# Construction and clustering properties of the 2 d non-linear $\sigma$-model form factors: $\mathrm{O}(3), \mathrm{O}(4)$, large $n$ examples 

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#### Abstract

Multi-particle form factors of local operators in integrable models in two dimensions seem to have the property that they factorize when one subset of the particles in the external states are boosted by a large rapidity with respect to the others. This remarkable property, which goes under the name of form factor clustering, was first observed by Smirnov in the O(3) non-linear $\sigma$-model and has subsequently found useful applications in integrable models without internal symmetry structure. In this paper we conjecture the nature of form factor clustering for the general $\mathrm{O}(n) \sigma$-model and make some tests in leading orders of the $1 / n$ expansion and for the special cases $n=3,4$.


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## 1. Introduction

In this paper we will investigate certain properties of form factors of integrable models in two dimensions. This class of models allow a unique field theoretical insight because they admit special non-perturbative methods to their solution known as the S-matrix bootstrap approach [1,2]. In particular once the spectrum of stable (massive) states has been identified a well-motivated S-matrix can be postulated. Going further one can then attempt to solve equations for the form factors of local operators and finally compute correlation functions of these operators by satu-

[^0]rating with a complete set of intermediate states. There are many examples of such applications. Of particular interest are the uncovering of structural relations which may have corresponding validity or inspire similar relations in models in higher dimensions.

We will in this paper consider mainly the non-linear $\mathrm{O}(n) \sigma$-models which have the additional interesting feature that they are asymptotically free. In particular for the case $n=3$ many form factors are explicitly known and one can compute vacuum 2-point functions up to rather high energies and compare the results with numerical MC and analytic perturbative results [3].

One of our main motivations for the present paper arose from our recent work on structure functions in these models [4]. A result that particularly intrigues us concerns the small Feynman $x$ behavior ( $Q^{2}$ fixed); we found that this was of the form $f(x) A\left(Q^{2}\right)$ where the behavior of $f(x)$ at small $x$ reflects the high energy behavior of the scattering amplitudes and the function $A\left(Q^{2}\right)$ is determined in terms of the vacuum 2-point function. We speculated that this structure may be universal in asymptotically free field theories, in particular for QCD. ${ }^{1}$ One of the key properties in the derivation of the result (for the $\sigma$-model) is the property of clustering of form factors, and it may be that at least this property has some analogy in QCD.

Roughly, form factor clustering says that if one considers a multi-particle form factor of a local operator and one boosts a subset of particles $A$ uniformly with respect to the rest $B$ by a large rapidity $\Delta$, then to leading order the form factor factorizes into a function of $\Delta$ times a product of two functions, one depending only on the rapidities of $A$ and the other only on the rapidities of $B$. The functions appearing here are again themselves form factors.

To our knowledge the first observation of FF clustering was by Smirnov [1] for the $\mathrm{O}(3)$ nonlinear $\sigma$-model. Thereafter investigations of this structure were mainly pursued for S-matrices without internal symmetry structure (see e.g. [5-12]). Recently it was used in the construction of local operators in the sinh-Gordon model by Delfino and Niccoli [13]. FF clustering appears to be an extra constraint on form factors which can be imposed in addition to the usual axioms. Its main application so far has been to identify the operator associated with a particular solution of the FF equations i.e. clustering can be useful in model building.

A systematic study of clustering properties for models with internal symmetry has not appeared in the literature so far. It is the purpose of this paper to make steps to fill this gap for the case of the $\mathrm{O}(n)$ non-linear $\sigma$-models. In order to be able to make non-trivial tests of the clustering properties of the form factors we first had to work out some $\sigma$-model form factors explicitly. In particular while considering the test in $\mathrm{O}(4)$ we had to work out details of the 3-particle spin form factor and this is presented in Appendix D. While the structure (a tensor product of $\mathrm{SU}(2)$ form factors) was given previously by Smirnov [14], he only considered the case of form factors with an even number of particles. We also work out a number of form factors in leading orders of the large $n$ expansion both by applying bootstrap techniques for this case and by the standard saddle point expansion of the functional integral. A by-product of our investigation is the verification of the (expected, but non-trivial) equivalence of the two methods.

The organization of the paper is as follows. In the next section we give a brief introduction to the model, in particular the S-matrix is described and form factors of some familiar operators are defined. In Section 3 we remind the reader of the FF axioms, give the 2-particle form factors of operators introduced in Section 2 and consider the specific case of the 3-particle spin form factor. In Section 4, for comparison, we first briefly review FF clustering for models without internal symmetry. Thereafter we consider the structure of FF clustering in the $\mathrm{O}(n)$ non-linear $\sigma$-models.

[^1]We motivate an ansatz for the general form for various cases encountered, and conjecture a relation of the leading FF clustering behavior to the anomalous dimensions of the operators which is reminiscent to the form of operator product expansions valid at short distances. In Section 5 we consider solutions for the form factors in leading orders of the $1 / n$ expansion, and in Section 6 we verify that these solutions are indeed identical with results derived from the quantum field theoretic formalism. Tests of the ansatz in the leading order of the $1 / n$-expansion are described in Section 7 and tests in the particular cases $n=3,4$ in Section 8. Various technical details are relegated to appendices.

## 2. $O(n)$ non-linear $\sigma$-model S-matrix and operators

### 2.1. The Zamolodchikov S-matrix

Particles in the $\mathrm{O}(n)$ model are characterized by their mass $M$ and the quantum numbers $(a, \theta)$, where $a=1,2, \ldots, n$ is an $\mathrm{O}(n)$ vector index and $\theta$ is the rapidity of the particle in terms of which the components of its momentum are $p^{0}=M \cosh \theta$ and $p^{1}=M \sinh \theta$. When two particles scatter there is no particle production and the bootstrap S-matrix proposed by Zamolodchikov and Zamolodchikov [15] is of the form

$$
\begin{equation*}
S_{a b}^{c d}(\theta)=\sigma_{1}(\theta) \delta^{c d} \delta_{a b}+\sigma_{2}(\theta) \delta_{a}^{c} \delta_{b}^{d}+\sigma_{3}(\theta) \delta_{a}^{d} \delta_{b}^{c} \tag{2.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma_{1}(\theta)=\frac{-2 \pi i \chi}{i \pi-\theta} \sigma_{2}(\theta), \quad \sigma_{3}(\theta)=\frac{-2 \pi i \chi}{\theta} \sigma_{2}(\theta) \tag{2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\sigma_{2}(\theta)=\frac{-\theta}{\theta-2 \pi i \chi} \exp \{i \delta(\theta)\} \tag{2.3}
\end{equation*}
$$

where the phase appearing here is given by

$$
\begin{equation*}
\delta(\theta)=2 \int_{0}^{\infty} \frac{\mathrm{d} \omega}{\omega} \sin (\theta \omega) \tilde{K}_{n}(\omega) \tag{2.4}
\end{equation*}
$$

with kernel

$$
\begin{equation*}
\tilde{K}_{n}(\omega)=\frac{\mathrm{e}^{-\pi \omega}+\mathrm{e}^{-2 \pi \chi \omega}}{1+\mathrm{e}^{-\pi \omega}} \tag{2.5}
\end{equation*}
$$

We have used the notation $\chi=\frac{1}{n-2}$ in the above formulae.
It is useful to introduce the invariant amplitudes corresponding to $s$-channel "isospin" $I=$ $0,2,1$ :

$$
\begin{align*}
& S_{0}(\theta)=n \sigma_{1}(\theta)+\sigma_{2}(\theta)+\sigma_{3}(\theta), \\
& S_{2}(\theta)=\sigma_{2}(\theta)+\sigma_{3}(\theta)=-\exp \{i \delta(\theta)\}, \\
& S_{1}(\theta)=-\sigma_{2}(\theta)+\sigma_{3}(\theta), \tag{2.6}
\end{align*}
$$

which obey unitarity $S_{I}(\theta) S_{I}(-\theta)=1$.
The particular cases $n=3,4$ are discussed in more detail in Appendix A.

### 2.2. Operators and form factors

In this subsection we discuss the form factors of the most important operators in the model, those of the $\mathrm{O}(n)$ spin field, the Noether current and the energy-momentum tensor. We also define the form factors of a symmetric, traceless scalar operator.

### 2.2.1. The $\mathrm{O}(n)$ field

The conventional normalization of the $\mathrm{O}(n)$ field is given by its one-particle matrix elements:

$$
\begin{equation*}
\langle 0| \Phi^{a}(0)|b, \theta\rangle=\delta^{a b} \tag{2.7}
\end{equation*}
$$

The general $r$-particle matrix elements define its form factors by

$$
\begin{equation*}
\langle 0| \Phi^{a}(0)\left|b_{1}, \theta_{1} ; \ldots ; b_{r}, \theta_{r}\right\rangle^{\text {in }}=\Lambda_{n} f_{b_{1} \ldots b_{r}}^{a}\left(\theta_{1}, \ldots, \theta_{r}\right), \tag{2.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\Lambda_{3}=\frac{2}{\sqrt{\pi}}, \quad \Lambda_{n}=1, \quad n>3 \tag{2.9}
\end{equation*}
$$

The physical "in" states correspond to the rapidity ordering $\theta_{1}>\theta_{2}>\cdots>\theta_{r}$. The form factors are originally defined for this ordered set of real rapidities but can be extended to the complete complex (multi-)rapidity space by analytic continuation. See Section 3. We use the state normalization

$$
\begin{equation*}
{ }^{\text {in }}\left\langle a_{1}^{\prime}, \theta_{1}^{\prime} ; \ldots ; a_{r}^{\prime}, \theta_{r}^{\prime} \mid a_{1}, \theta_{1} ; \ldots ; a_{r}, \theta_{r}\right\rangle^{\text {in }}=(4 \pi)^{r} \delta_{a_{1}^{\prime} a_{1}} \cdots \delta_{a_{r}^{\prime} a_{r}} \delta\left(\theta_{1}^{\prime}-\theta_{1}\right) \cdots \delta\left(\theta_{r}^{\prime}-\theta_{r}\right) \tag{2.10}
\end{equation*}
$$

### 2.2.2. The Noether current

The normalization of the Noether current operators $J_{\mu}^{a b}(x)$ is fixed by the equal time commutation relations

$$
\begin{equation*}
\left[J_{0}^{a b}(0, x), \Phi^{c}(0, y)\right]=i t_{c d}^{a b} \delta(x-y) \Phi^{d}(0, y) \tag{2.11}
\end{equation*}
$$

where

$$
\begin{equation*}
t_{a b}^{c d}=\delta_{a}^{c} \delta_{b}^{d}-\delta_{a}^{d} \delta_{b}^{c} \tag{2.12}
\end{equation*}
$$

The current form factors are given by

$$
\begin{equation*}
\langle 0| J_{\mu}^{a b}(0)\left|b_{1}, \theta_{1} ; \ldots ; b_{r}, \theta_{r}\right\rangle^{\text {in }}=-i \epsilon_{\mu \alpha} q^{\alpha} f_{b_{1} \ldots b_{r}}^{a b}\left(\theta_{1}, \ldots, \theta_{r}\right) \tag{2.13}
\end{equation*}
$$

where

$$
\begin{equation*}
q^{\alpha}=\left(p_{1}+p_{2}+\cdots+p_{r}\right)^{\alpha}, \quad p_{i}=\left(p_{i}^{0}, p_{i}^{1}\right)=\left(M \cosh \theta_{i}, M \sinh \theta_{i}\right) \tag{2.14}
\end{equation*}
$$

and $\epsilon_{01}=-\epsilon_{10}=1$. The normalization (2.11) implies $^{2}$ the following result for the one-particle expectation value:

$$
\begin{equation*}
\langle c, \theta| J_{\mu}^{a b}(0)|d, \theta\rangle=-2 i p_{\mu} t_{c d}^{a b} \tag{2.15}
\end{equation*}
$$

[^2]
### 2.2.3. The energy-momentum tensor

The energy-momentum tensor is normalized so that its space integral

$$
\begin{equation*}
H=\int_{-\infty}^{\infty} \mathrm{d} x T_{00}(0, x) \tag{2.16}
\end{equation*}
$$

is the Hamiltonian of the system with one-particle eigenvalues given by $H|b, \theta\rangle=M \cosh \theta|b, \theta\rangle$. The energy-momentum tensor form factors are

$$
\begin{equation*}
\langle 0| T_{\mu \nu}(0)\left|b_{1}, \theta_{1} ; \ldots ; b_{r}, \theta_{r}\right\rangle^{\text {in }}=\left(\eta_{\mu \nu} q^{2}-q_{\mu} q_{\nu}\right) f_{b_{1} \ldots b_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right), \tag{2.17}
\end{equation*}
$$

where $\eta_{\mu \nu}$ is the $(1+1)$-dimensional metric characterized by $\eta_{00}=-\eta_{11}=1$.
The case $n=3$ is discussed in further detail in Appendix B.

### 2.2.4. Symmetric, traceless tensor operator

Finally we define the form factors of a Lorenz scalar and symmetric, traceless iso-tensor operator $\Sigma^{c d}$

$$
\begin{equation*}
\langle 0| \Sigma^{c d}(0)\left|b_{1}, \theta_{1} ; \ldots ; b_{r}, \theta_{r}\right\rangle^{\text {in }}=\tilde{f}_{b_{1} \ldots b_{r}}^{c d}\left(\theta_{1}, \ldots, \theta_{r}\right) \tag{2.18}
\end{equation*}
$$

### 2.3. Two-particle form factors

Using $\mathrm{O}(n)$ symmetry and Poincaré invariance, the two-particle form factors can be parameterized as follows

$$
\begin{align*}
& \langle 0| J_{\mu}^{c d}(0)|a, \alpha ; b, \beta\rangle=i \epsilon_{\mu \nu} q^{\nu} \psi_{1}(\alpha-\beta) t_{a b}^{c d} \\
& \langle 0| \Sigma^{c d}(0)|a, \alpha ; b, \beta\rangle=-i \psi_{2}(\alpha-\beta) \tilde{t}_{a b}^{c d} \\
& \langle 0| T_{\mu \nu}(0)|a, \alpha ; b, \beta\rangle=\frac{i}{2}\left(q_{\mu} q_{\nu}-q^{2} \eta_{\mu \nu}\right) \psi_{0}(\alpha-\beta) \delta_{a b} \tag{2.19}
\end{align*}
$$

where

$$
\begin{equation*}
s_{a b}^{c d}=\delta_{a}^{c} \delta_{b}^{d}+\delta_{a}^{d} \delta_{b}^{c}, \quad \tilde{t}_{a b}^{c d}=s_{a b}^{c d}-\frac{2}{n} \delta^{c d} \delta_{a b} \tag{2.20}
\end{equation*}
$$

It can be shown that the normalization of the operators defined above implies the following singularity structure for the functions $\psi_{i}(\theta)$

$$
\begin{align*}
& \psi_{0}(\theta) \approx \frac{-4 i}{(\theta-i \pi)^{2}}, \quad \theta \approx i \pi, \\
& \psi_{1}(\theta) \approx \frac{2}{\theta-i \pi}, \quad \theta \approx i \pi, \\
& \psi_{2}(\theta) \quad \text { regular at } \theta=i \pi . \tag{2.21}
\end{align*}
$$

## 3. Form factor axioms

In this section we recall the functional equations [1] satisfied by the scalarized form factors, which we generically denote by $\mathcal{F}_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)$ in this section. It turns out to be convenient to introduce the Faddeev-Zamolodchikov operators $Z_{a}^{+}(\theta)$ satisfying the exchange relation

$$
\begin{equation*}
Z_{a}^{+}(\theta) Z_{b}^{+}\left(\theta^{\prime}\right)=S_{a b}^{y x}\left(\theta-\theta^{\prime}\right) Z_{x}^{+}\left(\theta^{\prime}\right) Z_{y}^{+}(\theta) \tag{3.1}
\end{equation*}
$$

Now we can define the multi-index matrix $S_{b a_{1} \ldots a_{r} ; b_{1} \ldots b_{r} a}\left(\beta \mid \theta_{1}, \ldots, \theta_{r}\right)$ by the relation

$$
\begin{equation*}
Z_{b}^{+}(\beta) Z_{a_{1}}^{+}\left(\theta_{1}\right) \cdots Z_{a_{r}}^{+}\left(\theta_{r}\right)=S_{b a_{1} \ldots a_{r} ; b_{1} \ldots b_{r} a}\left(\beta \mid \theta_{1}, \ldots, \theta_{r}\right) Z_{b_{1}}^{+}\left(\theta_{1}\right) \cdots Z_{b_{r}}^{+}\left(\theta_{r}\right) Z_{a}^{+}(\beta) \tag{3.2}
\end{equation*}
$$

The form factor axioms are the following five functional equations [1]

$$
\begin{align*}
& \mathcal{F}_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)=\mathcal{F}_{a_{1} \ldots a_{r}}\left(\theta_{1}+\lambda, \ldots, \theta_{r}+\lambda\right)  \tag{3.3}\\
& \mathcal{F}_{\ldots x y \ldots}\left(\cdots \theta, \theta^{\prime} \ldots\right)=S_{x y}^{v w}\left(\theta-\theta^{\prime}\right) \mathcal{F}_{\ldots w v \ldots}\left(\cdots \theta^{\prime}, \theta \ldots\right),  \tag{3.4}\\
& \mathcal{F}_{a_{1} a_{2} \ldots a_{r}}\left(\theta_{1}+2 \pi i, \theta_{2}, \ldots, \theta_{r}\right)=\mathcal{F}_{a_{2} \ldots a_{r} a_{1}}\left(\theta_{2}, \ldots, \theta_{r}, \theta_{1}\right),  \tag{3.5}\\
& \lim _{\varepsilon \rightarrow 0} \varepsilon \mathcal{F}_{a b a_{1} \ldots a_{r}}\left(\beta+i \pi+\varepsilon, \beta, \theta_{1}, \ldots, \theta_{r}\right) \\
& \quad=2 i\left\{\delta_{a b} \mathcal{F}_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)-S_{b a_{1} \ldots a_{r} ; b_{1} \ldots b_{r} a}\left(\beta \mid \theta_{1}, \ldots, \theta_{r}\right) \mathcal{F}_{b_{1} \ldots b_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)\right\},  \tag{3.6}\\
& \mathcal{F}_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)=w_{p} \mathcal{F}_{a_{r} \ldots a_{1}}\left(-\theta_{r}, \ldots,-\theta_{1}\right) \tag{3.7}
\end{align*}
$$

In the last equation $w_{p}$ is the parity of the scalarized form factors. It is equal to unity for all operators considered above except for the Noether current, for which it is equal to -1 .

Next we define a new type of reduced form factors ${ }^{3}$ by

$$
\begin{align*}
& \mathcal{F}_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)=\frac{F_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)}{C_{r}\left(\theta_{1}, \ldots, \theta_{r}\right)}  \tag{3.8}\\
& C_{r}\left(\theta_{1}, \ldots, \theta_{r}\right) \equiv \prod_{1 \leqslant i<j \leqslant r} \cosh \left(\frac{\theta_{i}-\theta_{j}}{2}\right) \tag{3.9}
\end{align*}
$$

Three of the form factor equations for $F_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)$ are of the same form as (3.3), (3.4) and (3.7) and the equation corresponding to (3.5) is only modified by a sign factor $(-1)^{r-1}$. Finally the residue axiom (3.6) is rewritten as

$$
\begin{align*}
& F_{a b a_{1} \ldots a_{r}}\left(\beta+i \pi, \beta, \theta_{1}, \ldots, \theta_{r}\right) \\
&=\left(\frac{i}{2}\right)^{r} \prod_{j=1}^{r} \sinh \left(\beta-\theta_{j}\right)\left\{S_{b a_{1} \ldots a_{r} ; b_{1} \ldots b_{r} a}\left(\beta \mid \theta_{1}, \ldots, \theta_{r}\right) F_{b_{1} \ldots b_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)\right. \\
&\left.-\delta_{a b} F_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)\right\} . \tag{3.10}
\end{align*}
$$

### 3.1. Two-particle form factors

Two of the form factor equations, (3.3) and (3.7), are automatically satisfied by the ansatz (2.19). The residue equation (3.6) does not apply to two-particle form factors, but one has to satisfy the normalization conditions (2.21) instead. Finally (3.4) and (3.5) become ( $I=0,1,2$ )

$$
\begin{equation*}
\psi_{I}(\theta)=S_{I}(\theta) \psi_{I}(-\theta) \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi_{I}(\theta+2 \pi i)=(-1)^{I} \psi_{I}(-\theta) \tag{3.12}
\end{equation*}
$$

respectively where the $S_{I}$ are defined in (2.6).

[^3]A useful building block in the construction of form factors is the function

$$
\begin{equation*}
\Delta(\theta)=\int_{0}^{\infty} \frac{\mathrm{d} \omega}{\omega} \tilde{K}_{n}(\omega) \frac{\cosh [(\pi+i \theta) \omega]-1}{\sinh \pi \omega} \tag{3.13}
\end{equation*}
$$

Its main properties are

$$
\begin{equation*}
\Delta(i \pi+\theta)=\Delta(i \pi-\theta), \quad \Delta(i \pi)=0, \quad \Delta(\theta)=\Delta(-\theta)+i \delta(\theta) \tag{3.14}
\end{equation*}
$$

and its asymptotic behavior for large positive $\theta$ is given by

$$
\begin{equation*}
\Delta(i \pi+\theta), \mathcal{R} e \Delta(\theta) \approx-\frac{\theta}{2} \tilde{K}_{n}(0)-\frac{\ln \theta}{\pi} \tilde{K}_{n}^{\prime}(0)+\mathrm{O}(1) \tag{3.15}
\end{equation*}
$$

If in (2.4) we substitute $\tilde{K}_{n}(\omega)$ by the function $\tilde{k}_{\alpha}(\omega)=-\mathrm{e}^{-\pi \alpha \omega}$ we get

$$
\begin{equation*}
\mathrm{e}^{i \delta_{\alpha}(\theta)}=\frac{i \alpha \pi+\theta}{i \alpha \pi-\theta} \tag{3.16}
\end{equation*}
$$

We denote the related building block by $\Delta_{\alpha}(\theta)$.
Using the building blocks defined above we find

$$
\begin{align*}
& \psi_{1}(\theta)=\tanh \frac{\theta}{2} \exp \left\{\Delta(\theta)+\Delta_{2 \chi}(\theta)\right\} \approx c_{1} \theta^{-\chi} \\
& \psi_{2}(\theta)=\sinh \frac{\theta}{2} \exp \{\Delta(\theta)\} \approx c_{2} \theta^{\chi} \\
& \psi_{0}(\theta)=\frac{-2 i}{i \pi-\theta} \psi_{1}(\theta) \approx c_{0} \theta^{-(1+\chi)} . \tag{3.17}
\end{align*}
$$

Here we also indicate the asymptotic behavior of the form factors $\psi_{I}(\theta)$ for large positive $\theta$.

### 3.2. Three-particle form factors

In this section we write down the form factor equations for the three-particle form factors of the $\mathrm{O}(n)$ field operators. We start with the definition

$$
\begin{equation*}
f_{a b c}^{d}(\alpha, \beta, \gamma)=\frac{F_{a b c}^{d}(\alpha, \beta, \gamma)}{\cosh \left(\frac{\alpha-\beta}{2}\right) \cosh \left(\frac{\alpha-\gamma}{2}\right) \cosh \left(\frac{\beta-\gamma}{2}\right)} . \tag{3.18}
\end{equation*}
$$

The form factor equations (3.3)-(3.7) in this special case become $(n>3)$

$$
\begin{align*}
& F_{a b c}^{d}(\alpha, \beta, \gamma)=F_{a b c}^{d}(\alpha+\lambda, \beta+\lambda, \gamma+\lambda),  \tag{3.19}\\
& F_{a b c}^{d}(\alpha, \beta, \gamma)=S_{b c}^{y x}(\beta-\gamma) F_{a x y}^{d}(\alpha, \gamma, \beta),  \tag{3.20}\\
& F_{a b c}^{d}(\alpha, \beta, \gamma)=F_{b c a}^{d}(\beta, \gamma, \alpha-2 \pi i),  \tag{3.21}\\
& F_{a b c}^{d}(\beta+i \pi, \beta, \gamma)=\frac{i}{2} \sinh (\beta-\gamma)\left\{S_{b c}^{a d}(\beta-\gamma)-\delta_{a b} \delta_{c}^{d}\right\},  \tag{3.22}\\
& F_{a b c}^{d}(\alpha, \beta, \gamma)=F_{c b a}^{d}(-\gamma,-\beta,-\alpha) . \tag{3.23}
\end{align*}
$$

(3.19) and (3.21) are satisfied by the following ansatz

$$
\begin{align*}
F_{a b c}^{d}(\alpha, \beta, \gamma)= & \delta_{a d} \delta_{b c} X(\alpha-\beta, \alpha-\gamma)+\delta_{b d} \delta_{a c} X(\beta-\gamma, \beta-\alpha+2 \pi i) \\
& +\delta_{c d} \delta_{a b} X(\gamma-\alpha+2 \pi i, \gamma-\beta+2 \pi i) . \tag{3.24}
\end{align*}
$$

We can also rewrite (3.22) and (3.23) in terms of this single function $X$. We get

$$
\begin{align*}
& X(u, v)=X(2 \pi i-v, 2 \pi i-u), \\
& X(\theta, i \pi)=\frac{i}{2} \sinh \theta \sigma_{3}(\theta), \\
& X(\theta, i \pi+\theta)=\frac{i}{2} \sinh \theta\left[\sigma_{2}(\theta)-1\right] . \tag{3.25}
\end{align*}
$$

Finally (3.20) is equivalent to

$$
\begin{align*}
X(u, v)= & {\left[n \sigma_{1}(v-u)+\sigma_{2}(v-u)+\sigma_{3}(v-u)\right] X(v, u) } \\
& +\sigma_{1}(v-u)[X(u-v, 2 \pi i-v)+X(2 \pi i-u, 2 \pi i+v-u)] \tag{3.26}
\end{align*}
$$

and

$$
\begin{align*}
X(v-u, 2 \pi i-u)= & \sigma_{2}(v-u) X(2 \pi i-u, 2 \pi i+v-u) \\
& +\sigma_{3}(v-u) X(u-v, 2 \pi i-v) . \tag{3.27}
\end{align*}
$$

## 4. Form factor clustering

### 4.1. Models without internal symmetry

We first briefly review clustering properties of form factors for models without internal symmetry for which the majority of detailed investigations have been carried out so far. In [9] Delfino, Simonetti and Cardy studied models defined as perturbations of conformal invariant theories i.e. those formally defined by the action

$$
\begin{equation*}
A=A_{\mathrm{CFT}}+g \int \mathrm{~d}^{2} x \phi(x) \tag{4.1}
\end{equation*}
$$

where the operator $\phi(x)$ is of dimension $2 \delta<2$. Further attention is restricted to those perturbations where an infinite number of integrals of motion survive and the resulting massive model is integrable.

