FACTORIZATION OF THE CLASSICAL AND THE QUANTUM S MATRIX AND CONSERVATION LAWS

P. P. Kulish

It is shown that the presence of a complete set of integrals of the motion that are a deformation of the free integrals leads to a factorization of the S matrix. The scattering characteristics of n identical particles are expressed in terms of the two-particle problem.

1. Considering a classical system of n one-dimensional particles with interparticle potential $1/x^2$ or $1/\sin^2 x$,* Moser [2] used the L–M pair method, which has been actively used in recent years to solve nonlinear equations [3]. This enabled him to construct n first integrals in involution and show that the set of particle momenta $\{p_i\}_1^n$ before and after the collisions (for the potential $1/x^2$) is the same. In this paper, we show that the classical and the quantum S matrix for these systems can be expressed solely in terms of the two-particle S matrix. The result is obtained as a consequence of the existence of a complete set of integrals of the motion and their simple structure (deformation of the free integrals), and an analogous situation obtains in the scattering of solitons. We also justify Calogero's assumption that the generalization of the Moser L–M pair method for n identical particles (preserving the structure of the matrices L and M and leaving the interaction a two-particle one) is possible only if the two-particle potential is the Weierstrass function [4].

The equations of the classical mechanics of n one-dimensional particles with two-particle potential v(x) and mass m = 1

$$\dot{x}_i=p_i,\quad \dot{p}_i=-\sum_{j=1}^{n} v'(x_i-x_j);\quad \dot{x}_i\equiv dx_i/dt, \tag{1}$$

are equivalent to an evolution equation for the matrices L and M:

$$\dot{L}=[M,L]=ML-LM, \tag{2}$$

where L and M are, respectively, n × n Hermitian and anti-Hermitian matrices with the elements [4]

$$L_{ij}=p_i\delta_{ij}+(1-\delta_{ij})\alpha(x_i-x_j);\quad i\leqslant j,\quad L_{ij}=L_{ji}^{*}, \tag{3}$$

$$M_{ij}=\delta_{ij}\sum_{m\neq i}\beta(x_i-x_m)-(1-\delta_{ij})\alpha'(x_i-x_j), \tag{4}$$

if the functions $\alpha(x)$ and $\beta(x)$ are related by the functional equation

$$\alpha'(y)\alpha(z)-\alpha(y)\alpha'(z)=\alpha(y+z)(\beta(y)-\beta(z)),\quad \beta(y)=\beta(-y),\quad v(x)=\alpha(x)\alpha(-x). \tag{5}$$

We assume that Eq. (5) has solutions with $v(x) > 0$ and $v(x) \to 0$ as $|x| \to \infty$. Examples of such potentials are $1/x^2$ and $1/\sinh^2 x$. A few words about the general solution will be said later.

2. We now turn to the analysis of classical scattering for such a system. Note that the arguments will have a general nature, i.e., they will be valid for systems that have a complete set of integrals of the motion that are nearly free ones. (These include, for example, the nonlinear Schrödinger equation of [7], the sine-Gordon equation [8], and various others.)

* The quantum problem for n one-dimensional particles with binary interaction $1/x^2$ was completely investigated by Calogero [1].

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The matrix M is anti-Hermitian, and therefore the eigenvalues λ_i of the matrix L are invariants of the motion. The asymptotic conditions for Eqs. (1) for $t \to -\infty$ and a decreasing potential have the form

$$x_i(t)=p_i^- t+x_{0i}^- +o(1), \tag{6}$$

i.e., they are determined by the set of 2n numbers $\{p_i^-\}_1^n$ and $\{x_{0i}^-\}_1^n$ (the particles can be labeled in such a way that $p_1^-<p_2^-<\ldots<p_n^-$). Thus, as $t \to -\infty$ the matrix L becomes diagonal (the choice of $\alpha(x)$ is nonunique [4], and for us it is convenient if $\alpha(x) \to 0$ as $|x| \to \infty$; for decreasing potentials $v(x)$ this can be done), $L_{ij}=p_i^-\delta_{ij}+o(1)$, $\lambda_i=p_i^-$. A similar argument in the limit $t \to +\infty$ and the condition that the particles do not pass through each other leads to $\lambda_i=p^+_{n-i+1}=p_i^-$ and $x_i(t)=p^-_{n-i+1}t+x_{0i}^+ +o(1)$. Besides the eigenvalues λ_i further invariants of the motion are the coefficients $I_m(p_i, x_j)$ of the characteristic polynomial

$$\det(L-\lambda E)=\sum_{m=0}^{n}(-\lambda)^{n-m}I_m(p_i,x_j), \quad E_{ij}=\delta_{ij}. \tag{7}$$

