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Structure functions of the 2d O(n) non-linear sigma models

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Abstract

We investigate structure functions in the 2-dimensional (asymptotically free) non-linear O(n) σ-models using the non-perturbative S-matrix bootstrap program. In particular, the *exact small (Bjorken) x behavior is derived*. Structure functions in the special case of the $n = 3$ model are accurately computed over the whole x range for $-q^2/M^2 < 10^5$, and some moments are compared with results from renormalized perturbation theory. Some results concerning the structure functions in the $1/n$ approximation are also presented.

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1. Introduction

In this paper we study structure functions in the asymptotically free O(n) sigma models in two dimensions. Due to the integrability of the model one has powerful tools to study various non-perturbative properties. In particular, one can derive the *exact small x* behavior (for all q^2) and for the case of $n = 3$ compute structure functions precisely up to very large values of q^2. Despite the fact that there are no transverse directions, the structure functions have a rather rich and non-trivial behavior. In a previous letter [1] we summarized our

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results and speculated on the possibility of discovering some similar structural features in QCD.

The purpose of this paper is to supply the derivation of the results presented in [1]. This paper is organized as follows. In the next section we give some basic definitions of the correlation functions of interest. In Section 3 we give the derivation of the (rather universal) exact small x behavior. Section 4 deals with certain general aspects concerning the relation of the high q^2 behavior of moments of structure functions to the operator product expansion (OPE). In Section 5 we consider the OPE for the cases of two spin fields and two currents in the framework of perturbation theory. More detailed results on the structure functions for the special case of $n = 3$ are presented in Section 6. Finally, in Section 7 we consider computations in the leading order of the $1/n$ expansion. Many technicalities and some conventions can be found in the appendices.

2. O(n) model and structure functions

The O(n) σ-model in 2d (formally described by the Lagrangian (5.1)) is perturbatively asymptotically free for $n \geqslant 3$. A special property is that these models have an infinite number of local [2] and non-local [3] classical conservation laws which survive quantization. At the quantum level they imply absence of particle production. Assuming the spectrum to consist of one stable O(n)-vector multiplet of mass M, the S-matrix has been proposed long ago by Zamolodchikov and Zamolodchikov [4]. Form factors of local operators can be computed using general principles [5,6]. The S-matrix bootstrap program for the construction of correlation functions involves summing the contributions over all intermediate states [7]. The possible equivalence of this construction to the continuum limit of the lattice regularized theory has been investigated in Ref. [8].

2.1. Current and spin operators, 2-point functions

The normalization of the conserved O(n) current operator $J_\mu^{ab}(x)$ $(a, b = 1, \ldots, n)$ is fixed, e.g., by the equal time commutation relation with the spin field $\Phi^c(y)$:

$$\left[J_0^{ab}\left(0, x^1\right), \Phi^c\left(0, y^1\right)\right] = i t_{cd}^{ab} \delta\left(x^1 - y^1\right) \Phi^d\left(0, y^1\right), \tag{2.1}$$

where the matrices t^{ab} given in (A.2) yield the vector representation of the O(n) Lie algebra. Its matrix elements are[1]

$$\langle 0|J_\mu^{ab}(0)|a_1, \theta_1; \ldots; a_r, \theta_r\rangle = -i\epsilon_{\mu\nu} P_r^\nu f_{a_1 \ldots a_r}^{ab}(\theta_1, \ldots, \theta_r). \tag{2.2}$$

Here the number of particles, r, has to be even and the form factors $f_{a_1 \ldots a_r}^{ab}$ depend on the rapidity differences only, making Lorentz invariance and current conservation manifest. The normalization of the r-particle states, the corresponding completeness relations, and

[1] $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$, $\epsilon_{01} = 1$.

other undefined kinematics encountered are given in Appendix B. We define

$$\sum_{a_1\dots a_r} f^{cd}_{a_1\dots a_r}(\theta_1,\dots,\theta_r) f^{*ef}_{a_1\dots a_r}(\theta_1,\dots,\theta_r) = \big(\delta^{ce}\delta^{df} - \delta^{cf}\delta^{de}\big)C^{(r)}(u), \tag{2.3}$$

where $C^{(r)}(u)$ is a symmetric function of the rapidity differences.

The normalization of the spin operator $\Phi^a(x)$ is fixed by its 1-particle matrix element:

$$\langle 0|\Phi^a(0)|b,\theta\rangle = \delta^{ab}. \tag{2.4}$$

Its r-particle matrix elements (r odd) are defined by

$$\langle 0|\Phi^a(0)|a_1,\theta_1;\dots;a_r,\theta_r\rangle = \Lambda_n f^a_{a_1\dots a_r}(\theta_1,\dots,\theta_r), \tag{2.5}$$

where the form factors $f^a_{a_1\dots a_r}$ depend on the rapidity differences only and the overall factor Λ_n is defined for later convenience. We choose

$$\Lambda_3 = \frac{2}{\sqrt{\pi}}, \qquad \Lambda_n = 1 \quad (n>3). \tag{2.6}$$

The analog of (2.3) for odd r is

$$\sum_{a_1\dots a_r} f^{a}_{a_1\dots a_r}(\theta_1,\dots,\theta_r) f^{*b}_{a_1\dots a_r}(\theta_1,\dots,\theta_r) = \delta^{ab}C^{(r)}(u). \tag{2.7}$$

We now make some further definitions:

$$I^{(r)}(z) = \frac{1}{(4\pi)^{r-1}}\int \mathcal{D}u^{(r)}\frac{C^{(r)}(u)}{z+[M^{(r)}(u)]^2}, \tag{2.8}$$

$$A^{(r)}(z) = -z^2\frac{\partial}{\partial z}I^{(r)}(z) = \frac{1}{(4\pi)^{r-1}}\int \mathcal{D}u^{(r)}\left(\frac{z}{z+[M^{(r)}(u)]^2}\right)^2 C^{(r)}(u) \tag{2.9}$$

and for $s=0,1$,

$$I_s(z) = \sum_{k=0}^{\infty} I^{(2k+1+s)}(z), \tag{2.10}$$

$$A_s(z) = \sum_{k=0}^{\infty} A^{(2k+1+s)}(z) = -z^2\frac{\partial}{\partial z}I_s(z). \tag{2.11}$$

The invariant functions I_s are related to the 2-point functions of the current and spin field operators by [9]

$$\begin{aligned}&\langle 0|T^* J^{cd}_\mu(x)J^{ef}_\nu(y)|0\rangle\\ &\quad= \big(\delta^{ce}\delta^{df} - \delta^{cf}\delta^{de}\big)\int\frac{\mathrm{d}^2p}{(2\pi)^2}\mathrm{e}^{-ip(x-y)}\big(p_\mu p_\nu - p^2\eta_{\mu\nu}\big)(-i)I_1\big(-p^2-i\varepsilon\big),\end{aligned} \tag{2.12}$$

valid up to contact terms and

$$\langle 0|T\Phi^a(x)\Phi^b(y)|0\rangle = \Lambda_n^2\delta^{ab}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\mathrm{e}^{-ip(x-y)}(-i)I_0\big(-p^2-i\varepsilon\big). \tag{2.13}$$

In (2.12) T^* denotes the covariantized T-product.

For $n = 3$ we can define $J^a_\mu(x) = \frac{1}{2}\epsilon^{abc} J^{bc}_\mu(x)$ and instead of (2.2) we have

$$\langle 0|J^a_\mu(0)|a_1,\theta_1;\ldots;a_r,\theta_r\rangle = -i\epsilon_{\mu\alpha} P^\alpha_r f^a_{a_1\ldots a_r}(\theta_1,\ldots,\theta_r). \tag{2.14}$$

In this case instead of (2.3) we can use (2.7) also for r even.

2.2. Structure functions, moments

The central object in DIS theory is

$$W^{ab;cdef}_{\mu\nu}(p,q) = \pi \sum_r \langle a,p|J^{cd}_\mu(0)|r\rangle\langle r|J^{ef}_\nu(0)|b,p\rangle \delta^{(2)}(p+q-P_r), \tag{2.15}$$

where $q^2 < 0$. We will use the parameterization

$$q^2 = -4\kappa^2 M^2 \tag{2.16}$$

and the Bjorken variable

$$x = -\frac{q^2}{2(pq)}. \tag{2.17}$$

Using Lorentz and O(n) invariance we have

$$W^{ab;cdef}_{\mu\nu}(p,q) = \left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \sum_{l=0}^{2} R^{ab;cdef}_l w_l\big(q^2,x\big), \tag{2.18}$$

where the projectors R_l corresponding to the 3 invariant t-channel "isospins" are defined in Appendix A. Note that in 2 dimensions there is only one independent structure function for each isospin channel.

Similarly we define the structure functions corresponding to the spin operator through

$$\Sigma^{ab;cd}(p,q) = -\pi q^2 \sum_r \langle a,p|\Phi^c(0)|r\rangle\langle r|\Phi^d(0)|b,p\rangle \delta^{(2)}(p+q-P_r), \tag{2.19}$$

and

$$\Sigma^{ab;cd}(p,q) = \Lambda_n^2 \sum_{l=0}^{2} P^{ab;cd}_l \tilde{w}_l\big(q^2,x\big), \tag{2.20}$$

where the t-channel projectors P_l for the vector representation are given in (A.5)–(A.7).

Separating the r-particle contributions we have

$$w_l\big(q^2,x\big) = \sum_{r\ \text{odd}} w^{(r)}_l\big(q^2,x\big) \quad \text{and} \quad \tilde{w}_l\big(q^2,x\big) = \sum_{r\ \text{even}} w^{(r)}_l\big(q^2,x\big) \tag{2.21}$$

with

$$w^{(r)}\big(q^2,x\big) = \frac{-\pi q^2}{(4\pi)^r} \int_{-\infty}^{\infty} \mathrm{d}\Lambda \int \mathcal{D}u^{(r)} \delta^{(2)}(p+q-P_r) J^{(r)}_l(\theta). \tag{2.22}$$

Here $p=(M,0)$ and for r odd

$$J_l^{(r)}(\theta)=\frac{1}{\pi_l\hat{r}_l}\sum_{\substack{abcdef\\ a_1\dots a_r}} R_l^{ab;cdef} f^{cd}_{aa_1\dots a_r}(i\pi,\theta_1,\dots,\theta_r) f^{*ef}_{ba_1\dots a_r}(i\pi,\theta_1,\dots,\theta_r), \quad (2.23)$$

while for r even

$$J_l^{(r)}(\theta)=\frac{1}{\pi_l}\sum_{\substack{abcd\\ a_1\dots a_r}} P_l^{ab;cd} f^{c}_{aa_1\dots a_r}(i\pi,\theta_1,\dots,\theta_r) f^{*d}_{ba_1\dots a_r}(i\pi,\theta_1,\dots,\theta_r). \quad (2.24)$$

By doing the Λ-integration we can further simplify (2.22):

$$w_l^{(r)}(q^2,x)=\frac{2\kappa^2}{(4\pi)^{r-1}}\int \mathcal{D}u^{(r)}\delta\left[\mu_r^2-1-\frac{4\kappa^2}{x}+4\kappa^2\right]J_l^{(r)}(\bar{\beta}_1,\dots,\bar{\beta}_r), \quad (2.25)$$

where

$$\bar{\beta}_j=\beta_j+b+2v_-^{(r)}, \quad (2.26)$$

$$b=\ln\left\{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{x^2}{\kappa^2}}+\frac{x}{4\kappa^2}\right\}-\ln\left\{1-x+\frac{x}{4\kappa^2}\right\}. \quad (2.27)$$

We define the structure function moments by

$$M_{l;N}(q^2)=\int_0^1 \mathrm{d}x\, x^{N-1}w_l(q^2,x) \quad \text{and} \quad \tilde{M}_{l;N}(q^2)=\int_0^1 \mathrm{d}x\, x^{N-1}\tilde{w}_l(q^2,x) \quad (2.28)$$

and similarly for fixed particle number

$$M^{(r)}_{l;N}(q^2)=\int_0^1 \mathrm{d}x\, x^{N-1}w^{(r)}_l(q^2,x). \quad (2.29)$$

Obviously,

$$M_{l;N}(q^2)=\sum_{r\ \text{odd}} M^{(r)}_{l;N}(q^2) \quad \text{and} \quad \tilde{M}_{l;N}(q^2)=\sum_{r\ \text{even}} M^{(r)}_{l;N}(q^2). \quad (2.30)$$

The r-particle moments can also be calculated directly from (2.25):

$$M^{(r)}_{l;N}(q^2)=\frac{1}{2(4\pi)^{r-1}}\int \mathcal{D}u^{(r)}\left[x^{N+1}J_l^{(r)}(\bar{\beta}_1,\dots,\bar{\beta}_r)\right]_{x=\bar{x}}, \quad (2.31)$$

$$\bar{x}=\frac{4\kappa^2}{4\kappa^2+\mu_r^2-1}. \quad (2.32)$$

For $n=3$ (2.15) can be written as

$$W^{ab;cd}_{\mu\nu}(p,q)=\pi\sum_r \langle a,p|J^c_\mu(0)|r\rangle\langle r|J^d_\nu(0)|b,p\rangle\delta^{(2)}(p+q-P_r) \quad (2.33)$$

and (2.18) becomes

$$W_{\mu\nu}^{ab;cd}(p,q) = \left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \sum_{l=0}^{2} P_l^{ab;cd} w_l\big(q^2, x\big). \tag{2.34}$$

In this case (2.24) is valid for odd as well as even r values (with $\pi_l = 2l+1$).

The 2-particle form factor can be written

$$f_{ab}^{cd}(\theta_1,\theta_2) = \phi(\theta_1-\theta_2)\big(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}\big), \tag{2.35}$$

with

$$\phi(\theta) = -\tanh\frac{\theta}{2}\exp\left\{-2\int_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{1-\mathrm{e}^{-\frac{2t}{n-2}}}{1+\mathrm{e}^t}\right]\frac{\sin^2([i\pi-\theta]t/2\pi)}{\sinh t}\right\}. \tag{2.36}$$

The 1-particle contribution to the structure functions is then given by

$$w_l^{(1)}\big(q^2,x\big) = m_l\delta(x-1)\big|\phi(i\pi-\alpha)\big|^2, \tag{2.37}$$

where

$$\sinh\frac{\alpha}{2} = \kappa, \quad \text{and} \quad m_0 = 1, \quad m_1 = -m_2 = \frac{1}{2}. \tag{2.38}$$

3. 2d structure functions at small x

In this section we derive a general formula describing the asymptotic behavior of the O(n) model structure functions at small x values. The derivation is based on general properties of the form factors and the scattering matrix elements and therefore the behavior we find here is expected to hold in other 2d integrable models as well.

For small $x \to 0$ the variable b in (2.27) behaves as

$$b = x + \mathrm{O}\big(x^2\big), \tag{3.1}$$

and if we do the u_{r-1}-integration in (2.25) with the help of the delta function we get

$$u_{r-1} = \ln\left(\frac{4\kappa^2}{x}\right) - 2v_+^{(r-1)} + \mathrm{O}(x), \tag{3.2}$$

and further

$$2v_-^{(r)} = \frac{x\mu_{r-1}^2}{4\kappa^2} + \mathrm{O}\big(x^2\big), \qquad \frac{\partial\mu_r^2}{\partial u_{r-1}} = \frac{4\kappa^2}{x} + \mathrm{O}(1). \tag{3.3}$$

Now putting all the above together we have

$$w_l^{(r)}\big(q^2,x\big) \cong \frac{x}{2(4\pi)^{r-1}}\int \mathcal{D}u^{(r-1)} J_l^{(r)}(-\varepsilon, -\beta-\varepsilon+\tilde\beta_{r-1}, \ldots, -\beta-\varepsilon+\tilde\beta_1), \tag{3.4}$$

where

$$\varepsilon = -\left(1 + \frac{\mu_{r-1}^2}{4\kappa^2}\right)x + \mathrm{O}(x^2) \quad \text{and} \quad \beta = \ln x + \mathrm{O}(1), \tag{3.5}$$

and $\tilde{\beta}_i$ are the variables for $r-1$ particles.

Eqs. (2.23) and (2.24) are both of the form

$$J_l^{(r)}(\theta) = \sum_{\substack{abAB \\ a_1 \dots a_r}} C^{ab;AB} f^A_{aa_1\dots a_r}(i\pi, \theta_1, \dots, \theta_r) f^B_{ba_1\dots a_r}(i\pi, \theta_1, \dots, \theta_r), \tag{3.6}$$

where for the current case

$$C^{ab;AB} = \frac{1}{\pi_l \hat{r}_l} R_l^{ab;cdef} \quad \text{with } A \sim cd,\ B \sim ef, \tag{3.7}$$

and for the spin case

$$C^{ab;AB} = \frac{1}{\pi_l} P_l^{ab;cd} \quad \text{with } A \sim c,\ B \sim d. \tag{3.8}$$

With this notation we can write $J_l^{(r)}$ in (3.4) as

$$J_l^{(r)} = \sum_{\substack{abAB \\ a_1 \dots a_r}} C^{ab;AB} f^A_{aa_r\dots a_1}(i\pi + \varepsilon + \beta, \beta, \tilde{\beta}_{r-1}, \dots, \tilde{\beta}_1) \times f^{*B}_{ba_r\dots a_1}(i\pi + \varepsilon + \beta, \beta, \tilde{\beta}_{r-1}, \dots, \tilde{\beta}_1). \tag{3.9}$$

The crucial point now is that since ε is small and β is large we can here use (D.4) which follows from general principles encoded in the Smirnov axioms. In leading order we get

$$J_l^{(r)} \cong \frac{(4\pi)^2}{(n-2)^2\varepsilon^2\beta^2} \sum_{\substack{abAB \\ a_1 \dots a_r}} C^{ab;AB} t^{aa_r}_{AA'} t^{ba_r}_{BB'} f^{A'}_{a_{r-1}\dots a_1}(\tilde{\beta}_{r-1}, \dots, \tilde{\beta}_1) \times f^{*B'}_{a_{r-1}\dots a_1}(\tilde{\beta}_{r-1}, \dots, \tilde{\beta}_1), \tag{3.10}$$

which can be further simplified with the help of (A.20), (A.22), (2.3) and (2.7) leading to

$$J_l^{(r)} \cong \frac{(4\pi)^2}{(n-2)^2\varepsilon^2\beta^2} G_l C^{(r-1)}(u), \tag{3.11}$$

where the constants G_l are equal to V_l and T_l for the spin and current cases, respectively, given in (A.21) and (A.23), and further

$$w_l^{(r)}(q^2, x) \cong \frac{1}{x \ln^2 x} \frac{8\kappa^4 G_l}{(4\pi)^{r-3}(n-2)^2} \int \mathcal{D}u^{(r-1)} \frac{C^{(r-1)}(u)}{(4\kappa^2 + \mu_{r-1}^2)^2}, \tag{3.12}$$

which can also be written as

$$w_l^{(r)}(q^2, x) \cong \frac{1}{x \ln^2 x} \frac{2\pi G_l}{(n-2)^2} A^{(r-1)}(-q^2), \tag{3.13}$$

where the Adler functions $A^{(r)}$ were defined in (2.9).

The final results for the complete structure functions are

$$w_l(q^2,x) \cong \frac{1}{x\ln^2 x}\frac{2\pi T_l}{(n-2)^2}A_1(-q^2), \tag{3.14}$$

$$\tilde{w}_l(q^2,x) \cong \frac{1}{x\ln^2 x}\frac{2\pi V_l}{(n-2)^2}A_0(-q^2). \tag{3.15}$$

Note that the structure of the asymptotic small x behavior, factorizing a part characteristic to the target and a part described by the vacuum 2-point function, is rather universal being independent of the operator, independent of n, and independent of the isospin channel.