Consider first the case when there is only one species of massive particle of mass $m$ and 2particle S-matrix element $S(\theta)$. The cluster hypothesis proposes that multi-particle form factors of a scaling operator $\Phi$ of dimension $2 \delta_{\Phi}<2$ in such a theory factorize according to

$$
\begin{align*}
& \lim _{\Lambda \rightarrow \infty} F_{r+l}^{\Phi}\left(\theta_{1}+\Lambda, \ldots, \theta_{r}+\Lambda, \theta_{r+1}, \ldots, \theta_{r+l}\right) \\
& \quad=\frac{1}{\langle\Phi\rangle} F_{r}^{\Phi}\left(\theta_{1}, \ldots, \theta_{r}\right) F_{l}^{\Phi}\left(\theta_{r+1}, \ldots, \theta_{r+l}\right) . \tag{4.2}
\end{align*}
$$

Actually, (4.2) is only valid for operators corresponding to primaries in the conformal limit. More generally, conformal operators can be classified as

$$
\begin{equation*}
\mathcal{L}_{m} \overline{\mathcal{L}}_{\bar{m}} \Phi, \tag{4.3}
\end{equation*}
$$

where $\mathcal{L}_{m}$ is a combination (at level $m$ ) of the left Virasoro operators and similarly $\overline{\mathcal{L}}_{\bar{m}}$ is built from right Virasoro operators. Identifying operators in the massive theory with their conformal
limit, the generalization of (4.2) for descendant operators reads [16]

$$
\begin{align*}
& \lim _{\Delta \rightarrow \infty} \mathrm{e}^{-m \Delta} F_{r+l}^{\mathcal{L}_{\mathcal{L}} \overline{\mathcal{L}}_{m} \Phi}\left(\theta_{1}+\Delta, \ldots, \theta_{r}+\Delta, \theta_{r+1}, \ldots, \theta_{r+l}\right) \\
& \quad=\frac{1}{\langle\Phi\rangle} F_{r}^{\mathcal{L}_{m} \Phi}\left(\theta_{1}, \ldots, \theta_{r}\right) F_{l}^{\overline{\mathcal{L}}_{m} \Phi}\left(\theta_{r+1}, \ldots, \theta_{r+l}\right) \tag{4.4}
\end{align*}
$$

Ref. [9] made the validity of (4.2) highly plausible by considering the massless limit to the UV critical point in which the mass $m \rightarrow 0$ and rapidities of the particles are simultaneously taken to $\infty$ such that momenta are fixed. The basic assumptions ${ }^{4}$ are that (i) $\lim _{\theta \rightarrow \infty} S(\theta)=1$ so that massless left and right movers decouple and (ii) that the operator space of the conformal point and that of the perturbed theory have the same basic structure. In particular to the scaling operator $\Phi(x)$ in the off-critical theory there is an associated conformal operator $\tilde{\Phi}$ of the same scaling dimension $2 \delta_{\Phi}$. The property has been noticed in the past to be fulfilled by various FF solutions in specific models [5-13], and many examples and tests of FF clustering have been successfully performed.

Tests of the hypothesis for a given S-matrix, involve solving the general functional equations together with the cluster constraints and seeing whether the number of independent solutions equals the number in the corresponding Kac table of the associated CFT. Once this has been established one can identify the operators corresponding to the solutions by computing the dimensions $\delta_{\Phi}$ either by studying the short distance behavior of the 2-point function computed by saturation by lowest states, or using the DSC [9] sum rule

$$
\begin{equation*}
\delta_{\Phi}^{\mathrm{UV}}-\delta_{\Phi}^{\mathrm{IR}}=-\frac{1}{4 \pi \delta} \int \mathrm{~d}^{2} x\langle\Theta(x) \Phi(0)\rangle_{c} \tag{4.5}
\end{equation*}
$$

where $\Theta(x)$ is the trace of the energy-momentum tensor which is related to the perturbing field by $\Theta=4 \pi g(1-\delta) \phi(x)$. In a massive theory $\delta_{\phi}^{\mathrm{IR}}=0$.

### 4.2. FF clustering in the non-linear sigma model

To our knowledge FF clustering in the $\mathrm{O}(3)$ non-linear sigma model was first discussed by Smirnov [1]. A more detailed exposition of this case was presented by Balog and Niedermaier [3]. In the following we consider the general $\mathrm{O}(n)$ case which exhibits a rather rich structure.

### 4.3. Clustering (leading term)

Let us divide the particles into two subsets and boost the particles in the first set by a large (positive) rapidity $\Delta$. Form factor clustering means that the scalarized, dimensionless form factors have universal large $\Delta$ asymptotics (which usually behave as a power in $\Delta$ instead of the constant behavior exhibited by the models considered in the previous subsection):

$$
\begin{align*}
& f_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \quad \cong h_{k ; l}(\Delta) g_{a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right) \tag{4.6}
\end{align*}
$$

[^4]where the clustering function
\[

$$
\begin{equation*}
h_{k ; l}(\Delta+\lambda) \cong h_{k ; l}(\Delta) \tag{4.7}
\end{equation*}
$$

\]

and also the functional form (and the dependence on $\Delta$ ) of the sub-leading terms depend on the type of the operator. These sub-leading terms are often suppressed by some negative power of $\Delta$, see Section 4.6.

Using the asymptotic properties of the S-matrix,

$$
\begin{equation*}
S_{c d}^{a b}(\theta) \cong \delta_{c}^{a} \delta_{d}^{b}+\frac{2 \pi i \chi}{\theta} t_{b d}^{a c}+\cdots \tag{4.8}
\end{equation*}
$$

we can show that the expressions

$$
\begin{equation*}
g_{a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right) \tag{4.9}
\end{equation*}
$$

satisfy the form factor axioms (3.3)-(3.6) in the variables corresponding to the first set while the dependence on the particles belonging to the second set are playing the role of dummy parameters. This is almost trivial for (3.3)-(3.5), and for (3.6) we get

$$
\begin{align*}
& f_{a b a_{1} \ldots a_{k} b_{1} \ldots b_{l}}\left(\gamma+i \pi+\epsilon+\Delta, \gamma+\Delta, \alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \cong h_{k+2 ; l}(\Delta) g_{a b a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}\left(\gamma+i \pi+\epsilon, \gamma, \alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right) \\
& \cong \frac{2 i}{\epsilon}\left\{\delta_{a b} f_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right)\right. \\
&-S_{b a_{1} \ldots a_{k} b_{1} \ldots b_{l} ; \tilde{a}_{1} \ldots \tilde{a}_{k} \tilde{b}_{1} \ldots \tilde{b}_{l} a}\left(\gamma+\Delta \mid \alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
&\left.\times f_{\tilde{a}_{1} \ldots \tilde{a}_{k} \tilde{b}_{1} \ldots \tilde{b}_{l}}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right)\right\} \\
& \cong \frac{2 i}{\epsilon} h_{k ; l}(\Delta)\left\{\delta_{a b} g_{a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right)\right. \\
& \quad-S_{b a_{1} \ldots a_{k} ; \tilde{a}_{1} \ldots \tilde{a}_{k} a}\left(\gamma \mid \alpha_{1}, \ldots, \alpha_{k}\right) g_{\left.\tilde{a}_{1} \ldots \tilde{a}_{k} ; b_{1} \ldots b_{l}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right)\right\} .} . \tag{4.10}
\end{align*}
$$

From this we see that, because of the uniqueness of the solution of the set of form factor axioms,

$$
\begin{equation*}
h_{k+2 ; l}(\Delta)=h_{k ; l}(\Delta) \tag{4.11}
\end{equation*}
$$

and for $k \geqslant 1$

$$
\begin{align*}
& g_{a b a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}\left(\gamma+i \pi+\epsilon, \gamma, \alpha_{1}, \ldots, \alpha_{k}, \beta_{1}, \ldots, \beta_{l}\right) \\
& \qquad \cong \frac{2 i}{\epsilon}\left\{\delta_{a b} g_{a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right)\right. \\
& \left.\quad-S_{b a_{1} \ldots a_{k} ; \tilde{a}_{1} \ldots \tilde{a}_{k} a}\left(\gamma \mid \alpha_{1}, \ldots, \alpha_{k}\right) g_{\tilde{a}_{1} \ldots \tilde{a}_{k} ; b_{1} \ldots b_{l}}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right)\right\}, \tag{4.12}
\end{align*}
$$

which is nothing but the residue axiom for the first set of particles where the quantum numbers of the particles belonging to the second set are dummy parameters.

Similarly we find that

$$
\begin{equation*}
h_{k ; l+2}(\Delta)=h_{k ; l}(\Delta) \tag{4.13}
\end{equation*}
$$

and the axioms (3.3)-(3.6) are satisfied in the variables corresponding to the second set, for fixed $\left\{a_{1} \ldots a_{k}\right\},\left\{\alpha_{1}, \ldots, \alpha_{k}\right\}$.

Using the recursion relations (4.11) and (4.13) we see that there are three clustering families, the cases of odd-odd, even-odd and even-even clustering.

The function $g_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}$ is essentially a product of two scalarized form factors, corresponding to the operators $B$ and $C$. Denoting the original operator by $A$, the clustering relations can be symbolically represented as

$$
\begin{equation*}
A \sim h(\Delta) B \bullet C \tag{4.14}
\end{equation*}
$$

or, since in the $\mathrm{O}(n)$ model $h(\Delta)$ is always a power, more explicitly as

$$
\begin{equation*}
A \sim \frac{1}{\Delta^{\kappa}} B \bullet C \tag{4.15}
\end{equation*}
$$

The residue axiom is applicable for clustering for $k \geqslant 3$ only but from [4] we see that for $\mathrm{O}(n)$ non-singlets

$$
\begin{equation*}
h_{2 ; l}(\Delta)=\frac{1}{\Delta} \tag{4.16}
\end{equation*}
$$

since

$$
\begin{equation*}
F_{a b ; b_{1} \ldots b_{l}}^{A}\left(\Delta+i \pi+\epsilon, \Delta, \beta_{1}, \ldots, \beta_{l}\right) \cong \frac{4 \pi \chi}{\epsilon \Delta} t_{A B}^{a b} F_{b_{1} \ldots b_{l}}^{B}\left(\beta_{1}, \ldots, \beta_{l}\right) \tag{4.17}
\end{equation*}
$$

where $t_{A B}^{a b}$ are $\mathrm{O}(n)$ generators in the representation under consideration. Similarly

$$
\begin{equation*}
h_{k ; 2}(\Delta)=\frac{1}{\Delta} \tag{4.18}
\end{equation*}
$$

and

$$
\begin{equation*}
g_{a_{1} \ldots a_{k} ; a b}^{A}\left(\alpha_{1}, \ldots, \alpha_{k} ; \gamma+i \pi+\epsilon, \gamma\right) \cong-\frac{4 \pi \chi}{\epsilon} t_{A B}^{a b} f_{a_{1} \ldots a_{k}}^{B}\left(\alpha_{1}, \ldots, \alpha_{k}\right) \tag{4.19}
\end{equation*}
$$

For $k$ odd, using the uniqueness of the solution, we can solve the problem step by step, starting from the $k=1$ case:

$$
\begin{equation*}
g_{a ; b_{1} \ldots b_{l}}\left(\alpha ; \beta_{1}, \ldots, \beta_{l}\right)=G_{a b_{1} \ldots b_{l}}\left(\beta_{1}, \ldots, \beta_{l}\right) \tag{4.20}
\end{equation*}
$$

Then

$$
\begin{equation*}
g_{a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right)=f_{a_{1} \ldots a_{k}}^{a}\left(\alpha_{1}, \ldots, \alpha_{k}\right) G_{a b_{1} \ldots b_{l}}\left(\beta_{1}, \ldots, \beta_{l}\right) \tag{4.21}
\end{equation*}
$$

where $f_{a_{1} \ldots a_{k}}^{a}\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ is the form factor of the basic field $\Phi^{a}$.
Similarly for $l$ odd

$$
\begin{equation*}
g_{a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right)=H_{a_{1} \ldots a_{k} b}\left(\alpha_{1}, \ldots, \alpha_{k}\right) f_{b_{1} \ldots b_{l}}^{b}\left(\beta_{1}, \ldots, \beta_{l}\right) \tag{4.22}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{a_{1} \ldots a_{k} ; b}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta\right)=H_{a_{1} \ldots a_{k} b}\left(\alpha_{1}, \ldots, \alpha_{k}\right) \tag{4.23}
\end{equation*}
$$

### 4.4. Odd-odd clustering

The simplest case is the odd-odd clustering. Starting from

$$
\begin{equation*}
g_{a ; b}(\alpha ; \beta)=T_{a b} \tag{4.24}
\end{equation*}
$$

we can build

$$
\begin{equation*}
g_{a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right)=T_{a b} f_{a_{1} \ldots a_{k}}^{a}\left(\alpha_{1}, \ldots, \alpha_{k}\right) f_{b_{1} \ldots b_{l}}^{b}\left(\beta_{1}, \ldots, \beta_{l}\right) \tag{4.25}
\end{equation*}
$$

In the case of the current operator $J^{c d}$

$$
\begin{equation*}
h(\Delta)=c_{1} \Delta^{-\chi} \tag{4.26}
\end{equation*}
$$

and

$$
\begin{equation*}
T_{a b} \rightarrow-t_{a b}^{c d} \tag{4.27}
\end{equation*}
$$

Similarly for the symmetric tensor $\Sigma^{c d}$

$$
\begin{equation*}
h(\Delta)=c_{2} \Delta^{\chi} \tag{4.28}
\end{equation*}
$$

and

$$
\begin{equation*}
T_{a b} \rightarrow-i \tilde{t}_{a b}^{c d} \tag{4.29}
\end{equation*}
$$

Finally for the energy-momentum tensor $T$ we have

$$
\begin{equation*}
h(\Delta)=c_{0} \Delta^{-(1+x)} \tag{4.30}
\end{equation*}
$$

and

$$
\begin{equation*}
T_{a b} \rightarrow-\frac{i}{2} \delta_{a b} \tag{4.31}
\end{equation*}
$$

We can represent the above clustering relations symbolically as

$$
\begin{align*}
& J^{a b} \sim \frac{1}{\Delta \chi} t_{c d}^{a b} \Phi^{c} \bullet \Phi^{d} \\
& \Sigma^{a b} \sim \Delta^{\chi} \tilde{t}_{c d}^{a b} \Phi^{c} \bullet \Phi^{d} \\
& T \sim \frac{1}{\Delta^{1+\chi}} \Phi^{a} \bullet \Phi^{a} \tag{4.32}
\end{align*}
$$

## 4.5. $3 \rightarrow 2+1$ clustering

This is the simplest case of even-odd clustering. In the odd-odd case we could afford the luxury of using the exact solution of the 2-particle form factors to obtain the $2 \rightarrow 1+1$ clustering relations. The exact 3-particle form factor is not known for general $n$ (except for $n=3,4$ and in the large $n$ limit), but the form factor equations can be solved in the clustering limit. We start from the representation

$$
\begin{equation*}
f_{a b c}^{x}(\Delta, \beta, \gamma)=\left(\Delta-\frac{\beta+\gamma}{2}\right)^{-\kappa}\left\{A_{a b c}^{x}(\beta-\gamma)+\frac{\tilde{A}_{a b c}^{x}(\beta-\gamma)}{\Delta-\frac{\beta+\gamma}{2}}+\cdots\right\} \tag{4.33}
\end{equation*}
$$

The functional form of the ansatz (4.33) is motivated by the following considerations. First of all, it is easy to show [8] using the known short distance asymptotics of the 2-point function and the spectral representation that the spin form factor must not grow faster than any power of the momenta, i.e. it is smaller than $\mathrm{e}^{\epsilon \Delta}$ for any $\epsilon$. This observation, the known behavior of the 2-particle form factors and the fact that the theory is asymptotically free, taken together make plausible that for large momenta also the 3-particle (and higher) form factors vary logarithmically.

Now we impose the form factor axioms in the large $\Delta$ limit. This gives for the leading coefficient $A_{a b c}^{x}$ the equations

$$
\begin{equation*}
A_{a b c}^{x}(2 \pi i-\theta)=A_{a c b}^{x}(\theta) \tag{4.34}
\end{equation*}
$$

and

$$
\begin{equation*}
A_{a b c}^{x}(\theta)=S_{b c}^{v u}(\theta) A_{a u v}^{x}(-\theta), \tag{4.35}
\end{equation*}
$$

which means that, as expected, $A_{a b c}^{x}$ satisfies the 2-particle form factor equations in the last two variables. The general solution is

$$
\begin{equation*}
A_{a b c}^{x}(\theta)=k_{0} \psi_{0}(\theta) \delta_{a}^{x} \delta_{b c}+k_{1} \psi_{1}(\theta) t_{b c}^{x a}+k_{2} \psi_{2}(\theta) \tilde{t}_{b c}^{x a} \tag{4.36}
\end{equation*}
$$

where the $k_{I}$ are constants. In the $\kappa=1$ case we also have to satisfy the residue axiom

$$
\begin{equation*}
A_{c a b}^{x}(i \pi+\epsilon) \cong-\frac{4 \pi \chi}{\epsilon} t_{x c}^{a b}, \tag{4.37}
\end{equation*}
$$

which gives $k_{1}=-2 \pi \chi$.
At the next order we find that $\tilde{A}_{a b c}^{x}$ also satisfies (4.35) but instead of (4.34) we have in this case

$$
\begin{equation*}
\tilde{A}_{a b c}^{x}(2 \pi i-\theta)-\tilde{A}_{a c b}^{x}(\theta)=-i \pi \kappa A_{a c b}^{x}(\theta)+2 \pi i \chi t_{u b}^{v a} A_{v c u}^{x}(\theta) \tag{4.38}
\end{equation*}
$$

Let us define

$$
\begin{equation*}
K_{a b c}^{x}(\theta) \equiv \tilde{A}_{a b c}^{x}(2 \pi i-\theta)-\tilde{A}_{a c b}^{x}(\theta) \tag{4.39}
\end{equation*}
$$

Obviously,

$$
\begin{equation*}
K_{a c b}^{x}(2 \pi i-\theta)+K_{a b c}^{x}(\theta)=0 \tag{4.40}
\end{equation*}
$$

There is no solution for $\tilde{A}_{a b c}^{x}$ unless the right-hand side of (4.38) is also antisymmetric under this operation. Thus the consistency of the next-to-leading order gives an additional condition on the leading order:

$$
\begin{equation*}
\chi t_{u a}^{v b} A_{v u c}^{x}+\chi t_{u c}^{v b} A_{v a u}^{x}-\kappa A_{b a c}^{x}=0 \tag{4.41}
\end{equation*}
$$

This purely algebraic relation restricts the possible solutions (4.36) so that only one of the coefficients $k_{I}$ can be different from zero and fixes the exponent $\kappa$ as follows:

$$
\begin{array}{ll}
I=0 \text { case }\left(k_{0} \neq 0\right): & \kappa=0 \\
I=1 \text { case }\left(k_{1} \neq 0\right): & \kappa=1 \\
I=2 \text { case }\left(k_{2} \neq 0\right): & \kappa=1+2 \chi \tag{4.42}
\end{array}
$$

Furthermore, the residue conditions give additional restrictions. First of all, since $\psi_{0}(\theta)$ has a double pole at $\theta=i \pi$, this excludes the $k_{0} \neq 0(\kappa=0)$ solution. In the $I=1(\kappa=1)$ case the residue conditions fix $k_{1}$ as given above. Finally, since $\psi_{2}(\theta)$ is regular, the coefficient $k_{2}$ cannot be determined by this method.

Putting everything together, we get for the $3 \rightarrow 2+1$ clustering the following result:

$$
\begin{equation*}
f_{a b c}^{x}(\Delta, \beta, \gamma) \cong-\frac{2 \pi \chi}{\Delta} \psi_{1}(\beta-\gamma) t_{b c}^{x a}+i \chi H_{2}\left(\frac{2 \pi}{\Delta}\right)^{1+2 \chi} \psi_{2}(\beta-\gamma) \tilde{t}_{b c}^{x a}+\cdots \tag{4.43}
\end{equation*}
$$

Here we have reparameterized the constant $k_{2}$ (in terms of the new constant $H_{2}$ ) for later convenience.
(4.43) can also be written in terms of the full 2-particle form factors as

$$
\begin{equation*}
f_{a b c}^{x}(\Delta, \beta, \gamma) \cong \frac{2 \pi \chi}{\Delta} f_{b c}^{x a}(\beta, \gamma)-\chi H_{2}\left(\frac{2 \pi}{\Delta}\right)^{1+2 \chi} \tilde{f}_{b c}^{x a}(\beta, \gamma)+\cdots \tag{4.44}
\end{equation*}
$$

Note that the second piece in the clustering formula (4.44) is always subleading to the first. Actually, it makes sense only for $n>4$, since for $n=3,4$ it is not dominant over the $1 / \Delta^{2}$ correction corresponding to the first term. On the other hand, the two terms are close for larger $n$ values and they become degenerate in the large $n$ limit. In Section 7 we check (4.44) in the large $n$ expansion.

Now we calculate these $1 / \Delta^{2}$ correction terms in the large rapidity expansion. We take $\kappa=1$ and get from (4.33)

$$
\begin{equation*}
f_{a b c}^{x}(\Delta, \beta, \gamma) \approx \frac{1}{\Delta-\frac{\beta+\gamma}{2}}\left\{A_{a b c}^{x}(\xi)+\frac{1}{\Delta} \tilde{A}_{a b c}^{x}(\xi)+\cdots\right\} \tag{4.45}
\end{equation*}
$$

where $\xi=\beta-\gamma$. The leading term is:

$$
\begin{equation*}
A_{a b c}^{x}(\xi)=-2 \pi \chi \psi_{1}(\xi) t_{b c}^{x a} . \tag{4.46}
\end{equation*}
$$

To calculate the next term we first write

$$
\begin{equation*}
\tilde{A}_{a b c}^{x}(\xi)=2 i \pi^{2} \chi \psi_{1}(\xi)\left\{\lambda_{1}(\xi) \delta_{a}^{x} \delta_{b c}+\lambda_{2}(\xi) \delta_{b}^{x} \delta_{a c}+\lambda_{3}(\xi) \delta_{c}^{x} \delta_{a b}\right\} \tag{4.47}
\end{equation*}
$$

We also introduce

$$
\begin{equation*}
\lambda_{ \pm}=\lambda_{2} \pm \lambda_{3}, \quad \bar{\lambda}=n \lambda_{1}+\lambda_{+} . \tag{4.48}
\end{equation*}
$$

We now rewrite (4.38) and (4.35) for $\tilde{A}_{a b c}^{x}$ in terms of these variables and get

$$
\begin{array}{ll}
\lambda_{+}(\xi)=\frac{2 \pi i \chi-\xi}{2 \pi i \chi+\xi} \lambda_{+}(-\xi), & \lambda_{+}(2 \pi i-\xi)+\lambda_{+}(\xi)=2 \\
\lambda_{-}(\xi)=\lambda_{-}(-\xi), & \lambda_{-}(2 \pi i-\xi)-\lambda_{-}(\xi)=0 \\
\bar{\lambda}(\xi)=\frac{i \pi+\xi}{i \pi-\xi} \bar{\lambda}(-\xi), & \bar{\lambda}(2 \pi i-\xi)+\bar{\lambda}(\xi)=4(1+\chi) \tag{4.49}
\end{array}
$$

From the residue equations we get

$$
\begin{equation*}
\lambda_{+}(i \pi)=1, \quad \lambda_{-}(i \pi)=-2, \quad \bar{\lambda}(i \pi)=2(1+\chi) . \tag{4.50}
\end{equation*}
$$

This results from expanding the residue equation

$$
\begin{equation*}
f_{a b c}^{x}(\alpha, \beta, \Delta) \approx \frac{2 i}{\xi-i \pi}\left\{\delta_{a b} \delta_{c}^{x}-S_{b c}^{a x}(\beta-\Delta)\right\} \tag{4.51}
\end{equation*}
$$

for large $\Delta$ using the expansion

$$
\begin{equation*}
S_{a b}^{x y}(\Delta)=\delta_{a}^{x} \delta_{b}^{y}+\frac{2 \pi i \chi}{\Delta} t_{y b}^{x a}-\frac{2 \pi^{2} \chi^{2}}{\Delta^{2}} t_{v b}^{u a} t_{y v}^{x u}+\cdots \tag{4.52}
\end{equation*}
$$

It is easy to find the general solution of (4.49) and (4.50). We get

$$
\begin{equation*}
\lambda_{-}(\xi)=\ell_{-}(\cosh \xi), \quad \ell_{-}(-1)=-2 \tag{4.53}
\end{equation*}
$$

Since we know that $\lambda_{-}(\xi)$ must not grow exponentially for large $\xi$ and it should be regular we conclude that

$$
\begin{equation*}
\lambda_{-}(\xi)=-2 . \tag{4.54}
\end{equation*}
$$

We also get

$$
\begin{equation*}
\bar{\lambda}(\xi)=\frac{\bar{\ell}(\cosh \xi)}{i \pi-\xi}-\frac{i(1+\chi)}{\pi}(i \pi+\xi) ; \quad \bar{\ell}(-1)=0 \tag{4.55}
\end{equation*}
$$

and requiring regularity here gives

$$
\begin{equation*}
\bar{\lambda}(\xi)=-\frac{i(1+\chi)}{\pi}(i \pi+\xi) \tag{4.56}
\end{equation*}
$$

Similarly we can find the regular solution of the $\lambda_{+}$equations:

$$
\begin{equation*}
\lambda_{+}(\xi)=i \frac{2 \pi i \chi-\xi}{\pi(1-2 \chi)}+\omega(\chi) \cosh \frac{\xi}{2} \mathrm{e}^{-\Delta_{2 \chi}(\xi)} \tag{4.57}
\end{equation*}
$$

where regularity at infinity requires that $\omega(\chi)$ is constant. (4.57) is only valid for $n \neq 4$, because for $n=4$ it becomes singular. For $n=4$ the solution of the $\lambda_{+}$equations is

$$
\begin{equation*}
\lambda_{+}(\xi)=\frac{2 i}{\pi}(i \pi-\xi)\left\{g_{0}+\frac{1}{4} \Psi\left(\frac{1}{2}+\frac{\xi}{2 \pi i}\right)+\frac{1}{4} \Psi\left(\frac{1}{2}-\frac{\xi}{2 \pi i}\right)\right\} \tag{4.58}
\end{equation*}
$$

where, again, regularity requires that $g_{0}$ cannot depend on the relative rapidity $\xi$.
$\omega(\chi)$ can be calculated for $\chi=1$ and $\chi=0$ (corresponding to $n=3$ and $n=\infty$ respectively) from the known solutions or for general $n \neq 4$ from the consistency of the next-to-next-to-leading order equations in the large rapidity expansion. Consistency would not fix the value of $g_{0}$ for $n=4$, and, as we will see later, it is actually not a constant, but depends (linearly) on $\log \Delta$. In the light of the $1 /(n-4)$ singularity in (4.57) the presence of the logarithmic term in (4.58) is not surprising. For $n=4$ (and only in this case) the $1 / \Delta^{2}$ subleading term is accompanied by a $\log (\Delta) / \Delta^{2}$ term. One can see that this is consistent here, because the term containing $g_{0}$ is exactly the same as the one corresponding to the constant $H_{2}$, the coefficient of the $\kappa=1+2 \chi$ term in (4.43). The only way to determine $g_{0}$ is to solve the full $\mathrm{O}(4)$ form factor equations explicitly. We will consider this problem in Section 8 and Appendix D.

