S-MATRIX OF THE YANG–LEE EDGE SINGULARITY IN TWO DIMENSIONS

John L. CARDY and G. MUSSARDO [1]

Department of Physics, University of California, Santa Barbara, CA 93106, USA

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The deformation by a relevant operator of the simplest, albeit nonunitary, conformal field theory, that of the Yang–Lee edge singularity, is analysed. It is shown that additional conserved currents exist which imply the factorization of the S-matrix, whose form is explicitly derived. The spectrum contains a single particle. It is proved that any such theory with only one particle must necessarily be nonunitary.

In statistical mechanics, or in the corresponding quantum field theory, the existence of an infinite number of conserved currents usually implies the exact solvability of the model. This is what happens in two-dimensional conformal field theories (CFTs), where the conservation law for a current $(J_{z\dots}, J_{\bar{z}\dots})$

$$\partial_{\bar{z}}J_{z\dots}+\partial_{z}J_{\bar{z}\dots}=0 \tag{1}$$

is satisfied for instance by taking $J_{z\dots}$ to be T, $:T^2:$, or any other operator in the conformal block of the stress tensor, since they are all analytic functions of z. Away from criticality, when the action of the critical point theory is perturbed by relevant or marginal fields, this structure is in general destroyed. However, it was pointed out by Zamolodchikov [1] that for particular deformations of the CFT there may still exist an infinite subset of these conserved currents. The corresponding massive field theories will then be integrable. The additional conserved currents will preclude the possibility of inelastic scattering, and the general n-particle S-matrix will factorize into a product of $\frac{1}{2}n(n-1)$ elastic 2-particle S-matrices. This 2-particle S-matrix satisfies the star–triangle equations [2] and in many cases its form may be obtained on the basis of symmetry and analyticity arguments alone.

In this letter we analyse the deformation of the simplest CFT, namely that of the Yang–Lee edge singularity, by its only relevant operator. We find that there exist conserved currents of weights (defined to be the canonical dimension of the corresponding conserved charge)

$$n=1,\,5,\,7,\,11,\,13,\,17,\,19,\,23\,, \tag{2}$$

and we conjecture the existence of a similar current for each value of n not divisible by 2 or 3. The corresponding conservation laws allow the existence of a massive particle A which appears as a bound state of itself, i.e. the amplitude S_{AA} has a pole at $s=m_{\mathrm{A}}^2$. In a unitary field theory this pole would have the form [3]

$$\mathcal{S}(s)\sim\frac{-\mathrm{i}\lambda^2}{s-m_{\mathrm{A}}^2}\,, \tag{3}$$

where $\mathcal{S}$ is the conventional S-matrix element obtained by the LSZ reduction formula, with the overall energy–momentum conserving δ-function factored out, and λ is the renormalized coupling constant. In the case of the Yang–Lee theory, since the theory is not unitary, the residue might have a different sign. This turns out to be true in the Yang–Lee case. Moreover, we shall show that a factorizing S-matrix involving only one particle necessarily leads to a sign for this residue inconsistent with unitarity.

This letter is organized as follows: after recalling some well-known features of factorizing S-matrices (an exercise which we go through in order to establish carefully the correct factors of i necessary for our discussion of unitarity), we discuss the Yang–Lee CFT, its deformation away from criticality and the

[1] INFN Fellow.

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corresponding S-matrix. We then describe briefly its features.

(1) In (1+1)-dimensional spacetime a convenient parametrization of the momentum of the particle A is

$$p^0 = m\cosh\theta, \quad p^1 = m\sinh\theta\,, \tag{4}$$

where θ is the rapidity. The 2-particle S-matrix is defined by

$$|A(\theta_1)A(\theta_2)\rangle_{\rm in} = S(\theta_1-\theta_2)\,|A(\theta_1)A(\theta_2)\rangle_{\rm out} \tag{5}$$

and it depends only on the difference of the rapidities, by Lorentz invariance. It is related to $\mathscr{S}$ by

$$\mathscr{S} = m^2\sinh(\theta_1-\theta_2)S\,, \tag{6}$$

where the factor carries from expressing the energy-momentum conserving δ-function in terms of rapidities. This factor has a square root branch point at threshold, and is purely imaginary on the real s-axis below threshold. For this reason, S itself has the same analyticity and reality properties as the T-matrix familiar from four-dimensional quantum field theory, as a function of the Mandelstam variable $s=4m^2\cosh\frac{1}{2}\theta$. In terms of θ these translate into analyticity in the strip $0<\mathrm{Im}\,\theta<\pi$, and reality on the imaginary θ axis, with possible poles there corresponding to bound states [4]. In particular, if there is a bound state at the mass of particle A there will occur a pole at $\theta=\frac{2}{3}\mathrm{i}\pi$.