This completes the derivation of the exact small x asymptotics first announced in Ref. [1]. The question of possible lessons that can be learned for QCD was addressed in the latter reference and will not be repeated here.

4. The operator product expansion

In the O(n) σ-models there does not seem to be a simple parton picture. This is even so for the case $n = 3$ where the model is equivalent to the $\mathbb{C}\mathrm{P}^1$ model. For although this model is formulated in terms of a complex doublet of fields which are analogous to quarks in that they are confined, it seems that they do not play a role more similar to partons than the elementary bare spin fields in the original formulation.[2] The question is related to that of understanding what are (if any) the "ultra-particles" in the sense of Buchholz and Verch [10], or to the associated question as to whether the σ-models have an underlying conformal field theory.

Although an intuitive parton description with suggestive DGLAP equations

$$q^2\frac{\partial}{\partial q^2}w_l(q^2,x) = \int_x^1 \frac{\mathrm{d}y}{y}p_l(x/y,q^2)w_l(q^2,y), \tag{4.1}$$

(where $p_l(z,q^2)$ would be the corresponding splitting functions) is still missing in these models, we still have the machinery of the operator product expansion (OPE) to give us information on the evolution of the moments (2.28) at large $-q^2$.

The OPE in the sigma model is surprisingly involved and hence we have decided to present the material as follows. In the next subsection we first summarize the results; readers who would prefer to skip the derivations can then jump to Section 6. The general structure of the product of two local operators (in this case the spins and currents) is described in the remaining part of this section. Our analysis extends that initiated, e.g., in Refs. [3,11]. So far too little is rigorously known about the detailed structure of the OPE from the general principles of the bootstrap approach to obtain the explicit results below.

[2] Perhaps the peculiar threshold behavior discussed in Section 6 is explained by the fact that (as opposed to QCD) with some probability the O(n) particle can consist of a single point-like parton that carries the same quantum numbers.

The extra required information is, however, supplied in the framework of renormalized perturbation theory which is presented in Section 5. Some comparisons of the moments with those from the bootstrap approach at high $-q^2$ are presented in Section 6.4.

4.1. Summary of results on the moments

For the current (N even) moments in the isospin 0 channel we have

$$M_{0;N}(q^2) = W_{0;N}\frac{n-2}{2(n-1)}\left\{1+\frac{1}{n-2}\lambda(q^2)+\mathrm{O}(\lambda^2)\right\}, \quad N \geqslant 2, \tag{4.2}$$

where $\lambda(q^2)$ is an effective running coupling function defined through

$$\frac{1}{\lambda(q^2)}+\frac{1}{n-2}\ln\lambda(q^2) = \ln\frac{\sqrt{|q^2|}}{\Lambda_{\overline{\mathrm{MS}}}}, \tag{4.3}$$

and the $W_{0;N}$ are renormalization group invariant, non-perturbative constants, corresponding to the matrix elements of spin N operators. In the $N=2$ case this is the energy-momentum tensor operator $T_{\mu\nu}$ for which we know the constant explicitly

$$\langle a,\, p|T_{\mu\nu}(0)|b,\, p\rangle = W_{0;2}p_\mu p_\nu \delta^{ab}, \qquad W_{0;2}=2. \tag{4.4}$$

In particular the "momentum sum rule" follows:

$$M_{0;2}(-\infty) = \frac{n-2}{n-1}. \tag{4.5}$$

Note that all the isospin 0 moments tend to constants as $-q^2\to\infty$. As a consequence these current structure functions in the O(n) models obey Bjorken scaling. Computations in the $n=3$ model (see Section 6 and, in particular, Fig. 3), indicate that the resulting limiting scaling functions are non-trivial. This is a special property of these models and we conjecture that this is due to the existence of an infinite set of local conserved quantities [2].

In the isospin $l=1$ channel for odd moments $N\geqslant 3$ we can only say that

$$M_{1;N}(q^2) = W_{1;N}\lambda(q^2)^{\frac{1}{n-2}}+\cdots, \quad N\geqslant 3, \tag{4.6}$$

but in the special case $N=1$ we have

$$M_{1;1}(q^2) = \frac{1}{2}\left\{1-\frac{1}{n-2}\lambda(q^2)+\mathrm{O}(\lambda^2)\right\}, \tag{4.7}$$

where the constant is known through the current normalization

$$\langle a,\, p|J_\mu^{cd}(0)|b,\, p\rangle = -4ip_\mu P_1^{ab;cd}. \tag{4.8}$$

From this follows the analogy to the Adler sum rule in QCD:

$$M_{1;1}(-\infty) = \frac{1}{2}. \tag{4.9}$$

For the spin field isospin 0 moments we have

$$\tilde{M}_{0;N}(q^2) = \frac{W_{0;N}\pi^2 n C_n}{(n-2)^2}\lambda(q^2)^{\frac{n-3}{n-2}}\{1+\mathrm{O}(\lambda)\}, \quad n \geqslant 4, \tag{4.10}$$

$$\tilde{M}_{0;N}(q^2) = \frac{W_{0;N}}{4}\{1+\lambda(q^2)+\mathrm{O}(\lambda^2)\}, \quad n = 3, \tag{4.11}$$

where the non-perturbative constants $W_{0;N}$ are the same as for the current, and where C_n is the non-perturbative constant appearing in the short distance expansion

$$\langle 0|\Phi^a(y)\Phi^b(0)|0\rangle \sim C_n\delta^{ab}(-\ln M|y|)^{\frac{n-1}{n-2}}. \tag{4.12}$$

So far the value of C_n is not known for general n; for the case $n = 3$ a (well tested) conjecture based on scaling [9] gives

$$C_3 = \frac{1}{3\pi^3}, \tag{4.13}$$

and we know for $n = \infty$:

$$C_\infty = \frac{1}{2\pi}. \tag{4.14}$$

We see that only for the case $n = 3$ do the moments of the field $l = 0$ structure function have the same leading asymptotic behavior as those of the current.

For the isospin $l = 1$ field (odd) moments we find to leading order PT

$$\tilde{M}_{1;1}(q^2) = \tilde{M}_{0;2}(q^2), \tag{4.15}$$

$$\tilde{M}_{1;N}(q^2) = \tilde{W}_{1;N}\lambda(q^2)^{\frac{2n-5}{n-2}}\{1+\mathrm{O}(\lambda^{\frac{1}{n-2}})\}, \quad N \geqslant 3, \tag{4.16}$$

where there is in general no obvious relation between the $\tilde{W}_{1;N}$ and the constants occurring in (4.6), except for $n = 3$ where they are equal ($\tilde{W}_{1;N} = W_{1;N}$, $n = 3$).

For isospin $l = 2$ moments we obtain (for all $n \geqslant 3$):

$$\tilde{M}_{2;N}(q^2) = \tilde{W}_{2;N}\lambda(q^2)^2\{1+\mathrm{O}(\lambda^{\frac{1}{n-2}})\}. \tag{4.17}$$

Finally using the exact ratio of the mass to the Λ-parameter

$$\frac{M}{\Lambda_{\overline{\mathrm{MS}}}} = \frac{(8/\mathrm{e})^{1/(n-2)}}{\Gamma[1+1/(n-2)]}, \tag{4.18}$$

obtained by Hasenfratz, Maggiore and Niedermayer [12], the perturbative results can be plotted as functions of $-q^2/M^2$.

4.2. Dispersion relations

For the discussion of the OPE it is convenient to work in the Euclidean formalism. For local operators $\mathcal{A}$ we have

$$\mathcal{A}(x^0, x^1) = \mathrm{e}^{i(Hx^0 - Px^1)}\mathcal{A}(0,0)\mathrm{e}^{-i(Hx^0 - Px^1)}, \tag{4.19}$$

where H, P are the time and spatial translation operators. We can similarly define Euclidean translation by

$$\mathcal{A}^{\mathrm{E}}(y_1, y_2) = \mathrm{e}^{Hy_2 - iPy_1}\mathcal{A}(0,0)\mathrm{e}^{-Hy_2 + iPy_1}, \tag{4.20}$$

which is formally $\mathcal{A}(-iy_2, y_1)$. For Euclidean vectors (and similarly for tensors) we define

$$V_2 = -iV_0. \tag{4.21}$$

The Euclidean time ordering is defined as

$$\begin{aligned} T_{\mathrm{E}}\big(\mathcal{A}^{\mathrm{E}}(y_1, y_2)\mathcal{B}^{\mathrm{E}}(z_1, z_2)\big) &= \Theta(y_2 - z_2)\mathcal{A}^{\mathrm{E}}(y_1, y_2)\mathcal{B}^{\mathrm{E}}(z_1, z_2) \\ &\quad + \Theta(z_2 - y_2)\mathcal{B}^{\mathrm{E}}(z_1, z_2)\mathcal{A}^{\mathrm{E}}(y_1, y_2), \end{aligned} \tag{4.22}$$

and the connected part of the product of two operators is

$$(\mathcal{O}_1\mathcal{O}_2)_c = \mathcal{O}_1\mathcal{O}_2 - \langle 0|\mathcal{O}_1\mathcal{O}_2|0\rangle - \mathcal{O}_1|0\rangle\langle 0|\mathcal{O}_2. \tag{4.23}$$

We now define Euclidean functions for the currents:

$$\begin{aligned} &\frac{1}{2}\int \mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{e}^{i(Q_1y_1 + Q_2y_2)}\langle a, 0|T^*_{\mathrm{E}}\big(J^{cd\mathrm{E}}_{\mu}(y_1, y_2)J^{ef}_{\nu}(0,0)\big)_c|b, 0\rangle \\ &\quad = \big(Q_\mu Q_\nu - Q^2\delta_{\mu\nu}\big)\sum_{l=0}^{2} R_l^{ab;cdef}\tau_l\big(Q^2, Q_2\big), \end{aligned} \tag{4.24}$$

where T^*_{E} stands for covariantized Euclidean time ordering, i.e., some non-covariant terms proportional to delta functions of the Euclidean time difference (and derivatives of the delta function) are dropped.

Similarly for the spin field:

$$\begin{aligned} &\frac{1}{2}\int \mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{e}^{i(Q_1y_1 + Q_2y_2)}\langle a, 0|T_{\mathrm{E}}\big(\Phi^{c\mathrm{E}}(y_1, y_2)\Phi^{d}(0,0)\big)_c|b, 0\rangle \\ &\quad = \Lambda_n^2\sum_{l=0}^{2} P_l^{ab;cd}\tilde{\tau}_l\big(Q^2, Q_2\big). \end{aligned} \tag{4.25}$$

The τ_l and $\tilde{\tau}_l$, as functions of Q_2 at fixed real Q^2 are real analytic

$$\tau_l\big(Q^2, Q_2\big)^* = \tau_l\big(Q^2, -Q_2^*\big), \qquad \tilde{\tau}_l\big(Q^2, Q_2\big)^* = \tilde{\tau}_l\big(Q^2, -Q_2^*\big), \tag{4.26}$$

and obey the crossing properties

$$\tau_l\big(Q^2, Q_2\big) = (-1)^l\tau_l\big(Q^2, -Q_2\big), \qquad \tilde{\tau}_l\big(Q^2, Q_2\big) = (-1)^l\tilde{\tau}_l\big(Q^2, -Q_2\big). \tag{4.27}$$

Further they have cuts along parts of the imaginary axis (with poles at $Q_2 = \pm iQ^2/2M$), and the discontinuities across the cuts are simply related to the structure functions:

$$w_l\big(-Q^2, x\big) = \frac{Q^2}{\pi}\,\mathrm{Im}\,\tau_l\bigg(Q^2, \varepsilon - i\frac{Q^2}{2Mx}\bigg), \tag{4.28}$$

$$\tilde{w}_l\big(-Q^2, x\big) = \frac{Q^2}{\pi}\,\mathrm{Im}\,\tilde{\tau}_l\bigg(Q^2, \varepsilon - i\frac{Q^2}{2Mx}\bigg). \tag{4.29}$$

Concerning the general singularity structure in the complex Q_2 plane away from the imaginary axis, little more is rigorously known except that the current function τ_1 has poles on the real Q_2 axis originating from the contribution from 1-particle intermediate states. The contribution from 1-particle states is easily computed:

$$\tau_l^{1\text{-part}}\big(Q^2, Q_2\big) = -\frac{m_l}{2M\cosh k}\phi(i\pi - k)\phi(i\pi + k) \times \left\{\frac{1}{M(\cosh k - 1) - iQ_2} + \frac{(-1)^l}{M(\cosh k - 1) + iQ_2}\right\}, \tag{4.30}$$

where $Q_1 = M\sinh k$, $\phi(\theta)$ is the form factor function (2.36) and the constants m_l are given in (2.38). Since for small k

$$\phi(i\pi + k) \cong -\frac{2}{k}, \tag{4.31}$$

the 1-particle contribution, for fixed Q^2 as function of Q_2, has poles at $Q_2 = \pm\sqrt{Q^2}$ with residue $-\frac{iM}{Q^2}\delta_{l1}$.

Assuming that no other singularities are generated by the higher intermediate states away from the imaginary Q_2 axis, from the usual Cauchy integral we conclude that for a circular contour centered at the origin with radius $\sqrt{Q^2} < R < Q^2/(2M)$

$$\frac{1}{2\pi i}\oint \frac{\mathrm{d}\zeta\ \tau_l(Q^2,\zeta)}{\zeta^{N+1}} = i^N\tau_{l;N}\big(Q^2\big) - i\frac{2M}{Q^2}\delta_{l1}\frac{1}{(\sqrt{Q^2})^{N+1}}, \tag{4.32}$$

where $\tau_{l;N}(Q^2)$ are the coefficients of the Taylor expansion

$$\tau_l\big(Q^2, Q_2\big) = \sum_{N=0}^{\infty}\tau_{l;N}\big(Q^2\big)(iQ_2)^N. \tag{4.33}$$

Now the structure function moments can be computed in the usual way by calculating the Cauchy integral along the deformed contour around the cuts. In this way we obtain expressions for the moments:

$$M_{l;N}\big(-Q^2\big) = M\left(\frac{Q^2}{2M}\right)^{N+1}\left\{\tau_{l;N}\big(Q^2\big) + \frac{2M}{Q^2}\delta_{l1}\frac{1}{(i\sqrt{Q^2})^{N+1}}\right\}. \tag{4.34}$$

The spin function $\tilde{\tau}$ has no 1-particle contribution, and so assuming no further singularities apart from the cuts we obtain for the moments

$$\tilde{M}_{l;N}\big(-Q^2\big) = M\left(\frac{Q^2}{2M}\right)^{N+1}\tilde{\tau}_{l:N}\big(Q^2\big), \tag{4.35}$$

where

$$\tilde{\tau}_l\big(Q^2, Q_2\big) = \sum_{N=0}^{\infty}\tilde{\tau}_{l;N}\big(Q^2\big)(iQ_2)^N. \tag{4.36}$$

Note in the equations above for $l = 0, 2$ N is even, positive and for $l = 1$ N is odd. It remains to extract information on the Taylor coefficients $\tau_{l;N}$, $\tilde{\tau}_{l;N}$ from the operator product expansions.

4.3. Operator product expansion for the spin field

Starting with the spin field, the connected part of the time-ordered product can be expanded as:

$$T_{\mathrm{E}}\big(\Phi^{c\mathrm{E}}(y_1,y_2)\Phi^{d}(0,0)\big)_c = \sum_{l,\omega}\mathcal{A}_{\omega}^{(l)cd}\gamma_{\omega}^{(l)}\big(y^2\big)+\sum_{J=1}^{\infty}\sum_{l,\omega}\gamma_{\omega}^{(J,l)}\big(y^2\big)\big\{\mathcal{B}_{\omega}^{(J,l)cd}y_+^J+\bar{\mathcal{B}}_{\omega}^{(J,l)cd}y_-^J\big\}, \tag{4.37}$$

where $y_{\pm}=\mp y_1-iy_2$. Employing a basis of hermitian operators

$$\mathcal{A}_{\omega}^{(l)cd\dagger}=\mathcal{A}_{\omega}^{(l)cd},\qquad \mathcal{B}_{\omega}^{(J,l)cd\dagger}=\mathcal{B}_{\omega}^{(J,l)cd} \tag{4.38}$$

and using Poincaré symmetry, parity and CPT invariance we have

$$\bar{\mathcal{B}}_{\omega}^{(J,l)cd}=V\mathcal{B}_{\omega}^{(J,l)cd}V,\qquad \mathcal{A}_{\omega}^{(l)cd}=V\mathcal{A}_{\omega}^{(l)cd}V, \tag{4.39}$$

where V is the parity operator and

$$\gamma_{\omega}^{(l)*}\big(y^2\big)=\gamma_{\omega}^{(l)}\big(y^2\big),\qquad \gamma_{\omega}^{(J,l)*}\big(y^2\big)=\gamma_{\omega}^{(J,l)}\big(y^2\big). \tag{4.40}$$

Further we define the matrix elements $B_{\omega}^{(J,l)}$ as in Appendix E

$$\langle a,\theta|\mathcal{B}_{\omega}^{(J,l)cd}|b,\theta\rangle=\left(-i\frac{M}{2}\mathrm{e}^{\theta}\right)^J P_l^{ab;cd}B_{\omega}^{(J,l)} \tag{4.41}$$

and we find

$$B_{\omega}^{(J,l)*}=B_{\omega}^{(J,l)}=(-1)^{J+l}B_{\omega}^{(J,l)}. \tag{4.42}$$

The "twist" of the operator is defined as

$$t_{\omega}^{(J,l)}=\dim\big(\mathcal{B}_{\omega}^{(J,l)cd}\big)-J \tag{4.43}$$

and the minimal possible twist value is zero. The contribution of these operators dominate for large momenta and we have

$$\tilde{M}_{l;N}\big(-Q^2\big)\approx\frac{1}{4\Lambda_n^2}\hat{\eta}^{(N,l)}\big(Q^2\big), \tag{4.44}$$

where

$$\hat{\eta}^{(J,l)}\big(Q^2\big)=\big(Q^2\big)^{J+1}\left(\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^J\int\mathrm{d}^2y\,\mathrm{e}^{iQy}\eta^{(J,l)}\big(y^2\big), \tag{4.45}$$

$$\eta^{(J,l)}\big(y^2\big)=\sum_{\omega}B_{\omega}^{(J,l)}\gamma_{\omega}^{(J,l)}\big(y^2\big). \tag{4.46}$$

4.4. Operator product expansion for the current

Using hermicity, Poincaré, O(n), parity and CPT symmetries and current conservation we can write

$$T_{\mathrm{E}}\big(J_\mu^{cd\mathrm{E}}(y)J_\nu^{ef}(0)\big) = \sum_{\substack{l=0,2\\ \omega}} R_l^{ab;cdef} H^{(l)}_{\mu\nu;\omega}(y)\mathcal{A}^{(l)ab}_\omega + \sum_{\substack{J\equiv l\\ \omega}} R_l^{ab;cdef}\big\{H^{(J,l)}_{\mu\nu;\omega}(y)\mathcal{B}^{(J,l)ab}_\omega + \bar{H}^{(J,l)}_{\mu\nu;\omega}(y)\bar{\mathcal{B}}^{(J,l)ab}_\omega\big\}\cdots, \tag{4.47}$$

where the dots indicate that we have omitted total derivative operators since they would not contribute to the diagonal expectation values. Otherwise the set of operators appearing here is as in (4.37) and the coefficient functions $H^{(l)}_{\mu\nu;\omega}$ take the form

$$H^{(l)}_{++;\omega}(y) = -\frac{y_-}{y_+}\mathcal{Y}(\mathcal{Y}+1)V^{(l)}_\omega\big(y^2\big), \tag{4.48}$$

$$H^{(l)}_{+-;\omega}(y) = H^{(l)}_{-+;\omega}(y) = (\mathcal{Y}+1)^2 V^{(l)}_\omega\big(y^2\big), \tag{4.49}$$

$$H^{(l)}_{--;\omega}(y) = -\frac{y_+}{y_-}\mathcal{Y}(\mathcal{Y}+1)V^{(l)}_\omega\big(y^2\big), \tag{4.50}$$

where $V^{(l)}_\omega(y^2)$ ($l=0,2$) are real functions unique up to $\frac{\text{const}}{y^2}$, and

$$\mathcal{Y} = y^2\frac{\mathrm{d}}{\mathrm{d}y^2}. \tag{4.51}$$

Similarly,

$$H^{(J,l)}_{++;\omega}(y) = \frac{c_\omega}{y_+}\delta_{J1} - y_- y_+^{J-1}(\mathcal{Y}+J)(\mathcal{Y}+J+1)V^{(J,l)}_\omega\big(y^2\big), \tag{4.52}$$

$$H^{(J,l)}_{+-;\omega}(y) = H^{(J,l)}_{-+;\omega}(y) = y_+^J(\mathcal{Y}+1)(\mathcal{Y}+J+1)V^{(J,l)}_\omega\big(y^2\big), \tag{4.53}$$

$$H^{(J,l)}_{--;\omega}(y) = -\frac{y_+^{J+1}}{y_-}\mathcal{Y}(\mathcal{Y}+1)V^{(J,l)}_\omega\big(y^2\big), \tag{4.54}$$

where $V^{(J,l)}_\omega(y^2)$ ($J\equiv l$) are real functions unique up to $\frac{\text{const}}{y^2}\delta_{J1}$ and c_ω are real constants. Finally

$$\bar{H}^{(J,l)}_{\mu\nu;\omega}(y_+,y_-) = H^{(J,l)}_{\bar{\mu}\bar{\nu};\omega}(y_-,y_+), \tag{4.55}$$

where $\bar{\mu}=-\mu$ for the light-cone index $\mu=\pm$.