### 4.6. General clustering

What we can learn from the three-particle example is that it is useful to include in (4.6) some of the subleading terms as well. Thus we have to allow for the occurrence of several terms (labeled by an index $\rho$ ) of the form

$$
\begin{align*}
& f_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \quad \cong \sum_{\rho} h_{k ; l}^{(\rho)}(\Delta) g_{a_{1} \ldots a_{k} ; b_{1} \ldots b_{l}}^{(\rho)}\left(\alpha_{1}, \ldots, \alpha_{k} ; \beta_{1}, \ldots, \beta_{l}\right) \tag{4.59}
\end{align*}
$$

where, as before, the coefficient functions $g^{(\rho)}$ have to solve the form factor equations for both sets of variable.

In the case of odd-odd clustering the sum contains only one term so the results in Section 4.4 do not change. But the general even-odd clustering formula ( $k$ odd, $l$ even) consists of two terms:

$$
\begin{align*}
& f_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}^{x}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \begin{array}{l}
\frac{2 \pi \chi}{\Delta} f_{a_{1} \ldots a_{k}}^{u}\left(\alpha_{1}, \ldots, \alpha_{k}\right) f_{b_{1} \ldots b_{l}}^{x u}\left(\beta_{1}, \ldots, \beta_{l}\right) \\
\quad-\chi H_{2}\left(\frac{2 \pi}{\Delta}\right)^{1+2 \chi} f_{a_{1} \ldots a_{k}}^{u}\left(\alpha_{1}, \ldots, \alpha_{k}\right) \tilde{f}_{b_{1} \ldots b_{l}}^{x u}\left(\beta_{1}, \ldots, \beta_{l}\right)+\cdots .
\end{array}
\end{align*}
$$

This can be symbolically represented as

$$
\begin{equation*}
\Phi^{x} \sim \frac{1}{\Delta} \Phi^{u} \bullet J^{x u}+\frac{1}{\Delta^{1+2 \chi}} \Phi^{u} \bullet \Sigma^{x u} \tag{4.61}
\end{equation*}
$$

### 4.7. A conjecture

We have seen that clustering relations can be represented in the form

$$
\begin{equation*}
A \sim \frac{1}{\Delta^{\kappa}} B \bullet C \tag{4.62}
\end{equation*}
$$

or as a sum of similar terms on the right-hand side. We found that the value of the exponent $\kappa$ depends on the anomalous dimensions of the operators involved. In all cases we studied so far we have the relation

$$
\begin{equation*}
\kappa=d_{B}+d_{C}-d_{A}(\bmod 1) \tag{4.63}
\end{equation*}
$$

where

$$
\begin{equation*}
d_{O}=\frac{\gamma_{O}}{2 \beta_{0}} \tag{4.64}
\end{equation*}
$$

Here $\gamma_{O}$ is the coefficient of the first term (in perturbation theory) of the anomalous dimension of the operator $O$ and $\beta_{0}$ is the coefficient of the first term of the perturbative $\beta$-function.

Explicitly,

$$
\begin{align*}
& d_{\Phi}=\frac{1}{2} \frac{\gamma_{0}}{2 \beta_{0}}=\frac{1}{2}(1+\chi), \\
& d_{J}=d_{T}=0, \\
& d_{\Sigma}=\frac{\gamma_{\Sigma}}{2 \beta_{0}}=2 \chi . \tag{4.65}
\end{align*}
$$

Here we used the results

$$
\begin{equation*}
\gamma_{0}=\frac{n-1}{2 \pi}, \quad \beta_{0}=\frac{n-2}{4 \pi}, \quad \gamma_{\Sigma}=\frac{1}{\pi} . \tag{4.66}
\end{equation*}
$$

Using this conjecture, it is possible to write down the formula for even-even clustering without doing any calculation. For the current operator we get

$$
\begin{align*}
& f_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}^{x y}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \qquad \cong \frac{2 \pi \chi}{\Delta}\left[f_{a_{1} \ldots a_{k}}^{x q}(\alpha) f_{b_{1} \ldots b_{l}}^{y q}(\beta)-f_{a_{1} \ldots a_{k}}^{y q}(\alpha) f_{b_{1} \ldots b_{l}}^{x q}(\beta)\right] \\
& \quad+\Omega \chi\left(\frac{2 \pi}{\Delta}\right)^{1+4 \chi}\left[\tilde{f}_{a_{1} \ldots a_{k}}^{x q}(\alpha) \tilde{f}_{b_{1} \ldots b_{l}}^{y q}(\beta)-\tilde{f}_{a_{1} \ldots a_{k}}^{y q}(\alpha) \tilde{f}_{b_{1} \ldots b_{l}}^{x q}(\beta)\right] . \tag{4.67}
\end{align*}
$$

Here we have used (beyond $\mathrm{O}(n)$ symmetry) the fact that isospin 0 form factors cannot occur here (they would give a double pole in the residue axiom) and that the residue axiom fixes the coefficient of the $1 / \Delta$ term containing the current form factors. The power $1+4 \chi$ is consistent with the conjecture and is uniquely fixed by the requirement that the two terms should be degenerate in the large $n$ limit. The constant $\Omega$ is not fixed by these considerations, but by studying the $k=l=2$ case in the large $n$ limit we can show that

$$
\begin{equation*}
\Omega=1+\mathrm{O}\left(\frac{1}{n}\right) \tag{4.68}
\end{equation*}
$$

Analogously for the tensor form factor we get

$$
\begin{align*}
& \tilde{f}_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}^{x y}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \cong-\frac{2 \pi \chi}{\Delta}\left[f_{a_{1} \ldots a_{k}}^{x q}(\alpha) \tilde{f}_{b_{1} \ldots b_{l}}^{y q}(\beta)+f_{a_{1} \ldots a_{k}}^{y q}(\alpha) \tilde{f}_{b_{1} \ldots b_{l}}^{x q}(\beta)\right] \\
&+\frac{2 \pi \chi}{\Delta}\left[\tilde{f}_{a_{1} \ldots a_{k}}^{x q}(\alpha) f_{b_{1} \ldots b_{l}}^{y q}(\beta)+\tilde{f}_{a_{1} \ldots a_{k}}^{y q}(\alpha) f_{b_{1} \ldots b_{l}}^{x q}(\beta)\right] \\
&+\tilde{\Omega} \chi\left(\frac{2 \pi}{\Delta}\right)^{1+2 \chi}\left[\tilde{f}_{a_{1} \ldots a_{k}}^{x q}(\alpha) \tilde{f}_{b_{1} \ldots b_{l}}^{y q}(\beta)+\tilde{f}_{a_{1} \ldots a_{k}}^{y q}(\alpha) \tilde{f}_{b_{1} \ldots b_{l}}^{x q}(\beta)\right. \\
&\left.-\frac{2}{n} \delta^{x y} \tilde{f}_{a_{1} \ldots a_{k}}^{p q}(\alpha) \tilde{f}_{b_{1} \ldots b_{l}}^{p q}(\beta)\right], \tag{4.69}
\end{align*}
$$

where

$$
\begin{equation*}
\tilde{\Omega}=-2+\mathrm{O}\left(\frac{1}{n}\right) \tag{4.70}
\end{equation*}
$$

## 5. Bootstrap form factors in leading orders $\mathbf{1} / \boldsymbol{n}$ expansion

In this section we consider the solution of the form factor equations in leading order $1 / n$ expansion. We start with the $1 / n$ expansion of the $S$-matrix and 2-particle form factors.

### 5.1. S-matrix and $1 / n$ expansion

The $1 / n$ expansion of the coefficients in (2.1) is of the form

$$
\begin{equation*}
\sigma_{i}(\theta)=\delta_{i 2}+\frac{a_{i}(\theta)}{n}+\frac{b_{i}(\theta)}{n^{2}}+\mathrm{O}\left(\frac{1}{n^{3}}\right), \quad i=1,2,3, \tag{5.1}
\end{equation*}
$$

with

$$
\begin{align*}
& a_{1}(\theta)=-\frac{2 \pi i}{i \pi-\theta}, \quad a_{2}(\theta)=-\frac{2 \pi i}{\sinh \theta}, \quad a_{3}(\theta)=-\frac{2 \pi i}{\theta}  \tag{5.2}\\
& b_{1}(\theta)=\frac{-4 \pi}{i \pi-\theta}\left(i+\frac{\pi}{\sinh \theta}\right), \quad b_{3}(\theta)=\frac{-4 \pi}{\theta}\left(i+\frac{\pi}{\sinh \theta}\right), \tag{5.3}
\end{align*}
$$

and $\left(\Psi(z)=\Gamma^{\prime}(z) / \Gamma(z)\right)$

$$
\begin{align*}
b_{2}(\theta)= & -\frac{4 \pi i}{\sinh \theta}-\frac{2 \pi^{2}}{\sinh ^{2} \theta}+\frac{1}{2}\left[\Psi^{\prime}\left(\frac{1}{2}+\frac{i \theta}{2 \pi}\right)-\Psi^{\prime}\left(\frac{1}{2}-\frac{i \theta}{2 \pi}\right)\right. \\
& \left.-\Psi^{\prime}\left(1+\frac{i \theta}{2 \pi}\right)+\Psi^{\prime}\left(-\frac{i \theta}{2 \pi}\right)\right] . \tag{5.4}
\end{align*}
$$

Later we will need the combination

$$
\begin{equation*}
b_{1}(\theta)+a_{2}(\theta)+a_{3}(\theta)=\frac{\theta+i \pi}{\theta-i \pi}\left(\frac{2 \pi i}{\theta}-\frac{2 \pi i}{\sinh \theta}\right) \tag{5.5}
\end{equation*}
$$

### 5.2. Large $n$ expansion of the two-particle form factors

The large $n$ expansion of the two-particle form factors is given by the expansion of the functions $\psi_{I}$ in (3.17):

$$
\begin{align*}
& \psi_{0}(\theta)=\frac{2 i \tanh \frac{\theta}{2}}{i \pi-\theta}\left\{1+\frac{2 \pi}{n}[a(\theta)+b(\theta)]+\mathrm{O}\left(\frac{1}{n^{2}}\right)\right\}, \\
& \psi_{1}(\theta)=\tanh \frac{\theta}{2}\left\{1+\frac{2 \pi}{n}[a(\theta)+b(\theta)]+\mathrm{O}\left(\frac{1}{n^{2}}\right)\right\} \\
& \psi_{2}(\theta)=i\left\{1+\frac{2 \pi}{n}[a(\theta)-b(\theta)]+\mathrm{O}\left(\frac{1}{n^{2}}\right)\right\}, \tag{5.6}
\end{align*}
$$

where

$$
\begin{equation*}
a(\theta)=\frac{1}{2 \pi}+\frac{\theta-i \pi}{2 \pi \sinh \theta} \tag{5.7}
\end{equation*}
$$

and

$$
\begin{equation*}
b(\theta)=\frac{i}{2 \theta}-\frac{1}{4 \pi}\left[\Psi\left(\frac{i \theta}{2 \pi}\right)+\Psi\left(-\frac{i \theta}{2 \pi}\right)-2 \Psi\left(\frac{1}{2}\right)\right] . \tag{5.8}
\end{equation*}
$$

### 5.3. Large $n$ expansion of the spin 3-particle form factor

We assume that for large $n$ the function $X$ appearing in (3.24) has an expansion of the form:

$$
\begin{equation*}
X(u, v)=\frac{f(u, v)}{n}+\frac{g(u, v)}{n^{2}}+\mathrm{O}\left(\frac{1}{n^{3}}\right) . \tag{5.9}
\end{equation*}
$$

The form factor equations (3.25) have to be satisfied order by order in the expansion. On the other hand, (3.26) and (3.27) mix the expansion coefficients (beyond leading order). At leading order they lead to

$$
\begin{align*}
& f(u, v)=\left[1+a_{1}(v-u)\right] f(v, u)  \tag{5.10}\\
& f(u, v)=f(v, 2 \pi i+u) \tag{5.11}
\end{align*}
$$

and at next-to-leading order we have

$$
\begin{align*}
g(u, v)= & {\left[1+a_{1}(v-u)\right] g(v, u) } \\
& +\left[b_{1}(v-u)+a_{2}(v-u)+a_{3}(v-u)\right] f(v, u) \\
& +a_{1}(v-u)[f(u-v, 2 \pi i-v)+f(2 \pi i-u, 2 \pi i+v-u)] \tag{5.12}
\end{align*}
$$

and

$$
\begin{align*}
g(v-u, 2 \pi i-u)= & g(2 \pi i-u, 2 \pi i+v-u)+a_{2}(v-u) f(2 \pi i-u, 2 \pi i+v-u) \\
& +a_{3}(v-u) f(u-v, 2 \pi i-v) . \tag{5.13}
\end{align*}
$$

### 5.3.1. Leading order solution

We here summarize the equations the leading order form factor $f(u, v)$ has to satisfy.

$$
\begin{align*}
& f(u, v)=f(2 \pi i-v, 2 \pi i-u)  \tag{5.14}\\
& f(\theta, i \pi)=\frac{i}{2} \sinh \theta a_{3}(\theta)=\frac{\pi \sinh \theta}{\theta}  \tag{5.15}\\
& f(\theta, i \pi+\theta)=\frac{i}{2} \sinh \theta a_{2}(\theta)=\pi  \tag{5.16}\\
& f(u, v)=\frac{v-u+i \pi}{v-u-i \pi} f(v, u)  \tag{5.17}\\
& f(u, v)=f(v, 2 \pi i+u) \tag{5.18}
\end{align*}
$$

Defining the function

$$
\begin{equation*}
R(\theta) \equiv \frac{\pi \sinh \theta}{i \pi-\theta} \tag{5.19}
\end{equation*}
$$

with properties

$$
\begin{equation*}
R(i \pi-\theta)=\frac{\pi \sinh \theta}{\theta}, \quad R(i \pi)=\pi, \quad R(2 \pi i-\theta)=R(\theta) \tag{5.20}
\end{equation*}
$$

we now take the ansatz

$$
\begin{equation*}
f(u, v)=R(v-u)[s(u, v)+1] \tag{5.21}
\end{equation*}
$$

and verify that (5.14)-(5.18) require

$$
\begin{align*}
& s(u, v)=s(v, u)=s(2 \pi i+u, v)=s(-u,-v) \\
& s(\theta, i \pi)=s(\theta, i \pi+\theta)=0 \tag{5.22}
\end{align*}
$$

It is easy to see that regularity and boundedness at infinity allows the trivial solution $s(u, v)=0$ only leading to the unique leading order solution

$$
\begin{equation*}
f(u, v)=R(v-u) \tag{5.23}
\end{equation*}
$$

### 5.3.2. Next-to-leading order solution

Using the leading order solution (5.23) the form factor equations (5.12) and (5.13) can be simplified a little. We list here the complete set of next-to-leading order (NLO) form factor equations after this simplification.

$$
\begin{align*}
g(u, v)= & {\left[1+a_{1}(v-u)\right] g(v, u) } \\
& +\left[b_{1}(v-u)+a_{2}(v-u)+a_{3}(v-u)\right] R(u-v) \\
& +a_{1}(v-u)[R(u)+R(v)],  \tag{5.24}\\
g(u, v)= & g(v, 2 \pi i+u)+a_{2}(u) R(v-u)+a_{3}(u) R(v), \tag{5.25}
\end{align*}
$$

$$
\begin{align*}
& g(u, v)=g(2 \pi i-v, 2 \pi i-u),  \tag{5.26}\\
& g(\theta, i \pi)=\frac{i}{2} \sinh \theta b_{3}(\theta),  \tag{5.27}\\
& g(\theta, i \pi+\theta)=\frac{i}{2} \sinh \theta b_{2}(\theta) . \tag{5.28}
\end{align*}
$$

We present the NLO solution in several steps in order to make the checking of Eqs. (5.24)(5.28) easier. We start with

$$
\begin{align*}
g(u, v)= & \left\{G(u, v)+\frac{i \pi}{v-u}-\frac{i \pi}{\sinh (v-u)}-\frac{\pi}{R(u)}-\frac{\pi}{R(v)}\right\} R(v-u) \\
& -R(u)-R(v) . \tag{5.29}
\end{align*}
$$

It is easy to show that (5.24) is satisfied if $G$ satisfies

$$
\begin{equation*}
G(u, v)=G(v, u) \tag{5.30}
\end{equation*}
$$

Next we write

$$
\begin{align*}
G(u, v)= & S(u, v)+\frac{v-u}{\sinh (v-u)}-k(u-v)-k(v-u) \\
& +\frac{\sinh v}{\sinh (v-u)}\left\{\frac{u-i \pi}{i \pi-v}+2 k(u)-k(v+i \pi)-k(v-i \pi)\right\} \\
& -\frac{\sinh u}{\sinh (v-u)}\left\{\frac{v-i \pi}{i \pi-u}+2 k(v)-k(u+i \pi)-k(u-i \pi)\right\} . \tag{5.31}
\end{align*}
$$

Here

$$
\begin{equation*}
k(\theta)=\frac{1}{2} \Psi\left(-\frac{i \theta}{2 \pi}\right) . \tag{5.32}
\end{equation*}
$$

Eqs. (5.24) and (5.25) are satisfied if

$$
\begin{equation*}
S(u, v)=S(v, u)=S(u+2 \pi i, v) \tag{5.33}
\end{equation*}
$$

In the next step we write

$$
\begin{align*}
S(u, v)= & \Sigma(u, v)+\frac{i \pi}{2 \sinh (v-u)}\left\{\cosh u-\cosh v+\frac{\sinh v(1+\cosh u)}{\sinh u}\right. \\
& \left.-\frac{\sinh u(1+\cosh v)}{\sinh v}\right\} . \tag{5.34}
\end{align*}
$$

In addition to (5.24) and (5.25), (5.26) is also satisfied if

$$
\begin{equation*}
\Sigma(u, v)=\Sigma(v, u)=\Sigma(u+2 \pi i, v)=\Sigma(-u,-v) \tag{5.35}
\end{equation*}
$$

Finally in the last step we represent $\Sigma$ as

$$
\begin{equation*}
\Sigma(u, v)=\frac{\cosh u+\cosh v}{1+\cosh (u-v)}+3+\Psi\left(\frac{1}{2}\right)+\sigma(u, v) \tag{5.36}
\end{equation*}
$$

It now follows that all Eqs. (5.24)-(5.28) are satisfied if $\sigma(u, v)$ satisfies the same equations as $s(u, v)$ in (5.22).

Although the form factors are bounded and regular functions for all $n$, the large $n$ expansion, as can be seen from (5.2), introduces some singularities at rapidity differences equal to 0 or $i \pi$.

Nevertheless, one can show that (5.22) has only trivial solution for $s(u, v)$ even if one allows (first order) poles at these special rapidity differences. Thus $\sigma(u, v)=0$ and the NLO solution is unique.

## 5.4. $n=\infty$ form factor equations

In this section we write down the form factor equations in the leading order of the large $n$ expansion. In this limit the homogeneous equations take the form

$$
\begin{align*}
& F_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)=F_{a_{1} \ldots a_{r}}\left(\theta_{1}+\lambda, \ldots, \theta_{r}+\lambda\right),  \tag{5.37}\\
& F_{\ldots x y \ldots}\left(\cdots, \theta^{\prime} \cdots\right)=\frac{1}{n} a_{1}\left(\theta-\theta^{\prime}\right) F_{\ldots z z \ldots}\left(\cdots \theta^{\prime}, \theta \cdots\right) \delta_{x y}+F_{\ldots y x \ldots}\left(\cdots \theta^{\prime}, \theta \cdots\right),  \tag{5.38}\\
& F_{a_{1} a_{2} \ldots a_{r}}\left(\theta_{1}+2 \pi i, \theta_{2}, \ldots, \theta_{r}\right)=(-1)^{r-1} F_{a_{2} \ldots a_{r} a_{1}}\left(\theta_{2}, \ldots, \theta_{r}, \theta_{1}\right),  \tag{5.39}\\
& F_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)=w_{p} F_{a_{r} \ldots a_{1}}\left(-\theta_{r}, \ldots,-\theta_{1}\right) . \tag{5.40}
\end{align*}
$$

In (5.38) the first term on the right-hand side is of $\mathrm{O}(1)$ if the contracted indices belong to the same Kronecker delta. Otherwise it is of order $1 / n$ and can be dropped.

To calculate the residue equation to leading order we first note the recursion relation

$$
\begin{equation*}
S_{b a_{1} \ldots a_{r} ; b_{1} \ldots b_{r} a}\left(\beta \mid \theta_{1}, \ldots, \theta_{r}\right)=S_{b a_{1} \ldots a_{r-1} ; b_{1} \ldots b_{r-1} x}\left(\beta \mid \theta_{1}, \ldots, \theta_{r-1}\right) S_{x a_{r}}^{a b_{r}}\left(\beta-\theta_{r}\right) . \tag{5.41}
\end{equation*}
$$

Starting from the $r=1$ case

$$
\begin{align*}
S_{b a_{1} ; b_{1} a}\left(\beta \mid \theta_{1}\right)= & \delta_{a b} \delta_{a_{1} b_{1}}+\frac{1}{n} a_{1}\left(\beta-\theta_{1}\right) \delta_{a b_{1}} \delta_{b a_{1}} \\
& +\frac{1}{n} a_{2}\left(\beta-\theta_{1}\right) \delta_{a b} \delta_{a_{1} b_{1}}+\frac{1}{n} a_{3}\left(\beta-\theta_{1}\right) \delta_{a a_{1}} \delta_{b b_{1}}+\mathrm{O}\left(\frac{1}{n^{2}}\right) \tag{5.42}
\end{align*}
$$

it is easy to show by induction that the residue equation takes the form

$$
\begin{align*}
& n F_{a b a_{1} \ldots a_{r}}\left(\beta+i \pi, \beta, \theta_{1}, \ldots, \theta_{r}\right) \\
&=\left(\frac{i}{2}\right)^{r}\left\{\prod_{j=1}^{r} \sinh \left(\beta-\theta_{j}\right)\right\}\left[\left\{\sum_{k=1}^{r} a_{2}\left(\beta-\theta_{k}\right)\right\} \delta_{a b} F_{a_{1} \ldots a_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)\right. \\
&+\sum_{k=1}^{r} a_{1}\left(\beta-\theta_{k}\right) \delta_{b a_{k}} F_{a_{1} \ldots a_{1} \ldots a_{r}}^{(k)}\left(\theta_{1}, \ldots, \theta_{k}, \ldots, \theta_{r}\right) \\
&+\sum_{k=1}^{r} a_{3}\left(\beta-\theta_{k}\right) \delta_{a a_{k}} F_{a_{1} \ldots b \ldots a_{r}}^{(k)}\left(\theta_{1}, \ldots, \theta_{k}, \ldots, \theta_{r}\right) \\
&\left.+\frac{1}{n} \sum_{l<k} a_{1}\left(\beta-\theta_{l}\right) a_{3}\left(\beta-\theta_{k}\right) \delta_{a a_{k}} \delta_{b a_{l}} F_{a_{1} \ldots z \ldots \ldots a_{r}}^{(l)(k)}\left(\theta_{1}, \ldots, \theta_{l}, \ldots, \theta_{k}, \ldots, \theta_{r}\right)\right] \tag{5.43}
\end{align*}
$$

in the leading order of the large $n$ expansion. Again, the last term is of the same order as the other terms only if the contracted indices belong to the same Kronecker delta and has to be dropped in all other cases.

