S Matrix for the One-Dimensional N-Body Problem with Repulsive or Attractive δ-Function Interaction

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(Received 14 December 1967)

For N particles with equal mass, interacting with repulsive or attractive δ-function interaction of the same strength, the S matrix is explicitly given and shown to be symmetrical and unitary. The incoming and outgoing states may consist of bound compounds as well as single particles. The momenta of the particles and compounds are not changed in the scattering, but particles are exchanged, such as $ABC+DE\rightarrow ADC+BE$. Only distinguishable particles are considered.

1. INTRODUCTION

FOR the one-dimensional N-body problem

$$H=-\sum_1^N \partial^2/\partial x_i^2+2c\sum_{i<j}\delta(x_i-x_j)\,, \tag{1}$$

with positive or negative c, the S matrix was discussed by McGuire[1] and by Zinn-Justine and Brezin.[2] (*Note added in proof.* K. Hepp kindly informed the author that F. A. Berezin and V. N. Sushko, Zh. Eksperim. i Teor. Fiz. **48**, 1293 (1965) [English transl.: Soviet Phys.—JETP **21**, 865 (1965)] have also discussed this problem.) We give in this paper a complete explicit expression for S. Only distinguishable particles are considered.

2. METHOD

The method used follows that of Sec. 1 of a recent paper.[3] We observe that all formulas there are also applicable to the case $c<0$.

If boundary conditions are not imposed, it is clear that all solutions of the Schrödinger equation are superpositions of solutions of the type ($Y2$). In other words, Bethe's hypothesis is proved in such a case.

3. INCOMING AND OUTGOING STATES

To construct scattering states, we need real values of the p's. Let us choose them so that

$$p_1<p_2<\cdots<p_N. \tag{2}$$

A term in ($Y2$) that has $P=$identity permutation$=I$, then, represents an *outgoing* wave. [A wave packet constructed out of such a term would have the left-most particle (at X_{Q1}) travel with velocity $2p_1$; the second left-most particle (at X_{Q2}) travel with velocity $2p_2$, etc. Thus the wave packet in *future* movement develops no collisions, meaning it is an outgoing wave packet.] A term in ($Y2$) that has $P=[N,N-1,\cdots,1]=I'$, i.e., the "reversed" permutation, represents an *incoming* wave.

Now each permutation Q represents a definite ordering of the coordinates and represents a scattering channel. A scattering state $Q_i\rightarrow Q_0$ is obtained if there are only incoming waves in channel Q_i:

$$\begin{aligned}&[Q_i,I']=1\,,\\ &[Q,I']=0 \quad\text{for}\quad Q\neq Q_i.\end{aligned} \tag{3}$$

In other words,

$$\begin{aligned}&\langle Q_i|\xi_{I'}\rangle=1\,,\\ &\langle Q|\xi_{I'}\rangle=0 \quad\text{for}\quad Q\neq Q_i.\end{aligned} \tag{4}$$

The amplitudes of the outgoing waves are the elements of ξ_I. Now ξ_I can be related to $\xi_{I'}$ through repeated use of ($Y2$):

$$\begin{aligned}\xi_I=&[Y_{21}{}^{12}Y_{31}{}^{23}Y_{41}{}^{34}\cdots Y_{N1}{}^{(N-1)N}]\\ &\times[Y_{32}{}^{12}Y_{42}{}^{23}\cdots Y_{N2}{}^{(N-2)(N-1)}]\cdots[Y_{N(N-1)}{}^{12}]\xi_{I'}.\end{aligned} \tag{5}$$

Thus the scattering amplitude for $Q_i\rightarrow Q_0$ is

$$\langle Q_0|S'|Q_i\rangle\,, \tag{6}$$

where S' is the right-hand side of (5) with $\xi_{I'}$ deleted.

4. OPERATOR: $\{ij\}$

We did not call the matrix S' in (6) the S matrix because it differs from the usual one in that the labeling of the columns is not in accordance with the usual rules. This is so because the incoming wave in Q_i, represented by the $[Q_i,I']$ term, describes particle $Q1$ with momentum p_N, $Q2$ with momentum p_{N-1}, etc. Thus the correct S matrix is

$$\begin{aligned}S&=S'[P^{N1}P^{(N-1)2}\cdots]\\ &=S'[P^{12}][P^{23}P^{12}][P^{34}P^{23}P^{12}]\cdots[P^{(N-1)N}\cdots P^{12}].\end{aligned} \tag{7}$$

If in (7) one explicitly writes S', as given in (5), one observed that the superscripts for the Y's are the same as those for the P's, but in reverse order. One now permutes the last factor P^{12} through to just behind the first factor $Y_{21}{}^{12}$; then the new last factor P^{23} through to just behind the second factor $Y_{31}{}^{23}$, etc. The final

[1] J. B. McGuire, J. Math. Phys. **5**, 622 (1964). This is a very interesting paper in which by geometrical construction many of the results of the present paper were obtained.