As a consequence of the asymptotic freedom of the O(n) model for small y^2

$$V^{(l)}_\omega\big(y^2\big) \sim |y|^{t^{(l)}_\omega-2}, \tag{4.56}$$

$$V^{(J,l)}_\omega\big(y^2\big) \sim |y|^{t^{(J,l)}_\omega-2}, \tag{4.57}$$

where $t^{(l)}_\omega$, $t^{(J,l)}_\omega$ are the twist of the corresponding operators.

4.4.1. Fourier transformation

Introducing

$$X_\omega^{(l)}(y) = y^2 V_\omega^{(l)}\big(y^2\big), \tag{4.58}$$

$$X_\omega^{(J,l)}(y) = y^2 y_+^J V_\omega^{(J,l)}\big(y^2\big), \tag{4.59}$$

$$\bar{X}_\omega^{(J,l)}(y) = y^2 y_-^J V_\omega^{(J,l)}\big(y^2\big) \tag{4.60}$$

the current operator product in Fourier space can be written

$$\begin{aligned}
&\int \mathrm{d}^2 y\, \mathrm{e}^{iQy} \langle a,0|T_{\mathrm{E}}\big(J_\mu^{cd\mathrm{E}}(y) J_\nu^{ef}(0)\big)|b,0\rangle \\
&\quad = -\sum_{\substack{l=0,2\\ \omega}} R_l^{ab;cdef} E_{\mu\nu}(Q) \tilde{X}_\omega^{(l)}(Q) A_\omega^{(l)} \\
&\qquad - \sum_{\substack{J\equiv l\\ \omega}} R_l^{ab;cdef} \Big\{ E_{\mu\nu}(Q)\big[\tilde{X}_\omega^{(J,l)}(Q) + \tilde{\bar{X}}_\omega^{(J,l)}(Q)\big] \\
&\qquad\qquad + i\pi c_\omega \delta_{J1} K_{\mu\nu}(Q) \Big\} \left(\frac{-iM}{2}\right)^J B_\omega^{(J,l)},
\end{aligned} \tag{4.61}$$

where the reduced matrix elements $A_\omega^{(l)}$, $B_\omega^{(J,l)}$ are real and $E_{\mu\nu}(Q)$ is the transversal tensor

$$E_{\mu\nu}(Q) = Q_\mu Q_\nu - Q^2 \delta_{\mu\nu}. \tag{4.62}$$

The complete expression (4.61), although conserved in coordinate space, is not transversal because of the anomalous terms proportional to the constants c_ω. These are multiplied by the tensor $K_{\mu\nu}(Q)$ with components

$$K_{++} = \frac{1}{Q_-}, \qquad K_{--} = \frac{1}{Q_+}, \qquad K_{+-} = K_{-+} = 0. \tag{4.63}$$

It is not quite trivial to see, but easy to check that

$$K_{\mu\nu}(Q) = -E_{\mu\nu}(Q)\frac{4iQ_2}{Q_1^2 Q^2} + \frac{\mu+\nu}{Q_1} - \mu\nu\frac{iQ_2}{Q_1^2} \qquad (\mu,\nu=\pm). \tag{4.64}$$

Thus $K_{\mu\nu}$ is transversal up to the last two terms, but these correspond to contact terms in coordinate space. Dropping these "seagulls", the coefficient of the transversal part in Fourier space becomes

$$\begin{aligned}
\tau_l(Q) &= -\frac{1}{2}\sum_\omega \tilde{X}_\omega^{(l)}(Q) A_\omega^{(l)} \\
&\quad - \frac{1}{2}\sum_{\substack{J\equiv l\\ \omega}} \left(\frac{-iM}{2}\right)^J B_\omega^{(J,l)} \left\{ \tilde{X}_\omega^{(J,l)}(Q) + \tilde{\bar{X}}_\omega^{(J,l)}(Q) - 4i\pi c_\omega \delta_{J1} \frac{iQ_2}{Q_1^2 Q^2} \right\}.
\end{aligned} \tag{4.65}$$

This can alternatively be written as

$$\tau_l(Q) = -\frac{1}{2}\sum_\omega \tilde{X}_\omega^{(l)}(Q)A_\omega^{(l)} - \frac{1}{2}\sum_{\substack{J\equiv l\\ \omega}}\left\{\left[(2MQ_-)^J + (2MQ_+)^J\right]\left(\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^J \tilde{W}_\omega^{(J,l)}(Q^2)B_\omega^{(J,l)}\right\} + M\pi\frac{iQ_2}{Q^2(Q^2-Q_2^2)}\delta_{l1}\sum_\omega c_\omega B_\omega^{(1,1)}, \tag{4.66}$$

where

$$W_\omega^{(J,l)}(y^2) = y^2 V_\omega^{(J,l)}(y^2). \tag{4.67}$$

Note that the anomalous contribution to (4.66) is regular on the imaginary Q_2 axis hence does not contribute to the structure functions. Let us also define

$$\xi^{(J,l)}(y^2) = \sum_\omega W_\omega^{(J,l)}(y^2)B_\omega^{(J,l)}. \tag{4.68}$$

If we now compute

$$\hat{\xi}^{(J,l)}(Q^2) = \sum_\omega (Q^2)^{J+1}\left(\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^J \tilde{W}_\omega^{(J,l)}(Q^2)B_\omega^{(J,l)} \tag{4.69}$$

we see (using asymptotic freedom) that the coefficient functions behave as $(Q^2)^{-t_\omega^{(J,l)}}$, up to logarithmic corrections. We will keep the contributions of the leading (twist 0) operators only. Note that for $J = l = 1$ the only twist 0 operator is $\mathcal{B}_1^{(1,1)ab} = J_+^{ab}$ with

$$B_1^{(1,1)} = 4. \tag{4.70}$$

From (4.66) we obtain for the Taylor coefficients

$$\tau_{l;N}(Q^2) = \hat{\tau}_{l;N}(Q^2) - \frac{4\pi M c_1}{Q^2}\frac{\delta_{l1}}{(i\sqrt{Q^2})^{N+1}}, \tag{4.71}$$

where (up to higher twist contributions)[3]

$$\hat{\tau}_{l;N}(Q^2) \cong -\frac{1}{2}\frac{(2M)^N}{(Q^2)^{N+1}}\hat{\xi}^{(N,l)}(Q^2). \tag{4.72}$$

Inserting this in (4.34) we obtain:

$$M_{l;N}(-Q^2) = M\left(\frac{Q^2}{2M}\right)^{N+1} \times \left\{\hat{\tau}_{l;N}(Q^2) + \frac{2M}{Q^2}\delta_{l1}\frac{1}{(i\sqrt{Q^2})^{N+1}}(1-2\pi c_1)\right\}. \tag{4.73}$$

[3] Note for large Q^2 the anomalous term dominates over $\hat{\tau}_{l;N}$.

Later we will see that $c_1 = 1/2\pi$. Thus these two subtle effects cancel each other, and so the final formula coincides with the naive one:

$$M_{l;N}(-Q^2) \cong -\frac{1}{4}\hat{\xi}^{(N,l)}(Q^2). \tag{4.74}$$

Up to now we have related the moments to the Taylor coefficients which we see are determined by the structure of the OPE. But to get quantitative results at this stage we need more dynamical input. This can be supplied by analyzing the OPE in the framework of renormalized PT, which is the topic of the next section.

5. Perturbation theory and operator product expansion

We consider the O(n) sigma model Lagrangian[4]

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{2g_0^2}\sum_{a=1}^{n}\partial_\mu S^a \partial_\mu S^a, \qquad S^2 = 1, \tag{5.1}$$

and work in $D = 2-\epsilon$ dimensions using dimensional regularization. Renormalized fields S^a_{R} and coupling g are given by

$$\begin{aligned} S^a &= Z^{1/2} S^a_{\mathrm{R}}, \qquad Z = 1 - \frac{\gamma_0 g^2}{\epsilon} + \cdots, \\ g_0^2 &= \mu^\epsilon g^2 Z_1, \qquad Z_1 = 1 - \frac{2\beta_0 g^2}{\epsilon} + \cdots. \end{aligned} \tag{5.2}$$

We denote the usual renormalization group (RG) derivative by

$$\mathcal{D} = \mu\frac{\mathrm{d}}{\mathrm{d}\mu} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}, \tag{5.3}$$

where the dimensional regularization beta function is

$$\beta(g) = -\frac{\epsilon}{2}g + \bar{\beta}(g), \qquad \bar{\beta}(g) = -\beta_0 g^3 - \beta_1 g^5 - \cdots \tag{5.4}$$

and

$$\beta_0 = \frac{n-2}{4\pi}, \qquad \beta_1 = \frac{n-2}{8\pi^2}. \tag{5.5}$$

The RG Λ-parameter in the $\overline{\mathrm{MS}}$ scheme satisfies $\mathcal{D}\Lambda_{\overline{\mathrm{MS}}} = 0$ and is written

$$\Lambda_{\overline{\mathrm{MS}}} = \mu \mathrm{e}^{f(g)}, \tag{5.6}$$

where

$$f(g) = -\frac{1}{2\beta_0 g^2} - \frac{\beta_1}{2\beta_0^2}\ln(2\beta_0 g^2) + \frac{\gamma}{2} + \mathrm{O}(g^2) \tag{5.7}$$

[4] In practical computations in infinite volume one usually adds a coupling to an external field $-(h_0/g_0^2)(S^n - 1)$ to serve as an intermediate IR-regulator. For IR-finite quantities the renormalized external field $h_{\mathrm{R}} = h_0\sqrt{Z}/Z_1$ is set to zero at the end of the computation.

with $\gamma = \ln 4\pi + \Gamma'(1)$ (note $\frac{\beta_1}{2\beta_0^2} = \frac{1}{n-2}$).

The spin field Φ^a differs from the renormalized O(n) field S_R^a only by a finite renormalization:

$$\Phi^{a\mathrm{E}} = \Omega_n S_R^a \tag{5.8}$$

and solving the RG equations for the vacuum two-point function the standard way we find

$$\Omega_n^2 \mathrm{e}^{-p(g)} = \left(\frac{2\pi}{n-2}\right)^{\frac{n-1}{n-2}} n C_n, \tag{5.9}$$

where the constant C_n is that appearing in Eq. (4.12) and $p(g)$ is the solution of

$$p'(g) = \frac{\gamma(g)}{\bar\beta(g)}, \tag{5.10}$$

where $\gamma(g)$ is the anomalous dimension of the spin field:

$$\gamma(g) = \mathcal{D} \ln Z = \gamma_0 g^2 + \cdots, \qquad \gamma_0 = \frac{n-1}{2\pi}. \tag{5.11}$$

The integration constant in (5.10) is fixed by requiring

$$\mathrm{e}^{-p(g)} = \big(g^2\big)^{\frac{\gamma_0}{2\beta_0}} \big\{1 + \mathrm{O}\big(g^2\big)\big\}. \tag{5.12}$$

5.1. Zero twist operators

We now introduce a basis for zero twist operators composed of an even number of spin fields. For isospin $l = 0$ we write

$$K^{(n_1,m_1)\ldots(n_k,m_k)} = \frac{1}{g_0^{2k}} \big(\partial_+^{n_1} S^{a_1} \cdot \partial_+^{m_1} S^{a_1}\big) \cdots \big(\partial_+^{n_k} S^{a_k} \cdot \partial_+^{m_k} S^{a_k}\big), \tag{5.13}$$

where we introduced the notation

$$\partial_\pm = \frac{1}{2}(i\partial_2 \mp \partial_1) = \frac{1}{2}(\partial_0 \mp \partial_1). \tag{5.14}$$

It is very important to notice that a complete basis can be chosen such that

$$n_i, m_i \geqslant 1, \quad i = 1, \ldots, k, \tag{5.15}$$

which can be achieved by using the identity

$$S^a \partial_+^m S^a = -\frac{1}{2} \sum_{i=1}^{m-1} \binom{m}{i} \partial_+^{m-i} S^a \cdot \partial_+^i S^a. \tag{5.16}$$

The spin of the above operators is $\sum_{i=1}^k (n_i + m_i) = J$, whereas the mass dimension is $J - k\epsilon$, i.e., the operators are of zero twist only in exactly two dimensions. For $l = 1, 2$ we can define the operators

$$K^{ab(n_0,m_0)(n_1,m_1)\ldots(n_k,m_k)} = \frac{1}{g_0^2} \partial_+^{n_0} S^a \cdot \partial_+^{m_0} S^b K^{(n_1,m_1)\ldots(n_k,m_k)} \tag{5.17}$$

with spin $\sum_{i=0}^{k}(n_i+m_i)=J$, and dimension $J-(k+1)\epsilon$. For $l=1$ we have to antisymmetrize the a,b indices, whereas for $l=2$ we take the symmetric, traceless part. Now

$$0\leqslant n_0\leqslant m_0,\qquad n_i,m_i\geqslant 1,\quad i=1,\ldots,k, \tag{5.18}$$

and correspondingly there are three types of operators:

type I: $l=0$ and $l=1,2$, $n_0>0$,

type II: $l=1,2$, $n_0=0$, $m_0\geqslant 1$,

type III: $l=2$, $n_0=m_0=0$.

It is now straightforward to calculate the potentially divergent matrix elements of these operators. At one-loop order, after wave function, charge and mass renormalization we find:

$l=0$: all matrix elements finite;

$l=1$: type I: $\left(1-\dfrac{g^2}{2\pi\epsilon}\right)\times$ lowest order,

type II: only type I operator matrix elements;

$l=2$: type I: $\left(1-\dfrac{g^2}{2\pi\epsilon}\right)\times$ lowest order,

type II: $\left(1-\dfrac{g^2}{\pi\epsilon}\right)\times$ lowest order + type I,

type III: $\left(1-\dfrac{g^2}{\pi\epsilon}\right)\times$ lowest order + type I, II.

5.1.1. Operator product expansion at tree level

The leading terms of the OPE in perturbation theory are simply given by Taylor expansion:

$$\begin{aligned}&T_{\mathrm{E}}\left(\frac{1}{g_0^2}S^a(y)S^b(0)\right)_c\\&\quad=\frac{1}{g_0^2}\big(S^aS^b\big)_c+\frac{1}{g_0^2}\sum_{J=1}^{\infty}\frac{1}{J!}\big[\big(\partial_+^JS^a\cdot S^b\big)y_+^J+\big(\partial_-^JS^a\cdot S^b\big)y_-^J\big],\end{aligned} \tag{5.19}$$

up to higher twist operators. The operators appearing in the sum over J can be written as a sum over operators of definite isospin:

$$\frac{1}{g_0^2}\partial_+^JS^a\cdot S^b=\sum_{l=0}^{2}\mathcal{O}_{(0)}^{(J,l)ab}, \tag{5.20}$$

$$\mathcal{O}_{(0)}^{(J,l)ab}=\frac{1}{g_0^2}P_l^{ab;cd}\partial_+^JS^c\cdot S^d. \tag{5.21}$$

Important operators with isospin 2 are

$$\tau^{ab}_{(0)} = \frac{1}{2g_0^2}\left(S^a S^b - \frac{1}{n}\delta^{ab}\right). \tag{5.22}$$

For isospin 1 we have the currents

$$J^{ab}_{\mu(0)} = \frac{1}{g_0^2}\left(S^a \partial_\mu S^b - S^b \partial_\mu S^a\right), \tag{5.23}$$

and for isospin 0 we have the energy-momentum tensor

$$T_{\mu\nu(0)} = \frac{1}{g_0^2}\left(\partial_\mu S^a \partial_\nu S^a - \frac{1}{D}\delta_{\mu\nu}\partial_\sigma S^a \partial_\sigma S^a\right). \tag{5.24}$$

In terms of these, the leading operators of the OPE can be written as

$$\mathcal{O}^{(J,2)ab}_{(0)} = \partial_+^J \tau^{ab}_{(0)} + \text{type I operators}, \tag{5.25}$$

$$\mathcal{O}^{(J,1)ab}_{(0)} = -\frac{1}{2}\partial_+^{J-1} J^{ab}_{+(0)} + \text{type I operators}, \tag{5.26}$$

$$\mathcal{O}^{(2,0)ab}_{(0)} = -\frac{\delta^{ab}}{n} T_{++(0)}. \tag{5.27}$$

5.1.2. Renormalization of the zero twist operators

We will now denote by $\mathcal{B}^{(J,l)ab}_{\alpha(0)}$ the zero twist operators introduced in the preceding section. Here α is a multi-index: it includes the operator type I, II, III and possible further indices. In D space–time dimensions the mass dimension of $\mathcal{B}^{(J,l)ab}_{\alpha(0)}$ is $J - \epsilon d^{(J,l)}_\alpha$. The corresponding renormalized (finite) operators of mass dimension J are:

$$\mathcal{B}^{(J,l)ab}_{\alpha} = \sum_\beta Z^{(J,l)}_{\alpha\beta} \mu^{\epsilon d^{(J,l)}_\beta} \mathcal{B}^{(J,l)ab}_{\beta(0)}, \tag{5.28}$$

where the operator renormalization constant matrix is

$$Z^{(J,l)}_{\alpha\beta} = \delta_{\alpha\beta} - \frac{g^2}{\epsilon} w^{(J,l)}_{\alpha\beta} + \cdots. \tag{5.29}$$

We now distinguish the types of operators by writing their multi-indices

$$\begin{aligned}
&\mathcal{B}^{(J,l)cd}_{a(0)}: \quad \text{for type I},\\
&\mathcal{B}^{(J,l)cd}_{\underline{A}(0)}: \quad \text{for type II} \quad (l = 1, 2),\\
&\mathcal{B}^{(J,2)cd}_{\underline{\underline{A}}(0)}: \quad \text{for type III}.
\end{aligned} \tag{5.30}$$

In this notation the one-loop results of the previous subsection are

$$w^{(J,l)}_{ab} = \frac{l(l-3)}{4\pi}\delta_{ab}, \qquad w^{(J,l)}_{a\underline{A}} = w^{(J,l)}_{a\underline{\underline{A}}} = 0, \tag{5.31}$$

$$w^{(J,l)}_{\underline{A}\,\underline{B}} = \frac{1-l}{\pi}\delta_{AB}, \qquad w^{(J,l)}_{\underline{A}\,\underline{\underline{B}}} = 0, \tag{5.32}$$

$$w^{(J,2)}_{\underline{\underline{A}}\,\underline{\underline{B}}} = -\frac{1}{\pi}\delta_{AB}. \tag{5.33}$$

Using the inverse matrix W:

$$\sum_{\beta} Z^{(J,l)}_{\alpha\beta} W^{(J,l)}_{\beta\gamma} = \delta_{\alpha\gamma}, \qquad W^{(J,l)}_{\alpha\beta} = \delta_{\alpha\beta} + \frac{g^2}{\epsilon} w^{(J,l)}_{\alpha\beta} + \cdots, \tag{5.34}$$

we define the anomalous dimension matrix

$$\nu^{(J,l)}_{\rho\sigma} = \sum_{\omega} Z^{(J,l)}_{\rho\omega}\big(\mathcal{D} W^{(J,l)}_{\omega\sigma} - \epsilon d^{(J,l)}_{\omega} W^{(J,l)}_{\omega\sigma}\big), \tag{5.35}$$

which is finite and is given by

$$\nu^{(J,l)}_{\rho\sigma} = g^2 w^{(J,l)}_{\rho\sigma}\big(d^{(J,l)}_{\sigma} - d^{(J,l)}_{\rho} - 1\big) + \mathrm{O}\big(g^4\big) \tag{5.36}$$

to leading order.