### 5.5. Solution of the leading order equations for the spin field operator

In this subsection the number of particles, $r$, is an odd number and we will use the notation $v=(r-1) / 2$. For the leading order form factor we take the following ansatz

$$
\begin{equation*}
F_{a_{1} \ldots a_{r}}^{x}\left(\theta_{1}, \ldots, \theta_{r}\right)=\mathcal{N}_{r} \sum_{\sigma} \delta_{a_{\sigma(1)}}^{x} R_{23}^{\sigma} \cdots R_{r-1 r}^{\sigma} Q_{r}\left(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(r)}\right), \tag{5.44}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{N}_{r}=\frac{1}{n^{\nu}} \frac{1}{2^{\nu} \nu!}\left(\frac{i}{2}\right)^{\nu(\nu-1)} \tag{5.45}
\end{equation*}
$$

and $\sigma$ runs over the $r$ ! permutations of the particles. Finally we used the shorthand notation ( $\epsilon$ is the sign function)

$$
\begin{equation*}
R_{i j}^{\sigma}=\delta_{a_{\sigma(i)} a_{\sigma(j)}} R\left[\epsilon(\sigma(j)-\sigma(i))\left(\theta_{\sigma(i)}-\theta_{\sigma(j)}\right)\right], \tag{5.46}
\end{equation*}
$$

which for physical (real, ordered) rapidities reduces to

$$
\begin{equation*}
\delta_{a_{\sigma(i)} a_{\sigma(j)}} R\left[\left|\theta_{\sigma(i)}-\theta_{\sigma(j)}\right|\right] . \tag{5.47}
\end{equation*}
$$

We require that the scalar function $Q_{r}\left(\theta_{1}, \ldots, \theta_{r}\right)$ is symmetric under the exchanges $2 \leftrightarrow 3$, $4 \leftrightarrow 5, \ldots, r-1 \leftrightarrow r$ and is totally symmetric under permutation of these pairs of variables. Further we require that $Q_{r}$ is $2 \pi i$ periodic in all variables and is even under simultaneous sign change of all variables. Then it is almost obvious that the ansatz (5.44) satisfies the homogeneous equations (5.37)-(5.40). ${ }^{5}$ It is also possible to write

$$
\begin{align*}
& F_{a_{1} \ldots a_{r}}^{x}\left(\theta_{1}, \ldots, \theta_{r}\right) \\
& \quad=\frac{1}{n^{v}}\left(\frac{i}{2}\right)^{v(v-1)} \delta_{a_{1}}^{x} \delta_{a_{2} a_{3}} R\left(\theta_{2}-\theta_{3}\right) \cdots \delta_{a_{r-1} a_{r}} R\left(\theta_{r-1}-\theta_{r}\right) Q_{r}\left(\theta_{1}, \ldots, \theta_{r}\right)+\cdots, \tag{5.48}
\end{align*}
$$

where the final dots stand for all similar terms, corresponding to such permutations of the variables not leaving the first term invariant. Finally (5.43) will also be satisfied by (5.44) if the set of scalar functions $Q_{r}$ obeys the following three relations for $r \geqslant 3$

$$
\begin{align*}
& Q_{r+2}\left(\theta_{1}, \beta+i \pi, \beta, \theta_{2}, \ldots, \theta_{r}\right) \\
& \quad=\left\{\prod_{j=1}^{r} \sinh \left(\beta-\theta_{j}\right)\right\}\left\{\sum_{k=1}^{r} \frac{1}{\sinh \left(\beta-\theta_{k}\right)}\right\} Q_{r}\left(\theta_{1}, \ldots, \theta_{r}\right),  \tag{5.49}\\
& Q_{r+2}\left(\beta+i \pi, \beta, \theta_{1}, \ldots, \theta_{r}\right)=\left\{\prod_{j=2}^{r} \sinh \left(\beta-\theta_{j}\right)\right\} Q_{r}\left(\theta_{1}, \ldots, \theta_{r}\right),  \tag{5.50}\\
& Q_{r+2}\left(\theta_{1}, \beta+i \pi, \theta_{2}, \beta, \theta_{3}, \ldots, \theta_{r}\right) \\
& \quad=\sinh \left(\theta_{2}-\theta_{3}\right)\left\{\prod_{j \neq 2,3}^{r} \sinh \left(\beta-\theta_{j}\right)\right\} Q_{r}\left(\theta_{1}, \ldots, \theta_{r}\right) . \tag{5.51}
\end{align*}
$$

[^5]Using the results of Section 5.3 we see that

$$
\begin{equation*}
Q_{3}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=1 \tag{5.52}
\end{equation*}
$$

For $r=5$ we have

$$
\begin{align*}
& Q_{5}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}\right) \\
&=-\frac{1}{2}\left[1+\sum_{k<l} \cosh \left(\theta_{k}-\theta_{l}\right)+\cosh \left(\theta_{2}+\theta_{3}-\theta_{4}-\theta_{5}\right)+\cosh \left(\theta_{1}+\theta_{3}-\theta_{4}-\theta_{5}\right)\right. \\
&\left.+\cosh \left(\theta_{2}+\theta_{3}-\theta_{1}-\theta_{5}\right)+\cosh \left(\theta_{2}+\theta_{3}-\theta_{4}-\theta_{1}\right)+\cosh \left(\theta_{2}+\theta_{1}-\theta_{4}-\theta_{5}\right)\right] . \tag{5.53}
\end{align*}
$$

It is easy to check that this satisfies (5.49)-(5.51) for $r=3$. Because of the $2 \pi i$ periodicity the scalar function $Q_{r}$ is really a function of the exponential variables $x_{k}=\mathrm{e}^{\theta_{k}}, k=1,2, \ldots, r$. Using the fact that the form factors are regular functions that are also regular at infinity and also taking into account the presence of the denominator in (3.8) we can show that $Q_{r}$, as function of one of the variables, say $x_{2}$, is a finite Laurent polynomial consisting of the terms $x_{2}^{\nu-1}, x_{2}^{\nu-2}, \ldots, x_{2}^{1-\nu}$. Applying this to the $r+2$ case, we see that $x_{2}^{\nu} Q_{r+2}\left(\theta_{1}, \ldots, \theta_{r+2}\right)$ is a polynomial of degree $r-1$ in $x_{2}$ hence it is determined by its values at $r$ different points. If $Q_{r}$ is given, these data are provided by (5.49) and (5.51) and we can use them as recursion relations to determine $Q_{r+2}$. The solution is given by the explicit formula

$$
\begin{align*}
& Q_{r+2}\left(\theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{r+2}\right) \\
& \quad=\sum_{k=3}^{r+2} Q_{r+2}\left(\theta_{1}, \theta_{k}+i \pi, \theta_{3}, \ldots, \theta_{r+2}\right)\left(-\frac{x_{k}}{x_{2}}\right)^{\frac{r-1}{2}} \prod_{l \neq 1,2, k} \frac{x_{l}+x_{2}}{x_{l}-x_{k}} \tag{5.54}
\end{align*}
$$

We have applied (5.54) to determine $Q_{7}$. We have checked that the function $Q_{7}$ constructed this way is also a polynomial in all the other variables and satisfies all symmetry requirements together with (5.50) (which was not used in the construction (5.54)). It would be interesting to show analogous results for general $r$.

### 5.6. Noether current and symmetric tensor form factors

The leading order form factors of the Noether current and the symmetric, traceless isotensor operator are given by the ansatz

$$
\begin{equation*}
F_{a_{1} \ldots a_{r}}^{x y}\left(\theta_{1}, \ldots, \theta_{r}\right)=\mathcal{M}_{r} \sum_{\sigma} T^{\sigma} R_{34}^{\sigma} \cdots R_{r-1 r}^{\sigma} Q_{r}\left(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(r)}\right), \tag{5.55}
\end{equation*}
$$

where $r$ is even, $\mu=(r-2) / 2$,

$$
\begin{equation*}
\mathcal{M}_{r}=\left(\frac{i}{n}\right)^{\mu} \frac{1}{2^{\mu+1} \mu!}\left(\frac{i}{2}\right)^{\mu^{2}} \tag{5.56}
\end{equation*}
$$

and

$$
\begin{equation*}
T^{\sigma}=T_{a_{\sigma(1)} a_{\sigma(2)}}^{x y}\left(\theta_{\sigma(1)}-\theta_{\sigma(2)}\right) \tag{5.57}
\end{equation*}
$$

where

$$
T_{a_{1} a_{2}}^{x y}(\theta)= \begin{cases}t_{t_{1} a_{2}}^{x y} \sinh \theta & \text { (current) }  \tag{5.58}\\ s_{a_{1} a_{2}} & \text { (tensor) }\end{cases}
$$

For even $r$ the scalar function $Q_{r}\left(\theta_{1}, \ldots, \theta_{r}\right)$ is symmetric under the exchanges $1 \leftrightarrow 2,3 \leftrightarrow$ $4, \ldots, r-1 \leftrightarrow r$ and is totally symmetric under permutation of the last $\mu$ pairs of variables. Further $Q_{r}$ is $2 \pi i$ antiperiodic in all variables and is invariant under simultaneous sign change of all variables. The ansatz (5.55) with $Q_{r}$ satisfying the above symmetry requirements satisfies the homogeneous equations (5.37)-(5.40). We introduce the functions $P_{r}^{(o)}$ for $o=c, t$

$$
Q_{r}\left(\theta_{1}, \ldots, \theta_{r}\right)= \begin{cases}P_{r}^{(c)}\left(\theta_{1}, \ldots, \theta_{r}\right) & \text { (current) }  \tag{5.59}\\ 2 \cosh \left(\frac{\theta_{1}-\theta_{2}}{2}\right) P_{r}^{(t)}\left(\theta_{1}, \ldots, \theta_{r}\right) & \text { (tensor) }\end{cases}
$$

(5.55) also satisfies the residue equation (5.43) if for $r \geqslant 4$

$$
\begin{align*}
& P_{r+2}^{(o)}\left(\theta_{1}, \theta_{2}, \beta+i \pi, \theta_{3}, \beta, \theta_{4}, \ldots, \theta_{r}\right) \\
& \quad=-i \sinh \left(\theta_{3}-\theta_{4}\right)\left\{\prod_{j \neq 3,4} \sinh \left(\beta-\theta_{j}\right)\right\} P_{r}^{(o)}\left(\theta_{1}, \ldots, \theta_{r}\right), \tag{5.60}
\end{align*}
$$

and for $r \geqslant 2$

$$
\begin{align*}
& P_{r+2}^{(o)}\left(\theta_{1}, \theta_{2}, \beta+i \pi, \beta, \theta_{3}, \theta_{4}, \ldots, \theta_{r}\right) \\
& \quad=-i\left\{\prod_{j=1}^{r} \sinh \left(\beta-\theta_{j}\right)\right\}\left\{\sum_{k=1}^{r} \frac{1}{\sinh \left(\beta-\theta_{k}\right)}\right\} P_{r}^{(o)}\left(\theta_{1}, \ldots, \theta_{r}\right),  \tag{5.61}\\
& P_{r+2}^{(c)}\left(\beta+i \pi, \theta_{1}, \beta, \theta_{2}, \ldots, \theta_{r}\right) \\
& \quad=-i \sinh \left(\theta_{1}-\theta_{2}\right)\left\{\prod_{j=3}^{r} \sinh \left(\beta-\theta_{j}\right)\right\} P_{r}^{(c)}\left(\theta_{1}, \ldots, \theta_{r}\right),  \tag{5.62}\\
& P_{r+2}^{(t)}\left(\beta+i \pi, \theta_{1}, \beta, \theta_{2}, \ldots, \theta_{r}\right) \\
& \quad=-2 \cosh \left(\frac{\theta_{1}-\theta_{2}}{2}\right) \cosh \left(\frac{\beta-\theta_{1}}{2}\right)\left\{\prod_{j=3}^{r} \sinh \left(\beta-\theta_{j}\right)\right\} P_{r}^{(t)}\left(\theta_{1}, \ldots, \theta_{r}\right) . \tag{5.63}
\end{align*}
$$

The $P_{2}^{(o)}$ functions are given by

$$
\begin{equation*}
P_{2}^{(c)}\left(\theta_{1}, \theta_{2}\right)=\frac{-1}{2 \cosh \left(\frac{\theta_{1}-\theta_{2}}{2}\right)}, \quad P_{2}^{(t)}\left(\theta_{1}, \theta_{2}\right)=\frac{1}{2} \tag{5.64}
\end{equation*}
$$

and for $r=4$ we have

$$
\begin{align*}
P_{4}^{(c)}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}\right)= & \cosh \left(\frac{\theta_{1}+\theta_{2}-\theta_{3}-\theta_{4}}{2}\right)  \tag{5.65}\\
P_{4}^{(t)}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}\right)= & -\cosh \left(\frac{\theta_{1}-\theta_{3}}{2}\right) \cosh \left(\frac{\theta_{1}-\theta_{4}}{2}\right) \\
& -\cosh \left(\frac{\theta_{2}-\theta_{3}}{2}\right) \cosh \left(\frac{\theta_{2}-\theta_{4}}{2}\right) . \tag{5.66}
\end{align*}
$$

Similarly to the case of field operators discussed in the previous subsection, the expression $x_{3}^{\frac{r-1}{2}} P_{r+2}^{(o)}$, treated as a function of the variable $x_{3}$, is a polynomial of degree $r-1$ and is determined by its values at $r$ different points. The recursion relations (5.60)-(5.63) can be used to determine this expression at $r$ different points and the Laurent polynomials $P_{r+2}^{(o)}$ can be calculated from a formula similar to (5.54).

## 6. $1 / n$ expansion of the functional integral

In this section we check that the bootstrap solutions for the multi-particle form factors found in leading order $1 / n$ expansion in Sections $5.5,5.6$ do indeed correspond to those obtained by quantum field theoretic calculations. ${ }^{6}$

The $1 / n$ expansion of the functional integral of the $\mathrm{O}(n)$ non-linear $\sigma$-model has been described in numerous papers. Starting from bare fields $q^{a}$ one imposes the constraint $q^{2}=n$ by introducing a Lagrange multiplier field $\lambda$. Here we just recall the resulting Feynman rules for computation of the correlation functions of the elementary field:

$$
\begin{align*}
q \text { propagator: } & =\delta^{a b} i D\left(p, m_{0}\right), \quad D\left(p, m_{0}\right)=\frac{1}{p^{2}-m_{0}^{2}+i \epsilon}  \tag{6.1}\\
\lambda \text { propagator: } & =2 J\left(p, m_{0}\right)^{-1}  \tag{6.2}\\
q-\lambda \text { vertex: } & =\frac{1}{\sqrt{n}} \delta^{a b} \tag{6.3}
\end{align*}
$$

with momentum conservation at each vertex

- for each external line a factor $Z^{-1 / 2}$;
- for each closed $q$-loop there is a factor $n$;
- only $q$-loops with more than 2 vertices should be drawn;
- integration $\int \frac{\mathrm{d}^{2} k}{(2 \pi)^{2}}$ over all internal momenta $k$ for which a cutoff $\Lambda$ is imposed (e.g. PauliVillars for the $q$-propagator).

Renormalization of the bare parameters order by order in $1 / n$ is given by

$$
\begin{align*}
& m_{0}^{2}=M^{2}\left(1-\sum_{s=1}^{\infty} \frac{\alpha_{s}}{n^{s}}\right), \quad \alpha_{s}=\alpha_{s}(\Lambda / M)  \tag{6.4}\\
& Z=1+\sum_{s=1}^{\infty} \frac{Z_{s}}{n^{s}}, \quad Z_{s}=Z_{s}(\Lambda / M) \tag{6.5}
\end{align*}
$$

The $\lambda$ inverse propagator function $J(q, m)$ is a special case of the 1-loop integrals:

$$
\begin{equation*}
J_{r}\left(q_{1}, \ldots, q_{r}, m\right)=\int \frac{\mathrm{d}^{2} k}{(2 \pi)^{2}} \prod_{j=1}^{r} D\left(k+l_{j}, m\right) \tag{6.6}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{j}=l_{j}-l_{j-1}, \quad l_{-1}=l_{r}, \tag{6.7}
\end{equation*}
$$

[^6]\[

$$
\begin{equation*}
\sum_{j=1}^{r} q_{j}=0 \tag{6.8}
\end{equation*}
$$

\]

which can be computed using the cutting rules. In particular we have

$$
\begin{equation*}
J(q, m) \equiv J_{2}(q,-q, m)=\frac{i}{4 m^{2} R(\theta)} \quad \text { for } q^{2}=4 m^{2} \cosh ^{2}\left(\frac{\theta}{2}\right) . \tag{6.9}
\end{equation*}
$$

We will also need the case $r=3$ :

$$
\begin{align*}
& J_{3}\left(q_{1}, q_{2}, q_{3}, m\right)=-\frac{\left(q_{1} q_{2}\right) q_{3}^{2} J\left(q_{3}, m\right)}{D_{3}\left(q_{1}, q_{2}, q_{3}, m\right)}+2 \text { perms }  \tag{6.10}\\
& D_{3}\left(q_{1}, q_{2}, q_{3}, m\right)=q_{1}^{2} q_{2}^{2} q_{3}^{2}+m^{2} \lambda\left(q_{1}^{2}, q_{2}^{2}, q_{3}^{2}\right)-i \epsilon q_{1} q_{2}  \tag{6.11}\\
& \lambda\left(x_{1}, x_{2}, x_{3}\right)=x_{1}^{2}+x_{2}^{2}+x_{3}^{2}-2 x_{1} x_{2}-2 x_{2} x_{3}-2 x_{3} x_{1} \tag{6.12}
\end{align*}
$$

### 6.1. 3-particle spin form factor

The 3-particle form factor in leading order is, using the Feynman rules above, simply obtained from a sum of tree graphs and amputating three of the external lines thereby obtaining:

$$
\begin{equation*}
f_{b_{1} b_{2} b_{3}}^{a}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=\frac{2}{n} \frac{i}{q^{2}-M^{2}}\left[\delta_{b_{1}}^{a} \delta_{b_{2} b_{3}} J\left(p_{2}+p_{3}\right)^{-1}+2 \text { perms }\right]+\mathrm{O}\left(1 / n^{2}\right) \tag{6.13}
\end{equation*}
$$

where $q=p_{1}+p_{2}+p_{3}$. Here and in the rest of this section we omit the argument $M$ in the functions $J_{r}, D$ i.e. $J(s)=J(s, M)$. For three incoming (on-shell $p_{i}^{2}=M^{2}$ ) particles one has

$$
\begin{equation*}
q^{2}-M^{2}=8 M^{2} C_{3}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \tag{6.14}
\end{equation*}
$$

thus producing Eq. (5.44) for the case $r=3$ with (5.52).

### 6.2. 4-particle current and isotensor form factors

Consider the current

$$
\begin{equation*}
J_{\mu}^{a b}=q^{a} \partial_{\mu} q^{b}-q^{b} \partial_{\mu} q^{a} \tag{6.15}
\end{equation*}
$$

whose 2-particle form factor is in leading order just given by the contact diagram ( $q=p_{1}+p_{2}$ ) :

$$
\begin{equation*}
-i \epsilon_{\mu \alpha} q^{\alpha} f_{b_{1} b_{2}}^{a b}\left(\theta_{1}, \theta_{2}\right)=i t_{b_{1} b_{2}}^{a b}\left(p_{1}-p_{2}\right)_{\mu}+\mathrm{O}(1 / n) \tag{6.16}
\end{equation*}
$$

yielding $f_{b_{1} b_{2}}^{a b}\left(\theta_{1}, \theta_{2}\right)=-t_{b_{1} b_{2}}^{a b} \tanh \frac{\left(\theta_{1}-\theta_{2}\right)}{2}$ as required.
The 4-particle current form factor in leading order is a sum of tree diagrams:

$$
\begin{align*}
-i \epsilon_{\mu \nu} q^{\nu} f_{b_{1} b_{2} b_{3} b_{4}}^{a b}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}\right)= & -\frac{2}{n} \sum_{1 \leqslant i<j \leqslant 4} t_{b_{i} b_{j}}^{a b} \delta_{b_{k} b_{l}} V_{\mu}\left(q, p_{i}, p_{j}\right) J\left(p_{k}+p_{l}\right)^{-1} \\
& +\mathrm{O}\left(1 / n^{2}\right) \tag{6.17}
\end{align*}
$$

where $k<l$ and $\{i, j\} \cup\{k, l\}=\{1,2,3,4\}, q=\sum_{j=1}^{4} p_{j}$, and

$$
\begin{equation*}
V_{\mu}\left(q, p_{i}, p_{j}\right) \equiv\left(2 p_{i}-q\right)_{\mu} D\left(q-p_{i}\right)-\left(2 p_{j}-q\right)_{\mu} D\left(q-p_{j}\right) \tag{6.18}
\end{equation*}
$$

For on-shell momenta $p_{i}$ it is clear that $q^{\mu} V_{\mu}\left(q, p_{i}, p_{j}\right)=0$ as required for current conservation, and one can check that

$$
\begin{equation*}
V_{\mu}\left(q, p_{i}, p_{j}\right)=\frac{1}{16 M^{2}} \epsilon_{\mu \nu} q^{\nu} \frac{\sinh \left(\theta_{i}-\theta_{j}\right) \cosh \frac{1}{2}\left(\theta_{i}+\theta_{j}-\theta_{k}-\theta_{l}\right)}{C_{4}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}\right)} \tag{6.19}
\end{equation*}
$$

thus deriving the 4-particle bootstrap solution in Section 5.6.
Similarly for the 4-particle isotensor form factor:

$$
\begin{align*}
\tilde{f}_{b_{1} b_{2} b_{3} b_{4}}^{a b}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}\right)= & \frac{2 i}{n} \sum_{1 \leqslant i<j \leqslant 4} s_{b_{i} b_{j}}^{a b} \delta_{b_{k} b_{l}}\left[D\left(q-p_{i}\right)+D\left(q-p_{j}\right)\right] \\
& \times J\left(p_{k}+p_{l}, m\right)^{-1}+\mathrm{O}\left(1 / n^{2}\right) \tag{6.20}
\end{align*}
$$

For on-shell momenta $p_{i}$ one has:

$$
\begin{align*}
D(q & \left.-p_{i}\right)+D\left(q-p_{j}\right) \\
= & \frac{1}{8 M^{2}} \frac{\cosh \left(\frac{\theta_{i}-\theta_{j}}{2}\right)}{C_{4}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}\right)} \\
& \times\left[\cosh \left(\frac{\theta_{i}-\theta_{k}}{2}\right) \cosh \left(\frac{\theta_{i}-\theta_{l}}{2}\right)+\cosh \left(\frac{\theta_{j}-\theta_{k}}{2}\right) \cosh \left(\frac{\theta_{j}-\theta_{l}}{2}\right)\right], \tag{6.21}
\end{align*}
$$

again consistent with the corresponding result in Section 5.6.

### 6.3. 5-particle spin field form factor

The leading contribution to the 5-particle (in state) spin form factor is $\mathrm{O}\left(1 / n^{2}\right)$. It is a little more complicated since there are two types of diagrams contributing: tree diagrams with two $\lambda$-propagators and others involving a closed $q$-triangle connected to the external lines by three $\lambda$-propagators. Using the $1 / n$ rules outlined above one gets $\left(q=\sum_{j=1}^{5} p_{j}\right)$ :

$$
\begin{align*}
& f_{b_{1} b_{2} b_{3} b_{4} b_{5}}^{a}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}\right) \\
& \quad=-\frac{4}{n^{2}} \frac{1}{\left(q^{2}-M^{2}\right)}\left[\delta_{b_{1}}^{c} \delta_{b_{2} b_{3}} \delta_{b_{4} b_{5}} f^{(5)}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}\right)+14 \text { perms }\right]+\mathrm{O}\left(1 / n^{3}\right), \tag{6.22}
\end{align*}
$$

where

$$
\begin{align*}
f^{(5)} & \left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}\right) \\
= & \frac{\left\{D\left(p_{1}+p_{4}+p_{5}\right)+D\left(p_{1}+p_{2}+p_{3}\right)\right\}}{J\left(p_{2}+p_{3}\right) J\left(p_{4}+p_{5}\right)} \\
& +\frac{\left\{D\left(p_{2}+p_{3}+p_{4}\right)+D\left(p_{2}+p_{3}+p_{5}\right)\right\}}{J\left(q-p_{1}\right) J\left(p_{2}+p_{3}\right)} \\
& +\frac{\left\{D\left(p_{2}+p_{4}+p_{5}\right)+D\left(p_{3}+p_{4}+p_{5}\right)\right\}}{J\left(q-p_{1}\right) J\left(p_{4}+p_{5}\right)} \\
& -2 \frac{J_{3}\left(p_{1}-q, p_{2}+p_{3}, p_{4}+p_{5}\right)}{J\left(q-p_{1}\right) J\left(p_{2}+p_{3}\right) J\left(p_{4}+p_{5}\right)} . \tag{6.23}
\end{align*}
$$

Using (6.10) this can be rewritten

$$
\begin{equation*}
f_{1}^{(5)}(\underline{\theta})=\frac{U_{1}}{J\left(q_{2}\right) J\left(q_{3}\right)}+\frac{U_{2}}{J\left(q_{1}\right) J\left(q_{3}\right)}+\frac{U_{3}}{J\left(q_{1}\right) J\left(q_{2}\right)}, \tag{6.24}
\end{equation*}
$$

with

$$
\begin{align*}
U_{1} & =D\left(p_{1}+q_{2}\right)+D\left(p_{1}+q_{3}\right)+\frac{2\left(q_{2} q_{3}\right) q_{1}^{2}}{D_{3}\left(q_{1}, q_{2}, q_{3}\right)},  \tag{6.25}\\
U_{2} & =D\left(p_{2}+q_{3}\right)+D\left(p_{3}+q_{3}\right)+\frac{2\left(q_{1} q_{3}\right) q_{2}^{2}}{D_{3}\left(q_{1}, q_{2}, q_{3}\right)}  \tag{6.26}\\
U_{3} & =D\left(p_{4}+q_{2}\right)+D\left(p_{5}+q_{2}\right)+\frac{2\left(q_{1} q_{2}\right) q_{3}^{2}}{D_{3}\left(q_{1}, q_{2}, q_{3}\right)} \tag{6.27}
\end{align*}
$$

where

$$
\begin{equation*}
q_{1}=p_{1}-q, \quad q_{2}=p_{2}+p_{3}, \quad q_{3}=p_{4}+p_{5} \tag{6.28}
\end{equation*}
$$

We note

$$
\begin{equation*}
D_{3}\left(q_{1}, q_{2}, q_{3}\right)=256 M^{6} C_{4}\left(\theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}\right) \cosh \left(\frac{\theta_{2}-\theta_{3}}{2}\right) \cosh \left(\frac{\theta_{4}-\theta_{5}}{2}\right) \tag{6.29}
\end{equation*}
$$

and then after some algebra it can be shown that

$$
\begin{equation*}
U_{2}=0=U_{3} \tag{6.30}
\end{equation*}
$$

and ${ }^{7}$

$$
\begin{equation*}
U_{1}=-\frac{\left(q^{2}-M^{2}\right) Q_{5}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}\right)}{256 M^{4} C_{5}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}\right)} \tag{6.31}
\end{equation*}
$$

with $Q_{5}$ defined in Eq. (5.53), thus reproducing the result in Section 5.5.