By virtue of the asymptotic conditions and the properties of the potential, I_m in the limit $|t| \to \infty$ are equal asymptotically to free symmetric functions of p_i:

$$I_m^{(0)}(p_i)=\sum_{1\leqslant i_1<\ldots<i_m\leqslant n} p_{i_1}p_{i_2}\ldots p_{i_m} \tag{8}$$

and since the Poisson brackets $\{I_m, I_k\}$ of the integrals of the motion is an integral of the motion, it follows from the vanishing of its asymptotic behavior that [2, 4]

$$\{I_m(p_i,x_j),\ I_k(p_i,x_j)\}=0, \quad 1\leqslant m, k\leqslant n. \tag{9}$$

In what follows, we shall require the explicit form of the third integral, and we shall therefore say a few words about the structure of $I_m(p_i, x_j)$. Using (7), we readily obtain the equations

$$I_1(p_i,x_j)=I_1^{(0)}(p_i)=\sum_{i=1}^{n}p_i=P, \tag{10a}$$

$$I_2(p_i,x_j)=I_2^{(0)}(p_i)-\sum_{1\leqslant i<k\leqslant n} v(x_i-x_k)={}^1/_2P^2-H, \tag{10b}$$

$$I_3(p_i,x_j)=I_3^{(0)}(p_i)-\sum_{\substack{1\leqslant i_1\leqslant n\\ i_2<i_3\\ i_2\neq i_1\neq i_3}} p_{i_1}v(x_{i_2}-x_{i_3}), \tag{10c}$$

while the structure of I_4 is more complicated, containing $\alpha(x_i - x_j)$ explicitly, and it cannot be expressed as a polynomial function of $v(x_i - x_j)$.

By virtue of the asymptotic behaviors of $x_i(t)$ as $t \to \pm\infty$, classical scattering is described by the transition from the set $\{p_i^-, x_{0i}^-\}_1^n$ to the set $\{p_i^+, x_{0i}^+\}_1^n$. As we have already mentioned, Moser and Calogero have shown [4] that $p_i^+=p^-_{n-i+1}$. We calculate the discontinuity of the asymptotic coordinates:

$$\Delta_i=x^+_{0\,n-i+1}-x_{0i}^-=\lim_{t\to\infty}(x_{n-i+1}(t)-x_i(-t)-2p_i^-t) \tag{11}$$

and show that it is entirely determined by the two-particle problem. We consider first a system of two particles. In the case when the particles cannot overtake each other, using the solution of Eqs. (1) in quadratures for two particles, we have

$$\Delta_1=x_{02}^+-x_{01}^-=a+\int_{-\infty}^{a}dx\left(\frac{k}{(k^2-4v(x))^{1/2}}-1\right)=\Delta(k), \tag{12}$$

where $k=p_2^--p_1^-$, $p_2^->p_1^-\geqslant 0$, $a<0$, $v(a)=k^2/4$.

We now consider three particles. Equations (1) correspond to motion defined by the energy function H. In addition, one can consider the motion of our system defined by the Poisson brackets of the function $I_3(p_i, x_j)$ (10c). We denote the parameter (time) of this motion by t_3. We obtain the equations of motion

$$\partial x_i/\partial t_3=p_jp_k-v(x_j-x_k), \quad (i,j,k)\leftrightarrow(1,2,3), \quad \partial p_i/\partial t_3=p_jv'(x_i-x_k)+p_kv'(x_i-x_j). \tag{13}$$

Thus, the coordinates x_i and the momenta p_i of our system depend on the two parameters t and t_3. In what follows, we shall use the fact that the motions defined by H and I_3 are independent because they are involutory.