We would like to go to a basis where the leading anomalous dimension matrix is diagonal. This basis is easily found due to the triangular structure of the leading anomalous dimension matrix. The renormalized operators in this new basis are denoted $\mathcal{B}^{(J,l)cd}_{\omega}$, where $\omega = a, A$ and $\mathcal{B}^{(J,l)cd}_{a}$ are (as before) the renormalized type I operators and $\mathcal{B}^{(J,l)cd}_{A}$ are operators of type II and (for $l = 2$) of type III mixed with lower type operators. In this basis we have

$$\nu^{(J,l)}_{ab} = \frac{l(3-l)}{4\pi}\delta_{ab} g^2 + \mathrm{O}\big(g^4\big), \tag{5.37}$$

$$\nu^{(J,l)}_{AB} = \frac{l-1}{\pi}\delta_{AB} g^2 + \mathrm{O}\big(g^4\big) \quad (l = 1, 2), \tag{5.38}$$

$$\nu^{(J,l)}_{aA} = \nu^{(J,l)}_{Aa} = \mathrm{O}\big(g^4\big) \quad (l = 1, 2). \tag{5.39}$$

We also note that the canonically dimensionless $l = 2$ operator $\tau^{ab}_{(0)}$ is multiplicatively renormalized:

$$\tau^{ab}_{(0)} = Y \mu^{-\epsilon} \tau^{ab}, \tag{5.40}$$

since there is no other operator with the same quantum numbers to mix with. Here

$$Y = 1 - \frac{g^2}{\pi\epsilon} + \cdots \tag{5.41}$$

leading to

$$\mathcal{D} \ln Y = \frac{g^2}{\pi} + \mathrm{O}\big(g^4\big). \tag{5.42}$$

It is clear that for the $l = 0$ operators we can chose the diagonal basis so that

$$\mathcal{O}^{(J,0)cd}_{(0)} = \mathcal{B}^{(J,0)cd}_{1(0)} \tag{5.43}$$

and for $l = 1, 2$ the $A = 1$ operators so that

$$\mathcal{B}^{(J,2)cd}_{1(0)} = \partial^{J}_{+} \tau^{cd}_{(0)} = \mu^{-\epsilon} Y \mathcal{B}^{(J,2)cd}_{1}, \quad \text{where } \mathcal{B}^{(J,2)cd}_{1} = \partial^{J}_{+} \tau^{cd}, \tag{5.44}$$

$$\mathcal{B}^{(J,1)cd}_{1(0)} = \partial_+^{J-1} J^{cd}_{+(0)} = \mu^{-\epsilon} \mathcal{B}^{(J,1)cd}_1, \quad \text{where } \mathcal{B}^{(J,1)cd}_1 = \partial_+^{J-1} J^{cd}_+. \tag{5.45}$$

We then have

$$\mathcal{O}^{(J,2)cd}_{(0)} = \mathcal{B}^{(J,2)cd}_{1(0)} + \sum_a \lambda^{(J,2)}_a \mathcal{B}^{(J,2)cd}_{a(0)}, \tag{5.46}$$

$$\mathcal{O}^{(J,1)cd}_{(0)} = -\frac{1}{2}\mathcal{B}^{(J,1)cd}_{1(0)} + \sum_a \lambda^{(J,1)}_a \mathcal{B}^{(J,1)cd}_{a(0)}. \tag{5.47}$$

Finally, for $(J,0) = (2,0)$ there is just one operator and we have

$$\mathcal{B}^{(2,0)cd}_{1(0)} = -\frac{\delta^{cd}}{n} T_{++(0)} = \mu^{-\epsilon} \mathcal{B}^{(2,0)cd}_1, \quad \text{where } \mathcal{B}^{(2,0)cd}_1 = -\frac{\delta^{cd}}{n} T_{++}, \tag{5.48}$$

and therefore, the 1-particle matrix element is known exactly:

$$B^{(2,0)}_1 = 2. \tag{5.49}$$

5.1.3. The operator product expansion in perturbation theory

In bare perturbation theory we have

$$\begin{aligned} &T_{\mathrm{E}} \frac{1}{g_0^2} \big(S^a(y) S^b(0)\big)_c \\ &\quad = \sum_{l,\omega} \mathcal{A}^{(l)ab}_{\omega(0)} k^{(l)}_{\omega(0)}\big(y^2\big) + \sum_{J=1}^{\infty} \sum_{l,\omega} k^{(J,l)}_{\omega(0)}\big(y^2\big)\big\{\mathcal{B}^{(J,l)ab}_{\omega(0)} y_+^J + \bar{\mathcal{B}}^{(J,l)ab}_{\omega(0)} y_-^J\big\}, \end{aligned} \tag{5.50}$$

which, after renormalization, becomes

$$\begin{aligned} &T_{\mathrm{E}} \big(S^a_R(y) S^b_R(0)\big)_c \\ &\quad = \sum_{l,\omega} \mathcal{A}^{(l)ab}_{\omega} k^{(l)}_{\omega}\big(y^2\big) + \sum_{J=1}^{\infty} \sum_{l,\omega} k^{(J,l)}_{\omega}\big(y^2\big)\big\{\mathcal{B}^{(J,l)ab}_{\omega} y_+^J + \bar{\mathcal{B}}^{(J,l)ab}_{\omega} y_-^J\big\}, \end{aligned} \tag{5.51}$$

where

$$k^{(J,l)}_{\omega}\big(y^2\big) = \mu^{\epsilon} \frac{g^2 Z_1}{Z} \sum_{\rho} k^{(J,l)}_{\rho(0)}\big(y^2\big) \mu^{-\epsilon d^{(J,l)}_{\rho}} W^{(J,l)}_{\rho\omega}, \tag{5.52}$$

which satisfies the renormalization group equation (RGE)

$$\big(\mathcal{D} + \gamma(g)\big) k^{(J,l)}_{\sigma} = \sum_{\omega} k^{(J,l)}_{\omega} \nu^{(J,l)}_{\omega\sigma}. \tag{5.53}$$

Finally, the original coefficient functions $\gamma^{(J,l)}_{\omega}$ of (4.37) are related to the renormalized coefficients $k^{(J,l)}_{\omega}$ by

$$\gamma^{(J,l)}_{\omega} = \Omega_n^2 k^{(J,l)}_{\omega}, \tag{5.54}$$

and so (4.46) can now be written as

$$\eta^{(J,l)}\big(y^2\big) = \Omega_n^2 \sum_{\omega} k^{(J,l)}_{\omega}\big(y^2\big) B^{(J,l)}_{\omega}. \tag{5.55}$$

The perturbative expansion of the renormalized coefficient functions is

$$k_\omega^{(J,l)}\big(\mu|y|,g\big) = g^2 K_\omega^{(J,l)} + g^4 \tilde{q}_\omega^{(J,l)}\big(\mu|y|\big) + \mathrm{O}\big(g^6\big). \tag{5.56}$$

We already computed the leading (tree-level) terms:

$$K_\omega^{(J,0)} = \frac{1}{J!}\delta_{\omega 1}, \tag{5.57}$$

$$K_A^{(J,1)} = -\frac{1}{2J!}\delta_{A1}, \qquad K_a^{(J,1)} = \frac{1}{J!}\lambda_a^{(J,1)}, \tag{5.58}$$

$$K_A^{(J,2)} = \frac{1}{J!}\delta_{A1}, \qquad K_a^{(J,2)} = \frac{1}{J!}\lambda_a^{(J,2)}. \tag{5.59}$$

A building block used in the solution of the RGE is the matrix $\hat{U}_{\omega\sigma}^{(J,l)}(g)$, which is a solution of the matrix differential equation

$$\bar{\beta}(g)\frac{\partial}{\partial g}\hat{U}_{\omega\sigma}^{(J,l)}(g) = -\sum_\rho \nu_{\omega\rho}^{(J,l)}(g)\hat{U}_{\rho\sigma}^{(J,l)}(g). \tag{5.60}$$

If we have such a solution and its matrix inverse U satisfying

$$\sum_\rho U_{\omega\rho}^{(J,l)}(g)\hat{U}_{\rho\sigma}^{(J,l)}(g) = \delta_{\omega\sigma}, \tag{5.61}$$

we can build the RG-invariant coefficient

$$G_\omega^{(J,l)} = \mathrm{e}^{p(\bar{g})}\sum_\rho k_\rho^{(J,l)}(1,\bar{g})\hat{U}_{\rho\omega}^{(J,l)}(\bar{g}) \tag{5.62}$$

and the RG-invariant numbers

$$V_\omega^{(J,l)} = \sum_\rho U_{\omega\rho}^{(J,l)}(g)B_\rho^{(J,l)}. \tag{5.63}$$

In (5.62) the running coupling $\bar{g}$ is defined as the solution of

$$f(\bar{g}) = f(g) + \ln\big(\mu|y|\big), \tag{5.64}$$

which, for small $|y|$, has the asymptotic expansion

$$2\beta_0\bar{g}^2 = \tilde{\lambda} + c\tilde{\lambda}^2 + \mathrm{O}\big(\tilde{\lambda}^3\big), \qquad c = \frac{1}{2}\big(\Gamma'(1) - \ln\pi\big), \tag{5.65}$$

where the effective coupling $\tilde{\lambda}$ is defined by

$$\frac{1}{\tilde{\lambda}} + \frac{1}{n-2}\ln\tilde{\lambda} = \ln\frac{2\mathrm{e}^{\Gamma'(1)}}{\Lambda_{\overline{\mathrm{MS}}}|y|}. \tag{5.66}$$

Putting the building blocks together we have

$$\eta^{(J,l)}\big(y^2\big) = \left(\frac{2\pi}{n-2}\right)^{\frac{n-1}{n-2}} nC_n \sum_\omega G_\omega^{(J,l)}V_\omega^{(J,l)}. \tag{5.67}$$

Note that if with some $\hat{Y}$

$$\mathcal{B}^{(J,l)cd}_{1(0)} = \mu^{-\epsilon}\hat{Y}\mathcal{B}^{(J,l)cd}_{1}, \tag{5.68}$$

which is the case for $l = 1, 2$ and also for $(J, l) = (2, 0)$, then

$$\nu^{(J,l)}_{1\omega} = \mathcal{D}\ln\hat{Y}\delta_{1\omega}. \tag{5.69}$$

5.1.4. Solution of the matrix problem

In this subsection we will omit the upper index $^{(J,l)}$ and use matrix notation. We want to solve

$$\bar{\beta}(g)\frac{\partial}{\partial g}\hat{U}(g) = -\nu(g)\hat{U}(g), \tag{5.70}$$

which is (5.60) in this notation. We know that in our basis

$$\nu(g) = 2\beta_0\Delta g^2 + \mathrm{O}\big(g^4\big), \tag{5.71}$$

where Δ is a diagonal matrix with diagonal elements:

$$\begin{aligned} &l = 0: \quad \Delta_a = 0, \\ &l = 1: \quad \Delta_A = 0, \quad \Delta_a = \frac{1}{n-2}, \\ &l = 2: \quad \Delta_A = \frac{2}{n-2}, \quad \Delta_a = \frac{1}{n-2}. \end{aligned} \tag{5.72}$$

Using the expansion

$$\frac{\nu(g)}{\bar{\beta}(g)} = -\frac{2\Delta}{g} - 2\sum_{p=1}^{\infty} g^{2p-1}A^{(p)}, \tag{5.73}$$

we can take the Ansatz

$$\hat{U}(g) = \big[1 + R(g)\big]g^{2\Delta} \tag{5.74}$$

with

$$R(g) = \sum_{s=1}^{\infty} g^{2s}R^{(s)}, \tag{5.75}$$

and put it into (5.70). We get

$$sR^{(s)} + \big[R^{(s)}, \Delta\big] = A^{(s)} + \sum_{p=1}^{s-1} A^{(s-p)}R^{(p)}, \quad s = 1, 2, \ldots, \tag{5.76}$$

which has a unique recursive solution unless $\Delta_\omega - \Delta_\sigma = s$ occurs for some ω, σ and s. In our case this is possible only for $s = 1$ and only if $n = 3$. For $n = 3$ we thus take the modified Ansatz

$$\hat{U}(g) = \big[1 + R(g) + \ln g^2\,\tilde{R}(g)\big]g^{2\Delta} \tag{5.77}$$

with

$$R(g) = \sum_{s=1}^{\infty} g^{2s} R^{(s)}, \qquad \tilde{R}(g) = \sum_{s=1}^{\infty} g^{2s} \tilde{R}^{(s)}. \tag{5.78}$$

In this case we start with

$$\begin{aligned} l=1: \quad & \tilde{R}^{(1)}_{aA} = A^{(1)}_{aA}, \quad \tilde{R}^{(1)}_{ab} = 0, \quad \tilde{R}^{(1)}_{Aa} = 0, \quad \tilde{R}^{(1)}_{AB} = 0, \\ & R^{(1)}_{aA} = 0, \quad R^{(1)}_{ab} = A^{(1)}_{ab}, \quad R^{(1)}_{Aa} = \frac{1}{2} A^{(1)}_{Aa}, \quad R^{(1)}_{AB} = A^{(1)}_{AB}, \end{aligned} \tag{5.79}$$

and

$$\begin{aligned} l=2: \quad & \tilde{R}^{(1)}_{Aa} = A^{(1)}_{Aa}, \quad \tilde{R}^{(1)}_{AB} = 0, \quad \tilde{R}^{(1)}_{ab} = 0, \quad \tilde{R}^{(1)}_{aA} = 0, \\ & R^{(1)}_{Aa} = 0, \quad R^{(1)}_{AB} = A^{(1)}_{AB}, \quad R^{(1)}_{ab} = A^{(1)}_{ab}, \quad R^{(1)}_{aA} = \frac{1}{2} A^{(1)}_{aA}, \end{aligned} \tag{5.80}$$

and after that there is a unique, recursive solution of the system

$$s\tilde{R}^{(s)} + \big[\tilde{R}^{(s)}, \Delta\big] = \sum_{p=1}^{s-1} A^{(s-p)} \tilde{R}^{(p)}, \tag{5.81}$$

$$s R^{(s)} + \big[R^{(s)}, \Delta\big] = A^{(s)} - \tilde{R}^{(s)} + \sum_{p=1}^{s-1} A^{(s-p)} R^{(p)} \tag{5.82}$$

for $s = 2, 3, \ldots$.

Note that from the recursion relations it follows that

$$\tilde{R}^{(s)}_{\omega a} = 0 \quad \text{for } l = 1, \tag{5.83}$$

$$\tilde{R}^{(s)}_{\omega A} = 0 \quad \text{for } l = 2. \tag{5.84}$$

We also note that because $\nu_{1\omega}$ is proportional to $\delta_{1\omega}$

$$\tilde{R}^{(s)}_{1\omega} = 0 \quad \text{and} \quad R^{(s)}_{1\omega} \sim \delta_{1\omega} \tag{5.85}$$

and therefore

$$\hat{U}_{1\omega} = \frac{1}{w} \delta_{1\omega}, \qquad U_{1\omega} = w \delta_{1\omega}, \tag{5.86}$$

where w is the solution of

$$\bar{\beta}(g) w'(g) = y(g) w(g), \tag{5.87}$$

where $y(g) = \mathcal{D} \ln \hat{Y}(g)$, the coefficient occurring in Eq. (5.69). This has the following important consequences:

$$V_1 = 0 \quad \text{for } l = 2, \text{ and for } l = 1, \ J > 1, \tag{5.88}$$

$$V_1 = B_1 \quad \text{for } l = 1, \ J = 1, \text{ and for } l = 0, \ J = 2. \tag{5.89}$$

5.1.5. Leading terms in coordinate space

Using the results of the preceding subsections we can calculate the leading terms in the short distance expansion of the functions (4.46). We find

$$\eta^{(J,0)} = \frac{1}{J!} f_0^{(J)} \tilde{\lambda}^{-\frac{1}{n-2}} \big\{1 + \mathrm{O}(\tilde{\lambda})\big\}, \tag{5.90}$$

$$\eta^{(1,1)} = f_1^{(1)} \tilde{\lambda}^{-\frac{1}{n-2}} \big\{1 + \mathrm{O}(\tilde{\lambda})\big\}, \tag{5.91}$$

where

$$f_0^{(J)} = \frac{2\pi n C_n}{n-2} V_1^{(J,0)}, \tag{5.92}$$

$$f_1^{(1)} = -\frac{\pi n C_n}{n-2} B_1^{(1,1)}. \tag{5.93}$$

For the case $l = 1$, $J > 1$ we have to distinguish between the cases $n > 3$ and $n = 3$. In the former case

$$\eta^{(J,1)} = \text{const} + \frac{1}{J!} f_1^{(J)} \tilde{\lambda}^{\frac{n-3}{n-2}} \big\{1 + \mathrm{O}\big(\lambda^{\frac{1}{n-2}}\big)\big\}, \tag{5.94}$$

where

$$f_1^{(J)} = \frac{4\pi^2 n C_n}{(n-2)^2} \sum_A \mathcal{L}_A^{(J)} V_A^{(J,1)}, \tag{5.95}$$

$$\mathcal{L}_A^{(J)} = -\frac{1}{2} R_{1A}^{(1)(J,1)} + \sum_a \lambda_a^{(J,1)} R_{aA}^{(1)(J,1)} + J! \tilde{q}_A^{(J,1)}. \tag{5.96}$$

In the $n = 3$ case we have

$$\eta^{(J,1)} = \text{const} + \frac{1}{J!} f_1^{(J)} \ln \tilde{\lambda} \big\{1 + \mathrm{O}(\tilde{\lambda})\big\}, \tag{5.97}$$

where

$$f_1^{(J)} = \frac{4}{\pi} \sum_A \hat{\mathcal{L}}_A^{(J)} V_A^{(J,1)}, \qquad \hat{\mathcal{L}}_A^{(J)} = \sum_a \lambda_a^{(J,1)} \tilde{R}_{aA}^{(1)(J,1)}. \tag{5.98}$$

Finally, for $l = 2$ we find

$$\eta^{(J,2)} = \text{const} + \frac{1}{J!} f_2^{(J)} \tilde{\lambda} \big\{1 + \mathrm{O}\big(\tilde{\lambda}^{\frac{1}{n-2}}\big)\big\}, \tag{5.99}$$

where

$$f_2^{(J)} = \left(\frac{2\pi}{n-2}\right)^{\frac{2n-3}{n-2}} n C_n \sum_a \mathcal{K}_a^{(J)} V_a^{(J,2)}, \tag{5.100}$$

$$\mathcal{K}_a^{(J)} = R_{1a}^{(1)(J,2)} + \sum_b \lambda_b^{(J,2)} R_{ba}^{(1)(J,2)} + J! \tilde{q}_a^{(J,2)}. \tag{5.101}$$

Using the coordinate space results above and the asymptotic formulae of Appendix G, we are now in a position to derive the results on leading large momentum behavior of the spin structure function moments given in Section 4.1.