### 6.4. 6-particle current and isotensor form factors

Inspecting the diagrams contributing to the 6-particle current form factor in the leading order $1 / n$ expansion we note that we can write these in terms of form factors of the spin field (here $q=\sum_{j=1}^{6} p_{j}$ ):

$$
\begin{align*}
& -i \epsilon_{\mu \nu} q^{\nu} f_{b_{1} b_{2} b_{3} b_{4} b_{5} b_{6}}^{a b}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) \\
& \quad=i\left[t_{b_{1} c}^{a b}\left(2 p_{1}-q\right)_{\mu} f_{b_{2} b_{3} b_{4} b_{5} b_{6}}^{c}\left(\theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)+5 \text { similar terms }\right] \\
& \quad+i\left[t_{c d}^{a b}\left(p_{1}+p_{2}+p_{3}-p_{4}-p_{5}-p_{6}\right)_{\mu} f_{b_{1} b_{2} b_{3}}^{c}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) f_{b_{4} b_{5} b_{6}}^{d}\left(\theta_{4}, \theta_{5}, \theta_{6}\right)\right. \\
& \quad+9 \text { similar terms }]+\mathrm{O}\left(1 / n^{3}\right) . \tag{6.32}
\end{align*}
$$

Using the expressions previously obtained we get

$$
\begin{aligned}
& -i \epsilon_{\mu \nu} q^{\nu} f_{b_{1} b_{2} b_{3} b_{4} b_{5} b_{6}}^{a b}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) \\
& = \\
& \quad-\frac{i}{4 n^{2}}\left\{t_{b_{1} b_{2}}^{a b} \delta_{b_{3} b_{4}} \delta_{b_{5} b_{6}} R\left(\theta_{34}\right) R\left(\theta_{56}\right)\right. \\
& \quad \times\left[\frac{\left(2 p_{1}-q\right)_{\mu} Q_{5}\left(\theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)}{C_{5}\left(\theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)}-\frac{\left(2 p_{2}-q\right)_{\mu} Q_{5}\left(\theta_{1}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)}{C_{5}\left(\theta_{1}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)}\right.
\end{aligned}
$$

[^7]\[

$$
\begin{align*}
& \left.-4\left(\frac{\left(p_{1}+p_{3}+p_{4}-p_{2}-p_{5}-p_{6}\right)_{\mu}}{C_{3}\left(\theta_{1}, \theta_{3}, \theta_{4}\right) C_{3}\left(\theta_{2}, \theta_{5}, \theta_{6}\right)}-\frac{\left(p_{2}+p_{3}+p_{4}-p_{1}-p_{5}-p_{6}\right)_{\mu}}{C_{3}\left(\theta_{2}, \theta_{3}, \theta_{4}\right) C_{3}\left(\theta_{1}, \theta_{5}, \theta_{6}\right)}\right)\right] \\
& +44 \text { similar terms }\}+\mathrm{O}\left(1 / n^{3}\right) \tag{6.33}
\end{align*}
$$
\]

One can check that contracting the rhs with $q_{\mu}$ is zero and then obtain the representation for the 6-particle current form factor with $P_{6}^{(c)}$ given in Appendix C.

Similarly for the isotensor:

$$
\begin{align*}
& \tilde{f}_{b_{1} b_{2} b_{3} b_{4} b_{5} b_{6}}^{a b}\left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) \\
& = \\
& =\left[s_{b_{1} c}^{a b} f_{b_{2}}^{c} b_{3} b_{4} b_{5} b_{6}\left(\theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)+5 \text { similar terms }\right]  \tag{6.34}\\
& \quad+\left[s_{c d}^{a b} f_{b_{1} b_{2} b_{3}}^{c}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) f_{b_{4} b_{5} b_{6}}^{d}\left(\theta_{4}, \theta_{5}, \theta_{6}\right)+9 \text { similar terms }\right]+\mathrm{O}\left(1 / n^{3}\right) \\
& = \\
& -\frac{1}{4 n^{2}}\left\{s_{b_{1} b_{2}}^{a b} \delta_{b_{3} b_{4} \delta_{b_{5} b_{6}} R\left(\theta_{34}\right) R\left(\theta_{56}\right)}\right. \\
& \quad \times\left[\frac{Q_{5}\left(\theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)}{C_{5}\left(\theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)}+\frac{Q_{5}\left(\theta_{1}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)}{C_{5}\left(\theta_{1}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right)}\right.  \tag{6.35}\\
& \left.\quad-4\left(\frac{1}{C_{3}\left(\theta_{1}, \theta_{3}, \theta_{4}\right) C_{3}\left(\theta_{2}, \theta_{5}, \theta_{6}\right)}+\frac{1}{C_{3}\left(\theta_{2}, \theta_{3}, \theta_{4}\right) C_{3}\left(\theta_{1}, \theta_{5}, \theta_{6}\right)}\right)\right] \\
& \\
& \quad+44 \text { similar terms }\}+\mathrm{O}\left(1 / n^{3}\right)
\end{align*}
$$

The expression for $P_{6}^{(t)}$ thus obtained agrees with the corresponding bootstrap solution as expected.

## 7. Large $\boldsymbol{n}$ clustering tests

## 7.1. $3 \rightarrow 2+1$ clustering in the large $n$ expansion

We have already computed the 2-particle and 3-particle form factors in the first two orders of the large $n$ expansion so we are able to check the clustering formula (4.44) in this limit.

For the 2-particle form factors we have

$$
\begin{equation*}
\psi_{I}(\theta)=F_{I}(\theta)+\frac{1}{n} G_{I}(\theta)+\cdots, \quad I=1,2, \tag{7.1}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{1}(\theta)=\tanh \frac{\theta}{2}, \quad F_{2}(\theta)=i \tag{7.2}
\end{equation*}
$$

and

$$
\begin{equation*}
G_{1}(\theta)=2 \pi \tanh \frac{\theta}{2}[a(\theta)+b(\theta)], \quad G_{2}(\theta)=2 \pi i[a(\theta)-b(\theta)] \tag{7.3}
\end{equation*}
$$

In the case of the 3-particle form factors we first write

$$
\begin{align*}
f_{a b c}^{x}(\Delta+\beta, \beta, \gamma) \cong & \frac{8 \mathrm{e}^{-\Delta}}{\mathrm{e}^{\theta}+1}\left\{\delta_{a}^{x} \delta_{b c} X(\Delta, \Delta+\theta)\right. \\
& \left.+\delta_{b}^{x} \delta_{a c} X(\Delta, 2 \pi i-\theta)+\delta_{c}^{x} \delta_{a b} X(\theta, \Delta+\theta)\right\} \tag{7.4}
\end{align*}
$$

where $\theta=\beta-\gamma$. From here it is easy to calculate that the coefficient of the leading $(1 / n)$ term on the right-hand side of (7.4) is

$$
\begin{equation*}
-\frac{2 \pi}{\Delta}\left\{F_{1}(\theta) t_{b c}^{x a}-i F_{2}(\theta) s_{b c}^{x a}\right\}+\cdots \tag{7.5}
\end{equation*}
$$

Comparing it to (4.43) we see that

$$
\begin{equation*}
H_{2}=1+\chi \omega+\mathrm{O}\left(\chi^{2}\right), \tag{7.6}
\end{equation*}
$$

where $\omega$ is a real number. Using this result and (4.43) the predicted form of the $\mathrm{NLO}\left(1 / n^{2}\right)$ coefficient on the right-hand side of (7.4) becomes

$$
\begin{equation*}
-\frac{2 \pi}{\Delta}\left\{\left(2 F_{1}+G_{1}\right) t_{b c}^{x a}-i s_{b c}^{x a}\left(2 F_{2}+G_{2}-2 F_{2} \ln \frac{\Delta}{2 \pi}+\omega F_{2}\right)+2 i F_{2} \delta_{a}^{x} \delta_{b c}\right\} . \tag{7.7}
\end{equation*}
$$

(Here the argument of all the functions $F_{I}, G_{I}$ is $\theta$.)
Now we use the large $\Delta$ asymptotic expansions

$$
\begin{align*}
& g(\Delta, \Delta+\theta) \cong \frac{\pi}{2 \Delta} \mathrm{e}^{\Delta}\left(1+\mathrm{e}^{\theta}\right), \\
& g(\Delta, 2 \pi i-\theta) \cong-\frac{\pi}{2 \Delta} \mathrm{e}^{\Delta}\left\{\mathrm{e}^{\theta}[2+2 \pi a(\theta)]+\left(\mathrm{e}^{\theta}+1\right)\left[\Psi\left(\frac{1}{2}\right)-\ln \left(\frac{\Delta}{2 \pi}\right)\right]-2 \pi b(\theta)\right\}, \\
& g(\theta, \Delta+\theta) \cong-\frac{\pi}{2 \Delta} \mathrm{e}^{\Delta}\left\{[2+2 \pi a(\theta)]+\left(\mathrm{e}^{\theta}+1\right)\left[\Psi\left(\frac{1}{2}\right)-\ln \left(\frac{\Delta}{2 \pi}\right)\right]-2 \pi b(\theta) \mathrm{e}^{\theta}\right\} \tag{7.8}
\end{align*}
$$

and calculate the large $\Delta$ asymptotics of our NLO large $n$ 3-particle form factor explicitly. We find that it is indeed exactly of the form (7.7), if we choose

$$
\begin{equation*}
\omega=2 \Psi\left(\frac{1}{2}\right) \tag{7.9}
\end{equation*}
$$

### 7.2. Odd-odd clustering

Specifying the odd-odd clustering formula for the current form factors at the leading order in the large $n$ expansion gives

$$
\begin{align*}
& F_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}^{c d}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \quad \cong-\left(\frac{\mathrm{e}^{\Delta / 2}}{2}\right)^{k l} t_{a b}^{c d} \exp \left\{\frac{l}{2} \sum_{i=1}^{k} \alpha_{i}-\frac{k}{2} \sum_{j=1}^{l} \beta_{j}\right\} F_{a_{1} \ldots a_{k}}^{a}\left(\alpha_{1}, \ldots, \alpha_{k}\right) F_{b_{1} \ldots b_{l}}^{b}\left(\beta_{1}, \ldots, \beta_{k}\right) . \tag{7.10}
\end{align*}
$$

Note the appearance of prefactors which arise because we are considering reduced form factors here.

Analogously for the leading large $n$ form factors of the symmetric tensor we get

$$
\begin{align*}
& \tilde{F}_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}^{c d}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \quad \cong\left(\frac{\mathrm{e}^{\Delta / 2}}{2}\right)^{k l} s_{a b}^{c d} \exp \left\{\frac{l}{2} \sum_{i=1}^{k} \alpha_{i}-\frac{k}{2} \sum_{j=1}^{l} \beta_{j}\right\} F_{a_{1} \ldots a_{k}}^{a}\left(\alpha_{1}, \ldots, \alpha_{k}\right) F_{b_{1} \ldots b_{l}}^{b}\left(\beta_{1}, \ldots, \beta_{k}\right) \tag{7.11}
\end{align*}
$$

Using the explicit solution of the current and symmetric tensor as well as the spin field form factors we can prove that the $k=1$ special case of the clustering relations implies

$$
\begin{equation*}
K_{l}\left(\beta_{1}, \beta_{2}, \ldots, \beta_{l}\right)=-\left(-\frac{1}{2}\right)^{\frac{l-1}{2}} \exp \left\{\frac{1}{2} \beta_{1}-\frac{1}{2} \sum_{j=2}^{l} \beta_{j}\right\} Q_{l}\left(\beta_{1}, \beta_{2}, \ldots, \beta_{l}\right) \tag{7.12}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{l}\left(\beta_{1}, \beta_{2}, \ldots, \beta_{l}\right)=-\left(-\frac{1}{2}\right)^{\frac{l+1}{2}} \exp \left\{-\frac{1}{2} \sum_{j=2}^{l} \beta_{j}\right\} Q_{l}\left(\beta_{1}, \beta_{2}, \ldots, \beta_{l}\right) \tag{7.13}
\end{equation*}
$$

Here the functions $K_{l}$ and $L_{l}$ are defined by the asymptotic relations

$$
\begin{equation*}
P_{l+1}^{(c)}\left(\Delta, \beta_{1}, \ldots, \beta_{l}\right) \cong \exp \left\{\frac{\Delta(l-2)}{2}\right\} K_{l}\left(\beta_{1}, \ldots, \beta_{l}\right) \tag{7.14}
\end{equation*}
$$

and

$$
\begin{equation*}
P_{l+1}^{(t)}\left(\Delta, \beta_{1}, \ldots, \beta_{l}\right) \cong \exp \left\{\frac{\Delta(l-1)}{2}\right\} L_{l}\left(\beta_{1}, \ldots, \beta_{l}\right) \tag{7.15}
\end{equation*}
$$

We have checked the clustering relations (7.12) and (7.13) for $l=3,5$ explicitly.

### 7.3. Odd-even clustering

Using the general odd-even clustering formula we can calculate the reduced form factor clustering in the large $n$ expansion at leading order:

$$
\begin{align*}
& F_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}^{x}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \cong \\
& \cong \frac{2 \pi}{n \Delta}\left(\frac{\mathrm{e}^{\Delta / 2}}{2}\right)^{k l} \exp \left\{\frac{l}{2} \sum_{i=1}^{k} \alpha_{i}-\frac{k}{2} \sum_{j=1}^{l} \beta_{j}\right\} F_{a_{1} \ldots a_{k}}^{a}\left(\alpha_{1}, \ldots, \alpha_{k}\right)  \tag{7.16}\\
& \quad \times\left\{F_{b_{1} \ldots b_{l}}^{x u}\left(\beta_{1}, \ldots, \beta_{l}\right)-\tilde{F}_{b_{1} \ldots b_{l}}^{x u}\left(\beta_{1}, \ldots, \beta_{l}\right)\right\} .
\end{align*}
$$

(Here $k$ is odd and $l$ is even.)
We define the function $M_{l}$ by the asymptotic formula

$$
\begin{equation*}
Q_{l+1}\left(\beta_{1}, \Delta, \beta_{2}, \ldots, \beta_{l}\right) \cong \exp \left\{\frac{\Delta(l-2)}{2}\right\} M_{l}\left(\beta_{1}, \beta_{2}, \ldots, \beta_{l}\right) \tag{7.17}
\end{equation*}
$$

In the special case $k=1$ (7.16) leads to the following recursion relation:

$$
\begin{align*}
& \mathrm{e}^{-\beta_{2}} M_{l}\left(\beta_{1}, \beta_{2}, \ldots, \beta_{l}\right) \\
& =\left(\frac{1}{2}\right)^{\frac{l-2}{2}} \exp \left\{-\frac{1}{2} \sum_{j=1}^{l} \beta_{j}\right\} \\
& \quad \times\left\{2 \cosh \left(\frac{\beta_{1}-\beta_{2}}{2}\right) P_{l}^{(t)}\left(\beta_{1}, \ldots, \beta_{l}\right)-\sinh \left(\beta_{1}-\beta_{2}\right) P_{l}^{(c)}\left(\beta_{1}, \ldots, \beta_{l}\right)\right\} \tag{7.18}
\end{align*}
$$

We have checked this relation for $l=2,4,6$.

Similarly the $l=2$ special case corresponds to the recursion relation

$$
\begin{equation*}
N_{k}\left(\beta_{1}, \beta_{2}, \alpha_{1}, \ldots, \alpha_{k}\right)=\left(-\frac{1}{4}\right)^{\frac{k-1}{2}} \exp \left\{\sum_{i=2}^{k} \alpha_{i}-\frac{k-1}{2}\left(\beta_{1}+\beta_{2}\right)\right\} Q_{k}\left(\alpha_{1}, \ldots, \alpha_{k}\right), \tag{7.19}
\end{equation*}
$$

where

$$
\begin{equation*}
Q_{k+2}\left(\beta_{1}-\Delta, \beta_{2}-\Delta, \alpha_{1}, \ldots, \alpha_{k}\right) \cong \exp \{(k-1) \Delta\} N_{k}\left(\beta_{1}, \beta_{2}, \alpha_{1}, \ldots, \alpha_{k}\right) \tag{7.20}
\end{equation*}
$$

We have checked (7.19) for $k=3,5$.

### 7.4. Even-even clustering

In this case the leading order reduced form factor clustering is of the form

$$
\begin{align*}
& F_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}^{x y}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \cong \\
& \left(\frac{2 \pi}{n \Delta}\right)\left(\frac{\mathrm{e}^{\Delta / 2}}{2}\right)^{k l} \exp \left\{\frac{l}{2} \sum_{i=1}^{k} \alpha_{i}-\frac{k}{2} \sum_{j=1}^{l} \beta_{j}\right\} \\
&  \tag{7.21}\\
& \\
& \times\left\{F_{a_{1} \ldots a_{k}}^{x q}(\alpha) F_{b_{1} \ldots b_{l}}^{y q}(\beta)-F_{a_{1} \ldots a_{k}}^{y q}(\alpha) F_{b_{1} \ldots b_{l}}^{x q}(\beta)\right. \\
& \\
& \left.\quad+\tilde{F}_{a_{1} \ldots a_{k}}^{x q}(\alpha) \tilde{F}_{b_{1} \ldots b_{l}}^{y q}(\beta)-\tilde{F}_{a_{1} \ldots a_{k}}^{y q}(\alpha) \tilde{F}_{b_{1} \ldots b_{l}}^{x q}(\beta)\right\}
\end{align*}
$$

and

$$
\begin{align*}
& \tilde{F}_{a_{1} \ldots a_{k} b_{1} \ldots b_{l}}^{x y}\left(\alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta, \beta_{1}, \ldots, \beta_{l}\right) \\
& \cong-\left(\frac{2 \pi}{n \Delta}\right)\left(\frac{\mathrm{e}^{\Delta / 2}}{2}\right)^{k l} \exp \left\{\frac{l}{2} \sum_{i=1}^{k} \alpha_{i}-\frac{k}{2} \sum_{j=1}^{l} \beta_{j}\right\} \\
& \times\left\{F_{a_{a_{1} \ldots a_{k}}^{x q}}(\alpha) \tilde{F}_{b_{1} \ldots b_{l}}^{y q}(\beta)+F_{a_{1} \ldots a_{k}}^{y q}(\alpha) \tilde{F}_{b_{1} \ldots b_{l}}^{x q}(\beta)\right. \\
&-\tilde{F}_{a_{1} \ldots a_{k}}^{x q}(\alpha) F_{b_{1} \ldots b_{l}}^{y q}(\beta)-\tilde{F}_{a_{1} \ldots a_{k}}^{y q}(\alpha) F_{b_{1} \ldots b_{l}}^{x q}(\beta) \\
&\left.+2 \tilde{F}_{a_{1} \ldots a_{k}}^{x q}(\alpha) \tilde{F}_{b_{1} \ldots b_{l}}^{y q}(\beta)+2 \tilde{F}_{a_{1} \ldots a_{k}}^{y q}(\alpha) \tilde{F}_{b_{1} \ldots b_{l}}^{x q}(\beta)\right\} . \tag{7.22}
\end{align*}
$$

Using the asymptotic relations

$$
\begin{align*}
& P_{l+2}^{(c)}\left(\alpha_{1}+\Delta, \beta_{1}, \alpha_{2}+\Delta, \beta_{2}, \ldots, \beta_{l}\right) \cong \exp \{(l-2) \Delta\} X_{l}\left(\alpha_{1}, \alpha_{2}, \beta_{1}, \ldots, \beta_{l}\right) \\
& P_{l+2}^{(t)}\left(\alpha_{1}+\Delta, \beta_{1}, \alpha_{2}+\Delta, \beta_{2}, \ldots, \beta_{l}\right) \cong \exp \left\{\left(l-\frac{3}{2}\right) \Delta\right\} Y_{l}\left(\alpha_{1}, \alpha_{2}, \beta_{1}, \ldots, \beta_{l}\right) \tag{7.23}
\end{align*}
$$

we have established the recursion relations

$$
\begin{aligned}
& X_{l}\left(\alpha_{1}, \alpha_{2}, \beta_{1}, \ldots, \beta_{l}\right) \\
& \quad=\left(-\frac{1}{4}\right)^{\frac{l-2}{2}} \exp \left\{\frac{l-2}{2}\left(\alpha_{1}+\alpha_{2}\right)-\sum_{j=3}^{l} \beta_{j}\right\}
\end{aligned}
$$

$$
\begin{align*}
& \times 2 \cosh \left(\frac{\beta_{1}-\beta_{2}}{2}\right)\left\{\cosh \left(\frac{\alpha_{1}-\alpha_{2}}{2}\right) P_{l}^{(t)}\left(\beta_{1}, \ldots, \beta_{l}\right)\right. \\
& \left.-\sinh \left(\frac{\alpha_{1}-\alpha_{2}}{2}\right) \sinh \left(\frac{\beta_{1}-\beta_{2}}{2}\right) P_{l}^{(c)}\left(\beta_{1}, \ldots, \beta_{l}\right)\right\} \tag{7.24}
\end{align*}
$$

and

$$
\begin{align*}
& \mathrm{e}^{\frac{1}{2}\left(\beta_{1}-\alpha_{1}\right)} Y_{l}\left(\alpha_{1}, \alpha_{2}, \beta_{1}, \ldots, \beta_{l}\right) \\
& =\left(-\frac{1}{4}\right)^{\frac{l-2}{2}} \exp \left\{\frac{l-2}{2}\left(\alpha_{1}+\alpha_{2}\right)-\sum_{j=3}^{l} \beta_{j}\right\} \\
& \quad \times \cosh \left(\frac{\beta_{1}-\beta_{2}}{2}\right)\left\{\left[\sinh \left(\frac{\alpha_{1}-\alpha_{2}}{2}\right)-2 \cosh \left(\frac{\alpha_{1}-\alpha_{2}}{2}\right)\right] P_{l}^{(t)}\left(\beta_{1}, \ldots, \beta_{l}\right)\right. \\
& \left.\quad+\cosh \left(\frac{\alpha_{1}-\alpha_{2}}{2}\right) \sinh \left(\frac{\beta_{1}-\beta_{2}}{2}\right) P_{l}^{(c)}\left(\beta_{1}, \ldots, \beta_{l}\right)\right\} \tag{7.25}
\end{align*}
$$

We have verified the above relations for $l=2,4$.

## 8. Clustering in the $O(3), O(4)$ models

8.1. $n=3$

We recall the discussion in [3]. One has for the reduced form factors (see Appendix B):

$$
\begin{align*}
& g_{b_{1} \ldots b_{m} a_{1} \ldots a_{k}}^{a}\left(\beta_{1}, \ldots, \beta_{m}, \alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta\right) \\
& \quad=\Delta^{k m-1} \epsilon_{a b c} g_{a_{1} \ldots a_{k}}^{b}\left(\alpha_{1}, \ldots, \alpha_{k}\right) g_{b_{1} \ldots b_{m}}^{c}\left(\beta_{1}, \ldots, \beta_{m}\right)+\mathrm{O}\left(\Delta^{k m-2}\right) \tag{8.1}
\end{align*}
$$

and similarly

$$
\begin{align*}
& h_{b_{1} \ldots b_{m} a_{1} \ldots a_{k}}\left(\beta_{1}, \ldots, \beta_{m}, \alpha_{1}+\Delta, \ldots, \alpha_{k}+\Delta\right) \\
& \quad=\Delta^{k m-2} g_{a_{1} \ldots a_{k}}^{a}\left(\alpha_{1}, \ldots, \alpha_{k}\right) g_{b_{1} \ldots b_{m}}^{a}\left(\beta_{1}, \ldots, \beta_{m}\right)+\mathrm{O}\left(\Delta^{k m-3}\right) \tag{8.2}
\end{align*}
$$

Note that in (8.1) members of the isospin 1 family are mapped into themselves, while in (8.2) members of the isospin 1 family are linked to members of the isospin 0 family. Observe also that there is no distinction between the factorization properties between even and odd members of the same family.