Using the translational and Galilean invariance, we go over to the system in which $x_{01}^-=p_1^-=0$, and then the asymptotic behaviors are $x_1(t)=o(1)$, $x_2(t)=p_2^-t+x_{02}^-+o(1)$, $x_3(t)=p_3^-t+x_{03}^-+o(1)$, and the collision time of the particles is of order $t_{12}=-x_{02}^-/p_2^-$, $t_{13}=-x_{03}^-/p_3^-$, $t_{23}=(x_{03}^--x_{02}^-)/(p_2^--p_3^-)$. For the chosen asymptotic momenta, the asymptotic behavior of the coordinates and the momenta as $t_3\to+\infty$ is

$$x_1(t,t_3)=p_2^-p_3^-t_3+o(1),\quad x_2(t,t_3)=p_2^-t+o(1),\quad x_3(t,t_3)=p_3^-t+o(1),\quad p_i(t,t_3)=p_i^-+o(1),\quad i=1,2,3. \tag{14}$$

The collision time t_{23} is the same, while $t_{12}=(-x_{02}^-+p_2^-p_3^-t_3)/p_2^-$, $t_{13}=(-x_{03}^-+p_2^-p_3^-t_3)/p_3^-$ and can be arbitrarily large. Thus, at the time of the collision of the second and the third particle the first particle may be arbitrarily far from them. After the scattering in the system (2, 3), we have

$$x_2(t)=p_3^-t+x_{03}^--\Delta(p_3^--p_2^-)+o(1),\quad x_3(t)=p_2^-t+x_{02}^-+\Delta(p_3^--p_2^-)+o(1), \tag{15}$$

where $\Delta(k)$ is given by Eq. (12). The scattering now takes place in the system (2, 1), and then again in the system (3, 2). After the transition to the limit $t\to+\infty$ it remains to set $t_3=0$, which, since H and I_3 are involutory, gives the result for the original equations (1):

$$\begin{gathered}x_1(t)=p_3^-t+x_{03}^--\Delta(p_3^--p_2^-)-\Delta(p_3^--p_1^-),\\ x_2(t)=p_2^-t+x_{02}^--\Delta(p_2^--p_1^-)+\Delta(p_3^--p_2^-),\ x_3(t)=p_1^-t+x_{01}^-+\Delta(p_2^--p_1^-)+\Delta(p_3^--p_1^-).\end{gathered} \tag{16}$$

For n particles, using the integral $I_n(p_i, x_j)$, we take the first particle from the group of $n-1$ particles and, arguing as before, we obtain finally for the discontinuity of the asymptotic coordinates

$$\Delta_i=x_{0n-i+1}^+-x_{0i}^-=-\sum_{j<i}\Delta(p_i^--p_j^-)+\sum_{j>i}(p_j^--p_i^-). \tag{17}$$

The transition from the set $\{p_i^-, x_{0i}^-\}_1^n$ to the set $\{p_i^+, x_{0i}^+\}_1^n$ during the scattering process is given by a canonical transformation whose generating function we denote by $\Phi(x_{0i}^-, p_i^+)$. The difference between $\Phi(x_{0i}^-, p_i^+)$ and the generating function of the identity transformation (to within relabeling of the particles):

$$\Phi_0(x_{0i}^-, p_i^+)=\sum_{i=1}^n x_{0i}^-p_{n-i+1}^+ \tag{18}$$

can be called naturally the classical S matrix:

$$\begin{gathered}\Phi(x_{0i}^-, p_i^+)=\sum_{i=1}^n x_{0i}^-p_{n-i+1}^++\sum_{i<j}\eta(p_i^+-p_j^+)=\Phi_0+S,\quad p_i^-=\partial\Phi/\partial x_{0i}^-=p_{n-i+1}^+,\\ x_{0n-i+1}^+=\partial\Phi/\partial p_i^-=x_{0i}^-+\sum_{j>i}\eta'(p_i^--p_j^-)-\sum_{j<i}\eta'(p_j^--p_i^-),\quad \eta(p)=-pa-\int_{-\infty}^{a}dx((p^2-4v(x))^{1/2}-p).\end{gathered} \tag{19}$$

Note the following fact. We require that the system of n one-dimensional particles with binary interaction have, besides the momentum and energy integrals, a third integral of the form (10c). This is equivalent to the vanishing of the Poisson brackets $\{H, I_3\}$, which are equal to

$$\{H,I_3\}=\sum_{\substack{i;j<k\\ j\neq i\neq k}}v(x_j-x_k)(v'(x_i-x_j)+v'(x_i-x_k)). \tag{20}$$

For three particles, introducing the notation $x_1-x_2=a$, $x_2-x_3=b$, $x_1-x_3=a+b$, we have

$$v(a)(v'(b)+v'(a+b))-v(b)(v'(a)+v'(a+b))+v(a+b)(v'(a)-v'(b))=\begin{vmatrix}v(a) & v'(a) & 1\\ v(b) & v'(b) & 1\\ v(a+b) & -v'(a+b) & 1\end{vmatrix}=0. \tag{21}$$