[2] E. Brezin and J. Zinn-Justine, Compt. Rend. Acad. Sci. Paris **B263**, 670 (1966).

[3] C. N. Yang, Phys. Rev. Letters, **19**, 1312 (1967). Formula (m) of this paper will be called (Ym) in the present paper.

result is

$$S=[\{21\}\{31\}\{41\}\cdots\{N1\}]\times[\{32\}\{42\}\cdots\{N2\}]\cdots[\{N(N-1)\}], \quad (8)$$

where

$$\{ij\}\equiv X_{ij}=P^{ij}Y_{ij}{}^{ij}=(1-P^{ij}x_{ij})(1+x_{ij})^{-1}. \quad (9)$$

5. S MATRIX

In (8) we have *an explicit formula for the S matrix* (for both $c\geq 0$ and $c<0$, $p_1<p_2<p_3\cdots<p_N$ being all real). S is an $N!\times N!$ matrix. The scattering only exchanges particle momenta. The elements of S have the following meaning:

$\langle A'B'C'\cdots|S|ABC\cdots\rangle$
= matrix element of S for
[State: particle A with p_1, B with p_2, etc.] $\rightarrow$
[State: particle A' with p_1, B' with p_2, etc.].

In (9) the permutation operator P^{ij} is defined so that, e.g.,

$$P^{31}|CDBA\rangle=|BDCA\rangle=P^{41}|ADCB\rangle.$$

It is easy to verify that each $\{ij\}$ is unitary. Hence S is unitary. S is a symmetrical matrix, as required by the time-reversal invariance of the interaction we have, because each $\{ij\}$ is symmetrical and the order of the operators $\{ij\}$ in (8) can be reversed by repeated application of Eq. (Y12). For example, for $N=4$,

$$\begin{aligned} S&=\{21\}\{31\}\{41\}\{32\}\{42\}\{43\}\\ &=\{21\}\{31\}\{41\}\{43\}\{42\}\{32\}\\ &=\{21\}\{43\}\{41\}\{31\}\{42\}\{32\}\\ &=\{43\}\{21\}\{41\}\{42\}\{31\}\{32\}\\ &=\{43\}\{42\}\{41\}\{21\}\{31\}\{32\}\\ &=\{43\}\{42\}\{41\}\{32\}\{31\}\{21\}\\ &=\{43\}\{42\}\{32\}\{41\}\{31\}\{21\}=\tilde{S}. \end{aligned}$$

6. ATTRACTIVE CASE

For the case $c<0$, there are bound states[1] for the system of N particles. The wave function for the bound state is

$$\psi=\exp[\tfrac{1}{2}c\sum_{i<j}|x_i-x_j|]. \quad (10)$$

It is easy to show directly that (10) satisfies the Schrödinger equation.

It is clear that (10) is of Bethe's form (Y2) with

$$p_1=\tfrac{1}{2}ic(N-1),\quad p_2=\tfrac{1}{2}ic(N-3),\ \cdots,\quad p_N=-\tfrac{1}{2}ic(N-1), \quad (11)$$

and with

$$\xi_I=(\text{a column with all elements equal}), \quad (12a)$$

$$\xi_P=0 \quad \text{for all } P\neq I, \quad (12b)$$

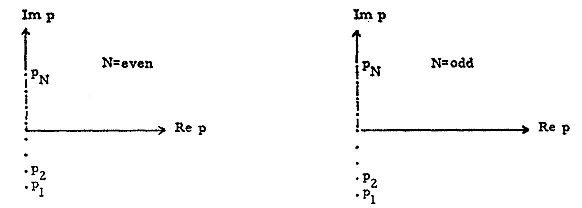

FIG. 1. The p's for the N-particle bound state. The p's are pure imaginary, and the difference between successive p's is $-ic$.

The numbers p_1, p_2, $\cdots$, p_N are plotted in Fig. 1. Equation (12a) can also be written as

$$p^{ab}\xi_I=\xi_I \quad \text{for any } a \text{ and } b. \quad (12c)$$

The energy of this bound state is

$$E=\sum_j p_j{}^2=-c^2N(N^2-1)/12, \quad (13)$$

a result already given by[1] McGuire.