For the special case of $k=1, m=r-1$ the clustering relations read

$$
\begin{align*}
& g_{a_{1} \ldots a_{r}}^{a}\left(\beta_{1}, \ldots, \beta_{r}\right)=\beta_{r}^{r-2} \epsilon_{a a_{r} b} g_{a_{1} \ldots a_{r-1}}^{b}\left(\beta_{1}, \ldots, \beta_{r-1}\right)+\mathrm{O}\left(\beta_{r}^{r-3}\right),  \tag{8.3}\\
& h_{a_{1} \ldots a_{r}}\left(\beta_{1}, \ldots, \beta_{r}\right)=\beta_{r}^{r-3} g_{a_{1} \ldots a_{r-1}}^{a_{r}}\left(\beta_{1}, \ldots, \beta_{r-1}\right)+\mathrm{O}\left(\beta_{r}^{r-4}\right), \tag{8.4}
\end{align*}
$$

which is in accordance with the property that the reduced form factors are polynomials of partial degree $(r-2)$ and $(r-3)$ in the isospin 1 and 0 case, respectively. Since the product $\Psi_{r}$ also factorizes under clustering, the full (scalarized) form factors also satisfy clustering relations, which are similar to (8.1) and (8.2). For the $l=1$ family they can be found in Smirnov's book [1].

The clustering relations closely resemble some classical equations satisfied by the operators. For example dividing an even number of particles into two odd clusters, (8.1) can be interpreted as the quantum counterpart of the current in terms of the spin operators. The division of an even number of particles into two even clusters on the other hand, resembles the classical equation
$\partial_{\mu} J_{v}^{a}-\partial_{\nu} J_{\mu}^{a} \propto \epsilon_{a b c} J_{\mu}^{b} J_{v}^{c}$. Finally the clustering of an odd number of particles corresponds to $\partial_{\mu} s^{a} \propto \epsilon_{a b c} s^{b} J_{\mu}^{c}$. Similarly (8.2) corresponds to the defining equation for the energy-momentum tensor in terms of the spin fields or equivalently to its Sugawara form $T_{\mu \nu} \propto J_{\mu}^{a} J_{\nu}^{a}-\frac{1}{2} \eta_{\mu \nu} J_{\rho}^{a} J_{\rho}^{a}$.

In Ref. [3] the clustering properties above were used in particular to deduce the clustering properties of absolute squares of the form factors, summed over internal symmetry indices which enter in the expressions for spectral densities.

## 8.2. $\mathrm{O}(4)$ form factors: A three-particle example

Just as the $\mathrm{O}(4) \mathrm{S}$-matrix (A.19) is (minus) the tensor product of two chiral Gross-Neveu S-matrices, the $\mathrm{O}(4)$ form factors can be written as tensor products of two chiral Gross-Neveu form factors. More precisely, the $\mathrm{O}(4)$ form factors can be written as linear combinations of several such tensor products. This solution of the $\mathrm{O}(4)$ form factor equations, for the case of even particle numbers, was given by F.A. Smirnov [14]. The odd particle form factors must have a similar structure. The solution for the three-particle form factors of the $\mathrm{O}(4)$ field operator was found by M. Karowski [18]:

$$
\begin{equation*}
f_{P ; A B C}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=D\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \sum_{\omega} \tilde{F}_{p_{1} ; a_{1} b_{1} c_{1}}^{(\omega)}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \tilde{F}_{p_{2} ; a_{2} b_{2} c_{2}}^{(\bar{\omega})}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) . \tag{8.5}
\end{equation*}
$$

Here $\tilde{F}_{p ; a b c}^{(\omega)}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)$ for $\omega= \pm$ are the spin $s= \pm \frac{1}{4} \mathrm{SU}(2)$-symmetric chiral Gross-Neveu model form factors discussed in Appendix D. They satisfy the following homogeneous bootstrap equations:

$$
\begin{align*}
& \tilde{F}_{p ; i_{1} i_{2} i_{3}}^{(\omega)}\left(\theta_{1}+\lambda, \theta_{2}+\lambda, \theta_{3}+\lambda\right)=\mathrm{e}^{\omega \lambda / 4} \tilde{F}_{p ; i_{1} i_{2} i_{3}}^{(\omega)}\left(\theta_{1}, \theta_{2}, \theta_{3}\right),  \tag{8.6}\\
& \tilde{F}_{p ; i_{1} i_{2} i_{3}}^{(\omega)}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=\tilde{S}_{i_{2} i_{3}}^{u v}\left(\theta_{2}-\theta_{3}\right) \tilde{F}_{p ; i_{1} v u}^{(\omega)}\left(\theta_{1}, \theta_{3}, \theta_{2}\right),  \tag{8.7}\\
& \tilde{F}_{p ; i_{1} i_{2} i_{3}}^{(\omega)}\left(\theta_{1}+2 \pi i, \theta_{2}, \theta_{3}\right)=-i \omega \tilde{F}_{p ; i_{2} i_{3} i_{1}}^{(\omega)}\left(\theta_{2}, \theta_{3}, \theta_{1}\right) \tag{8.8}
\end{align*}
$$

and the residue equations

$$
\begin{equation*}
\tilde{F}_{p ; i_{1} i_{2} i_{3}}^{(\omega)}\left(\alpha, \beta, \theta_{3}\right) \approx \frac{-4}{\alpha-\beta-i \pi}\left\{\tilde{c}_{i_{1} i_{2}} \tilde{F}_{p ; i_{3}}^{(\omega)}\left(\theta_{3}\right)+i \omega \tilde{c}_{i_{1} k} \tilde{S}_{i_{2} i_{3}}^{k l}\left(\beta-\theta_{3}\right) \tilde{F}_{p ; l}^{(\omega)}\left(\theta_{3}\right)\right\} \tag{8.9}
\end{equation*}
$$

Here the chiral Gross-Neveu S-matrix $\tilde{S}_{i j}^{k l}(\theta)$, the anti-symmetric charge conjugation matrix $\tilde{c}_{i_{1} i_{2}}$ and the one-particle form factors $\tilde{F}_{p ; l}^{(\omega)}(\theta)$ are all defined in Appendix D.

It is easy to show that the homogeneous form factor equations are satisfied by (8.5) if the scalar prefactor $D$ is shift-invariant, anti-symmetric under the exchange of any pair of rapidities and is $2 \pi i$-periodic in all rapidity variables. It also has to have a first order zero at points where two rapidities differ by $i \pi$ in order to satisfy the residue equation as well. The solution is

$$
\begin{equation*}
D\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=\frac{-i}{32} \prod_{i<j} \operatorname{coth}\left(\frac{\theta_{i}-\theta_{j}}{2}\right) \tag{8.10}
\end{equation*}
$$

which also has the right normalization. With this choice we have

$$
\begin{aligned}
f_{P ; A B C}(\alpha, \beta, \theta) \approx & \frac{i}{\alpha-\beta-i \pi} \sum_{\omega}\left\{\tilde{c}_{a_{1} b_{1}} \tilde{c}_{p_{1} c_{1}}+i \omega \tilde{c}_{a_{1} k_{1}} \tilde{c}_{p_{1} l_{1}} \tilde{S}_{b_{1} c_{1}}^{k_{1} l_{1}}(\beta-\theta)\right\} \\
& \times\left\{\tilde{c}_{a_{2} b_{2}} \tilde{c}_{p_{2} c_{2}}+i \bar{\omega} \tilde{c}_{a_{2} k_{2}} \tilde{c}_{p_{2} l_{2}} \tilde{S}_{b_{2} c_{2}}^{k_{2} l_{2}}(\beta-\theta)\right\} \\
= & \frac{2 i}{\alpha-\beta-i \pi}\left\{\bar{C}_{A B} \bar{C}_{P C}-\bar{C}_{A K} \bar{C}_{P L} S_{B C}^{K L}(\beta-\theta)\right\}
\end{aligned}
$$

Here the $\mathrm{O}(4)$ charge conjugation matrix $\bar{C}_{A B}$ is defined in (D.10).
We parameterize the $\mathrm{O}(4)$ form factors as follows:

$$
\begin{align*}
f_{P ; A B C}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)= & \bar{C}_{P A} \bar{C}_{B C} g_{1}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)+\bar{C}_{P B} \bar{C}_{A C} g_{2}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \\
& +\bar{C}_{P C} \bar{C}_{A B} g_{3}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \tag{8.11}
\end{align*}
$$

Using the tensor product solution we can write this as

$$
\begin{align*}
& g_{2}+g_{3}=2 D F_{-}^{(+)} F_{-}^{(-)}, \\
& g_{2}-g_{3}=D\left(F_{+}^{(+)} F_{-}^{(-)}+F_{-}^{(+)} F_{+}^{(-)}\right), \\
& 4 g_{1}+g_{2}+g_{3}=2 D F_{+}^{(+)} F_{+}^{(-)} . \tag{8.12}
\end{align*}
$$

For $\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=(\Delta, \alpha, \beta)$ and using the large $\Delta$ expansion of the components $F_{ \pm}^{(\omega)}$ described in Appendix D we find for the $\mathrm{O}(4)$ form factors

$$
\begin{aligned}
& g_{2}+g_{3} \approx-\frac{2 \pi}{\Delta^{2}}(i \pi-\xi) \psi_{1}(\xi)\left\{g_{0}+\frac{1}{4}\left[\Psi\left(\frac{1}{2}+\frac{\xi}{2 \pi i}\right)+\Psi\left(\frac{1}{2}-\frac{\xi}{2 \pi i}\right)\right]\right\}+\cdots \\
& g_{2}-g_{3} \approx-\frac{2 \pi \psi_{1}(\xi)}{\Delta-\frac{\alpha+\beta}{2}}\left\{1+\frac{i \pi}{\Delta}+\cdots\right\} \\
& 4 g_{1}+g_{2}+g_{3} \approx \frac{3 \pi}{2 \Delta^{2}} \psi_{1}(\xi)(\xi+i \pi)+\cdots
\end{aligned}
$$

where $\xi=\alpha-\beta$. We see that this is exactly the same expansion as the one we found by directly solving the $\mathrm{O}(4)$ form factor equations asymptotically in Section 4.5 . Only the value of $g_{0}$ cannot be determined from the asymptotic solution. This we found in Appendix D by expanding the complete solution for large $\Delta$ :

$$
\begin{equation*}
g_{0}=1-\frac{1}{2} \ln \frac{8 \Delta}{\pi} . \tag{8.13}
\end{equation*}
$$

## 9. Concluding remarks

In the course of this work various intriguing relations concerning form factor clustering in the $\mathrm{O}(n)$ sigma-models were discussed and new structures revealed. The relationship of the pattern of clustering to the classical field equations in the case of $\mathrm{O}(3)$ has been previously known [1, 3]. Some of these patterns in particular those involving the Sugawara structure of the energymomentum tensor and those involving the (non-Abelian) curl-freeness of the Noether current (which is so important for integrability) probably extend to general $n$. We have further formulated a conjecture in Section 4.7 concerning the (on-shell) nature of clustering to the operator product expansion.

We have tested our ansatz in various examples. Firstly we checked that the solutions obtained by solving the form factor equations in leading order $1 / n$ coincided with those obtained by the
field theoretical approach to the model. Although this is as generally expected, it constitutes yet another test of the proposed equivalence of the S-matrix bootstrap construction and functional integral definition of the models. We also found that the large $n$ limit and limit of large rapidity commute. Although this is observed in previous studies it is not an obvious fact (recall that there are many examples where the large $n$ limit and limit of small rapidity do not commute).

The case $n=4$ is a special case. Here we studied in detail (to our knowledge for the first time) the 3-particle spin form factor. The tensor product structure is probably particular to this case but its general form involving the hypergeometric functions may give a hint to the outstanding unsolved problem of the construction of form factors for general $n>4$. Moreover in this case we found that subleading terms in the form factor clustering involved also logarithms of the (large) rapidity shift.

As mentioned in the introduction it is not completely implausible that some of the structural properties found concerning form factor clustering in integrable 2d asymptotically free models have their analog in 4-dimensional models in particular for processes when the kinematics effectively reduces the dimension to $1+1$.

## Acknowledgements

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## Appendix A. Explicit S-matrices

## A.1. The $n=3$ case

For $n=3$ the kernel $\tilde{K}_{3}(\omega)=\mathrm{e}^{-\pi \omega}$ and the integral in (2.3) is easily done. The result is well known [15]:

$$
\begin{align*}
\sigma_{1}(\theta) & =\frac{2 \pi i \theta}{(\theta+i \pi)(\theta-2 \pi i)}, \\
\sigma_{2}(\theta) & =\frac{\theta(\theta-i \pi)}{(\theta+i \pi)(\theta-2 \pi i)}, \\
\sigma_{3}(\theta) & =\frac{2 \pi i(i \pi-\theta)}{(\theta+i \pi)(\theta-2 \pi i)} . \tag{A.1}
\end{align*}
$$

A.2. The $n=4$ case

In this case (2.2) and (2.3) simplify to

$$
\begin{align*}
\sigma_{1}(\theta) & =\frac{i \pi \theta}{(i \pi-\theta)^{2}} S^{(2)}(\theta) \\
\sigma_{2}(\theta) & =\frac{\theta}{\theta-i \pi} S^{(2)}(\theta) \\
\sigma_{3}(\theta) & =\frac{i \pi}{i \pi-\theta} S^{(2)}(\theta), \quad S^{(2)}(\theta)=-\mathcal{A}^{2}(\theta) \tag{A.2}
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{A}(\theta)=-\exp \left\{2 i \int_{0}^{\infty} \frac{d \omega}{\omega} \frac{\sin (\theta \omega)}{1+\mathrm{e}^{\tau \omega}}\right\} \tag{A.3}
\end{equation*}
$$

The $\mathrm{O}(4)$ S-matrix is here given in the real basis

$$
\begin{equation*}
|a, \theta\rangle, \quad a=1,2,3,4 \tag{A.4}
\end{equation*}
$$

It is useful to transform the particles into a complex $\mathrm{SU}(2) \times \mathrm{SU}(2)$ basis

$$
\begin{equation*}
|A, \theta\rangle, \quad A=++,--,+-,-+. \tag{A.5}
\end{equation*}
$$

The transformation and its inverse are given by

$$
\begin{equation*}
|a, \theta\rangle=\Omega_{a}{ }^{A}|A, \theta\rangle, \quad|A, \theta\rangle=\mathcal{K}_{A}{ }^{a}|a, \theta\rangle, \tag{A.6}
\end{equation*}
$$

with $\mathcal{K}_{A}{ }^{a} \Omega_{a}{ }^{B}=\delta_{A}^{B}, \Omega_{a}{ }^{A} \mathcal{K}_{A}{ }^{b}=\delta_{a}^{b}$.
The transformation rule of the S-matrix is

$$
\begin{equation*}
S_{A B}^{C D}(\theta)=\mathcal{K}_{A}{ }^{a} \mathcal{K}_{B}{ }^{b} \Omega_{c}{ }^{C} \Omega_{d}{ }^{D} S_{a b}^{c d}(\theta) \tag{A.7}
\end{equation*}
$$

Using the $\mathrm{O}(4) \mathrm{S}$-matrix explicitly we get

$$
\begin{equation*}
S_{A B}^{C D}(\theta)=\sigma_{1}(\theta) P^{C D} R_{A B}+\sigma_{2}(\theta) \delta_{A}^{C} \delta_{B}^{D}+\sigma_{3}(\theta) \delta_{B}^{C} \delta_{A}^{D}, \tag{A.8}
\end{equation*}
$$

where

$$
\begin{equation*}
P^{C D}=\Omega_{x}{ }^{C} \Omega_{x}{ }^{D}, \quad R_{A B}=\mathcal{K}_{A}{ }^{x} \mathcal{K}_{B}^{x} . \tag{A.9}
\end{equation*}
$$

Next we define the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ basis explicitly. The $\mathrm{O}(4)$ generators in the vector representation are

$$
\begin{equation*}
\left(\tau^{a b}\right)_{x y}=i\left(\delta^{a x} \delta^{b y}-\delta^{a y} \delta^{b x}\right), \quad a, b=1,2,3,4 . \tag{A.10}
\end{equation*}
$$

We now define

$$
\begin{equation*}
V^{k}=\frac{1}{2} \epsilon^{k l m} \tau^{l m} \quad \text { and } \quad A^{k}=\tau^{k 4}, \quad k, l, m=1,2,3 . \tag{A.11}
\end{equation*}
$$

Further

$$
\begin{equation*}
W_{ \pm}^{k}=\frac{1}{2}\left(V^{k} \pm A^{k}\right) \tag{A.12}
\end{equation*}
$$

These are the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ generators since

$$
\begin{equation*}
\left[W_{+}^{k}, W_{-}^{l}\right]=0, \quad\left[W_{ \pm}^{k}, W_{ \pm}^{l}\right]=-i \epsilon^{k l m} W_{ \pm}^{m} \tag{A.13}
\end{equation*}
$$

We now define $\mathrm{SU}(2) \times \mathrm{SU}(2)$ particle states such that

$$
\begin{align*}
& |++, \theta\rangle \quad \text { has eigenvalue } \frac{1}{2} \text { w.r.t. } W_{+}^{3} \text { and } \frac{1}{2} \text { w.r.t. } W_{-}^{3},  \tag{A.14}\\
& |+-, \theta\rangle \quad \text { has eigenvalue } \frac{1}{2} \text { w.r.t. } W_{+}^{3} \text { and }-\frac{1}{2} \text { w.r.t. } W_{-}^{3}, \tag{A.15}
\end{align*}
$$

and so on. Here the phases are also important. We make the following choice

$$
\begin{aligned}
& |++, \theta\rangle=\frac{1}{\sqrt{2}}\{|1, \theta\rangle-i|2, \theta\rangle\}, \\
& |--, \theta\rangle=\frac{1}{\sqrt{2}}\{|1, \theta\rangle+i|2, \theta\rangle\}, \\
& |+-, \theta\rangle=\frac{1}{\sqrt{2}}\{i|3, \theta\rangle+|4, \theta\rangle\}, \\
& |-+, \theta\rangle=\frac{1}{\sqrt{2}}\{i|3, \theta\rangle-|4, \theta\rangle\} .
\end{aligned}
$$

Now we calculate

$$
\begin{equation*}
P^{C D}=\omega_{C} \delta^{C \bar{D}} \quad(\text { no sum }), \quad R_{A B}=\omega_{A} \delta_{A \bar{B}} \quad(\text { no sum }) \tag{A.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega_{++}=\omega_{--}=1 \quad \text { and } \quad \omega_{+-}=\omega_{-+}=-1 \tag{A.17}
\end{equation*}
$$

and $\bar{B}$ is the charge conjugate of $B$.

## A.3. Tensor product S-matrix

The S-matrix of the $\mathrm{SU}(2)$ chiral Gross-Neveu model is [1]

$$
\begin{equation*}
\tilde{S}_{\alpha \beta}^{\gamma \delta}(\theta)=\frac{\mathcal{A}(\theta)}{i \pi-\theta}\left\{i \pi \delta_{\beta}^{\gamma} \delta_{\alpha}^{\delta}-\theta \delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta}\right\}, \tag{A.18}
\end{equation*}
$$

where $\alpha, \beta, \gamma, \delta=+,-$. Taking the tensor product of two such S-matrices we get

$$
\begin{align*}
-\tilde{S}_{\alpha_{1} \beta_{1}}^{\gamma_{1} \delta_{1}}(\theta) \tilde{S}_{\alpha_{2} \beta_{2}}^{\gamma_{2} \delta_{2}}(\theta)= & \sigma_{2}(\theta) \delta_{\alpha_{1}}^{\gamma_{1}} \delta_{\alpha_{2}}^{\gamma_{2}} \delta_{\beta_{1}}^{\delta_{1}} \delta_{\beta_{2}}^{\delta_{2}}+\sigma_{3}(\theta) \delta_{\beta_{1}}^{\gamma_{1}} \delta_{\beta_{2}}^{\gamma_{2}} \delta_{\alpha_{1}}^{\delta_{1}} \delta_{\alpha_{2}}^{\delta_{2}} \\
& +\sigma_{1}(\theta)\left(\delta_{\beta_{1}}^{\gamma_{1}} \delta_{\alpha_{1}}^{\delta_{1}}-\delta_{\alpha_{1}}^{\gamma_{1}} \delta_{\beta_{1}}^{\delta_{1}}\right)\left(\delta_{\beta_{2}}^{\gamma_{2}} \delta_{\alpha_{2}}^{\delta_{2}}-\delta_{\alpha_{2}}^{\gamma_{2}} \delta_{\beta_{2}}^{\delta_{2}}\right) \tag{A.19}
\end{align*}
$$

which is the same as (A.8) if we make the identification $A=\left(\alpha_{1}, \alpha_{2}\right)$, etc.

## Appendix B. Some form factors for the case $\boldsymbol{n}=3$

For $n=3$ we can define $J_{\mu}^{a}=\frac{1}{2} \epsilon^{a b c} J_{\mu}^{b c}$ and rewrite (2.13) as

$$
\begin{equation*}
\langle 0| J_{\mu}^{a}(0)\left|b_{1}, \theta_{1} ; \ldots ; b_{r}, \theta_{r}\right\rangle^{\text {in }}=-i \epsilon_{\mu \alpha} q^{\alpha} f_{b_{1} \ldots b_{r}}^{a}\left(\theta_{1}, \ldots, \theta_{r}\right) . \tag{B.1}
\end{equation*}
$$

Using this unified notation ${ }^{8}$ for the form factors of the $\mathrm{O}(3)$ field and the Noether current we can introduce reduced form factors $g_{b_{1} \ldots b_{r}}^{a}$ by

$$
\begin{equation*}
f_{b_{1} \ldots b_{r}}^{a}\left(\theta_{1}, \ldots, \theta_{r}\right)=\Psi_{r}\left(\theta_{1}, \ldots, \theta_{r}\right) g_{b_{1} \ldots b_{r}}^{a}\left(\theta_{1}, \ldots, \theta_{r}\right), \tag{B.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\Psi_{r}\left(\theta_{1}, \ldots, \theta_{r}\right)=\frac{1}{2} \pi^{\left(\frac{3 r}{2}-1\right)} \prod_{1 \leqslant i<j \leqslant r} \psi\left(\theta_{i}-\theta_{j}\right) \tag{B.3}
\end{equation*}
$$

[^8]with
\[

$$
\begin{equation*}
\psi(\theta)=\frac{\theta-i \pi}{\theta(2 \pi i-\theta)} \tanh ^{2}\left(\frac{\theta}{2}\right) \tag{B.4}
\end{equation*}
$$

\]

Similarly for the energy-momentum tensor we define

$$
\begin{equation*}
f_{b_{1} \ldots b_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right)=\Psi_{r}\left(\theta_{1}, \ldots, \theta_{r}\right) g_{b_{1} \ldots b_{r}}\left(\theta_{1}, \ldots, \theta_{r}\right) \tag{B.5}
\end{equation*}
$$

The advantage of using these reduced form factors is that they, with only one exception, are polynomial expressions in the particle rapidities [3].

The first reduced form factors for the $\mathrm{O}(3)$ field are $g_{b_{1}}^{a}\left(\theta_{1}\right)=\delta^{a b_{1}}$ and

$$
\begin{align*}
& g_{b_{1} b_{2} b_{3}}^{a}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \\
& \quad=\delta^{a b_{3}} \delta^{b_{1} b_{2}}\left(\theta_{2}-\theta_{1}\right)+\delta^{a b_{2}} \delta^{b_{1} b_{3}}\left(\theta_{1}-\theta_{3}-2 \pi i\right)+\delta^{a b_{1}} \delta^{b_{2} b_{3}}\left(\theta_{3}-\theta_{2}\right) . \tag{B.6}
\end{align*}
$$

For the current we have

$$
\begin{equation*}
g_{b_{1} b_{2}}^{a}\left(\theta_{1}, \theta_{2}\right)=\epsilon^{a b_{1} b_{2}} \tag{B.7}
\end{equation*}
$$

and finally for the energy-momentum tensor ${ }^{9}$

$$
\begin{equation*}
g_{b_{1} b_{2}}\left(\theta_{1}, \theta_{2}\right)=\frac{\delta^{b_{1} b_{2}}}{\theta_{1}-\theta_{2}-i \pi} \tag{B.8}
\end{equation*}
$$

Note that for $n=3$ the functions $\psi_{0}, \psi_{1}$ defined in (2.19) are given by

$$
\begin{equation*}
\psi_{1}(\theta)=-\frac{\pi^{2}}{2} \psi(\theta), \quad \psi_{0}(\theta)=\frac{i \pi^{2} \psi(\theta)}{\theta-i \pi} \tag{B.9}
\end{equation*}
$$

Many further explicit examples can be found in Ref. [3].