If the potential satisfies the functional equation (21) for three particles, then $\{H, I_3\}=0$ for any number of particles. Equation (21) is none other than the composition theorem for the Weierstrass function $\wp(z)$ [5], which determines this function to within a factor and a constant. As Calogero has shown, the Weierstrass function as potential satisfies the original equations (5), and, therefore, in such a system there exist n integrals of the motion. This assertion is analogous to the one obtained in [6]. There, the sine-Gordon equation was recovered uniquely from the free Klein–Fock equation on the basis of the requirement that, besides the momentum and energy integrals, there should exist a third integral for the equation with interaction as a modification of the free one by the addition of a power series in the field and its derivative (see the Appendix).

3. We now turn to the quantum case for n one-dimensional particles. The general formulation of the problem in quantum mechanics for the system (10a), (10b), ... does not present difficulties. The expressions $I_m(p_i, x_j)$ in each term contain different p_i and $\alpha(x_j - x_k)$, the numbers of the coordinates do not coincide with the numbers of the momenta, and there is no operator ordering problem. The set of operators $I_m(p_i, x_j)$, $1 \le m \le n$, is a complete set for our system, and the eigenfunctions satisfy the equations

$$I_m(p_i, x_j)\Psi(x_1, \ldots, x_n) = E_m \Psi(x_1, \ldots, x_n), \quad 1 \leqslant m \leqslant n. \tag{22}$$

The admissible potentials $v(x)$ have the singulatiry $1/x^2$ in the neighborhood of the origin, and we shall therefore consider the eigenfunctions in the region $x_1 \ge x_2 \ge \ldots \ge x_n$ with null conditions on the boundaries $x_i = x_{i+1}$. It is convenient to represent the eigenvalues E_m as symmetric functions of the parameters k_i, $1 \le i \le n$, which describe the common eigenfunctions of the operators

$I_m^{(0)}(p_i)\exp\left(i\sum_{j=1}^{n} k_{\sigma_j} x_j\right)$, where $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is an arbitrary permutation of $(1, 2, \ldots, n)$. Remembering that the operators $I_m(p_i, x_j)$ are given asymptotically for $x_i = a_i r$, $a_1 > a_2 > \ldots > a_n$, $r \to \infty$, by $I_m^{(0)}(p_i)$, the asymptotic behavior of the eigenfunction $\Psi(x_1, \ldots, x_n) \equiv \Psi(x_i | k_j)$ is a linear combination of exponentials:

$$\Psi(x_i | k_j) = \sum_{\sigma} A_\sigma \exp\left(i \sum_{j=1}^{n} k_{\sigma_j} x_j\right), \tag{23}$$

where the sum is taken over all $n!$ permutations $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. We determine the coefficients A_σ by considering the asymptotic behavior of the operators $I_m(p_i, x_j)$ along the different boundaries of the chosen region $x_i \ge x_{i+1}$, $1 \le i \le n-1$. Suppose, as before, $x_i = a_i r$, $r \to \infty$, but $a_j = a_{j+1} + y/r$, i.e., $x_j - x_{j+1}$ can take any positive values. Then

$$I_m(p_i, x_k) = I_m^{(0)}(n-2) + (p_j + p_{j+1}) I_{m-1}^{(0)}(n-2) + (p_j p_{j+1} - v(x_j - x_{j+1})) I_{m-2}^{(0)}(n-2), \tag{24}$$

where $I_m^{(0)}(n-2)$ is a symmetric function of $n-2$ momentum operators without the momenta p_j and p_{j+1}. In this direction, the asymptotic behavior of the eigenfunction $\Psi(x_i | k_m)$ is

$$\sum_{\tilde{\sigma}} A_{\tilde{\sigma}} \exp\left(i \sum_{j \neq i \neq j+1} k_{\sigma_i} x_i\right) \exp\left(i(k_{\sigma_j} + k_{\sigma_{j+1}})(x_j + x_{j+1})/2\right) \psi\left(x_j - x_{j+1} \,|\, (k_{\sigma_j} - k_{\sigma_{j+1}})/2\right), \tag{25}$$