The physical region is the positive real θ-axis. Because of the real analyticity of S, we have $S^\dagger(\theta) = S(-\theta)$, so that the unitarity condition $SS^\dagger=1$ can be written

$$S(\theta)S(-\theta)=1\,, \tag{7}$$

an equation which may be analytically continued to the whole strip. In addition, crossing symmetry implies that $S(\theta)=S(\mathrm{i}\pi-\theta)$.

The bootstrap equation is derived by using the factorization properties of the $3\to3$ S-matrix, illustrated in fig. 1a. Taking the residue of the pole $\theta'=\frac{2}{3}\pi\mathrm{i}$ in the (12)-channel, as indicated in fig. 1b, leads to the equation

$$S(\theta)=S\,(\theta-\tfrac{1}{3}\mathrm{i}\pi)\;S(\theta+\tfrac{1}{3}\mathrm{i}\pi)\,. \tag{8}$$

(2) The Yang–Lee singularity [5,6] describes the critical behavior of an Ising model in a pure imaginary magnetic field ih. For $h>h_c$ the zeros of the partition function are dense on the imaginary h-axis. The lagrangian, or Landau–Ginzburg form, of this model, which is an appropriate description close to its upper critical dimension $d=6$, was found by Fisher [7] to be

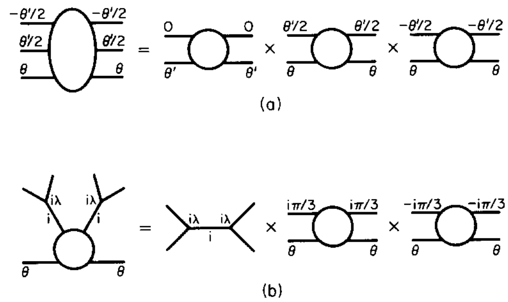

Fig. 1. (a) Factorization of the $3\to3$ S-matrix. (b) Residue of the $3\to3$ S-matrix at the pole at $\theta=\frac{2}{3}\mathrm{i}\pi$.

$$\mathscr{L}=\int\left[\tfrac{1}{2}(\partial\phi)^2-\mathrm{i}(h-h_c)\phi-\mathrm{i}g\phi^3\right]\mathrm{d}^dx\,. \tag{9}$$

This theory at criticality ($h=h_c$) has only one relevant operator, namely the field ϕ itself. It was shown in ref. [8] that this property is satisfied by the minimal CFT with central charge $c=-\frac{22}{5}$, which has only one relevant operator with scaling dimension $x=-\frac{2}{5}$, corresponding to the equivalent positions (1,2) and (1,3) in the Kac table. The negative values of c and x indicate that this theory is not unitary, as could be expected from the explicit factor of i in eq. (9). Now suppose that this theory is perturbed away from criticality by the operator $\phi_{1,2}$. Following Zamolodchikov [1], the existence of additional conserved currents may be inferred from a counting argument. If the dimension of the space $\hat{\Lambda}_{n+1}$ of quasiprimary operators of scaling dimension $n+1$ in the conformal block of the identity is greater than the dimension of the corresponding space $\hat{\Phi}_n$ of quasiprimary operators of dimension n in the conformal block of $\phi_{1,2}$, then there must exist elements $T_{n+1}\in\hat{\Lambda}_{n+1}$ and $Q_{n-1}\in\hat{\Phi}_{n-1}$ such that

$$\partial_{\bar z}T_{n+1}=(h-h_c)\,\partial_z Q_{n-1}\,. \tag{10}$$

The generating functions for the dimensions of these spaces are given [1] in terms of the characters [8]

$$\sum_{n=0}^{\infty} q^n \dim \hat{\Lambda}_n = (1-q) q^{c/24} \chi_{1,1}(q) + q\,,$$

$$\sum_{n=0}^{\infty} q^n \dim \hat{\phi}_n = (1-q) q^{c/24-h_{1,2}} \chi_{1,2}(q)\,. \tag{11}$$

The results of this calculation for $n \leqslant 26$ are shown in table 1. We see that conserved currents exist for those values of n given in eq. (2). The pattern of allowed values of n leads us to conjecture the existence of such conserved currents for all values of n not divisible by 2 or 3. Note that although this counting argument does not indicate the existence of such a current for $n=25$, since it gives only a sufficient condition there is no contradiction with the above conjecture. Similar features occur when the counting argument is applied to other theories [1]. As discussed by Zamolodchikov [1], a theory with the above weights for the conserved currents will possess the "ϕ^3-property" that the particel A can appear as a bound state of a pair of the same kind of particle. In this case, the nth power of the left components of the momentum will be conserved,

$$e^{n\theta_1} + e^{n\theta_2} = (e^{\theta_1} + e^{\theta_2})^n\,, \tag{12}$$

when $\theta_1 - \theta_2 = \frac{2}{3}\pi i$. This is only satisfied when n is not divisible by 2 or 3. In fact, on the basis of the lagrangian eq. (9), we expect no further bound states in the theory. In the nonrelativistic Born approximation, the exchange of the A particle in the t-channel will lead to a *repulsive* potential, because of the imaginary coupling constant. In addition, the work of ref. [1] shows that the existence of further bound states coupling to the (AA)-channel tends to rule out some of the values of n for which we know there exist conserved currents.