It can be shown that for the N-particle problem, (10) gives the *only* bound state. This fact was already noted by McGuire.[1]

7. S MATRIX FOR BOUND STATES

If one multiplies the wave function (10) by $\exp(ik\sum x)$, one obtains a new one describing the bound state moving with a momentum Nk (k = real). The wave function is again of the form (Y2) with the p's equal to those of Fig. 1 displaced by k along the real p axis.

Would such bound particles scatter each other? To study this problem, we evidently need to fuse the considerations of Secs. 3 and 4 with those of Sec. 6.

Consider as an example the scattering of a two-particle bound state by a three-particle bound state. The p's for such a problem are plotted in Fig. 2(a). Note

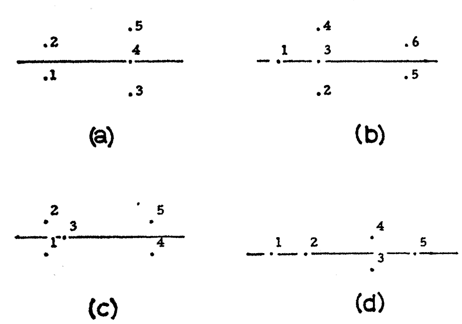

FIG. 2. Some scattering states. (a) Scattering between a bound doublet with momentum p_1+p_2 and a bound triplet with momentum $p_3+p_4+p_5$. (b) Scattering between a particle with momentum p_1, a bound triplet with momentum $p_2+p_3+p_4$, and a bound doublet with momentum p_5+p_6. (c) Scattering between a doublet of momentum p_1+p_2, a particle with momentum p_3, and a doublet of momentum p_4+p_5. (d) Scattering between three particles of momenta p_1, p_2, and p_5 and a doublet of momentum p_3+p_4. The difference between two successive p's in any vertical column is $-ic$. Note that if the p at the top of the left-most column is p_a, then the S matrix is $S=[\{\cdot a\}\{\cdot a\}\{\cdot a\}\cdots\{\cdot a\}]\cdots$.

that

$$p_2-p_1=-ic\,,\quad p_5-p_4=p_4-p_3=-ic\,. \tag{14}$$

The operators $Y_{ij}{}^{ab}$ are all *defined* and *have nonzero eigenvalues,* except for the following:

$$Y_{12}{}^{ab}=Y_{34}{}^{ab}=Y_{45}{}^{ab}=\tfrac{1}{2}(P^{ab}-1)\,;$$

$$Y_{21}{}^{ab},\ Y_{43}{}^{ab},\ Y_{54}{}^{ab} \text{ are not defined,} \tag{15}$$

$$y_{21}=y_{43}=y_{54}=0\,. \tag{16}$$

For the wave function $(Y2)$ to be bounded, such columns as $\xi_2\cdots$ must be zero; for otherwise as $x_{Q1}\to\infty$, the terms in $(Y2)$ with the elements of $\xi_2\cdots$ as coefficients will diverge exponentially. Considerations like this and a reexamination of $(Y2)$, which remains valid except for the cases where $Y_{ij}{}^{ab}$ is not defined, finally lead to

$$\xi_P\neq 0 \quad \text{if } P \text{ is of type } A\,, \tag{17a}$$

$$\xi_P=0 \quad \text{if } P \text{ is not of type } A\,, \tag{17b}$$

where P is defined to be of type A if in

$$P=[P1,P2,P3,P4,P5]$$

1 and 2 are in that order and 3, 4, 5 are in that order (e.g., [23145] is not in A, [34152] is in A). Furthermore,

$$\xi_I=\xi_{12345}$$

satisfies

$$\xi_I=P^{12}\xi_I=P^{34}\xi_I=P^{45}\xi_I\,. \tag{18}$$

Because of (18), we have, e.g.,

$$\begin{aligned}\xi_{21345}&=Y_{12}{}^{12}\xi_{12345}=\tfrac{1}{2}(P^{12}-1)\xi_{12345}=0\,,\\ \xi_{54321}&=Y_{45}{}^{12}Y_{35}{}^{23}\cdots Y_{12}{}^{12}\xi_{12345}=0\,.\end{aligned} \tag{19}$$

ξ_{12345} still gives the outgoing waves, but the incoming waves are not given by ξ_{54321}, which is zero by (19). Instead, it is given by ξ_{34512}. Thus, instead of the S' of (6), we have now

$$\xi_{12345}=S'\xi_{34512}\,, \tag{20}$$

$$S'=(Y_{32}{}^{23}Y_{42}{}^{34}Y_{52}{}^{45})(Y_{31}{}^{12}Y_{41}{}^{23}Y_{51}{}^{34})\,. \tag{21}$$

