soliton in the quasiclassical approximation.

Proceeding as in Sec. 2 [see (8)], we obtain $G(p_1,t_1|p_2,t_2)=\delta(p_1-p_2)F_{-1}(p_1,\theta)$, where

$$F_{-1}(p,\theta)=U(t_1,t_2,\theta,p)\exp\{-i2M\sin\theta\sqrt{1-v^2}\,T\},\ p=\frac{2M\sin\theta v}{\sqrt{1-v^2}}.$$

We do not need the explicit form of U, and we note that |U| = 1, and in the center of mass system U is quasiperiodic in t_1 and t_2:

$$U\left(t_1+\frac{2\pi}{m\cos\theta},t_2,\theta,0\right)=e^{i\varepsilon}U(t_1,t_2,\theta,0),$$

where $\varepsilon = 32\pi\theta/\gamma$.

The function U does not contribute to the coefficient of T and therefore does not modify the mass only in the case when U is exactly periodic, i.e.,

$$\varepsilon=2\pi k.\quad \theta_k=\gamma k/16. \tag{32}$$

Note that the condition (32) is equivalent to the quantization of periodic Bohr–Sommerfeld orbits. We have

obtained one more confirmation for the quasiclassical spectrum of the double soliton:

$$M_k=\frac{16m}{\gamma}\sin\theta_k,\qquad \theta_k=\frac{\gamma}{16}k.$$

We now turn to the expression for the S matrix and give without derivation the quasiclassical result for the scattering of double solitons with internal momenta θ_k and θ_n:

$$\exp\left\{-i\left[K(\zeta e^{i(\theta_k-\theta_n)})+K(-\zeta e^{-i(\theta_k+\theta_n)})+K(-\zeta e^{i(\theta_k+\theta_n)})+K(\zeta e^{i(\theta_n-\theta_k)})-\frac{16\pi^2}{\gamma}\right]\right\}$$

$$\left(=\exp\left\{2\frac{N}{\pi}\int_0^{\pi/N}d\theta\ln\frac{\zeta^2-1+2i\zeta\cdot\sin\theta}{\zeta^2-1-2i\zeta\cdot\sin\theta}\right\}\text{ for }k=n=1\right),\quad \zeta=\frac{s-M_k^2-M_n^2}{2M_kM_n}+\sqrt{\left(\frac{s-M_k^2-M_n^2}{2M_kM_n}\right)^2-1}, \tag{33}$$

where K is the same as in Eq. (27).

We also give the quasiclassical scattering matrix of a double soliton with internal momentum θ_k and of a simple soliton:

$$\exp\left\{-iK(\zeta e^{i(\pi/2-\theta_k)})-iK(\zeta e^{i(\theta_k-\pi/2)})+i\frac{8\pi^2}{\gamma}\right\},\quad \zeta=\frac{s-M^2-M_k^2}{2MM_k}+\sqrt{\left(\frac{s-M^2-M_k^2}{2MM_k}\right)^2-1}. \tag{34}$$

Finally, we consider the scattering of a particle of the basic field on a soliton. The S-matrix element is proportional to the identity operator (6). Solving the analyticity and unitarity condition [12], we obtain for the coefficient of it

$$S=\frac{g(x)\sqrt{m^2-x}+1}{g(x)\sqrt{m^2-x}-1},\qquad x=\left(\frac{s-m^2-M^2}{2M}\right)^2 \tag{35}$$

where g(x) can be expanded in positive powers of γ. As a function of its argument, g(x) has the form

$$g(x)=R_0-\sum_n\frac{R_n}{e_n^2-x},\qquad R_n>0.$$

We show that in the lowest order in γ the value of g(x) is g = 1/m. The expression for the generating functional of the Green's function has the form $\delta(p_{in}^{sol}-p_{out}^{sol})F$ (8) where

$$F=\int\exp\left\{i\int d^2x[\pi u_t-H]+\frac{i}{\gamma}\int d^2x\,j(x)\,(u-u_1)\right\}\prod_t\delta(P_1)\,\delta(\chi)\,\{P_1,\chi\}\prod_x du\,d\pi.$$

We have taken a fixed soliton. Calculating the integral by the method of stationary phase and taking into account only the first order, we obtain

$$F_{-1}=\exp\left\{i\int dt\,\mathcal{L}(u^{cl})+\frac{i}{\gamma}\int d^2x(u^{cl}-u_1)j(x)\right\},\quad \Box u^{cl}+m^2\sin u^{cl}=-j(x),\quad u^{cl}(\infty,t)-u^{cl}(-\infty,t)=2\pi.$$

We differentiate twice with respect to the current and obtain for the Green's function of a soliton–basic particle the expression $F_{-1}=\exp\{-iMT\}\Delta_2(x_1,t_1|x_2,t_2)$, where

$$\left(\Box+m^2-\frac{2m^2}{\operatorname{ch}^2 mx}\right)\Delta_2=\delta(x-y)-\frac{1}{\gamma M}u_{1x}(x)u_{1y}(y),\quad\int u_{1x}(x)\Delta_2(x,y)\,dx=0.$$

We go over to the S matrix by means of reduction formulas and find that it is

$$\hat{S}=I\frac{p+im}{p-im},$$

where p is the momentum of the particle. From this we obtain directly the expression for g(x).

We now return to the conjecture of Dashen, Hasslacher, and Neveu that the basic particle and the first state of the double soliton are identical. In the spirit of this conjecture, the following must coincide: 1) the S matrix of two double solitons with $\theta=\pi/2N$ and the S matrix of two basic particles, 2) the S matrix of a soliton and a double soliton with $\theta=\pi/2N$ and the S matrix of a soliton and the basic particle. Let us verify this in the lowest order in γ. Three of the S matrices have been given above, and the fourth (for two basic particles) in [7]. The verification reduces to expanding expressions of the form $K(-\zeta e^{-i\pi/N})$ in powers of $1/N=\gamma/8\pi$; for example,

$$K(-\zeta e^{-i\pi/N})=K(-\zeta)-2i\ln\frac{\zeta-1}{\zeta+1}+O\left(\frac{1}{N}\right).$$

Calculating (33) in the lowest order in γ, we obtain

$$1+i\frac{m^2\gamma}{2\sqrt{s(s-4m^2)}},$$

which obviously agrees with the Born expression for the S matrix of basic particles. The expression (34) in the lowest order in γ is

$$\frac{\sqrt{m^2-x}+m}{\sqrt{m^2-x}-m}$$

which agrees with (35).

Conclusions

We hope that we have succeeded in convincing the reader that this one-dimensional nonlinear model of field theory has a number of attractive properties. Let us list some of them.

1. The Lagrangian of the theory contains only one field, but a complete spectrum of particles is manifested. In the weak interaction approximation the solitons are heavy particles and they interact strongly.
2. The solitons have a quantum number which has a topological nature, and this can be interpreted as a charge. Solitons with the same charge repel each other, while solitons with different charge attract each other.
3. In the weak interaction approximation a prescription exists for calculating in perturbation theory. The quantum corrections are small for small coupling constants, and the quasiclassical treatment determines the entire nonanalytic contribution to the physical quantities.

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