where the sum is over pairs of permutations $\tilde{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\tilde{\sigma}(j \leftrightarrow j+1) = (\sigma_1, \ldots, \sigma_{j+1}, \sigma_j, \ldots, \sigma_n)$ that differ from one another by a transposition, and $\psi(y | k)$ is an eigenfunction of the operator $-d^2/dy^2 + v(y)$. Its asymptotic behavior is $(k = (k_{\sigma_j} - k_{\sigma_{j+1}})/2)$

$$\psi(x_j - x_{j+1} | k) = \exp(ik(x_j - x_{j+1})) + \exp(2i\eta(k) - ik(x_j - x_{j+1})) + o(1), \tag{26}$$

and, therefore, the coefficients $A_{\tilde{\sigma}}$ and $A_{\tilde{\sigma}(j \leftrightarrow j+1)}$ are related by the equation

$$A_{\tilde{\sigma}(j \leftrightarrow j+1)} = \exp\left(2i\eta\left((k_{\sigma_j} - k_{\sigma_{j+1}})/2\right)\right) A_{\tilde{\sigma}}. \tag{27}$$

Taking into account the antisymmetry property of the two-particle phase shift $\eta(k) = \eta(-k)$, ordering the k_i's: $k_1 < k_2 < \ldots < k_n$, and choosing the coefficient of the incident wave $\exp(i\Sigma k_i x_i)$ equal to unity, we obtain for A_σ

$$A_\sigma = \exp\left(2i \sum_{(ij)} \eta((k_i - k_j)/2)\right), \tag{28}$$

where the sum is over all transpositions (i, j) whose product carries $(1, 2, \ldots, n)$ into the given permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$. Thus, the coefficient of

$$\exp\left(i \sum_{i=1}^{n} k_{n-i+1} x_i\right) \tag{29}$$

is

$$\exp\left(2i \sum_{1 \leqslant i < j \leqslant n} \eta((k_i - k_j)/2)\right). \tag{30}$$

For single-channel scattering, which holds in our case, the S matrix must be determined by considering the behavior in the limit $r \to \infty$ of the eigenfunction corresponding to the fixed set $\{k_i\}_1^n$ integrated with respect to the angular coordinate variables with a smooth function. The coefficient of the reflected wave determined by this asymptotic behavior then gives the S matrix.

In the integration with respect to the angular variables in our region $x_1 \ge x_2 \ge \ldots \ge x_n$ and for the chosen ordering of the set $\{k_i\}_1^n$, a contribution to the incident wave* $\exp(-i|k|r)/(|k|r)^{(n-1)/2}$ and to

* In these expressions $r^2 = \sum_{i=1}^{n} x_i^2$, $|k|^2 = \sum_{i=1}^{n} k_i^2$.

the reflected wave $\exp(i|k|r)/(|k|r)^{(n-1)/2}$ from the sum (23) come from only the terms $\exp\left(i\sum_{i=1}k_ix_i\right)$ and (29), respectively. Thus, for the S matrix of the system of n one-dimensional particles with binary interaction $v(x)=c\wp(x)$ admitting the formulation of the scattering problem we have obtained the formula

$$S(k_1',\ldots,k_n';k_1,\ldots,k_n)=\exp\left(2i\sum_{1\leqslant i<j\leqslant n}\eta((k_i-k_j)/2)\right)\prod_{i=1}^{n}\delta(k'_{n-i+1}-k_i), \tag{31}$$

where $\eta((k_i-k_j)/2)$ is the phase shift of the two-particle problem with total momentum k_i+k_j and energy $(k_i^2+k_j^2)/2$.

Note that the Weierstrass function – in the general case it is a doubly periodic meromorphic function – defines a more interesting class of potentials for statistical physics rather than scattering theory.

The question of the factorization of the S matrix for Moser–Calogero systems was suggested to me by L. D. Faddeev. I am also very grateful to V. S. Buslaev, F. Calogero, and M. A. Semenov-Tyan-Shan'skii for helpful discussions.