## Appendix C. Current 6-particle function

$$
\begin{aligned}
P_{6}^{(c)} & \left(\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5}, \theta_{6}\right) \\
= & \frac{1}{32}\left[\left\{3 \cosh \frac{1}{2}\left(\theta_{1}+\theta_{2}+\theta_{3}-\theta_{4}-\theta_{5}-\theta_{6}\right)+3 \cosh \frac{1}{2}\left(\theta_{1}+\theta_{2}+\theta_{5}-\theta_{3}-\theta_{4}-\theta_{6}\right)\right.\right. \\
& +3 \cosh \frac{1}{2}\left(\theta_{1}+\theta_{3}+\theta_{4}-\theta_{2}-\theta_{5}-\theta_{6}\right)+4 \cosh \frac{1}{2}\left(\theta_{1}+\theta_{3}+\theta_{5}-\theta_{2}-\theta_{4}-\theta_{6}\right) \\
& +2 \cosh \frac{1}{2}\left(\theta_{1}+3 \theta_{3}-\theta_{2}-\theta_{4}-\theta_{5}-\theta_{6}\right)+2 \cosh \frac{1}{2}\left(\theta_{3}+3 \theta_{1}-\theta_{2}-\theta_{4}-\theta_{5}-\theta_{6}\right) \\
& +2 \cosh \frac{1}{2}\left(\theta_{1}+3 \theta_{5}-\theta_{2}-\theta_{3}-\theta_{4}-\theta_{6}\right)+2 \cosh \frac{1}{2}\left(\theta_{5}+3 \theta_{1}-\theta_{2}-\theta_{3}-\theta_{4}-\theta_{6}\right) \\
& +2 \cosh \frac{1}{2}\left(\theta_{3}+3 \theta_{5}-\theta_{1}-\theta_{2}-\theta_{4}-\theta_{6}\right)+2 \cosh \frac{1}{2}\left(\theta_{5}+3 \theta_{3}-\theta_{1}-\theta_{2}-\theta_{4}-\theta_{6}\right) \\
& +2 \cosh \frac{1}{2}\left(\theta_{3}+3 \theta_{4}-\theta_{1}-\theta_{2}-\theta_{5}-\theta_{6}\right)+2 \cosh \frac{1}{2}\left(\theta_{5}+3 \theta_{6}-\theta_{1}-\theta_{2}-\theta_{3}-\theta_{4}\right) \\
& +2 \cosh \frac{1}{2}\left(\theta_{1}+3 \theta_{2}-\theta_{3}-\theta_{4}-\theta_{5}-\theta_{6}\right)+2 \cosh \frac{1}{2}\left(\theta_{1}+\theta_{2}+3 \theta_{3}-\theta_{4}-\theta_{5}-3 \theta_{6}\right)
\end{aligned}
$$

[^9]\[

$$
\begin{align*}
& +2 \cosh \frac{1}{2}\left(\theta_{1}+\theta_{2}+3 \theta_{6}-\theta_{4}-\theta_{5}-3 \theta_{3}\right)+4 \cosh \frac{1}{2}\left(\theta_{1}+3 \theta_{2}+\theta_{3}-\theta_{4}-\theta_{5}-3 \theta_{6}\right) \\
& +2 \cosh \frac{1}{2}\left(\theta_{1}+3 \theta_{2}+\theta_{3}-\theta_{5}-\theta_{6}-3 \theta_{4}\right)+4 \cosh \frac{1}{2}\left(\theta_{1}+3 \theta_{2}+\theta_{5}-\theta_{3}-\theta_{6}-3 \theta_{4}\right) \\
& +2 \cosh \frac{1}{2}\left(\theta_{1}+3 \theta_{2}+\theta_{5}-\theta_{3}-\theta_{4}-3 \theta_{6}\right)+4 \cosh \frac{1}{2}\left(\theta_{1}+\theta_{3}+3 \theta_{4}-\theta_{2}-\theta_{5}-3 \theta_{6}\right) \\
& +2 \cosh \frac{1}{2}\left(\theta_{1}+\theta_{3}+3 \theta_{4}-3 \theta_{2}-\theta_{5}-\theta_{6}\right)+2 \cosh \frac{1}{2}\left(\theta_{1}+\theta_{5}+3 \theta_{6}-3 \theta_{2}-\theta_{3}-\theta_{4}\right) \\
& +\cosh \frac{1}{2}\left(3 \theta_{1}+3 \theta_{2}-\theta_{3}-\theta_{4}-\theta_{5}-3 \theta_{6}\right)+\cosh \frac{1}{2}\left(3 \theta_{1}+3 \theta_{2}-\theta_{3}-\theta_{5}-\theta_{6}-3 \theta_{4}\right) \\
& +\cosh \frac{1}{2}\left(3 \theta_{3}+3 \theta_{4}-\theta_{1}-\theta_{2}-\theta_{5}-3 \theta_{6}\right)+\cosh \frac{1}{2}\left(3 \theta_{3}+3 \theta_{4}-\theta_{1}-\theta_{5}-\theta_{6}-3 \theta_{2}\right) \\
& +\cosh \frac{1}{2}\left(3 \theta_{5}+3 \theta_{6}-\theta_{1}-\theta_{2}-\theta_{3}-3 \theta_{4}\right)+\cosh \frac{1}{2}\left(3 \theta_{5}+3 \theta_{6}-\theta_{1}-\theta_{3}-\theta_{4}-3 \theta_{2}\right) \\
& +\cosh \frac{1}{2}\left(\theta_{1}+3 \theta_{3}+3 \theta_{4}-\theta_{2}-3 \theta_{5}-3 \theta_{6}\right) \\
& +\cosh \frac{1}{2}\left(\theta_{3}+3 \theta_{1}+3 \theta_{2}-\theta_{4}-3 \theta_{5}-3 \theta_{6}\right) \\
& \left.+\cosh \frac{1}{2}\left(\theta_{5}+3 \theta_{1}+3 \theta_{2}-\theta_{6}-3 \theta_{3}-3 \theta_{4}\right)\right\}+\left\{\theta_{3} \leftrightarrow \theta_{4}\right\}+\left\{\theta_{5} \leftrightarrow \theta_{6}\right\} \\
& \left.+\left\{\theta_{3} \leftrightarrow \theta_{4}, \theta_{5} \leftrightarrow \theta_{6}\right\}\right]+\left[\theta_{1} \leftrightarrow \theta_{2}\right] . \tag{C.1}
\end{align*}
$$
\]

## Appendix D. XY model and chiral Gross-Neveu model form factors

The bootstrap solution of the XY model [19] is based on the extremal sine-Gordon S-matrix, which is given by

$$
\begin{align*}
& S_{--}^{--}(\theta)=S_{++}^{++}(\theta)=\mathcal{A}(\theta)  \tag{D.1}\\
& S_{+-}^{+-}(\theta)=S_{-+}^{-+}(\theta)=\frac{\kappa \theta}{i \pi-\theta} \mathcal{A}(\theta),  \tag{D.2}\\
& S_{-+}^{+-}(\theta)=S_{+-}^{-+}(\theta)=\frac{i \pi}{i \pi-\theta} \mathcal{A}(\theta) \tag{D.3}
\end{align*}
$$

where $\mathcal{A}(\theta)$ is given in (A.3) and $\kappa=1$ for the XY model S-matrix. The choice $\kappa=-1$ gives the $\mathrm{SU}(2)$ symmetric chiral Gross-Neveu S-matrix (A.18). We will keep the notation $S_{\alpha \beta}^{\gamma \delta}(\theta)$ for the XY model S-matrix and will denote the chiral Gross-Neveu S-matrix by $\tilde{S}_{\alpha \beta}^{\gamma \delta}(\theta)$.

For the XY model the crossing relation is

$$
\begin{equation*}
S_{\alpha \beta}^{\gamma \delta}(i \pi-\theta)=c_{\beta \mu} c_{\delta \nu} S_{\alpha \nu}^{\gamma \mu}(\theta) \tag{D.4}
\end{equation*}
$$

where the charge conjugation matrix $c_{\alpha \beta}$ has non-vanishing components

$$
\begin{equation*}
c_{+-}=c_{-+}=1 \tag{D.5}
\end{equation*}
$$

whereas for the chiral Gross-Neveu case crossing is given by

$$
\begin{equation*}
\tilde{S}_{\alpha \beta}^{\gamma \delta}(i \pi-\theta)=\tilde{c}_{\beta \mu} \tilde{c}_{\delta \nu} \tilde{S}_{\alpha \nu}^{\gamma \mu}(\theta), \tag{D.6}
\end{equation*}
$$

with

$$
\begin{equation*}
\tilde{c}_{+-}=-\tilde{c}_{-+}=i \tag{D.7}
\end{equation*}
$$

As discussed in Appendix A, the $\mathrm{O}(4) \mathrm{S}$-matrix is (minus) the tensor product of two chiral GrossNeveu S-matrices:

$$
\begin{equation*}
S_{A B}^{C D}(\theta)=-\tilde{S}_{\alpha_{1} \beta_{1}}^{\gamma_{1} \delta_{1}}(\theta) \tilde{S}_{\alpha_{2} \beta_{2}}^{\gamma_{2} \delta_{2}}(\theta) \tag{D.8}
\end{equation*}
$$

where $A=\left(\alpha_{1}, \alpha_{2}\right)$, etc. The crossing relation is

$$
\begin{equation*}
S_{A B}^{C D}(i \pi-\theta)=\bar{C}_{B M} \bar{C}_{D N} S_{A N}^{C M}(\theta) \tag{D.9}
\end{equation*}
$$

with

$$
\begin{equation*}
\bar{C}_{A B}=\tilde{c}_{\alpha_{1} \beta_{1}} \tilde{c}_{\alpha_{2} \beta_{2}} . \tag{D.10}
\end{equation*}
$$

## D.1. $\mathrm{SU}_{-1}$ (2) symmetry

As is well known, the sine-Gordon model has quantum group symmetry $\mathrm{SU}_{q}(2)$, which becomes $\mathrm{SU}_{-1}(2)$ in the extremal case. In this case the algebra of generators $\tau_{ \pm}, j$ is identical to the ordinary $\mathrm{SU}(2)$ algebra:

$$
\begin{equation*}
\left[\tau_{+}, \tau_{-}\right]=2 j, \quad\left[j, \tau_{ \pm}\right]= \pm \tau_{ \pm} \tag{D.11}
\end{equation*}
$$

it is only the co-product $\Delta$ that is different from the classical case:

$$
\begin{equation*}
\Delta(j)=j \otimes 1+1 \otimes j, \quad \Delta\left(\tau_{ \pm}\right)=\tau_{ \pm} \otimes(-1)^{2 j}+1 \otimes \tau_{ \pm} \tag{D.12}
\end{equation*}
$$

This means that if we build tensor product representations from the basic doublet representation, the representation matrices are given by

$$
\begin{align*}
& \Delta_{2}\left(\tau_{ \pm}\right)=-\tau_{ \pm} \otimes 1+1 \otimes \tau_{ \pm}  \tag{D.13}\\
& \Delta_{3}\left(\tau_{ \pm}\right)=\tau_{ \pm} \otimes 1 \otimes 1-1 \otimes \tau_{ \pm} \otimes 1+1 \otimes 1 \otimes \tau_{ \pm} \tag{D.14}
\end{align*}
$$

etc. These are representation matrices of the classical $\mathrm{SU}(2)$ algebra and they are simply related to the usual ones. For the two-particle case the relation is
where the $|\alpha \beta\rangle_{\mathrm{cl}}$ states transform according to the usual two-particle representation. Similarly for higher states we have

$$
\begin{equation*}
\left|\alpha_{r} \ldots \alpha_{2} \alpha_{1}\right\rangle_{\mathrm{cl}}=\prod_{l=1}^{r}\left(\alpha_{l}\right)^{l+1}\left|\alpha_{r} \ldots \alpha_{2} \alpha_{1}\right\rangle \tag{D.16}
\end{equation*}
$$

We denote the $\mathrm{SU}(2)$ generators acting in the Hilbert space by $\hat{J}, \hat{T}_{ \pm}$. We will be looking for local fields $\phi_{ \pm}(z)$ transforming as elements of an $\mathrm{SU}(2)$ doublet:

$$
2\left[\hat{J}, \phi_{ \pm}(z)\right]= \pm \phi_{ \pm}(z), \quad\left[\hat{T}_{+}, \phi_{+}(z)\right]=\left[\hat{T}_{-}, \phi_{-}(z)\right]=0
$$

$$
\begin{equation*}
\left[\hat{T}_{+}, \phi_{-}(z)\right]=\phi_{+}(z), \quad\left[\hat{T}_{-}, \phi_{+}(z)\right]=\phi_{-}(z) \tag{D.17}
\end{equation*}
$$

Note that $\left(\phi_{+}\right)^{\dagger} \neq c \phi_{-}$for any constant $c$ (the above equations cannot have such solutions) i.e. the doublet fields must be genuinely complex.
$\mathrm{SU}(2)$ symmetry restricts the form factors of doublet operators. For the 1-particle form factors we have

$$
\begin{equation*}
F_{p ; a}(\theta)=\langle 0| \phi_{p}(0)|a, \theta\rangle=i G(\theta) \tilde{c}_{p a} . \tag{D.18}
\end{equation*}
$$

For the 3-particle ones

$$
\begin{equation*}
F_{p ; a b c}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=\langle 0| \phi_{p}(0)\left|a, \theta_{1} ; b, \theta_{2} ; c, \theta_{3}\right\rangle \tag{D.19}
\end{equation*}
$$

we introduce the notation

$$
\begin{align*}
& F_{-;++-}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=F_{1}\left(\theta_{1}, \theta_{2}, \theta_{3}\right), \\
& F_{-;+-+}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=F_{2}\left(\theta_{1}, \theta_{2}, \theta_{3}\right), \\
& F_{-;-++}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=F_{3}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) . \tag{D.20}
\end{align*}
$$

Note that all other components either vanish by charge conservation or are related to these ones by charge conjugation:

$$
\begin{equation*}
F_{\bar{p} ; \bar{a} \bar{b} \bar{c}}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=-F_{p ; a b c}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) . \tag{D.21}
\end{equation*}
$$

The restriction coming from $\mathrm{SU}(2)$ symmetry is

$$
\begin{equation*}
F_{1}-F_{2}+F_{3}=0 \tag{D.22}
\end{equation*}
$$

We also introduce the form factors corresponding to the manifestly $\mathrm{SU}(2)$ invariant basis (D.16):

$$
\begin{equation*}
\tilde{F}_{p ; a b c}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=\langle 0| \phi_{p}(0)\left|a, \theta_{1} ; b, \theta_{2} ; c, \theta_{3}\right\rangle_{\mathrm{cl}} . \tag{D.23}
\end{equation*}
$$

For these form factors we have

$$
\begin{align*}
& \tilde{F}_{1}=F_{1}, \quad \tilde{F}_{2}=-F_{2}, \quad \tilde{F}_{3}=F_{3}  \tag{D.24}\\
& \tilde{F}_{\bar{p} ; \bar{a} \bar{b} \bar{c}}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=\tilde{F}_{p ; a b c}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \tag{D.25}
\end{align*}
$$

and

$$
\begin{equation*}
\tilde{F}_{1}+\tilde{F}_{2}+\tilde{F}_{3}=0 \tag{D.26}
\end{equation*}
$$

We note that the basic spin fields [19] of the XY model,

$$
\begin{equation*}
S^{ \pm}(z)=S^{1}(z) \pm S^{2}(z) \tag{D.27}
\end{equation*}
$$

obviously satisfy $\left(S^{+}\right)^{\dagger}=S^{-}$and hence cannot be elements of a doublet.
In the following we will consider the form factors of not only the doublet operators but also more general, charge -1 , spin $s$ fields. We will use the notation (D.20) also for these more general form factors. Of course, (D.22) only holds for the $\mathrm{SU}(2)$ doublet case.

## D.2. 3-particle form factor equations for general spin

We recall the bootstrap equations satisfied by the form factors of a charge -1 , spin $s$ operator (which may or may not be the lower component of an $\mathrm{SU}(2)$ doublet). The homogeneous equations are:

$$
\begin{align*}
& F_{i_{1} i_{2}{ }_{3}}\left(\theta_{1}+\lambda, \theta_{2}+\lambda, \theta_{3}+\lambda\right)=\mathrm{e}^{s \lambda} F_{i_{1} i_{2} i_{3}}\left(\theta_{1}, \theta_{2}, \theta_{3}\right),  \tag{D.28}\\
& F_{i_{1} i_{2} i_{3}}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=S_{i_{2} i_{3}}^{u v}\left(\theta_{2}-\theta_{3}\right) F_{i_{1} v u}\left(\theta_{1}, \theta_{3}, \theta_{2}\right),  \tag{D.29}\\
& F_{i_{1} i_{2} i_{3}}\left(\theta_{1}+2 \pi i, \theta_{2}, \theta_{3}\right)=\eta_{i_{1}} F_{i_{2} i_{3} i_{1}}\left(\theta_{2}, \theta_{3}, \theta_{1}\right) . \tag{D.30}
\end{align*}
$$

This is supplemented by the residue equation

$$
\begin{equation*}
F_{i_{1} i_{2} i_{3}}\left(\alpha, \beta, \theta_{3}\right) \approx \frac{4 i}{\alpha-\beta-i \pi}\left\{c_{i_{1} i_{2}} F_{i_{3}}\left(\theta_{3}\right)-\eta_{i_{1}} c_{i_{1} k} S_{i_{2} i_{3}}^{k l}\left(\beta-\theta_{3}\right) F_{l}\left(\theta_{3}\right)\right\} . \tag{D.31}
\end{equation*}
$$

Here and in the cyclic equation (D.30) $\eta_{i_{1}}$ is a phase factor that expresses the relative nonlocality between the field, whose form factors we are constructing and the basic spin fields $S^{ \pm}$ that create the asymptotic particles using the LSZ asymptotic formula. Consistency between the cyclic equation (D.30) and the shift equation (D.28) requires that

$$
\begin{equation*}
\eta_{+}=\eta=\mathrm{e}^{2 \pi i s}, \quad \eta_{-}=\mathrm{e}^{-2 \pi i s} \tag{D.32}
\end{equation*}
$$

and this is sufficient to determine the one-point function

$$
\begin{equation*}
F_{j}(\theta)=g \delta_{j}^{+} \mathrm{e}^{s \theta} \tag{D.33}
\end{equation*}
$$

up to the normalization constant $g$.
If we write the shift and cyclic equations in terms of the independent components $F_{1}, F_{2}, F_{3}$ we get

$$
\begin{equation*}
F_{k}\left(\theta_{1}+\lambda, \theta_{2}+\lambda, \theta_{3}+\lambda\right)=\mathrm{e}^{s \lambda} F_{k}\left(\theta_{1}, \theta_{2}, \theta_{3}\right), \quad k=1,2,3, \tag{D.34}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{k}\left(\theta_{1}+2 \pi i, \theta_{2}, \theta_{3}\right)=\eta F_{k+1}\left(\theta_{2}, \theta_{3}, \theta_{1}\right), \quad k=1,2 \tag{D.35}
\end{equation*}
$$

The $k=3$ equation $F_{3}\left(\theta_{1}+2 \pi i, \theta_{2}, \theta_{3}\right)=\eta^{-1} F_{1}\left(\theta_{2}, \theta_{3}, \theta_{1}\right)$ is already a consequence of the above two.

We have seen that $\mathrm{SU}(2)$ symmetry requires

$$
\begin{equation*}
\zeta=F_{1}-F_{2}+F_{3}=0 \tag{D.36}
\end{equation*}
$$

For later purposes we introduce

$$
\begin{equation*}
F_{ \pm}=F_{1} \pm F_{2} \tag{D.37}
\end{equation*}
$$

in terms of which (D.36) can also be written as $F_{3}=-F_{-}$. From (D.35) it follows that

$$
\begin{equation*}
\zeta\left(\theta_{1}+2 \pi i, \theta_{2}, \theta_{3}\right)=-\eta \zeta\left(\theta_{2}, \theta_{3}, \theta_{1}\right)+\left(\eta+\frac{1}{\eta}\right) F_{1}\left(\theta_{2}, \theta_{3}, \theta_{1}\right) \tag{D.38}
\end{equation*}
$$

thus an $\mathrm{SU}(2)$ doublet field must have $\eta= \pm i$ i.e. spin $s= \pm 1 / 4(\bmod 1)$. We will consider two such doublet solutions $\phi_{p}^{(\omega)}(z)$ with $s=\omega / 4, \eta=\omega i(\omega= \pm)$. It is natural to write the form factors in this case using the manifestly symmetric basis vectors (D.16). We choose the
normalization $\left(g^{( \pm)}=2\right)$ such that the 1-particle form factors are given by

$$
\begin{equation*}
\tilde{F}_{p ; a}^{(\omega)}(\theta)=\langle 0| \phi_{p}^{(\omega)}(0)|a, \theta\rangle=2 i \tilde{c}_{p a} \mathrm{e}^{\omega \theta / 4} \tag{D.39}
\end{equation*}
$$

Written in terms of the form factors of these operators, the three-particle equations (D.28)(D.31) become the form factor equations (8.6)-(8.9), discussed in the main text.

## D.3. Reduced form factors

To simplify the solution of the 3-particle form factor equations we introduce a set of "reduced" form factors $f_{m}(m=1,2,3)$ by writing [19]

$$
\begin{equation*}
F_{m}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=-2 \pi^{2} \mathcal{N} Y\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \mathrm{e}^{s\left(\theta_{1}+\theta_{2}+\theta_{3}\right)} f_{m}\left(\theta_{1}, \theta_{2}, \theta_{3}\right), \tag{D.40}
\end{equation*}
$$

where the prefactor $Y$ is

$$
\begin{equation*}
Y\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=\prod_{i<j} y\left(\theta_{i}-\theta_{j}\right) \tag{D.41}
\end{equation*}
$$

with

$$
\begin{equation*}
y(\theta)=\sinh \left(\frac{\theta}{2}\right) \mathrm{e}^{E(\theta)}, \tag{D.42}
\end{equation*}
$$

where

$$
\begin{equation*}
E(\theta)=\int_{0}^{\infty} \frac{\mathrm{d} \omega}{\omega} \frac{[\cosh \omega(\pi+i \theta)-1]}{\sinh \pi \omega} \frac{1}{\left(1+\mathrm{e}^{\pi \omega}\right)} \tag{D.43}
\end{equation*}
$$

Finally the normalization constant is

$$
\begin{equation*}
\mathcal{N}=\frac{i}{\pi^{11 / 2}} \mathrm{e}^{-E(0)} \mathrm{e}^{-i \pi s} \tag{D.44}
\end{equation*}
$$

Note that the function $E(\theta)$ is related to $\psi_{1}(\theta)$ (defined in (3.17)) for the case $n=4$ by

$$
\begin{equation*}
\left.\psi_{1}(\theta)\right|_{n=4}=\frac{2 i \sinh \left(\frac{\theta}{2}\right)}{i \pi-\theta} \mathrm{e}^{2 E(\theta)} \tag{D.45}
\end{equation*}
$$

and for large $\theta$ it behaves as

$$
\begin{equation*}
4 E(\theta)=-\theta+\ln (2 \theta)+2 E(0)+i \pi\left(1-\theta^{-1}\right)+\mathrm{O}\left(\theta^{-2}\right) \tag{D.46}
\end{equation*}
$$

For later use we introduce the function

$$
\begin{equation*}
\Phi(\theta)=\Gamma\left(\frac{1}{2}+\frac{\theta}{2 \pi i}\right) \Gamma\left(-\frac{\theta}{2 \pi i}\right) . \tag{D.47}
\end{equation*}
$$

With the help of this function we can write the S -matrix element $\mathcal{A}(\theta)$ as

$$
\begin{equation*}
\mathcal{A}(\theta)=\frac{\Phi(\theta)}{\Phi(-\theta)} \tag{D.48}
\end{equation*}
$$

For completeness, we give here the form factor equations, rewritten in terms of the reduced form factors $f_{m}$ :

$$
\begin{equation*}
f_{m}\left(\theta_{1}+\lambda, \theta_{2}+\lambda, \theta_{3}+\lambda\right)=\mathrm{e}^{-2 s \lambda} f_{m}\left(\theta_{1}, \theta_{2}, \theta_{3}\right), \tag{D.49}
\end{equation*}
$$

$$
\begin{align*}
& f_{3}\left(\alpha, \theta, \theta^{\prime}\right)=f_{3}\left(\alpha, \theta^{\prime}, \theta\right), \\
& f_{-}\left(\alpha, \theta, \theta^{\prime}\right)=f_{-}\left(\alpha, \theta^{\prime}, \theta\right), \\
& f_{+}\left(\alpha, \theta, \theta^{\prime}\right)=\frac{i \pi+\theta-\theta^{\prime}}{i \pi-\theta+\theta^{\prime}} f_{+}\left(\alpha, \theta^{\prime}, \theta\right), \quad f_{ \pm}=f_{1} \pm f_{2},  \tag{D.50}\\
& f_{m}\left(\theta_{1}+2 \pi i, \theta_{2}, \theta_{3}\right)=f_{m+1}\left(\theta_{2}, \theta_{3}, \theta_{1}\right) \quad(m=1,2),  \tag{D.51}\\
& f_{1}(\alpha, \beta, \theta) \approx \frac{-2 i g \eta \pi^{2} \mathrm{e}^{-2 s \beta}}{\alpha-\beta-i \pi}\left\{\frac{-i \pi}{i \pi-\beta+\theta} \Phi(\beta-\theta)\right\}, \\
& f_{2}(\alpha, \beta, \theta) \approx \frac{-2 i g \pi^{2} \mathrm{e}^{-2 s \beta}}{\alpha-\beta-i \pi}\left\{\Phi(\theta-\beta)-\frac{\eta(\beta-\theta)}{i \pi-\beta+\theta} \Phi(\beta-\theta)\right\}, \\
& f_{3}(\alpha, \beta, \theta) \approx \frac{-2 i g \pi^{2} \mathrm{e}^{-2 s \beta}}{\alpha-\beta-i \pi}\left\{\Phi(\theta-\beta)-\frac{1}{\eta} \Phi(\beta-\theta)\right\} . \tag{D.52}
\end{align*}
$$

Later we will explicitly solve the form factor equations for the reduced form factors and calculate their large rapidity limit. But even before having the complete solution, a lot of information about their large rapidity behavior can already be obtained by expanding the equations themselves. We take the ansatz

$$
\begin{equation*}
f_{m}(\Delta, \alpha, \beta) \approx \frac{\mathrm{e}^{-\frac{1}{2} \Delta} \mathrm{e}^{k \alpha}}{(\Delta-\alpha)^{p}}\left\{U_{m}(\alpha-\beta)+\frac{W_{m}(\alpha-\beta)}{\Delta-\alpha}+\cdots\right\} \quad(m=1,2,3) \tag{D.53}
\end{equation*}
$$

for large $\Delta$, where $k=\frac{1}{2}-2 s$. We get restrictions on the leading power $p$ and the expansion coefficients $U_{m}, W_{m}$ by substituting this ansatz into the reduced form factor equations. In particular, from the residue equations, using the asymptotic expansion of $\Phi$, we get for large $\Delta$

$$
f_{m}(\alpha, \beta, \Delta) \approx \frac{-i c}{\alpha-\beta-i \pi} \mathrm{e}^{k \beta} \mathrm{e}^{-\frac{1}{2} \Delta}\left\{\begin{array}{l}
\frac{\eta}{(\Delta-\beta)^{3 / 2}}-\frac{3 i \pi \eta}{4 \Delta^{5 / 2}}+\cdots  \tag{D.54}\\
\frac{1+i \eta}{\pi} \frac{1}{(\Delta-\beta)^{1 / 2}}+\frac{3 \eta-i}{4} \frac{1}{\Delta^{3 / 2}}+\cdots \\
\frac{1}{\pi}\left(1-\frac{i}{\eta}\right) \frac{1}{(\Delta-\beta)^{1 / 2}}+\frac{1}{4}\left(\frac{1}{\eta}-i\right) \frac{1}{\Delta^{3 / 2}}+\cdots
\end{array}\right.
$$

where

$$
\begin{equation*}
c=4 \pi^{4} \sqrt{2 \pi} \mathrm{e}^{-\frac{i \pi}{4}} g \tag{D.55}
\end{equation*}
$$