5.2. OPE for the currents

It is straightforward to calculate the leading operator product coefficients in perturbation theory:

$$W^{(J,0)}_{\omega(0)}(y^2) = -\frac{2}{J!}\frac{1}{(n-1)g_0^2}\delta_{\omega 1} + \mathrm{O}(1), \qquad J \geqslant 2, \tag{5.102}$$

$$W^{(J,1)}_{\omega(0)}(y^2) = \frac{1}{2J!}\frac{1}{(n-2)g_0^2}\delta_{\omega k} + \mathrm{O}(1), \qquad J \geqslant 3, \tag{5.103}$$

$$\mathcal{Y}W^{(1,1)}_{1(0)}(y^2) = -\frac{1}{8\pi}\left(1 - \frac{g_0^2}{2\pi}\right) + \mathrm{O}(g_0^4), \tag{5.104}$$

where the operator associated with (5.103) is

$$\mathcal{B}^{(J,1)ab}_{k(0)} = \frac{1}{g_0^2}\left(\partial_+^{J-1}S^a \cdot \partial_+ S^b - \partial_+^{J-1}S^b \cdot \partial_+ S^a\right). \tag{5.105}$$

Eqs. (5.102) and (5.103) can be obtained by tree-level perturbation theory, while the results necessary to write down the one-loop formula (5.104) can be found in [11]. Also the results of [3,11] show that the coefficient in (4.71) is given by

$$c_1 = \frac{1}{2\pi}. \tag{5.106}$$

Using renormalization group improved perturbation theory we can write

$$\xi^{(J,l)}(y^2) = \sum_{\omega} \Gamma^{(J,l)}_{\omega}(y^2) V^{(J,l)}_{\omega}, \tag{5.107}$$

where, as in (5.63),

$$V^{(J,l)}_{\omega} = \sum_{\rho} U^{(J,l)}_{\omega\rho}(g) B^{(J,l)}_{\rho} \tag{5.108}$$

are renormalization group invariant constants and

$$\Gamma^{(J,l)}_{\omega}(y^2) = \sum_{\rho} W^{(J,l)}_{\rho}(y^2) \hat{U}^{(J,l)}_{\rho\omega}(\bar{g}) \tag{5.109}$$

are renormalization group invariant coefficient functions. Putting everything together we arrive at the results already given in Section 4.1.

6. Structure functions for $n = 3$

In this section we consider the case $n = 3$, where it is possible to compute the structure functions accurately in the whole range of x for a given q^2. The case $n = 3$ is rather special for various reasons, e.g., the spin and current 2-point functions exhibit in this case very similar properties and there are miraculous scaling relations [9] which relate them.[5] In the

[5] See also the OPE in Section 4.

S-matrix bootstrap approach its distinguishing feature is that it is the model for which the r-particle form factors can most easily be obtained explicitly. They take the simple form

$$f^a_{b_1\dots b_r}(\theta_1,\dots,\theta_r) = \Psi_r(\theta_1,\dots,\theta_r) g^a_{b_1\dots b_r}(\theta_1,\dots,\theta_r), \tag{6.1}$$

where

$$\Psi_r(\theta_1,\dots,\theta_r) = \frac{1}{2}\pi^{3r/2-1} \prod_{1\leqslant i<j\leqslant r} \psi(\theta_i-\theta_j), \tag{6.2}$$

$$\psi(\theta) = \frac{\theta-\pi i}{\theta(2\pi i-\theta)} \tanh^2\frac{\theta}{2}, \tag{6.3}$$

and the reduced form factors $g^a_{b_1\dots b_r}$ are polynomials in the rapidities. There are well defined recursive procedures for computing the form factors, the only practical limitation being that they become very involved. So far the record we have achieved is the 7-particle form factor of the spin field [13]; already its algebraic expression in MAPLE involves many megabytes of storage. Fortunately, for the structure functions we only require sums over bilinear factors of the form factors which are computationally more manageable. In correspondence to (6.1) we define reduced form factor squares $j_l^{(r)}$ through

$$J_l^{(r)}(\bar\beta_1,\dots,\bar\beta_r) = \left|\Psi_{r+1}(i\pi,\bar\beta_1,\dots,\bar\beta_r)\right|^2 j_l^{(r)}(\bar\beta_1,\dots,\bar\beta_r) \tag{6.4}$$

$$= \frac{1}{4}\pi^{3r+1}\left[\prod_{i=1}^{r} A(\bar\beta_i)\right]\left[\prod_{1\leqslant j<k\leqslant r} B(u_{jk})\right] j_l^{(r)}(\bar\beta_1,\dots,\bar\beta_r), \tag{6.5}$$

where we have introduced two new functions

$$A(\theta) \equiv \psi(i\pi-\theta)^2 = \frac{\theta^2}{(\theta^2+\pi^2)^2}\frac{1}{\tanh^4\frac{\theta}{2}}, \tag{6.6}$$

$$B(\theta) \equiv \left|\psi(\theta)\right|^2 = \frac{\theta^2+\pi^2}{\theta^2(\theta^2+4\pi^2)}\tanh^4\frac{\theta}{2}. \tag{6.7}$$

The reduced form factor squares for $r=2,3,4$ are given in Appendix H. For $r>4$ the expressions are too lengthy to exhibit in print; the results for $r=5,6$ can be obtained in the form of files from the authors.

For $r=1$ we then have (noting that $\sinh\frac{1}{2}b=\kappa$ for $x=1$),

$$w_l^{(1)} = \frac{m_l\pi^4}{4}A(b)\delta(1-x), \tag{6.8}$$

where the factors m_l are given in (2.38).

6.1. Case $r=2$

For the case $r=2$ the delta-function constraint in the integral representation is simply solved and we obtain the analytic expression

$$w_l^{(2)}\big(q^2,x\big) = \theta(\omega-2)\frac{\pi^6\kappa^2}{8\omega\sqrt{\omega^2-4}}A(\Lambda+\phi/2)A(\Lambda-\phi/2)B(\phi)\mathcal{C}_l(\Lambda,\phi), \tag{6.9}$$

where the kinematic variables ω, Λ, ϕ are given by

$$2\cosh\frac{\phi}{2} = \omega, \tag{6.10}$$

$$\omega = \frac{W}{M}, \qquad W^2 = (p+q)^2, \tag{6.11}$$

$$\cosh\Lambda = \frac{M^2 + pq}{MW}, \tag{6.12}$$

and

$$\mathcal{C}_l(\Lambda,\phi) = j_l^{(2)}(\Lambda+\phi/2, \Lambda-\phi/2). \tag{6.13}$$

Using the expression for $j_l^{(2)}$ in (H.2)–(H.4) we have

$$\mathcal{C}_0 = 8\pi^2 + 4\Lambda^2 + 3\phi^2, \tag{6.14}$$

$$\mathcal{C}_1 = -2\pi^2 + 2\Lambda^2 + \frac{1}{2}\phi^2, \tag{6.15}$$

$$\mathcal{C}_2 = 2\pi^2 - 2\Lambda^2 + \frac{3}{2}\phi^2. \tag{6.16}$$

Despite its relative simplicity, this case exhibits many features in common with higher r. The structure function approaches its asymptotic values very slowly, e.g., for q^2 fixed

$$\omega_l^{(2)}\big(q^2,x\big) \sim e_l \frac{2\pi^2}{\big(1+\frac{1}{4\kappa^2}\big)^2} \frac{1}{x\ln^2\big(\frac{4\kappa^2}{x}\big)} \quad \text{for } x\to 0, \tag{6.17}$$

with $e_0 = 1$, $e_1 = e_2 = 1/4$ (consistent with the small x behavior derived in Section 3), while for $-q^2 \to \infty$, x fixed we have

$$\omega_l^{(2)} \sim e_l \frac{\pi^6 x A(-\ln(1-x))}{8(1-x)(\ln(-q^2/M^2))^2}. \tag{6.18}$$

6.2. *Results for the entire x range*

Just as for the 2-point functions [9] we find that for a fixed $-q^2$ only states with a limited number of particles contribute significantly. To appreciate this better we consider the sum of the field and current structure functions, which is a rather peculiar thing to do in general, but which is in fact rather natural in the special case $n = 3$. Figs. 1 and 2 illustrate how the structure function $x(w_0 + \tilde{w}_0)$ is built up from states with increasing particle number for the cases $-q^2/M^2 = 10^2$ and $-q^2/M^2 = 10^4$, respectively. We see that the higher states contribute very little and that we obtain nearly exact values for the structure functions for all values of $-q^2/M^2 < 10^5$ by including only intermediate states with $\leqslant 5$ particles for the current and $\leqslant 6$ particles for the spin field.

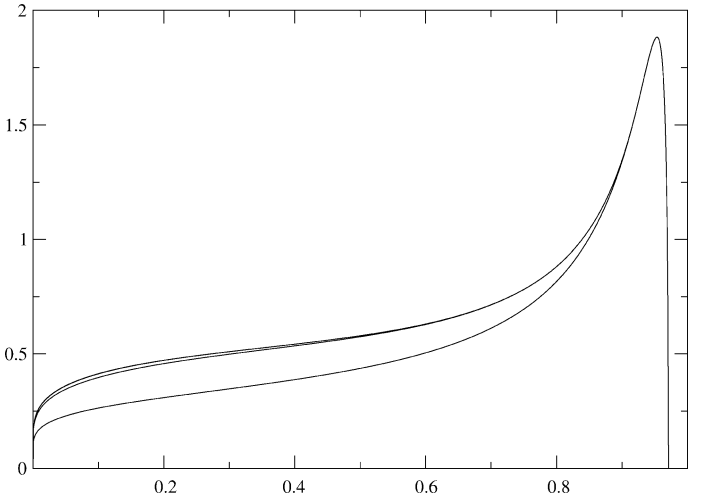

Fig. 1. Approximations to $x(w_0+\tilde{w}_0)$ as functions of $0<x<1$ for $-q^2/M^2=100$. Curves correspond to sums up to and including 2, 3, 4, 5, 6-particle intermediate states. The last 3 curves are indistinguishable on this scale.

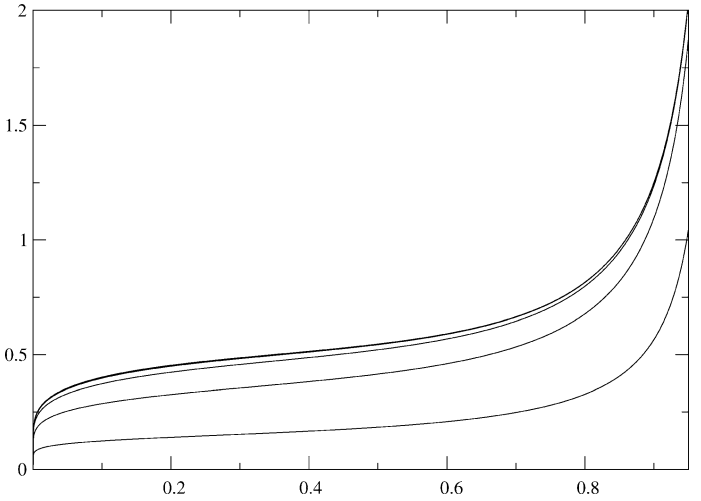

Fig. 2. Approximations to $x(w_0+\tilde{w}_0)$ as functions of $0<x\leqslant 0.95$ for $-q^2/M^2=10^4$. Curves correspond to sums up to and including 2,3,4,5,6-particle states.

In Fig. 3 we plot $xw_0(q^2,x)$ as a function of $\log_{10}(-q^2/M^2)$, for a selection of x-values.[6] The function increases as $-q^2$ increases for all values of x in this range and seems consistent with Bjorken scaling as mentioned in Section 4.

[6] For this model we prefer to show this rather than the typical HERA plot where one adds $-\log_{10}(x)$ to separate the x-values, because the latter would obscure the $-q^2$ variation which is rather small compared to the variation of $-\log_{10}(x)$.

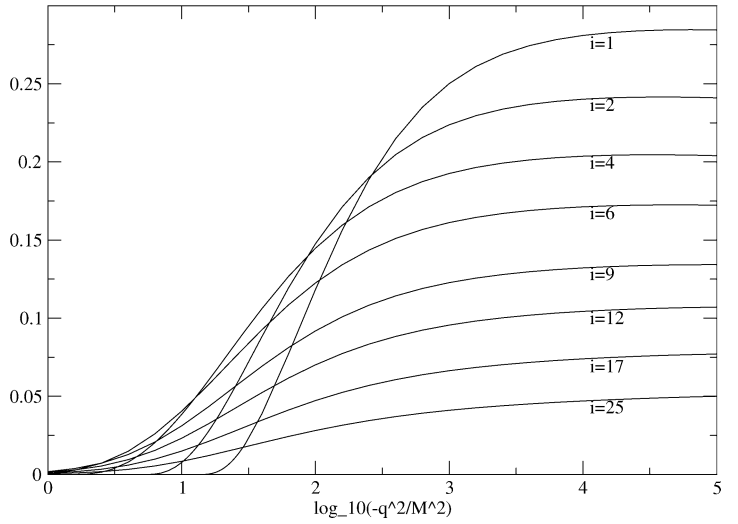

Fig. 3. $x w_0(q^2, x)$ for various values of $x = 10^{-i/5}$.

6.3. Threshold behavior

Note that in Fig. 2 we have cut off the plot at $x = 0.95$. This is because near $x = 1$ the function develops a big bump with a peak ~ 70 which, if included in the same plot, would obscure the features we wanted to show there. The behavior of the σ-model structure functions near $x = 1$ is indeed rather involved. For a fixed $-q^2$ the contribution to the structure function from the r-particle state $w^{(r)}$ vanishes for x greater than some threshold value

$$x_r(-q^2) = \left[1 - (r^2 - 1)M^2/q^2\right]^{-1}. \tag{6.19}$$

The big bump in $x(w_0 + \tilde{w}_0)$ referred to above is at this value of $-q^2/M^2 = 10^4$ practically entirely due to the 2-particle contribution. For this contribution:

$$w_l^{(2)} \sim E_l(q^2)\sqrt{x_2(-q^2) - x}, \quad x \to x_2, \quad -q^2 \text{ fixed}, \tag{6.20}$$

$$w_l^{(2)} \sim \frac{F_l(x)}{\ln^2(-q^2/M^2)}, \quad -q^2 \to \infty, \quad x \text{ fixed}, \tag{6.21}$$

where E_l, F_l are some (known) functions. The bump arises because F_l is quite singular near threshold, $F_l \sim [(1-x)\ln^2(1-x)]^{-1}$. The analytic behavior as $x \to x_2$ sets in only extremely close to threshold, e.g., for $-q^2/M^2 = 10^4$ the position of the peak of the bump is at $x = 0.99954$ whereas the function vanishes at $x_2 = 0.99970$. At $-q^2/M^2 = 10^4$ the 3-particle contribution also has a bump but it is less pronounced (peak value ~ 2.5 at $x \sim 0.9953$). We conjecture that the threshold behavior of $w_0^{(r)}$ in the O(3) model is $(x_r - x)^{(r^2-3)/2}$.

6.4. Moments

For $r = 1$ the moments (2.29) are simply given by

$$M^{(1)}_{l;N}(q^2) = \frac{m_l \pi^4}{4} A(2\,\mathrm{asinh}\,\kappa). \tag{6.22}$$

The moments with $N > 1$ are quite simple to evaluate numerically, but for $N = 1$ some care must be taken to obtain accurate results.

The problem arises already for $r = 2$ where we have integration just over u_1. The ψ-factor together with the x^2 factor in the integrand is

$$F^{(2)}_{\psi} = A(\bar\beta_1)A(\bar\beta_2)B(u_1)\bar x^2. \tag{6.23}$$

Now for u_1 very large $\bar\beta_2 \sim (-q^2 + M^2)\mathrm{e}^{-u_1}$ is exponentially small and so also $\bar x \sim -q^2\mathrm{e}^{-u_1}$. Noting (i) for u_1 large $\bar\beta_1 \sim u_1$ and (ii) $A(\theta) \sim 16/(\pi^4\theta^2)$ for $\theta \to 0$ we have for large u_1

$$F^{(2)}_{\psi} \sim \left(\frac{-4q^2}{\pi^2(-q^2 + M^2)}\right)^2 A_B(u_1), \tag{6.24}$$

where

$$A_B(u) = A(u)B(u) = \frac{1}{(u^2 + \pi^2)(u^2 + 4\pi^2)}. \tag{6.25}$$

The integral over large u_1 gives a sizeable contribution because the integrand decays only as u_1^{-2}. The integral is thus broken up into two parts where for the large u_1 region the substitution (6.24) is made and there computation of exponential functions of large argument are not necessary.

For the case of higher r the procedure is similar. Here the integrations over $u_1, \ldots, u_{r-2}$ can be done safely by introducing for them (large) cutoffs (and monitoring the dependence on them), since the integrands are exponentially suppressed. But for the integration over large u_{r-1} the integrand is not exponentially suppressed and in this region one replaces the corresponding ψ-factor by

$$F^{(r)}_{\psi} \sim \left(\frac{-4q^2}{\pi^2[-q^2 + M^{(r-1)}(\mathbf{u})^2]}\right)^2 \prod_{j=1}^{r-1} A_B(u_{jr}) \prod_{1 \leqslant k < l \leqslant r-1} B(u_{kl}). \tag{6.26}$$

In Figs. 4 and 5 we plot the separate r-particle contributions $M^{(r)}_{0;2}$ and $M^{(r)}_{1;1}$ respectively; some corresponding numbers are given in Tables 1–3 in Appendix I. They are typically bell-shaped (except for $r = 1$) and perhaps obey scaling relations similar to those of the spectral functions examined in Ref. [9]. The figures show how they build up the sum of moments $M_{0;2} + \tilde M_{0;2}$ and $M_{1;1} + \tilde M_{1;1}$. Using the exact ratio of the mass to the Λ-parameter Eq. (4.18), we also exhibit the perturbative results up to and including terms of order $\lambda(q^2)$. The agreement of the summed terms and PT is impressive for $-q^2/M^2 \sim 10^5$. For values of $-q^2/M^2 >\sim 10^6$ contributions from states with $\geqslant 7$ particles must be taken into ac-

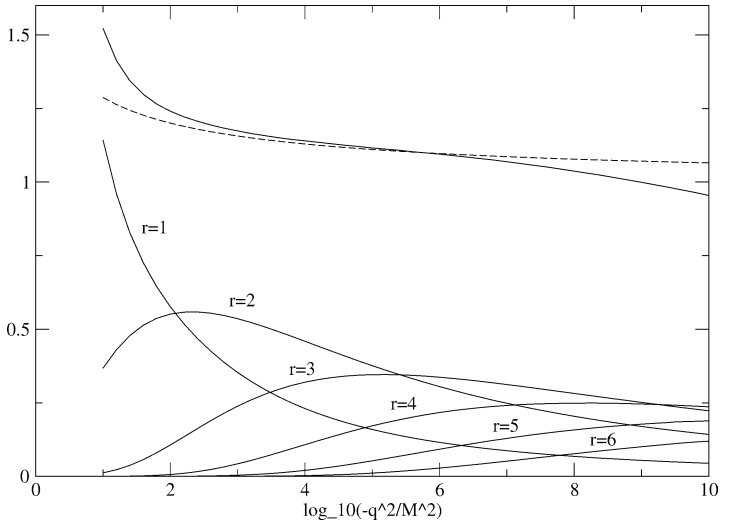

Fig. 4. Contributions $M^{(r)}_{0;2}$ for $n = 3$ from $(r = 1, \ldots, 6)$-particle states. The upper full line is their sum. The dashed line is the perturbative expansion of $M_{0;2} + \tilde{M}_{0;2} = 1 + \lambda$ up to and including terms of order $\lambda(q^2)$.