Again, the S matrix is obtained by a permutation of the columns of S':

$$S=\{32\}\{42\}\{52\}\{31\}\{41\}\{51\}\,. \tag{22}$$

8. ALLOWED STATES

Equation (22) gives explicitly the S matrix for a two-particle bound state scattered by a three-particle bound state. Because of (18), S should only operate between states Φ, satisfying

$$\Phi=P^{12}\Phi=P^{34}\Phi=P^{45}\Phi\,. \tag{23}$$

We shall call such states "allowed" states. Among the $5!=120$ components of the column Φ, there are only $5!/2!3!=10$ independent allowed ones. For example,

$$\begin{aligned}\langle ABCDE|\Phi\rangle&=\langle ABCED|\Phi\rangle=\langle ABDCE|\Phi\rangle\\ &=\text{etc.}=\langle ABEDC|\Phi\rangle\\ &=\langle BACDE|\Phi\rangle=\langle BACED|\Phi\rangle\\ &=\langle BADCE|\Phi\rangle=\text{etc.}\\ &=\langle BAEDC|\Phi\rangle=1/(12)^{1/2}\end{aligned}$$

together describe the allowed incoming state

$$AB+CDE\,, \tag{24}$$

where AB is the symmetrical bound state of A and B with momentum p_1+p_2, and CDE is the symmetrical bound state of C, D, and E with momentum $p_3+p_4+p_5$.

9. SOME IDENTITIES

We shall prove in Sec. 10 three important properties of the S of Eq. (22). A few mathematical preliminaries will be given in this section.

We note that

$$\begin{aligned}\{12\}=\tfrac{1}{2}(1-P^{12})\,,\quad \{34\}=\tfrac{1}{2}(1-P^{34})\,,\\ \{45\}=\tfrac{1}{2}(1-P^{45})\,,\end{aligned} \tag{25}$$

so that Eq. (23) is equivalent to

$$0=\{12\}\Phi=\{34\}\Phi=\{45\}\Phi\,. \tag{26}$$

$(Y12)$ remains valid, or rather the following hold true:

$$\{ij\}\{ji\}=1\,, \tag{27a}$$

$$\{ij\}\{kj\}\{ki\}=\{ki\}\{kj\}\{ij\}\,, \tag{27b}$$

$$\{ij\}\{kl\}=\{kl\}\{ij\} \quad \text{if } i,\ j,\ k,\ l \text{ are all different}\,, \tag{27c}$$

provided the undefined $\{21\}$, $\{43\}$, and $\{54\}$ do not appear. [Note that $\{35\}$ and $\{53\}$ are defined.]

Although $\{21\}$ is not defined, we can try to define y_{21} ($\{21\}$) so that $y_{21}=0$ does not appear any more in the denominator. In other words, we define

$$\begin{aligned}\{21'\}&=(1-y_{21})P^{21}+1=P^{21}+1\,,\\ \{43'\}&=P^{43}+1\,,\\ \{54'\}&=P^{54}+1\,.\end{aligned} \tag{28}$$

With this definition, (27b) is true also for those cases where $\{21\}$, $\{43\}$, and/or $\{54\}$ appear, provided we replace them by $\{21'\}$, $\{43'\}$, and $\{54'\}$. For example,

$$\begin{aligned}\{43'\}\{53\}\{54'\}&=\{54'\}\{53\}\{43'\}\,,\\ \{21'\}\{51\}\{52\}&=\{52\}\{51\}\{21'\}\,.\end{aligned} \tag{29}$$

Φ is allowed if, and only if,

$$2\Phi=\{21'\}\Phi=\{43'\}\Phi=\{54'\}\Phi\,. \tag{30}$$

Equations (23), (26), and (30) are equivalent.

10. UNITARITY AND SYMMETRY OF S

We now first prove that if Φ is allowed, so is $S\Phi$. This follows from

$$\{12\}S\Phi=\{12\}(\{32\}\{31\})(\{42\}\{41\})(\{52\}\{51\})\Phi$$
$$=(\{31\}\{32\})(\{41\}\{42\})(\{51\}\{52\})\{12\}\Phi$$

and

$$\{34\}S\Phi=\{45\}S\Phi=0.$$

Next we shall prove that S is symmetrical for allowed states, i.e., if Φ_1 and Φ_2 are both allowed,