Appendix

The free Klein–Fock equation $(-\infty<x,\ t<+\infty,\ \varphi_t\equiv\partial\varphi/\partial t)$

$$\varphi_{tt}-\varphi_{xx}+m^2\varphi=\varphi_{\sigma\tau}+m^2\varphi=0;\quad \tau=(t-x)/2,\ \sigma=(t+x)/2, \tag{32}$$

has, besides the energy–momentum tensor [which we write in the form of the pair of conserved currents $(\varphi_\sigma^2,-m^2\varphi^2)$, $(m^2\varphi^2,-\varphi_\tau^2)$], an infinite number of conserved currents by virtue of the symmetry under the substitution $\sigma\leftrightarrow\tau$ and the fact that in addition to $\varphi(\sigma,\tau)$ the derivative with respect to σ of any order is also a solution of Eq. (32). We denote the conserved currents by $(J_\tau^{(n)},J_\sigma^{(n)})$ and $(j_\tau^{(n)},j_\sigma^{(n)})$, where n = 1, 2, ..., and

$$J_\tau^{(n)}=(\partial_\sigma{}^n\varphi)^2,\qquad J_\sigma^{(n)}=-m^2(\partial_\sigma^{n-1}\varphi)^2;\quad j_\tau^{(n)}=m^2(\partial_\tau^{n-1}\varphi)^2,\qquad j_\sigma^{(n)}=-(\partial_\tau{}^n\varphi)^2. \tag{33}$$

By virtue of Eq. (32), $\partial_\tau J_\tau^{(n)}-\partial_\sigma J_\sigma^{(n)}=0$, $\partial_\tau j_\tau^{(n)}-\partial_\sigma j_\sigma^{(n)}=0$. Let us consider what interaction terms can be added to Eq. (32) if we require, first, that the interaction contain only powers of the field φ and, second, the conserved currents acquire corrections of higher than second degree in φ. We add to (32) an infinite series with arbitrary constants a_n that are to be determined:

$$\varphi_{\sigma\tau}=-m^2\varphi+\sum_{n=2}^{\infty}a_n\varphi^n. \tag{34}$$

The currents $J^{(1)}$ and $j^{(1)}$ can be readily found in this case too (the energy–momentum tensor exists for any local interaction). Consider the current $J_\tau^{(2)}$; for its conservation we must have the equation

$$\partial_\tau\varphi_{\sigma\sigma}{}^2=2\varphi_{\sigma\sigma}\varphi_{\sigma\sigma\tau}=\partial_\sigma J_\sigma^{(2)}+\partial_\tau F(\varphi,\varphi_\sigma,\ldots), \tag{35}$$

where the right-hand side can contain an additional term in the form of a derivative with respect to τ, which, modifying the current, we transfer to the left-hand side. Using Eq. (25), we obtain

$$\varphi_{\sigma\sigma\tau}=-m^2\varphi_\sigma+\sum_{n=2}^{\infty}na_n\varphi^{n-1}\varphi_\sigma,$$

and we rewrite Eq. (35) in the form

$$\partial_\tau\varphi_{\sigma\sigma}{}^2=2\varphi_{\sigma\sigma}\varphi_\sigma\left(-m^2+\sum_{n=2}^{\infty}na_n\varphi^{n-1}\right)=\partial_\sigma\left(\varphi_\sigma{}^2\left(-m^2+\sum_{n=2}^{\infty}na_n\varphi^{n-1}\right)\right)-\varphi_\sigma{}^3\sum_{n=2}^{\infty}n(n-1)a_n\varphi^{n-2}. \tag{36}$$

If this equation is to express a conservation law, the last term must be a total derivative with respect to τ. We express the term with n = 3 in terms of the equation of motion (34):

$$-m^2\varphi=\varphi_{\sigma\tau}-\sum_{n=2}^{\infty}a_n\varphi^n,$$

$$-\varphi_\sigma{}^3 2a_2+6a_3m^{-2}\varphi_\sigma{}^3\left(\varphi_{\sigma\tau}-\sum_{n=2}^{\infty}a_n\varphi^n\right)-\varphi_\sigma{}^3\sum_{n=4}^{\infty}n(n-1)a_n\varphi^{n-2}=\frac{6a_3}{4m^2}\partial_\tau\varphi_\sigma{}^4-\varphi_\sigma{}^3\left(2a_2+\sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2}+6m^{-2}a_3a_n]\varphi^n\right).$$

Thus, the last term in (36) has the form of a total derivative with respect to τ plus a series in φ, which cannot be represented in the form of a derivative with respect to σ or with respect to τ. Therefore, this series must vanish identically, which gives relations between its coefficients:

$$a_2=a_{2n}=0,\ a_{2n+1}=-m^2A^n/(2n+1)!,\ A=-6a_3m^{-2},$$

and Eq. (25) can be rewritten in a compact form with arbitrary constant A:

$$\varphi_{\sigma\tau}+m^2A^{-1/2}\,\mathrm{sh}\,(A^{1/2}\varphi)=0.$$

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