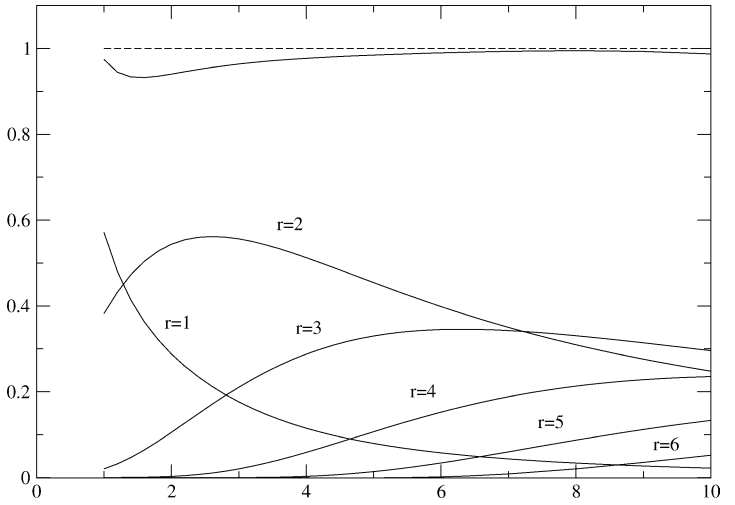

Fig. 5. As for Fig. 4 but for the moment $l = 1$, $N = 1$; here the PT result is $1 + \mathrm{O}(\lambda^2)$.

count. Note we have also included the contribution of the one particle states in the sums; these tend to improve the agreement at lower values of $-q^2$ and fall asymptotically as $M^{(1)}_{l;N} \sim m_l \pi^4/[4 \ln^2(-q^2/M^2)]$.

7. Sigma model structure functions in the $1/n$ approximation

7.1. The spin field structure functions

In the framework of the $1/n$ approximation the spin amplitude (2.19) has an expansion of the form

$$\Sigma^{ab;cd}(p,q) = \sum_{r=1}^{\infty} \frac{1}{n^r} \big[S_1^{[r]} \delta^{ac}\delta^{bd} + S_2^{[r]} \delta^{ad}\delta^{bc} + S_3^{[r]} \delta^{ab}\delta^{cd} \big], \tag{7.1}$$

and so for the various isospin channels

$$\tilde{w}_l(q^2,x) = \sum_{r=1}^{\infty} \frac{1}{n^r} \tilde{w}_l^{[r]}(q^2,x), \tag{7.2}$$

with

$$\tilde{w}_0^{[r]} = S_1^{[r]} + S_2^{[r]} + S_3^{[r+1]}, \tag{7.3}$$

$$\tilde{w}_1^{[r]} = S_1^{[r]} - S_2^{[r]}, \tag{7.4}$$

$$\tilde{w}_2^{[r]} = S_1^{[r]} + S_2^{[r]}. \tag{7.5}$$

The Feynman rules for the $1/n$ expansion of the σ-model has been described in many places (see, e.g., Ref. [14] and references therein) and will not be repeated here. We only mention that the diagrams involve the bare propagator of the fundamental spin field, and the bare propagator B^{-1} of an auxiliary isospin scalar composite field, which we call λ, which is the inverse of the scalar 1-loop integral given in Appendix J.

In leading order $1/n$ the only contribution to the scalar structure function is the tree diagram involving λ exchange in the "s-channel"; one thus gets an amplitude proportional to the imaginary part of B^{-1}:

$$S_2^{[1]} = S_3^{[1]} = 0, \tag{7.6}$$

$$S_1^{[1]} = 4\pi\theta(\omega-2) \frac{-q^2M^2}{(-q^2+M^2)^2} \frac{\operatorname{sh}\phi}{\phi^2+\pi^2}, \tag{7.7}$$

where ϕ is as in (6.10). Note we already anticipated $S_3^{[1]} = 0$ by starting the sum over r at 1 in (7.2).

In the limit of small x we have

$$S_1^{[1]} \sim \frac{2\pi}{(1-M^2/q^2)^2} \frac{1}{x \ln^2 x}, \quad x \to 0, \quad -q^2 \text{ fixed}, \tag{7.8}$$

consistent with the general result (3.15) for $l = 1, 2$, the scalar Adler function in the leading order $1/n$ being just $A_0(z) = (1+M^2/z)^{-2}$. Note the limit is approached extremely slowly. E.g., denoting the asymptotic function on the rhs of (7.8) by S_{asympt} one has $S_1^{[1]}/S_{\text{asympt}} = 0.245, 0.441, 0.638, 0.907$ for $x = 10^{-5}, 10^{-10}, 10^{-20}, 10^{-100}$, respectively.

One also observes the limit

$$S_1^{[1]} \sim \frac{2\pi(1-x)}{x} \frac{1}{\ln^2(-q^2/M^2)}, \quad -q^2 \to \infty, \quad x \text{ fixed}, \tag{7.9}$$

and the threshold behavior:

$$S_1^{[1]} \sim \frac{4(-q^2/M^2)^{3/2}}{\pi(1-q^2/M^2)^2 x_2}\sqrt{x_2-x}, \quad x \to x_2, \quad -q^2 \text{ fixed.} \tag{7.10}$$

We caution that the limits $n \to \infty$ and $x \to x_2$ may not commute.

The moments (2.28) have an $1/n$ expansion of the form

$$\tilde{M}_{l;N}(q^2) = \sum_{r=1}^{\infty} \frac{1}{n^r} \tilde{M}_{l;N}^{[r]}(q^2). \tag{7.11}$$

One then shows (e.g., numerically) that for $-q^2 \to \infty$

$$\tilde{M}_{l;1}^{[1]}(q^2) \sim \frac{2\pi}{\ln(-q^2/M^2)}, \quad l = 1, 2, \tag{7.12}$$

$$\tilde{M}_{l;N}^{[1]}(q^2) \sim \frac{2\pi}{N(N-1)\ln^2(-q^2/M^2)} \quad \text{for } N > 1,\ l = 1, 2, \tag{7.13}$$

consistent with the results (4.15) and (4.16). The $1/\ln(-q^2)$ behavior for the $N = 1$ moment comes from the singular behavior at $x = 0$.

So far we have only obtained the leading order for the isospin $l = 1, 2$ channels. This is because Eq. (7.3) shows that to obtain the leading order $1/n$ approximation for $l = 0$, one has to take into account also the amplitude $S_3^{[2]}$. To this amplitude the only contribution comes from a diagram with two λ-exchanges in the "t-channel"

$$S_3^{[2]} = \frac{-4q^2}{2\pi(-q^2+M^2)^2} \operatorname{Im} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} d\big(k+q^{\mathrm{E}}\big) d\big(k-p^{\mathrm{E}}\big) B(k)^{-2}, \tag{7.14}$$

where $d(k)$ is the Euclidean bare spin propagator

$$d(k) = \big(k^2 + M^2\big)^{-1}, \tag{7.15}$$

and $q^{\mathrm{E}} = (iq_0, q_1)$ and similarly p^{E} with $p^{\mathrm{E}2} = -p^2 = -M^2$. Using the spectral representation of $B(k)^{-1}$ and the cutting rules[7] (see Appendix J) we get

$$S_3^{[2]} = \frac{-q^2}{M^2(-q^2+M^2)^2} \frac{\theta(\omega-2)}{\omega\sqrt{\omega^2-4}} \frac{1}{2\pi} \big[B(k_+)^{-2} + B(k_-)^{-2}\big], \tag{7.16}$$

where

$$k_\pm^2 = \frac{RM^2}{y}\left[1 - \frac{y}{R} \pm \sqrt{\frac{(1-y)(1+2x_2y^2/R)}{1-y+2y/R}}\right], \tag{7.17}$$

$$y = \frac{x}{x_2(-q^2)}, \qquad R = \frac{-q^2}{2M^2 x_2(-q^2)}. \tag{7.18}$$

Again the small x limit

$$S_3^{[2]} \sim \frac{2\pi}{(1-M^2/q^2)^2} \frac{1}{x\ln^2 x}, \quad x \to 0, \quad -q^2 \text{ fixed}, \tag{7.19}$$

[7] One can initially introduce a UV cutoff in the spectral integral and remove it after invoking the cutting rules.

is as expected. For the large q^2 and threshold behaviors we have

$$S_3^{[2]} \sim \frac{2\pi}{x(1-x)} \frac{1}{\ln^2(-q^2/M^2)}, \quad -q^2 \to \infty, \quad x \text{ fixed}, \tag{7.20}$$

$$S_3^{[2]} \sim \frac{8\pi(-q^2/M^2)^{1/2} x_2}{(1-q^2/M^2)^2} \frac{\sinh^2\varphi}{\varphi^2} \frac{1}{\sqrt{x_2 - x}}, \quad x \to x_2, \quad -q^2 \text{ fixed}, \tag{7.21}$$

where

$$\varphi = 2\,\mathrm{asinh}\left(\frac{\sqrt{R-1}}{2}\right). \tag{7.22}$$

For the leading isospin 0 moments we then have

$$\tilde{M}_{0;N}^{[1]}(q^2) = \frac{2\pi x_2^{N-1}}{R^2(1-M^2/q^2)^2} \int_0^1 \mathrm{d}z \frac{z^N}{S(z,q^2)} \left[\frac{\sinh^2\theta_+}{\theta_+^2} + \frac{\sinh^2\theta_-}{\theta_-^2}\right], \tag{7.23}$$

where

$$k_\pm^2 = 4M^2 \sinh^2 \frac{\theta_\pm}{2}, \tag{7.24}$$

$$S(z,q^2) = \sqrt{(1-z)(1-z+2z/R)}. \tag{7.25}$$

Numerically one extracts the behavior

$$\tilde{M}_{0;N}^{[1]}(q^2) \sim \frac{2\pi}{\ln(-q^2/M^2)} \quad \text{for } N \geqslant 2, \tag{7.26}$$

in perfect agreement with (4.10). The dominant asymptotic piece comes from the "k_+" contribution in (7.23), and the dominant large q^2 behavior originates from the singularity near threshold.

7.2. The current structure functions

We now turn to the current structure functions for which the non-trivial parts are more complicated to compute in the $1/n$ approximation than those for the spins. We have

$$w^{ab;cdef}(p,q) = \sum_{r=0}^{\infty} \frac{1}{n^r} \left[-Y^{ab;cdef} W_1^{[r]} - Y^{ba;cdef} W_2^{[r]} + X^{ab;cdef} W_3^{[r]}\right], \tag{7.27}$$

and so for the various invariant isospin channels

$$w_I(q^2,x) = \sum_{r=0}^{\infty} \frac{1}{n^r} w_I^{[r]}(q^2,x), \tag{7.28}$$

with

$$w_0^{[r]} = 2W_1^{[r]} + 2W_2^{[r]} + W_3^{[r+1]}, \tag{7.29}$$

$$w_1^{[r]} = W_1^{[r]} - W_2^{[r]}, \tag{7.30}$$

$$w_2^{[r]} = -W_1^{[r]} - W_2^{[r]}. \tag{7.31}$$

At leading order one has the simple diagram with the 1-particle pole in the s-channel; this contributes to the structure functions only terms $\propto \delta(1-x)$.

We denote the amputated two-current two-spin correlation function by $T_{\mu\nu}^{ab;cdef}$ with coefficients in the $1/n$ expansion $T_{s;\mu\nu}^{[r]}$ whose imaginary parts are proportional to the $W_s^{[r]}$. There are 3 types of diagrams contributing to $T_{1;\mu\nu}^{[1]}$ which involve one λ propagator. One is the box diagram, another involves a vertex correction, and the last involves a spin self energy diagram. They can be conveniently combined together[8] to yield (in Euclidean space)

$$T_{1;\mu\nu}^{[1]}\big(p^{\mathrm{E}}, q^{\mathrm{E}}\big) = 8\int \frac{\mathrm{d}^2k}{(2\pi)^2} D_\mu\big(p^{\mathrm{E}}, q^{\mathrm{E}}, k\big) D_\nu\big(p^{\mathrm{E}}, q^{\mathrm{E}}, k\big) d\big(k+q^{\mathrm{E}}\big) B\big(p^{\mathrm{E}}-k\big)^{-1}, \tag{7.32}$$

with

$$D_\mu(p,q,k) = (k+q/2)_\mu d(k) + (p+q/2)_\mu d(p+q). \tag{7.33}$$

So contracting over μ, ν

$$\sum_\mu T_{1;\mu\mu}^{[1]}(p,q) = F(p) + 8\int \frac{\mathrm{d}^2k}{(2\pi)^2} D(p,q,k) d(k+q) B(p-k)^{-1}, \tag{7.34}$$

$$\begin{aligned} D(p,q,k) &= (p+q/2)^2 d(p+q)^2 + \frac{1}{2}\big[d(p+q) + d(k)\big], \\ &\quad + \big[-(p-k)^2 + p^2 + pq - M^2\big] d(p+q) d(k) - \frac{1}{4}\big[q^2 + 4M^2\big] d(k)^2, \end{aligned} \tag{7.35}$$

where F is a function of p^2 alone. Since we are only interested in the structure function only one term appearing in the cutting rule is relevant. Still in Euclidean space, and omitting pole terms $\propto \delta(1-x)$, the relevant term is

$$\sum_\mu T_{1;\mu\mu}^{[1]}\big(p^{\mathrm{E}}, q^{\mathrm{E}}\big) \sim -16\pi M^4 \int_0^\Lambda \mathrm{d}\kappa \frac{\sinh^2\kappa}{\kappa^2+\pi^2} B\big(p^{\mathrm{E}} + q^{\mathrm{E}}; M, m\big)(I_+ + I_-), \tag{7.36}$$

where Λ is some ultraviolet cutoff and

$$m = 2M\cosh\kappa/2, \tag{7.37}$$

$$I_\pm = 2D\big(p^{\mathrm{E}}, q^{\mathrm{E}}, k_\pm\big(p^{\mathrm{E}}, q^{\mathrm{E}}\big)\big), \tag{7.38}$$

$$\begin{aligned} k_\pm^2(p,q) &= -qp - \frac{[M^2(p^2+qp) + m^2(q^2+qp)]}{(q+p)^2} \\ &\quad \mp \frac{i\epsilon qp}{(q+p)^2}\sqrt{\big[(p+q)^2 + (m+M)^2\big]\big[(p+q)^2 + (m-M)^2\big]}. \end{aligned} \tag{7.39}$$

[8] One can conveniently use lattice UV regularization at intermediate stages.

So going over to Minkowski space

$$W_1^{[1]} = 4M^4\theta(W-3M)\int_0^{\kappa_0} \mathrm{d}\kappa \frac{(C_+ + C_-)\sinh^2\kappa}{(\kappa^2+\pi^2)\sqrt{[W^2-(m+M)^2][W^2-(m-M)^2]}} \tag{7.40}$$

with κ_0 defined through

$$\cosh\frac{\kappa_0}{2} = \frac{W-M}{2M} \tag{7.41}$$

and

$$\begin{aligned} C_\pm &= -\frac{4M^2+4pq+q^2}{2(W^2-M^2)^2} - \frac{1}{W^2-M^2}\big[1+2\big(-pq+m^2-2M^2\big)d(K_\pm)\big] \\ &\quad + d(K_\pm) - \frac{1}{2}\big(-q^2+4M^2\big)d(K_\pm)^2, \end{aligned} \tag{7.42}$$

$$\begin{aligned} K_\pm^2 &= qp - \frac{[M^2(M^2+qp)+m^2(q^2+qp)]}{W^2} \\ &\quad \mp \frac{\sqrt{(pq)^2-M^2q^2}}{W^2}\sqrt{\big[W^2-(m+M)^2\big]\big[W^2-(m-M)^2\big]}. \end{aligned} \tag{7.43}$$

Numerically for small x we find consistency[9] with the general result (3.14):

$$w_l^{[1]} \sim \frac{2\pi}{x\ln^2 x} a_1\big(-q^2\big), \quad l=1,2, \tag{7.44}$$

with

$$a_1\big(-q^2\big) = \frac{1}{2\pi}\left[3 - \frac{\theta}{\sinh\theta}(2+\cosh\theta)\right], \tag{7.45}$$

$$-q^2 = 4M^2\sinh^2\frac{\theta}{2}, \tag{7.46}$$

since $a_1(z)$ is the leading order $1/n$ contribution to the Adler function $A_1(z)$. In leading order $1/n$ the current vacuum 2-point function amplitude $I_1 = i_1$ with[10]

$$i_1(z) = \frac{1}{\pi z} - \frac{(z+4M^2)}{z}B(r), \quad r^2 = z. \tag{7.47}$$

We have not yet computed the $1/n$ contribution to the isospin 0 structure function w_0. This also involves computing $T_{3\mu\nu}^{[2]}$ which is more complicated because it requires the evaluation of 2-loop graphs involving also λ propagators.

[9] We did not yet confirm this analytically.

[10] For $z=-q^2$ we have

$$-z^2\frac{\partial}{\partial z}i_1(z) = \frac{1}{\pi} + \frac{z^2}{2M^4\sinh\theta}\frac{\partial}{\partial\theta}\left[\frac{\theta\cosh\theta/2}{8\pi\sinh^3\theta/2}\right] = a_1(z).$$

8. Summary and conclusions

In this paper we calculated the DIS structure functions (and their moments) in the family of the 2-dimensional O(n) non-linear sigma models using standard field theory techniques available in any asymptotically free field theory model. In the special case of the O(3) model we compared the results to the non-perturbative (bootstrap) determination of the same structure functions.

The very good agreement between the results (in the intermediate energy range, where both the perturbative field theory results and the non-perturbative bootstrap results are expected to be valid) strongly indicates—once again—that standard field theory and the bootstrap define the same model. On the other hand this agreement provides some further, indirect proof for all the assumptions that are used in the derivations in both methods.

The study of the structure functions has lead us to two interesting findings. First, we found that the isospin 0 structure functions exhibit exact Bjorken scaling: for $-q^2 \to \infty$ the structure functions go to some non-trivially x-dependent limits. Second, the exact small x asymptotics of the structure functions are shown to be different from the soft-Pomeron like fractional power behavior: the asymptotics of (x times) the structure function is logarithmic.

In the first case we have obtained concrete results in the O(3) model only but we think that our findings are more generally valid: it is probable that exact Bjorken scaling is due to the presence of the (infinitely many) higher spin conserved charges characteristic to integrable models. Also in the second case we believe that the small x asymptotics we found here is valid in a more general setting. Whether something similar applies to QCD remains to be seen.

Acknowledgements

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Appendix A. O(n) notations and identities

The O(n) generators Q^{ab} act in the defining (vector) representation as

$$\left[Q^{ab}, V^c\right] = i t^{ab}_{cd} V^d, \tag{A.1}$$

where the generator matrix is

$$t^{ab}_{cd} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}. \tag{A.2}$$

This corresponds to the usual relation

$$\left[Q^a, V^b\right] = i\epsilon^{abc} V^c \tag{A.3}$$

in the $n = 3$ case with $Q^a = \frac{1}{2}\epsilon^{abc} Q^{bc}$.