From here we see that for most spin values for which $\eta \neq i\left(-\frac{1}{4} \leqslant s<\frac{1}{4}\right)$, the leading power is $p=1 / 2$ but for $s=\frac{1}{4}$ we have $\eta=i$ and the leading power is $p=3 / 2$. It is easy to solve the form factor equations in this expanded form. For the case $\eta=-i\left(s=-\frac{1}{4}\right)$ we get

$$
U_{m}(\xi)=\left(\begin{array}{l}
1  \tag{D.56}\\
1 \\
0
\end{array}\right) U(\xi), \quad W_{3}(\xi)=-W_{-}(\xi)
$$

where

$$
\begin{aligned}
& U(\xi)=-\frac{2 c}{\pi} \frac{\mathrm{e}^{-\frac{1}{2} \xi}}{(\pi i-\xi)} \\
& W_{+}(\xi)=\frac{c}{\pi} \frac{\mathrm{e}^{-\frac{1}{2} \xi}}{(\pi i-\xi)}(\xi-2 \pi i)
\end{aligned}
$$

$$
\begin{equation*}
W_{-}(\xi)=-\frac{2 c}{\pi} \mathrm{e}^{-\frac{1}{2} \xi}\left\{g_{0}+\frac{1}{4}\left[\Psi\left(\frac{1}{2}+\frac{\xi}{2 \pi i}\right)+\Psi\left(\frac{1}{2}-\frac{\xi}{2 \pi i}\right)\right]\right\} \tag{D.57}
\end{equation*}
$$

whereas for the case $\eta=i\left(s=\frac{1}{4}\right)$ we get

$$
U_{m}(\xi)=\left(\begin{array}{c}
-1 / 2  \tag{D.58}\\
1 / 2 \\
1
\end{array}\right) U(\xi), \quad W_{3}(\xi)=-W_{-}(\xi)
$$

and

$$
\begin{align*}
& U(\xi)=\frac{i c}{2 \cosh \frac{\xi}{2}} \\
& W_{-}(\xi)=\frac{3 i c}{8} \frac{(\xi-2 \pi i)}{\cosh \frac{\xi}{2}} \\
& W_{+}(\xi)=\frac{3 i c}{16} \frac{(\xi+\pi i)}{\cosh \frac{\xi}{2}} \tag{D.59}
\end{align*}
$$

As discussed in Section 4, there is a logarithmic piece in the subleading term of the $\mathrm{O}(4)$ form factors. As we will see later, such terms are also present in the asymptotic expansion of the spin $\pm 1 / 4$ form factors. Anticipating this fact we extend (D.53) by a logarithmic piece of the form

$$
\begin{equation*}
\frac{\mathrm{e}^{-\frac{1}{2} \Delta} \mathrm{e}^{k \alpha}}{(\Delta-\alpha)^{\tilde{p}}} \ln (\Delta-\alpha)\left\{\tilde{U}_{m}(\alpha-\beta)+\frac{\tilde{W}_{m}(\alpha-\beta)}{\Delta-\alpha}+\cdots\right\} \tag{D.60}
\end{equation*}
$$

The functions $\tilde{U}_{m}$ and $\tilde{W}_{m}$ satisfy very similar equations to the ones discussed above for $U_{m}$ and $W_{m}$. The main difference is that the residue equations are free of logs. We find that nonvanishing solutions are only possible for $\tilde{p}=1 / 2$ or $\tilde{p}=3 / 2$. Finally requiring regularity (also at infinity) eliminates all but one possibility: $\tilde{p}=3 / 2$ for $s=-1 / 4$ with solution

$$
\begin{equation*}
\tilde{U}_{-}^{(-)}(\xi)=\ell_{0} \mathrm{e}^{-\frac{1}{2} \xi} \tag{D.61}
\end{equation*}
$$

with arbitrary constant $\ell_{0}$. This means that the full asymptotic expansion (the sum of (D.53) and (D.60)) is correctly given by (D.53) alone, with solution (D.56)-(D.59), provided only that we allow the "constant" $g_{0}$ depend (linearly) on $\ln \Delta$. The actual value of $g_{0}$ can be calculated from the large $\Delta$ expansion of the exact solution. We now turn to this calculation.

## D.4. Contour integral solution

H. Babujian et al. [20] found the solution of the reduced form factor equations in terms of a contour integral:

$$
\begin{equation*}
f_{m}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=-\frac{1}{2 \pi^{2}} \int_{\mathcal{C}} \mathrm{d} u \mathrm{e}^{-2 s u} t_{m}\left(\theta_{1}, \theta_{2}, \theta_{3} ; u\right) \prod_{j=1}^{3} \Phi\left(\theta_{j}-u\right) \tag{D.62}
\end{equation*}
$$

where

$$
t_{1}\left(\theta_{1}, \theta_{2}, \theta_{3} ; u\right)=\frac{\theta_{1}-u}{i \pi-\theta_{1}+u} \frac{\theta_{2}-u}{i \pi-\theta_{2}+u} \frac{i \pi}{i \pi-\theta_{3}+u}
$$

$$
\begin{align*}
t_{2}\left(\theta_{1}, \theta_{2}, \theta_{3} ; u\right) & =\frac{\theta_{1}-u}{i \pi-\theta_{1}+u} \frac{i \pi}{i \pi-\theta_{2}+u} \\
t_{3}\left(\theta_{1}, \theta_{2}, \theta_{3} ; u\right) & =\frac{i \pi}{i \pi-\theta_{1}+u} \tag{D.63}
\end{align*}
$$

and the contour $\mathcal{C}$ (for real rapidities $\theta_{i}$ ) comes from $-\infty$ along a line parallel to the real axis and going somewhat below the singular points $\theta_{i}-i \pi$, then turns back and goes around the points $\theta_{i}$ before it goes to $+\infty$ again parallel to the real line. Precisely this integral along such a contour is the special function known as Meijer's G-function [21]:

$$
f_{m}\left(\theta_{1}, \theta_{2}, \theta_{3}\right)=G_{33}^{33}\left(\mathrm{e}^{-4 \pi i s} \left\lvert\, \begin{array}{ccc}
a_{1}^{(m)} & a_{2}^{(m)} & a_{3}^{(m)}  \tag{D.64}\\
b_{1}^{(m)} & b_{2}^{(m)} & b_{3}^{(m)}
\end{array}\right.\right) .
$$

The parameters depend on the rapidities:

$$
\begin{align*}
a_{1}^{(1)}=-\frac{i \theta_{1}}{2 \pi}, & a_{2}^{(1)}=-\frac{i \theta_{2}}{2 \pi}, & a_{3}^{(1)}=1-\frac{i \theta_{3}}{2 \pi}, \\
b_{1}^{(1)}=-\frac{1}{2}-\frac{i \theta_{1}}{2 \pi}, & b_{2}^{(1)}=-\frac{1}{2}-\frac{i \theta_{2}}{2 \pi}, & b_{3}^{(1)}=-\frac{1}{2}-\frac{i \theta_{3}}{2 \pi}, \\
a_{1}^{(2)}=-\frac{i \theta_{1}}{2 \pi}, & a_{2}^{(2)}=1-\frac{i \theta_{2}}{2 \pi}, & a_{3}^{(2)}=1-\frac{i \theta_{3}}{2 \pi}, \\
b_{1}^{(2)}=-\frac{1}{2}-\frac{i \theta_{1}}{2 \pi}, & b_{2}^{(2)}=-\frac{1}{2}-\frac{i \theta_{2}}{2 \pi}, & b_{3}^{(2)}=\frac{1}{2}-\frac{i \theta_{3}}{2 \pi}, \\
a_{1}^{(3)}=1-\frac{i \theta_{1}}{2 \pi}, & a_{2}^{(3)}=1-\frac{i \theta_{2}}{2 \pi}, & a_{3}^{(3)}=1-\frac{i \theta_{3}}{2 \pi}, \\
b_{1}^{(3)}=-\frac{1}{2}-\frac{i \theta_{1}}{2 \pi}, & b_{2}^{(3)}=\frac{1}{2}-\frac{i \theta_{2}}{2 \pi}, & b_{3}^{(3)}=\frac{1}{2}-\frac{i \theta_{3}}{2 \pi} . \tag{D.65}
\end{align*}
$$

Finally we note that Meijer's G-function $G_{33}^{33}$ can be expressed in terms of Gamma functions and hypergeometric functions as follows [21]:

$$
\begin{align*}
& G_{33}^{33}\left(z \left\lvert\, \begin{array}{lll}
a_{1} & a_{2} & a_{3} \\
b_{1} & b_{2} & b_{3}
\end{array}\right.\right) \\
& \quad=z^{b_{1}} \Omega\left(1-a_{1}+b_{1}, 1-a_{2}+b_{1}, 1-a_{3}+b_{1} ; b_{2}-b_{1}, b_{3}-b_{1} ;-z\right)+2 \text { perms } \tag{D.66}
\end{align*}
$$

where

$$
\begin{equation*}
\Omega\left(u_{1}, u_{2}, u_{3} ; v_{1}, v_{2} ; z\right)=\Gamma\left(u_{1}\right) \Gamma\left(u_{2}\right) \Gamma\left(u_{3}\right) \Gamma\left(v_{1}\right) \Gamma\left(v_{2}\right)_{3} F_{2}\left(u_{1}, u_{2}, u_{3} ; 1-v_{1}, 1-v_{2} ; z\right) \tag{D.67}
\end{equation*}
$$

This can be used to express the three-particle form factors as follows.

$$
\begin{align*}
& f_{1}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \\
& =\mathrm{e}^{2 \pi i s} \mathrm{e}^{-2 s \theta_{1}} \Omega\left(\frac{1}{2}, \frac{1}{2}-\frac{i \theta_{12}}{2 \pi},-\frac{1}{2}-\frac{i \theta_{13}}{2 \pi} ; \frac{i \theta_{12}}{2 \pi}, \frac{i \theta_{13}}{2 \pi} ;-\mathrm{e}^{-4 \pi i s}\right) \\
& \quad+\mathrm{e}^{2 \pi i s} \mathrm{e}^{-2 s \theta_{2}} \Omega\left(\frac{1}{2}+\frac{i \theta_{12}}{2 \pi}, \frac{1}{2},-\frac{1}{2}-\frac{i \theta_{23}}{2 \pi} ;-\frac{i \theta_{12}}{2 \pi}, \frac{i \theta_{23}}{2 \pi} ;-\mathrm{e}^{-4 \pi i s}\right) \\
& \quad+\mathrm{e}^{2 \pi i s} \mathrm{e}^{-2 s \theta_{3}} \Omega\left(\frac{1}{2}+\frac{i \theta_{13}}{2 \pi}, \frac{1}{2}+\frac{i \theta_{23}}{2 \pi},-\frac{1}{2} ;-\frac{i \theta_{13}}{2 \pi},-\frac{i \theta_{23}}{2 \pi} ;-\mathrm{e}^{-4 \pi i s}\right), \tag{D.68}
\end{align*}
$$

$$
\begin{align*}
& f_{2}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \\
& =\mathrm{e}^{2 \pi i s} \mathrm{e}^{-2 s \theta_{1}} \Omega\left(\frac{1}{2},-\frac{1}{2}-\frac{i \theta_{12}}{2 \pi},-\frac{1}{2}-\frac{i \theta_{13}}{2 \pi} ; \frac{i \theta_{12}}{2 \pi}, 1+\frac{i \theta_{13}}{2 \pi} ;-\mathrm{e}^{-4 \pi i s}\right) \\
& \quad+\mathrm{e}^{2 \pi i s} \mathrm{e}^{-2 s \theta_{2}} \Omega\left(\frac{1}{2}+\frac{i \theta_{12}}{2 \pi},-\frac{1}{2},-\frac{1}{2}-\frac{i \theta_{23}}{2 \pi} ;-\frac{i \theta_{12}}{2 \pi}, 1+\frac{i \theta_{23}}{2 \pi} ;-\mathrm{e}^{-4 \pi i s}\right) \\
& \quad+\mathrm{e}^{-2 \pi i s} \mathrm{e}^{-2 s \theta_{3}} \Omega\left(\frac{3}{2}+\frac{i \theta_{13}}{2 \pi}, \frac{1}{2}+\frac{i \theta_{23}}{2 \pi}, \frac{1}{2} ;-1-\frac{i \theta_{13}}{2 \pi},-1-\frac{i \theta_{23}}{2 \pi} ;-\mathrm{e}^{-4 \pi i s}\right), \tag{D.69}
\end{align*}
$$

$$
\begin{align*}
& f_{3}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \\
& =\mathrm{e}^{2 \pi i s} \mathrm{e}^{-2 s \theta_{1}} \Omega\left(-\frac{1}{2},-\frac{1}{2}-\frac{i \theta_{12}}{2 \pi},-\frac{1}{2}-\frac{i \theta_{13}}{2 \pi} ; 1+\frac{i \theta_{12}}{2 \pi}, 1+\frac{i \theta_{13}}{2 \pi} ;-\mathrm{e}^{-4 \pi i s}\right) \\
& \quad+\mathrm{e}^{-2 \pi i s} \mathrm{e}^{-2 s \theta_{2}} \Omega\left(\frac{1}{2}+\frac{i \theta_{12}}{2 \pi}, \frac{1}{2}, \frac{1}{2}-\frac{i \theta_{23}}{2 \pi} ;-1-\frac{i \theta_{12}}{2 \pi}, \frac{i \theta_{23}}{2 \pi} ;-\mathrm{e}^{-4 \pi i s}\right) \\
& \quad+\mathrm{e}^{-2 \pi i s} \mathrm{e}^{-2 s \theta_{3}} \Omega\left(\frac{1}{2}+\frac{i \theta_{13}}{2 \pi}, \frac{1}{2}+\frac{i \theta_{23}}{2 \pi}, \frac{1}{2} ;-1-\frac{i \theta_{13}}{2 \pi},-\frac{i \theta_{23}}{2 \pi} ;-\mathrm{e}^{-4 \pi i s}\right) . \tag{D.70}
\end{align*}
$$

We are interested in the asymptotics of these form factors in the limit $\theta_{1} \rightarrow+\infty$. The exponential part of the form factor asymptotics, which comes entirely from the Gamma functions, is

$$
\begin{equation*}
\mathrm{e}^{-(1+2 s) \theta_{1}} \tag{D.71}
\end{equation*}
$$

for the first of the three terms for all $f_{m}$ and is

$$
\begin{equation*}
\mathrm{e}^{-\frac{1}{2} \theta_{1}} \tag{D.72}
\end{equation*}
$$

for the second and third terms. Thus the exponential part of the asymptotics is given by (D.72), which comes from the second and third terms in almost all cases, except for $s=-1 / 4$, in which case also the first terms contribute.

To calculate the leading asymptotics of our form factors we will need the asymptotic behavior of the generalized hypergeometric functions ${ }_{3} F_{2}$ in the case of some of its parameters large. We will use a simple integral representation [22] of this function to establish the asymptotic formulae we need in this calculation.

$$
\begin{align*}
& { }_{3} F_{2}\left(a_{1}, a_{2}, a_{3} ; b_{1}, b_{2} ; z\right) \\
& \quad=\frac{\Gamma\left(b_{2}\right)}{\Gamma\left(a_{3}\right) \Gamma\left(b_{2}-a_{3}\right)} \int_{0}^{1} \mathrm{~d} t t^{a_{3}-1}(1-t)^{b_{2}-a_{3}-1}{ }_{2} F_{1}\left(a_{1}, a_{2} ; b_{1} ; t z\right), \tag{D.73}
\end{align*}
$$

which is valid for $\operatorname{Re}\left(b_{2}\right)>\operatorname{Re}\left(a_{3}\right)>0$ in the range $|z|<1$, but can be extended to the limit $|z| \rightarrow 1$. The Gauss hypergeometric function ${ }_{2} F_{1}$ can in turn be expressed as the integral

$$
\begin{equation*}
{ }_{2} F_{1}(a, b ; c ; z)=\frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{1} \mathrm{~d} t t^{b-1}(1-t)^{c-b-1}(1-t z)^{-a} \tag{D.74}
\end{equation*}
$$

valid for $\operatorname{Re}(c)>\operatorname{Re}(b)>0$.

Using both integral representations above simultaneously we can show that for large (real) $\lambda$

$$
\begin{align*}
& { }_{3} F_{2}\left(a_{1}, \alpha_{2}+i \lambda, \alpha_{3}+i \lambda ; \beta_{1}+i \lambda, \beta_{2}+i \lambda ; 1\right) \\
& \quad \approx \frac{\Gamma\left(\beta_{1}+\beta_{2}-\alpha_{2}-\alpha_{3}-a_{1}\right)}{\Gamma\left(\beta_{1}+\beta_{2}-\alpha_{2}-\alpha_{3}\right)}|\lambda|^{a_{1}} \mathrm{e}^{\frac{i \pi a_{1}}{2} \operatorname{sgn} \lambda} \tag{D.75}
\end{align*}
$$

valid for $\operatorname{Re}\left(\beta_{2}\right)>\operatorname{Re}\left(\alpha_{3}\right)>0$ and $\operatorname{Re}\left(\beta_{1}\right)>\operatorname{Re}\left(\alpha_{2}\right)>0$. We are sure however that this estimate holds in a larger range of parameters. In particular to get the contribution of the first terms for $f_{1}$, $f_{2}$ we need an estimate of the lhs of (D.75) for $a_{1}=1 / 2, \alpha_{2}=\frac{1}{2} \rho+\frac{i \theta_{2}}{2 \pi}, \alpha_{3}=-\frac{1}{2}+\frac{i \theta_{3}}{2 \pi}, \beta_{1}=$ $1+\frac{i \theta_{2}}{2 \pi}, \beta_{2}=\frac{1}{2}(1+\rho)+\frac{i \theta_{3}}{2 \pi}$ for the two cases $\rho= \pm 1$. We have numerically checked that the estimate (D.75) is indeed valid for these cases in the range $\left|\theta_{23}\right|<4 \pi$. It is plausible that it can also be proved for arbitrary values of $\mathfrak{R} \theta_{23}$ for some range of $\Im \theta_{23}$ by assuming analyticity of the formula in this variable. Applying then (D.75) to the first terms of (D.68)-(D.70) (for $s=-1 / 4$ ) we find

$$
f_{m}^{(I)}\left(\theta_{1}, \theta_{2}, \theta_{3}\right) \approx \frac{i c}{2} \mathrm{e}^{-\frac{1}{2} \theta_{1}} \frac{\mathrm{e}^{\frac{1}{2}\left(\theta_{2}+\theta_{3}\right)}}{\theta_{12}^{3 / 2}}\left(\begin{array}{c}
-1  \tag{D.76}\\
1 \\
2
\end{array}\right)
$$

Thus the first terms only contribute to $W_{-}$for $s=-1 / 4$. Their contribution is:

$$
\begin{equation*}
\left[W_{-}^{(-)}(\xi)\right]^{(I)}=-i c \mathrm{e}^{-\frac{1}{2} \xi} \tag{D.77}
\end{equation*}
$$

For the second and third terms (of $f_{1}$ and $f_{2}$ ) we have to consider, for large positive $\lambda=\frac{\theta_{12}}{2 \pi}$, the following product:

$$
\begin{align*}
& \Gamma\left(a_{1}\right) \Gamma\left(a_{2}\right) \Gamma\left(-a_{1}-a_{2}\right) \Gamma\left(\alpha_{3}+i \lambda\right) \Gamma\left(\frac{1}{2}-\alpha_{3}-i \lambda\right) \\
& \quad \times{ }_{3} F_{2}\left(a_{1}, a_{2}, \alpha_{3}+i \lambda ; 1+a_{1}+a_{2}, \frac{1}{2}+\alpha_{3}+i \lambda ; 1\right) . \tag{D.78}
\end{align*}
$$

Its asymptotic form can be established using the integral representation (D.73) together with the formula [23]

$$
\begin{align*}
& { }_{2} F_{1}\left(a_{1}, a_{2} ; 1+a_{1}+a_{2} ; t\right) \\
& \qquad \begin{array}{l}
\approx \frac{\Gamma\left(1+a_{1}+a_{2}\right)}{\Gamma\left(a_{1}\right) \Gamma\left(a_{2}\right)}\left\{\frac{1}{a_{1} a_{2}}+(1-t)[\ln (1-t)-\Psi(1)-\Psi(2)\right. \\
\left.\left.\quad+\Psi\left(1+a_{1}\right)+\Psi\left(1+a_{2}\right)\right]\right\},
\end{array} .
\end{align*}
$$

valid in the vicinity of $t=1$. With the help of this formula we can calculate the large $\lambda$ expansion of (D.78):

$$
\begin{equation*}
\frac{2 \pi^{2} i}{\sqrt{\lambda}} \mathrm{e}^{\frac{i \pi}{4}} \frac{\mathrm{e}^{-\pi \lambda} \mathrm{e}^{i \pi \alpha_{3}}}{\sin \pi\left(a_{1}+a_{2}\right)}\left\{\frac{1}{a_{1} a_{2}}-\frac{i}{2 \lambda}\left[X-\ln \lambda-\frac{i \pi}{2}+\Psi\left(\frac{3}{2}\right)\right]\right\}+\cdots \tag{D.80}
\end{equation*}
$$

where

$$
\begin{equation*}
X=\Psi\left(1+a_{1}\right)+\Psi\left(1+a_{2}\right)-\Psi(1)-\Psi(2)+\frac{\frac{1}{4}-\alpha_{3}}{a_{1} a_{2}} . \tag{D.81}
\end{equation*}
$$

Using (D.80) in (D.68) and (D.69) we find the following results. For $s=1 / 4$ the $\frac{\mathrm{e}^{-\frac{1}{2} \theta_{12}}}{\theta_{12}^{1 / 2}}$ terms cancel and only the $\frac{\mathrm{e}^{-\frac{1}{2} \theta_{12}}}{\theta_{12}^{3 / 2}}$ terms remain. In this case these contribute to the leading terms $U_{m}$ and we find

$$
\begin{equation*}
U^{(+)}(\xi)=\frac{i c}{2 \cosh \frac{\xi}{2}} \tag{D.82}
\end{equation*}
$$

the same result as we found in the previous subsection. For $s=-1 / 4$ (D.80) gives contributions both to the leading $U_{m}$ and the subleading $W_{m}$ terms. We find

$$
\begin{align*}
U^{(-)}(\xi) & =\frac{2 c}{\pi} \frac{\mathrm{e}^{-\frac{1}{2} \xi}}{\xi-i \pi}  \tag{D.83}\\
W_{+}^{(-)}(\xi) & =-\frac{c}{\pi} \mathrm{e}^{-\frac{1}{2} \xi} \frac{\xi-2 \pi i}{\xi-i \pi} \tag{D.84}
\end{align*}
$$

and

$$
\begin{align*}
& {\left[W_{-}^{(-)}(\xi)\right]^{(I I)+(I I I)} } \\
&=-\frac{2 c}{\pi} \mathrm{e}^{-\frac{1}{2} \xi}\left[\frac{1}{4} \Psi\left(\frac{1}{2}+\frac{i \xi}{2 \pi}\right)+\frac{1}{4} \Psi\left(\frac{1}{2}-\frac{i \xi}{2 \pi}\right)\right. \\
&\left.+\frac{1}{2} \Psi\left(\frac{3}{2}\right)+\frac{1}{2} \Psi\left(\frac{1}{2}\right)-\Psi(1)-\frac{1}{2} \ln \frac{\Delta}{2 \pi}-\frac{i \pi}{2}\right] . \tag{D.85}
\end{align*}
$$

Again, the above results are in agreement with the ones obtained in the previous subsection solving the asymptotic form factor equations. Finally we get

$$
\begin{equation*}
g_{0}=\Psi\left(\frac{1}{2}\right)-\Psi(1)+1-\frac{1}{2} \ln \frac{\Delta}{2 \pi}=1-\frac{1}{2} \ln \frac{8 \Delta}{\pi} . \tag{D.86}
\end{equation*}
$$

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[^1]:    ${ }^{1}$ It holds for example in approximations like naive vector meson dominance.

[^2]:    ${ }^{2}$ Recall that for particles with rapidity $\theta$ corresponding to 'bra' vectors in the expectation value the form factor functions have to be analytically continued to the complex rapidity value $\theta+i \pi$.

[^3]:    ${ }^{3}$ Here "new" is w.r.t. those usually defined by factoring out the product of 2-particle scalar form factors e.g as in Appendix B.

[^4]:    4 Note that assumption (i) is not always satisfied in specific models.

[^5]:    5 Note the relation $R(\theta)=R(-\theta)\left[1+a_{1}(\theta)\right]$.

[^6]:    ${ }^{6}$ Checks of the S-matrix itself to leading orders in $1 / n$ were performed much earlier [17].

[^7]:    ${ }^{7}$ The vanishing of $U_{1}$ as $q$ goes on-shell is consistent with the absence of particle production.

[^8]:    ${ }^{8}$ Note that no confusion can arise here since the $\mathrm{O}(n)$ spin field has non-vanishing form factors for odd number of particles only whereas the current form factors are non-vanishing for an even number of particles only.

[^9]:    9 This is the only exceptional, non-polynomial reduced form factor.