Table 1
Dimensions of the spaces $\hat{\Lambda}_{n+1}$ and $\hat{\phi}_n$. For each value of n for which the former is larger than the latter, there must exist a conserved current.

n	$\hat{\Lambda}_{n+1}$	$\hat{\phi}_n$
1	1	0
2	0	0
3	0	0
4	0	1
5	1	0
6	0	1
7	1	0
8	0	1
9	1	1
10	0	1
11	2	1
12	0	2
13	2	1
14	1	2
15	2	2
16	1	3
17	3	2
18	1	4
19	4	3
20	2	5
21	4	4
22	3	6
23	6	5
24	3	8
25	7	7
26	5	9

We may now construct the S-matrix. The s-channel pole is at $\theta = \frac{2}{3}i\pi$. Crossing symmetry then implies the existence of a t-channel pole at $\theta = \frac{1}{3}i\pi$. We expect no further poles. From the bootstrap equation (8), the unique solution is then

$$S(\theta) = \frac{(e^{\theta} - e^{-2i\pi/3})(e^{\theta} - e^{-i\pi/3})}{(e^{\theta} - e^{2i\pi/3})(e^{\theta} - e^{i\pi/3})}$$
$$= \tanh(\tfrac{1}{2}\theta + \tfrac{1}{6}\pi i)\coth(\tfrac{1}{2}\theta - \tfrac{1}{6}\pi i)\,. \tag{13}$$

The residue of this S-matrix at $s=m^2$ is found to be $+6m^2$. Comparing with eq. (3), and using the relation (6), we see that the renormalized coupling is given by

$$-i\lambda^2 = 6m^4 \sinh \tfrac{2}{3}i\pi = i3\sqrt{3}\,m^4\,. \tag{14}$$

We see that λ^2 is of the opposite sign to that expected in a unitary theory but of the correct sign expected at lowest order in perturbation theory for the Yang–Lee lagrangian (9).

However, it is easy to check that eq. (13) does satisfy the unitary equation (7). The resolution of this paradox is as follows. The hamiltonian H corresponding to eq. (9) is not hermitian. Instead, if we define the operator C which takes $\phi \to -\phi$, we have $H^{\dagger} = CHC$. Note that the Fock space states of the theory are all eigenstates of C with eigenvalue $(-1)^N$ where N is the particle number. Since H is not hermitian, its left eigenstates $\langle n_L|$ are not the adjoints of the right eigenstates $|n_R\rangle$. Instead we have $\langle n_L| = \langle n_R|C$. Completeness of the eigenstates of H then reads

$$\sum_n |n_R\rangle\langle n_L| = \sum_n |n_R\rangle\langle n_R|C = 1 . \qquad (15)$$

Now the unitarity of the S-matrix $SS^\dagger = 1$ follows solely from the fact that the in-kets and the out-kets both give a basis for the space, and is independent of whether or not H is hermitian. However, when we insert a complete set of states eq. (15) into this equation, each term will pick up a factor of $(-1)^N$. This is why the residue of the pole has the wrong sign. Alternatively, we can choose to make H hermitian by defining a new pseudo-inner product $\langle a|b\rangle' \equiv \langle a|C|b\rangle$, but then some of the physical states will have negative norm.

Finally, we may prove that any unitary theory with a factorizing S-matrix has at least two particles. For suppose the converse were true, i.e. there is only one particle in the spectrum. Then its S-matrix $\tilde{S}(\theta)$ would have exactly the same poles and zeros as eq. (13), and thus $\tilde{S}(\theta)/S(\theta)$ would be a constant. But the bootstrap equation (8) would then fix this constant to be unity. Hence $\tilde{S}$ would have a pole with a residue of the wrong sign, contradicting the assumption that the theory is unitary. This argument assumes, of course, that the theory is interacting. The argument may also be modified if there exist singularities (for example, higher order poles) of the S-matrix which do not correspond to physical particles. We remark that in ref. [9] an S-matrix for a unitary theory with a single particle and its distinct antiparticle was given. According to Zamolodchikov [10], this represents the noncritical 3-state Potts model.

In conclusion, we have shown that the simplest possible conformal field theory leads, away from criticality, to the simplest possible S-matrix. Although unphysical from the quantum field theory point of view, this model has several important applications in statistical mechanics. It would be very interesting to discover other exact properties of the noncritical theory.

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