The generator matrix in the 2-index tensor representation is

$$t^{ab}_{cd;uv} = t^{ab}_{cu}\delta^{dv} + \delta^{cu}t^{ab}_{dv}, \tag{A.4}$$

and similarly for higher representations.

Projector matrices in the 2-index tensor representation are

$$P_0^{ab;a'b'} = \frac{1}{n}\delta^{ab}\delta^{a'b'}, \tag{A.5}$$

$$P_1^{ab;a'b'} = \frac{1}{2}\big(\delta^{aa'}\delta^{bb'} - \delta^{ab'}\delta^{ba'}\big), \tag{A.6}$$

$$P_2^{ab;a'b'} = \frac{1}{2}\big(\delta^{aa'}\delta^{bb'} + \delta^{ab'}\delta^{ba'}\big) - \frac{1}{n}\delta^{ab}\delta^{a'b'}. \tag{A.7}$$

They satisfy

$$P_k^{ab;a'b'}P_l^{a'b';a''b''} = \delta_{kl}P_l^{ab;a''b''}, \tag{A.8}$$

$$\sum_l P_l^{ab;a'b'} = \delta^{aa'}\delta^{bb'}, \tag{A.9}$$

and

$$P_k^{ab;ab} = \pi_l \tag{A.10}$$

with

$$\pi_0 = 1, \qquad \pi_1 = \frac{n(n-1)}{2}, \qquad \pi_2 = \frac{(n-1)(n+2)}{2}. \tag{A.11}$$

In the $n = 3$ case $\pi_l = 2l + 1$.

6-index invariant tensors, antisymmetric in the last two index pairs are

$$X^{ab;cdef} = \delta^{ab}\big(\delta^{ce}\delta^{df} - \delta^{cf}\delta^{de}\big), \tag{A.12}$$

$$Y^{ab;cdef} = \delta^{ac}\delta^{be}\delta^{df} - \delta^{ad}\delta^{be}\delta^{cf} - \delta^{ac}\delta^{bf}\delta^{de} + \delta^{ad}\delta^{bf}\delta^{ce}, \tag{A.13}$$

(and $Y^{ba;cdef}$). The irreducible combinations are

$$R_0^{ab;cdef} = \frac{1}{n}X^{ab;cdef}, \tag{A.14}$$

$$R_1^{ab;cdef} = \frac{1}{2}\big(Y^{ab;cdef} - Y^{ba;cdef}\big), \tag{A.15}$$

$$R_2^{ab;cdef} = \frac{2}{n}X^{ab;cdef} - \frac{1}{2}\big(Y^{ab;cdef} + Y^{ba;cdef}\big), \tag{A.16}$$

which satisfy

$$R_k^{ab;cdef}R_l^{a'b';cdef} = \delta_{kl}\hat{r}_l P_l^{ab;a'b'} \tag{A.17}$$

with

$$\hat{r}_0 = 2(n-1), \qquad \hat{r}_1 = \hat{r}_2 = 4(n-2). \tag{A.18}$$

Note that in the $n = 3$ case the antisymmetric tensor representation coincides with the vector representation and indeed in this case

$$\frac{1}{4}\epsilon^{cxy}\epsilon^{duv}R_l^{ab;xyuv} = P_l^{ab;cd}. \tag{A.19}$$

Finally, we note the following identities. The generators of the vector representation satisfy

$$t_{cu}^{ax}t_{du}^{bx} = \sum_{l=0}^{2} V_l P_l^{ab;cd} \tag{A.20}$$

with

$$V_0 = 2(n-1), \qquad V_1 = V_2 = n-2, \tag{A.21}$$

and similarly for the antisymmetric tensor representation

$$t_{cd;uv}^{ax}t_{ef;uv}^{bx} - t_{cd;uv}^{ax}t_{ef;vu}^{bx} = \sum_{l=0}^{2} T_l R_l^{ab;cdef}, \tag{A.22}$$

where

$$T_0 = 4(n-2), \qquad T_1 = n-2, \qquad T_2 = 4-n. \tag{A.23}$$

Appendix B. Particle states, rapidity integrals

The r-particle "in" states are characterized by the O(n) labels $a_1, \ldots, a_r$ and the decreasing set of rapidities $\theta_1, \ldots, \theta_r$ and their normalization is

$$\begin{aligned} &{}^{\text{in}}\langle a_1', \theta_1'; \ldots; a_r', \theta_r' | a_1, \theta_1; \cdots; a_r, \theta_r\rangle^{\text{in}} \\ &\quad = (4\pi)^r \delta^{a_1 a_1'} \cdots \delta^{a_r a_r'} \delta(\theta_1' - \theta_1) \cdots \delta(\theta_r' - \theta_r), \end{aligned} \tag{B.1}$$

corresponding to the completeness relation in the r-particle sector

$$\begin{aligned} \Pi^{(r)} = \frac{1}{(4\pi)^r} \sum_{a_1 \ldots a_r} \int_{-\infty}^{\infty} \mathrm{d}\theta_1 \int_{-\infty}^{\theta_1} \mathrm{d}\theta_2 \cdots \\ \times \int_{-\infty}^{\theta_{r-1}} \mathrm{d}\theta_r \, |a_1, \theta_1; \ldots; a_r, \theta_r\rangle^{\text{in}\,\text{in}}\langle a_1, \theta_1; \ldots; a_r, \theta_r|. \end{aligned} \tag{B.2}$$

As usual, we introduce the set of positive rapidity differences

$$u_1 = \theta_1 - \theta_2, \qquad u_2 = \theta_2 - \theta_3, \quad \ldots, \quad u_{r-1} = \theta_{r-1} - \theta_r \tag{B.3}$$

and the r-particle invariant mass $M^{(r)}(u)$ with the definition

$$M^{(r)}(u)\mathrm{e}^{\pm\Lambda} = M\sum_{i=1}^{r}\mathrm{e}^{\pm\theta_i} = E_r \pm P_r = P_r^0 \pm P_r^1, \tag{B.4}$$

where M is the mass of the O(n) particles.

The r-particle rapidity integral can now be written

$$\int_{-\infty}^{\infty} \mathrm{d}\theta_1 \int_{-\infty}^{\theta_1} \mathrm{d}\theta_2 \cdots \int_{-\infty}^{\theta_{r-1}} \mathrm{d}\theta_r = \int_{-\infty}^{\infty} \mathrm{d}\Lambda \int \mathcal{D}u^{(r)}, \tag{B.5}$$

where

$$\mathcal{D}u^{(r)} = \int_0^{\infty} \mathrm{d}u_1 \int_0^{\infty} \mathrm{d}u_2 \cdots \int_0^{\infty} \mathrm{d}u_{r-1}. \tag{B.6}$$

The inverse transformation is

$$\theta_i = \beta_i + \Lambda - v_+^{(r)} + v_-^{(r)}, \tag{B.7}$$

where

$$\beta_j = u_{jr}, \quad j = 1, \ldots, r-1, \tag{B.8}$$

$$u_{jk} = u_j + u_{j+1} + \cdots + u_{k-1}, \quad 1 \leqslant j < k \leqslant r, \tag{B.9}$$

$$\beta_r = 0, \tag{B.10}$$

and

$$v_{\pm}^{(r)} = \frac{1}{2} \ln \left[1 + \sum_{i=1}^{r-1} \mathrm{e}^{\pm\beta_i} \right]. \tag{B.11}$$

We note that

$$v_+^{(r)} + v_-^{(r)} = \ln \mu_r, \tag{B.12}$$

where $\mu_r = M^{(r)}(u)/M$ is the dimensionless invariant mass.

Appendix C. S-matrix asymptotics

The Zamolodchikov O(n) S-matrix is [4]

$$S_{ab;cd}(\theta) = \sigma_1(\theta)\delta^{ab}\delta^{cd} + \sigma_2(\theta)\delta^{ac}\delta^{bd} + \sigma_3(\theta)\delta^{ad}\delta^{bc}, \tag{C.1}$$

where

$$\sigma_1(\theta) = \frac{-2\pi i\theta}{i\pi - \theta} \frac{s_2(\theta)}{(n-2)\theta - 2\pi i}, \tag{C.2}$$

$$\sigma_2(\theta) = (n-2)\theta \frac{s_2(\theta)}{(n-2)\theta - 2\pi i}, \tag{C.3}$$

$$\sigma_3(\theta) = -2\pi i \frac{s_2(\theta)}{(n-2)\theta - 2\pi i}, \tag{C.4}$$

and the "isospin 2" phase shift is given by

$$s_2(\theta) = -\exp\left\{2i\int_0^\infty \frac{\mathrm{d}\omega}{\omega}\sin(\theta\omega)\tilde{K}_n(\omega)\right\} \tag{C.5}$$

with

$$\tilde{K}_n(\omega) = \frac{\mathrm{e}^{-\pi\omega} + \mathrm{e}^{-2\pi\frac{\omega}{n-2}}}{1+\mathrm{e}^{-\pi\omega}}. \tag{C.6}$$

Specially for $n = 3$

$$s_2(\theta) = \frac{\theta - i\pi}{\theta + i\pi}. \tag{C.7}$$

Using the asymptotic formula

$$\int_0^\infty \frac{\mathrm{d}\omega}{\omega}\sin(\theta\omega)k(\omega) \cong \frac{\pi}{2}k(0) + \frac{k'(0)}{\theta} + \mathrm{O}\left(\frac{1}{\theta^2}\right), \tag{C.8}$$

we get the large θ asymptotics of the S-matrix, which can be written as

$$S_{ab;cd}(\theta) \cong \delta^{ac}\delta^{bd} + \frac{2\pi i}{(n-2)\theta}t^{ac}_{bd} + \mathrm{O}\left(\frac{1}{\theta^2}\right). \tag{C.9}$$

Appendix D. Residue asymptotics

For any $(r+2)$-particle form factor in the O(n) model Smirnov's residue axiom [6] can be written as

$$\begin{aligned}
&\mathcal{F}^A_{aba_1\ldots a_r}(\beta + i\pi + \varepsilon, \beta, \theta_1, \ldots, \theta_r) \\
&\quad\cong \frac{2i}{\varepsilon}\big\{\delta_{ab}\mathcal{F}^A_{a_1\ldots a_r}(\theta_1, \ldots, \theta_r) \\
&\qquad - S_{ba_1\ldots a_r;b_1\ldots b_r a}(\theta_1, \ldots, \theta_r|\beta)\mathcal{F}^A_{b_1\ldots b_r}(\theta_1, \ldots, \theta_r)\big\},
\end{aligned} \tag{D.1}$$

where

$$\begin{aligned}
&S_{ba_1\ldots a_r;b_1\ldots b_r a}(\theta_1, \ldots, \theta_r|\beta) \\
&\quad= S_{ba_1;c_1b_1}(\beta - \theta_1)S_{c_1a_2;c_2b_2}(\beta - \theta_2)\cdots \\
&\qquad\times S_{c_{r-2}a_{r-1};c_{r-1}b_{r-1}}(\beta - \theta_{r-1})S_{c_{r-1}a_r;ab_r}(\beta - \theta_r).
\end{aligned} \tag{D.2}$$

If β is large we can use (C.9) to get

$$\mathcal{F}^A_{aba_1\ldots a_r}(\beta + i\pi + \varepsilon, \beta, \theta_1, \ldots, \theta_r) \cong -\frac{4\pi}{(n-2)\varepsilon\beta}\sum_{i=1}^{r} t^{ab}_{a_ib_i}\mathcal{F}^A_{a_1\ldots b_i\ldots a_r}(\theta_1, \ldots, \theta_r). \tag{D.3}$$

So far the operator index A did not play any role. For the case of tensor operators, where A is an $\mathrm{O}(n)$ (multi) index, the form factors are invariant tensors and (D.3) can equivalently be written

$$\mathcal{F}^{A}_{aba_1\ldots a_r}(\beta + i\pi + \varepsilon, \beta, \theta_1, \ldots, \theta_r) \cong \frac{4\pi}{(n-2)\varepsilon\beta} t^{ab}_{AB} \mathcal{F}^{B}_{a_1\ldots a_r}(\theta_1, \ldots, \theta_r), \tag{D.4}$$

where t^{ab}_{AB} is the $\mathrm{O}(n)$ generator in the appropriate representation.

Appendix E. Operator basis

In the operator product expansions we use a basis spanned by $\mathcal{A}^{(l)ab}_{\omega}$, $\mathcal{B}^{(J,l)ab}_{\omega}$ and $\bar{\mathcal{B}}^{(J,l)ab}_{\omega}$, where these basis elements are hermitian local operators at $(0,0)$ and $l = 0, 1$ or 2 tensor operators (in the index pair ab) under $\mathrm{O}(n)$. Under the action of the parity operator V they transform as

$$V\mathcal{A}^{(l)ab}_{\omega}V = \mathcal{A}^{(l)ab}_{\omega}, \qquad V\mathcal{B}^{(J,l)ab}_{\omega}V = \bar{\mathcal{B}}^{(J,l)ab}_{\omega}. \tag{E.1}$$

Their Lorentz spin can be read off the relations

$$\left[\mathcal{M}, \mathcal{A}^{(l)ab}_{\omega}\right] = 0, \tag{E.2}$$

$$\left[\mathcal{M}, \mathcal{B}^{(J,l)ab}_{\omega}\right] = iJ\mathcal{B}^{(J,l)ab}_{\omega}, \tag{E.3}$$

$$\left[\mathcal{M}, \bar{\mathcal{B}}^{(J,l)ab}_{\omega}\right] = -iJ\bar{\mathcal{B}}^{(J,l)ab}_{\omega}, \tag{E.4}$$

where J is a positive integer and $\mathcal{M}$ is the Lorentz boost operator. Finally, under the action of the (anti-linear) CPT operator Θ,

$$\Theta\mathcal{A}^{(l)ab}_{\omega}\Theta = (-1)^l\mathcal{A}^{(l)ab}_{\omega}, \tag{E.5}$$

$$\Theta\mathcal{B}^{(J,l)ab}_{\omega}\Theta = (-1)^l\mathcal{B}^{(J,l)ab}_{\omega}, \tag{E.6}$$

$$\Theta\bar{\mathcal{B}}^{(J,l)ab}_{\omega}\Theta = (-1)^l\bar{\mathcal{B}}^{(J,l)ab}_{\omega}. \tag{E.7}$$

The one-particle matrix elements of the above operators are parametrized as

$$\langle a, \theta|\mathcal{A}^{(l)cd}_{\omega}|b, \theta\rangle = P^{ab;cd}_{l} A^{(l)}_{\omega}, \tag{E.8}$$

$$\langle a, \theta|\mathcal{B}^{(J,l)cd}_{\omega}|b, \theta\rangle = \left(-\frac{iM}{2}\mathrm{e}^{\theta}\right)^J P^{ab;cd}_{l} B^{(J,l)}_{\omega}, \tag{E.9}$$

$$\langle a, \theta|\bar{\mathcal{B}}^{(J,l)cd}_{\omega}|b, \theta\rangle = \left(-\frac{iM}{2}\mathrm{e}^{-\theta}\right)^J P^{ab;cd}_{l} B^{(J,l)}_{\omega}. \tag{E.10}$$

Note that we have considered operators with non-vanishing one-particle matrix elements only.

Appendix F. Notations and conventions

We will use the notation

$$W_2 = -iW_0 \tag{F.1}$$

for any vector (and higher tensor) index. The light-cone components are

$$W_\pm = \frac{1}{2}(W_0 \mp W_1) = \frac{1}{2}(iW_2 \mp W_1) \tag{F.2}$$

and similarly

$$\partial_\pm = \frac{1}{2}(\partial_0 \mp \partial_1) = \frac{1}{2}(i\partial_2 \mp \partial_1). \tag{F.3}$$

We treat the two-dimensional Euclidean coordinates exceptionally since here we use

$$y_\pm = \mp y_1 - iy_2, \tag{F.4}$$

which gives

$$y_+ y_- = -y^2, \tag{F.5}$$

as opposed to the Euclidean square of vectors in (F.2), which is given by

$$\big(W^2\big)_{\mathrm{E}} = W_1^2 + W_2^2 = -4W_+W_-. \tag{F.6}$$

Two-dimensional Fourier transformation is indicated by tilde:

$$\tilde{f}(Q) = \int \mathrm{d}^2 y\, \mathrm{e}^{iQy} f(y). \tag{F.7}$$

For functions depending on y^2 only we also define

$$\hat{f}(Q) = \big(Q^2\big)^{J+1}\left(\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^J \int \mathrm{d}^2 y\, \mathrm{e}^{ipy} f\big(y^2\big). \tag{F.8}$$

Appendix G. Asymptotic expansions

Assume that $S(y)$ has an asymptotic expansion

$$S(y) = f_0 \tilde{\lambda}^{\sigma-1} + \mathrm{O}\big(\tilde{\lambda}^\sigma\big) \tag{G.1}$$

in terms of the effective coupling $\tilde{\lambda}$ defined in (5.66). Then $\hat{S}(Q)$ can be asymptotically expanded as

$$\hat{S}(Q) = (2\pi)(-1)^J J!(1-\sigma) f_0 \lambda^\sigma + \mathrm{O}\big(\lambda^{\sigma+1}\big) \tag{G.2}$$

in terms of the effective coupling λ defined by

$$\frac{1}{\lambda} + \frac{1}{n-2}\ln\lambda = \ln\frac{|Q|}{\Lambda_{\overline{\mathrm{MS}}}}. \tag{G.3}$$

In the special case

$$S(y) = f_0 \frac{1}{\tilde{\lambda}} + \mathrm{O}(1), \tag{G.4}$$

we have

$$\hat{S}(Q) = (2\pi)(-1)^J J! f_0 \left(1 + \frac{1}{n-2}\lambda\right) + \mathrm{O}\big(\lambda^2\big), \tag{G.5}$$

i.e., in this case we also know the coefficient of the sub-leading term. Finally, if

$$S(y) = f_0 \ln \tilde{\lambda} + \mathrm{O}(\tilde{\lambda}) \tag{G.6}$$

then

$$\hat{S}(Q) = -(2\pi)(-1)^J J! f_0 \lambda + \mathrm{O}\big(\lambda^2\big). \tag{G.7}$$

An alternative way of presenting the above results is as follows. If the derivative of the function $W(y^2)$ has the asymptotic expansion

$$y^2 \frac{\mathrm{d}}{\mathrm{d}y^2} W\big(y^2\big) = \alpha \tilde{\lambda}^{\sigma} + \mathrm{O}\big(\tilde{\lambda}^{\sigma+1}\big) \tag{G.8}$$

then in Fourier space we have

$$\hat{W}\big(Q^2\big) = -4\pi\alpha(-1)^J J! \lambda^{\sigma} + \mathrm{O}\big(\lambda^{\sigma+1}\big). \tag{G.9}$$

In the special case $\sigma = 0$ if

$$y^2 \frac{\mathrm{d}}{\mathrm{d}y^2} W\big(y^2\big) = \alpha + \beta\tilde{\lambda} + \mathrm{O}\big(\tilde{\lambda}^2\big) \tag{G.10}$$

then

$$\hat{W}\big(Q^2\big) = -4\pi(-1)^J J! \big\{\alpha + \beta\lambda + \mathrm{O}\big(\lambda^2\big)\big\}. \tag{G.11}$$

Appendix H. Reduced spin and current form factor squares

The space of homogeneous symmetric polynomials in r variables θ_i, $i = 1, \ldots, r$, is spanned by products of $\sigma_k^{(r)}$, $1 \leqslant k \leqslant r$,

$$\sigma_k^{(r)} = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant r} \theta_{i_1} \cdots \theta_{i_k}. \tag{H.1}$$