$$\Phi_1^\dagger S\Phi_2=\Phi_1^\dagger\tilde{S}\Phi_2. \quad (31)$$

To prove this, we note that $\{ij\}$ is symmetrical. Thus

$$\tilde{S}=\{51\}\{41\}\{31\}\{52\}\{42\}\{32\}. \quad (23)$$

Now

$$\begin{aligned}
S\{21'\}&=(\{32\}\{31\})(\{42\}\{41\})(\{52\}\{51\})\{21'\}=\{21'\}(\{31\}\{32\})(\{41\}\{42\})(\{51\}\{52\}),\\
S\{21'\}\{54'\}&=\{21'\}\{54'\}(\{31\}\{32\})(\{51\}\{41\})(\{52\}\{42\}),\\
S\{21'\}\{54'\}\{53\}&=\{21'\}\{54'\}\{53\}(\{51\}\{31\})(\{52\}\{32\})(\{41\}\{42\}),\\
S\{21'\}\{54'\}\{53\}\{43'\}&=\{21'\}\{54'\}\{53\}\{43'\}(\{51\}\{52\})(\{41\}\{31\})(\{42\}\{32\})=\{21'\}\{54'\}\{53\}\{43'\}\tilde{S}.
\end{aligned} \quad (33)$$

But

$$\{21'\}\{54'\}\{53\}\{43'\}\Phi_2=8(2y_{53}^{-1}-1)\Phi_2=24\Phi_2,$$
$$\Phi_1^\dagger\{21'\}\{54'\}\{53\}\{43'\}=8(2y_{53}^{-1}-1)\Phi_1^\dagger=24\Phi_1^\dagger.$$

Thus (33) yields directly (31).

Last we shall prove that S is unitary for allowed states, i.e., if Φ_2 is allowed,

$$\Phi_2^\dagger S^\dagger S\Phi_2=\Phi_2^\dagger\Phi_2. \quad (34)$$

To prove this we find that

$$S^\dagger=\{51\}^\dagger\{41\}^\dagger\{31\}^\dagger\{52\}^\dagger\{42\}^\dagger\{32\}^\dagger.$$

Now

$$P^{12}P^{53}\{51\}^\dagger P^{53}P^{12}=\{23\} \text{ etc.}$$

Thus

$$S^\dagger P^{53}P^{12}=P^{53}P^{12}\{23\}\{24\}\{25\}\{13\}\{14\}\{15\}.$$

By (32),

$$S^\dagger P^{53}P^{12}\tilde{S}=P^{53}P^{12}.$$

Thus

$$\Phi_2^\dagger S^\dagger P^{53}P^{12}\tilde{S}\Phi_2=\Phi_2^\dagger\Phi_2. \quad (35)$$

Put

$$\Phi_1=S\Phi_2.$$

Thus Φ_1 is allowed, and $\Phi_1=P^{12}P^{53}\Phi_1$. Equations (35) and (31) together give

$$\Phi_2^\dagger\Phi_2=\Phi_1^\dagger\tilde{S}\Phi_2=\Phi_1^\dagger S\Phi_2=\Phi_2^\dagger S^\dagger S\Phi_2.$$

11. GENERAL CASE

The results of Secs. 7–10 can be generalized in a straightforward way to the scattering between any number of particles or compounds, each of which may be a bound state of any number of particles. The S matrix can be easily written down. For example, we write down the S matrix for a scattering between a single particle of momentum p_1, a bound triplet of momentum $p_2+p_3+p_4$, and a bound doublet of momentum p_5+p_6. These p's are plotted in Fig. 2(b). We have, like Eq. (22),

$$S=(\{21\}\{31\}\{41\}\{51\}\{61\})(\{54\}\{64\})\times(\{53\}\{63\})(\{52\}\{62\}). \quad (36)$$

For the case where the p's are given by Fig. 2(c), we have

$$S=(\{32\}\{42\}\{52\})(\{31\}\{41\}\{51\})(\{43\}\{53\}). \quad (37)$$

For the case where the p's are given by Fig. 2(d), we have

$$S=(\{21\}\{31\}\{41\}\{51\})(\{32\}\{42\}\{52\})\times(\{54\})(\{53\}). \quad (38)$$

All these S matrices are unitary and symmetrical for the allowed Φ in each case.

12. REDUNDANT POLES

The S matrix discussed above has evidently matrix elements that are rational functions of the relative momenta of the particles involved. For real values of these relative momenta, S is regular. But for complex values of these relative momenta, S may have poles. For example, in the reaction $AB+CDE$ discussed in Secs. 7–10, for which the S matrix is given by (22), there are poles when y_{32}, y_{42}, y_{52}, y_{31}, y_{41}, or y_{51} vanishes. However, only the pole $y_{32}=0$ corresponds to a bound state (the 5-particle bound state). The others are *redundant poles.* This point was already realized by McGuire.[1]

ACKNOWLDGMENTS

It is a pleasure to acknowledge useful discussions with T. T. Wu, M. Moshinsky, and J. Groeneveld.