For the reduced spin and current structure functions we have for $r = 2$:

$$j_0^{(2)} = 4\big(\sigma_1^2 - 3\sigma_2\big) + 8\pi^2, \tag{H.2}$$

$$j_1^{(2)} = \sigma_1^2 - 2\sigma_2 - 2\pi^2, \tag{H.3}$$

$$j_2^{(2)} = \sigma_1^2 - 6\sigma_2 + 2\pi^2. \tag{H.4}$$

For $r = 3$:

$$
\begin{aligned}
j_0^{(3)} &= 4\big(-6\sigma_1^3\sigma_3 + 2\sigma_1^2\sigma_2^2 + 19\sigma_1\sigma_2\sigma_3 - 6\sigma_2^3 - 9\sigma_3^2\big) \\
&\quad + 4\pi^2\big(4\sigma_1^4 - 21\sigma_1^2\sigma_2 - 19\sigma_1\sigma_3 + 34\sigma_2^2\big) \\
&\quad + 8\pi^4\big(9\sigma_1^2 - 22\sigma_2\big) + 64\pi^6, \qquad \text{(H.5)} \\
j_1^{(3)} &= -2\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 + 2\sigma_1\sigma_2\sigma_3 - 2\sigma_2^3 + 9\sigma_3^2 \\
&\quad - 2\pi^2\big(\sigma_1^4 - 3\sigma_1^2\sigma_2 + 10\sigma_1\sigma_3 - 4\sigma_2^2\big) - 2\pi^4\big(3\sigma_1^2 - \sigma_2\big) - 8\pi^6, \qquad \text{(H.6)} \\
j_2^{(3)} &= 6\sigma_1^3\sigma_3 - \sigma_1^2\sigma_2^2 - 38\sigma_1\sigma_2\sigma_3 + 6\sigma_2^3 + 99\sigma_3^2 \\
&\quad - 2\pi^2\big(\sigma_1^4 - 9\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 16\sigma_2^2\big) - 2\pi^4\big(3\sigma_1^2 - 17\sigma_2\big) - 8\pi^6. \qquad \text{(H.7)}
\end{aligned}
$$

For $r = 4$:

$$
\begin{aligned}
j_0^{(4)} &= 16\sigma_1^2\sigma_2^2\sigma_3^2 - 48\sigma_2^3\sigma_3^2 - 48\sigma_1^3\sigma_3^3 + 152\sigma_1\sigma_2\sigma_3^3 - 72\sigma_3^4 \\
&\quad - \big(48\sigma_1^2\sigma_2^3 - 144\sigma_2^4 - 152\sigma_1^3\sigma_2\sigma_3 + 476\sigma_1\sigma_2^2\sigma_3 - 56\sigma_1^2\sigma_3^2 - 52\sigma_2\sigma_3^2\big)\sigma_4 \\
&\quad - \big(72\sigma_1^4 - 52\sigma_1^2\sigma_2 - 352\sigma_2^2 - 128\sigma_1\sigma_3\big)\sigma_4^2 - 640\sigma_4^3 \\
&\quad + 4\pi^2\big[8\sigma_1^2\sigma_2^4 - 24\sigma_2^5 - 42\sigma_1^3\sigma_2^2\sigma_3 + 131\sigma_1\sigma_2^3\sigma_3 + 68\sigma_1^4\sigma_3^2 \\
&\quad - 249\sigma_1^2\sigma_2\sigma_3^2 + 73\sigma_2^2\sigma_3^2 + 55\sigma_1\sigma_3^3 \\
&\quad - \big(38\sigma_1^4\sigma_2 - 279\sigma_1^2\sigma_2^2 + 528\sigma_2^3 + 175\sigma_1^3\sigma_3 - 669\sigma_1\sigma_2\sigma_3 + 133\sigma_3^2\big)\sigma_4 \\
&\quad - \big(157\sigma_1^2 + 32\sigma_2\big)\sigma_4^2\big] \\
&\quad + 4\pi^4\big[36\sigma_1^4\sigma_2^2 - 184\sigma_1^2\sigma_2^3 + 260\sigma_2^4 - 88\sigma_1^5\sigma_3 + 447\sigma_1^3\sigma_2\sigma_3 \\
&\quad + 253\sigma_1^2\sigma_3^2 - 696\sigma_1\sigma_2^2\sigma_3 - 248\sigma_2\sigma_3^2 \\
&\quad + \big(185\sigma_1^4 - 1156\sigma_1^2\sigma_2 + 2120\sigma_2^2 - 740\sigma_1\sigma_3\big)\sigma_4 + 96\sigma_4^2\big] \\
&\quad + 4\pi^6\big[32\sigma_1^6 - 232\sigma_1^4\sigma_2 + 780\sigma_1^2\sigma_2^2 - 992\sigma_2^3 - 532\sigma_1^3\sigma_3 + 1269\sigma_1\sigma_2\sigma_3 \\
&\quad + 295\sigma_3^2 + \big(1211\sigma_1^2 - 3220\sigma_2\big)\sigma_4\big] \\
&\quad + 16\pi^8\big[60\sigma_1^4 - 278\sigma_1^2\sigma_2 + 405\sigma_2^2 - 212\sigma_1\sigma_3 + 434\sigma_4\big] \\
&\quad + 128\pi^{10}\big[17\sigma_1^2 - 37\sigma_2\big] + 1280\pi^{12}, \qquad \text{(H.8)}
\end{aligned}
$$

$$
\begin{aligned}
j_1^{(4)} &= \sigma_1^2\sigma_2^2\sigma_3^2 - 2\sigma_2^3\sigma_3^2 - 2\sigma_1^3\sigma_3^3 + 2\sigma_1\sigma_2\sigma_3^3 + 9\sigma_3^4 \\
&\quad + \big(-2\sigma_1^2\sigma_2^3 + 4\sigma_2^4 + 2\sigma_1^3\sigma_2\sigma_3 + 2\sigma_1\sigma_2^2\sigma_3 + 18\sigma_1^2\sigma_3^2 - 62\sigma_2\sigma_3^2\big)\sigma_4 \\
&\quad + \big(9\sigma_1^4 - 62\sigma_1^2\sigma_2 + 80\sigma_2^2\big)\sigma_4^2 + 128\sigma_4^3 \\
&\quad - 2\pi^2\big[\sigma_1^2\sigma_2^4 - 2\sigma_2^5 - 3\sigma_1^3\sigma_2^2\sigma_3 + 3\sigma_1\sigma_2^3\sigma_3 \\
&\quad - 4\sigma_1^4\sigma_3^2 + 33\sigma_1^2\sigma_2\sigma_3^2 - 39\sigma_2^2\sigma_3^2 + 7\sigma_1\sigma_3^3 \\
&\quad + \big(10\sigma_1^4\sigma_2 - 45\sigma_1^2\sigma_2^2 + 78\sigma_2^3 - 15\sigma_1^3\sigma_3 - 36\sigma_1\sigma_2\sigma_3 - 118\sigma_3^2\big)\sigma_4 \\
&\quad + \big(-14\sigma_1^2 + 264\sigma_2\big)\sigma_4^2\big] \\
&\quad + 2\pi^4\big[-3\sigma_1^4\sigma_2^2 + 12\sigma_1^2\sigma_2^3 - 16\sigma_2^4 + \sigma_1^5\sigma_3 + \sigma_1^3\sigma_2\sigma_3 + 58\sigma_1^2\sigma_3^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 9\sigma_1\sigma_2^2\sigma_3 - 174\sigma_2\sigma_3^2 \\
&+ \big(15\sigma_1^4 - 202\sigma_1^2\sigma_2 + 456\sigma_2^2 - 256\sigma_1\sigma_3\big)\sigma_4 + 1184\sigma_4^2\big] \\
&- 2\pi^6\big[4\sigma_1^6 - 24\sigma_1^4\sigma_2 + 57\sigma_1^2\sigma_2^2 - 26\sigma_2^3 + 37\sigma_1^3\sigma_3 \\
&- 198\sigma_1\sigma_2\sigma_3 + 26\sigma_3^2 + \big(-272\sigma_1^2 + 1100\sigma_2\big)\sigma_4\big] \\
&+ 8\pi^8\big[-3\sigma_1^4 - 2\sigma_1^2\sigma_2 + 13\sigma_2^2 + 3\sigma_1\sigma_3 + 126\sigma_4\big] \\
&- 256\pi^{10}\sigma_2 + 128\pi^{12}, \qquad \text{(H.9)}
\end{aligned}
$$

$$
\begin{aligned}
j_2^{(4)} = {}& \sigma_1^2\sigma_2^2\sigma_3^2 - 6\sigma_2^3\sigma_3^2 - 6\sigma_1^3\sigma_3^3 + 38\sigma_1\sigma_2\sigma_3^3 - 99\sigma_3^4 \\
&+ \big(-6\sigma_1^2\sigma_2^3 + 36\sigma_2^4 + 38\sigma_1^3\sigma_2\sigma_3 - 242\sigma_1\sigma_2^2\sigma_3 + 2\sigma_1^2\sigma_3^2 + 670\sigma_2\sigma_3^2\big)\sigma_4 \\
&+ \big(-99\sigma_1^4 + 670\sigma_1^2\sigma_2 - 656\sigma_2^2 - 1600\sigma_1\sigma_3\big)\sigma_4^2 + 3584\sigma_4^3 \\
&- 2\pi^2\big[-\sigma_1^2\sigma_2^4 + 6\sigma_2^5 + 9\sigma_1^3\sigma_2^2\sigma_3 - 55\sigma_1\sigma_2^3\sigma_3 - 16\sigma_1^4\sigma_3^2 \\
&+ 99\sigma_1^2\sigma_2\sigma_3^2 + 103\sigma_2^2\sigma_3^2 - 257\sigma_1\sigma_3^3 \\
&+ \big(-8\sigma_1^4\sigma_2 + 21\sigma_1^2\sigma_2^2 + 78\sigma_2^3 + 125\sigma_1^3\sigma_3 - 762\sigma_1\sigma_2\sigma_3 + 1664\sigma_3^2\big)\sigma_4 \\
&+ \big(248\sigma_1^2 - 824\sigma_2\big)\sigma_4^2\big] \\
&+ 2\pi^4\big[3\sigma_1^4\sigma_2^2 - 32\sigma_1^2\sigma_2^3 + 64\sigma_2^4 - 17\sigma_1^5\sigma_3 + 165\sigma_1^3\sigma_2\sigma_3 \\
&- 280\sigma_1^2\sigma_3^2 - 303\sigma_1\sigma_2^2\sigma_3 + 632\sigma_2\sigma_3^2 \\
&+ \big(55\sigma_1^4 - 284\sigma_1^2\sigma_2 + 160\sigma_2^2 + 632\sigma_1\sigma_3\big)\sigma_4 - 960\sigma_4^2\big] \\
&- 2\pi^6\big[-4\sigma_1^6 + 56\sigma_1^4\sigma_2 - 201\sigma_1^2\sigma_2^2 + 238\sigma_2^3 - 37\sigma_1^3\sigma_3 \\
&- 36\sigma_1\sigma_2\sigma_3 + 472\sigma_3^2 + \big(-82\sigma_1^2 + 260\sigma_2\big)\sigma_4\big] \\
&+ 8\pi^8\big[3\sigma_1^4 - 46\sigma_1^2\sigma_2 + 93\sigma_2^2 + 35\sigma_1\sigma_3 + 58\sigma_4\big] \\
&+ 64\pi^{10}\big[\sigma_1^2 - 8\sigma_2\big] + 128\pi^{12}. \qquad \text{(H.10)}
\end{aligned}
$$

Appendix I. Structure function moments

Table 1
Values of moment $M_{0;2}^{(r)}$

$\log_{10}(-q^2/M^2)$	$M_{0;2}^{(1)}$	$M_{0;2}^{(2)}$	$M_{0;2}^{(3)}$	$M_{0;2}^{(4)}$	$M_{0;2}^{(5)}$	$M_{0;2}^{(6)}$
1	1.1434	0.3675	0.01199	0.0001163	7.7E-7	5.13E-9
2	0.5766	0.5516	0.1065	0.006140	0.00015	2.31E-6
3	0.3531	0.5360	0.2384	0.04244	0.0036	0.000177
4	0.2305	0.4591	0.3200	0.1076	0.020	0.002358
5	0.1592	0.3762	0.3456	0.1716	0.053	0.01093
6	0.1154	0.3051	0.3370	0.2168	0.092	0.02811
7	0.08700	0.2484	0.3121	0.2410	0.129	0.05156
8	0.06777	0.2041	0.2818	0.2490	0.158	0.07691
9	0.05419	0.1696	0.2513	0.2457	0.177	0.1006
10	0.04427	0.1425	0.2229	0.2363	0.188	0.1196

Table 2
Values of moment $M^{(r)}_{1;1}$

$\log_{10}(-q^2/M^2)$	$M^{(1)}_{1;1}$	$M^{(2)}_{1;1}$	$M^{(3)}_{1;1}$	$M^{(4)}_{1;1}$	$M^{(5)}_{1;1}$	$M^{(6)}_{1;1}$
1	0.5717	0.3822	0.02045	0.000075	−1.7E-6	−5.0E-8
2	0.2883	0.5439	0.1055	0.00273	−5.6E-5	−3.3E-6
3	0.1766	0.5565	0.2107	0.02029	−8.4E-5	−8.3E-5
4	0.1153	0.5127	0.2879	0.05878	0.00286	−0.00050
5	0.0796	0.4545	0.3301	0.1072	0.0139	−0.00069
6	0.0577	0.3986	0.3451	0.1526	0.0342	0.00162
7	0.04350	0.3501	0.3427	0.1885	0.0601	0.00848
8	0.03389	0.3096	0.3307	0.2133	0.0873	0.02014
9	0.02710	0.2760	0.3141	0.2282	0.1123	0.03531
10	0.02214	0.2481	0.2959	0.2354	0.1333	0.05215

Table 3
Values of sums of moments $M^{(r)}_{0;2}$ and $M^{(r)}_{1;1}$, in the even and odd channels

$\log_{10}(-q^2/M^2)$	$\sum_{k=1}^{3} M^{(2k-1)}_{0;2}$	$\sum_{k=1}^{3} M^{(2k)}_{0;2}$	$\sum_{k=1}^{3} M^{(2k-1)}_{1;1}$	$\sum_{k=1}^{3} M^{(2k)}_{1;1}$
1	1.155	0.3676	0.5921	0.3823
2	0.683	0.5577	0.3937	0.5466
3	0.595	0.5786	0.3872	0.5767
4	0.571	0.5690	0.4061	0.5710
5	0.558	0.5587	0.4236	0.5610
6	0.544	0.5500	0.4370	0.5528
7	0.528	0.5410	0.4463	0.5471
8	0.508	0.5300	0.4519	0.5430
9	0.482	0.5161	0.4535	0.5395
10	0.455	0.4984	0.4513	0.5357

Appendix J. One-loop 2d integrals

We start with the 1-loop Euclidean integral with 2 internal scalar propagators with masses m_1, m_2:

$$B(k; m_1, m_2) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{[(q+k)^2 + m_1^2][q^2 + m_2^2]}. \tag{J.1}$$

The integral can be done analytically to obtain

$$B(k; m_1, m_2) = \frac{1}{2\pi\sqrt{(k^2 + m_-^2)(k^2 + m_+^2)}} \ln\left\{ \frac{\sqrt{k^2 + m_+^2} + \sqrt{k^2 + m_-^2}}{\sqrt{k^2 + m_+^2} - \sqrt{k^2 + m_-^2}} \right\}, \tag{J.2}$$

where

$$m_\pm = m_1 \pm m_2. \tag{J.3}$$

For the equal mass case $m_1 = m_2 = M$ we have

$$B(k) \equiv B(k; M, M) = \frac{1}{2\pi\sqrt{k^2(k^2+4M^2)}} \ln \frac{\sqrt{k^2+4M^2}+\sqrt{k^2}}{\sqrt{k^2+4M^2}-\sqrt{k^2}}$$
$$= b(\theta) = \frac{\theta}{4\pi M^2 \sinh\theta} \quad \text{for } k^2 = 4M^2 \sinh^2 \frac{\theta}{2}. \tag{J.4}$$

Note $B(k)$ is analytic in k^2 with a cut from $-\infty$ to $-4M^2$. Also $B(k) \neq 0$ for all k^2 and

$$B(k) \sim \frac{\ln k^2}{2\pi k^2} \quad \text{for } k^2 \to \infty, \tag{J.5}$$

$$B(0) = \frac{1}{4\pi M^2}. \tag{J.6}$$

It can be represented by the dispersion relation

$$B(k) = \frac{1}{2\pi i} \int_{-\infty}^{-4M^2} \mathrm{d}z \frac{B(z+i\epsilon) - B(z-i\epsilon)}{z-k^2}$$
$$= \frac{1}{2\pi} \int_0^{\infty} \mathrm{d}\kappa \frac{1}{k^2 + 4M^2 \cosh^2 \frac{\kappa}{2}}, \tag{J.7}$$

where we have substituted $z = -4M^2 \cosh^2 \frac{\kappa}{2}$ and noted $z \pm i\epsilon$ corresponds to setting $\theta = i\pi \pm \kappa$ with $\kappa > 0$:

$$\frac{1}{2\pi i}\big[b(i\pi+\kappa) - b(i\pi-\kappa)\big] = \frac{-1}{4\pi M^2 \sinh\kappa}. \tag{J.8}$$

The inverse of B satisfies a once subtracted dispersion relation

$$B(k)^{-1} = B(0)^{-1} + \frac{k^2}{2\pi i} \int_{-\infty}^{-4M^2} \mathrm{d}z \frac{B(z+i\epsilon)^{-1} - B(z-i\epsilon)^{-1}}{z(z-k^2)}. \tag{J.9}$$

Noting

$$\frac{1}{2\pi i}\left[\frac{1}{b(i\pi+\kappa)} - \frac{1}{b(i\pi-\kappa)}\right] = \frac{4\pi M^2 \sinh\kappa}{\kappa^2+\pi^2}, \tag{J.10}$$

we have

$$B(k)^{-1} = 4\pi M^2 \left[1 + 2k^2 \int_0^{\infty} \mathrm{d}\kappa \frac{\sinh^2 \kappa/2}{(\kappa^2+\pi^2)(k^2+4M^2\cosh^2\kappa/2)}\right]. \tag{J.11}$$

J.1. General 1-loop integrals ("cutting rule")

We consider an arbitrary 1-loop integral ($\sum_{i=1}^{n} k_i = 0$):

$$I(\mathbf{k}) = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \prod_{i=1}^{n} \left[(q + l_i)^2 + m_i^2\right]^{-1}, \tag{J.12}$$

where

$$l_i = \sum_{j=1}^{i} k_j \quad (l_n = 0). \tag{J.13}$$

The result is simply

$$I(\mathbf{k}) = \sum_{i<j} \frac{1}{2}\left(I_{ij}^{+} + I_{ij}^{-}\right) B(l_{ij}; m_i, m_j), \tag{J.14}$$

where

$$I_{ij}^{\pm} = \prod_{r=1, r\neq i,j}^{n} \left[(q + l_r)^2 + m_r^2\right]^{-1} \Big|_{q=q_{ij}^{\pm}}, \tag{J.15}$$

and the momenta $q_{ij}^{\pm}$ are given by[11]

$$2q_{ij}^{\pm} = -(l_i + l_j) - \frac{m_i^2 - m_j^2}{l_{ij}^2} l_{ij} \mp \frac{i}{l_{ij}^2} \sqrt{s_{ij}^4 + 4m_j^2 l_{ij}^2}\, \epsilon l_{ij}, \tag{J.16}$$

$$l_{ij} = l_i - l_j, \tag{J.17}$$

$$s_{ij}^2 = l_{ij}^2 + m_i^2 - m_j^2. \tag{J.18}$$

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