# FORM FACTORS OF DESCENDENT OPERATORS IN PERTURBED CONFORMAL FIELD THEORIES 

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#### Abstract

Using the Ising model with a thermal perturbation as an example, we show that the solution space of the linear equations satisfied by the off-shell form factors of an integrable perturbed conformal field theory admits a structure which is isomorphic to that of the Virasoro irreducible representations characterizing the critical theory.


## 1. Introduction

One of the most powerful results of conformal symmetry in two-dimensional field theories is the classification of the operator content of a given theory according to irreducible representations of two commuting Virasoro algebras [1]. According to this scheme, every scaling operator is either a primary operator, or a descendent thereof. Primary operators with scaling dimensions ( $h, \bar{h}$ ) correspond to highest weight states $|h, \bar{h}\rangle$. Descendent operators at level ( $n, \bar{n}$ ), with scaling dimensions ( $h+n, \bar{h}+\bar{n}$ ) then correspond to linear combinations of states of the form

$$
\begin{equation*}
L_{-n}^{k_{n}} L_{-n+1}^{k_{n}-1} \ldots L_{-1}^{k_{1}} \bar{L}_{-\bar{n}}^{k_{\bar{n}}} \bar{L}_{-\bar{n}_{\bar{n}+1} \bar{k}_{1}} \ldots \bar{L}_{-1}^{\bar{k}_{1}}|h, \bar{h}\rangle \tag{1}
\end{equation*}
$$

where $\sum r k_{r}=n$ and $\sum r \bar{k}_{r}=\bar{n}$. In general the number of such independent states at this level is $p(n) p(\bar{n})$, where $p(n)$ is the number of partitions of $n$ into positive integers. However, for the minimal models, the actual number is less than this owing to the existence of null states which must be projected out to obtain an irreducible representation. The generating function for the dimensions of the space of states at a given level is the Virasoro character [2]. The correlation functions of descendent operators are given, by the conformal Ward identities, in

[^0]terms of differential operators acting on the correlation functions of the primary operators [1].

Now suppose that the theory is perturbed by some relevant operator which drives it away from criticality, so that it corresponds to a massive two-dimensional field theory. We would expect the set of local operators in this theory to be in one-to-one correspondence with those in the conformal theory. However, the perturbation will destroy the conformal symmetry and therefore, in general, split the degeneracies. For example, the two-point functions of two different operators belonging to the same level of a representation are identical in the conformal field theory, but away from criticality this will no longer be true.

In general, the computation of correlation functions away from criticality is a formidable problem. However, some progress has been made in the case of theories possessing an infinite number of conservation laws [3]. In these massive field theories the $S$-matrices connecting the asymptotic in- and out-states obey the properties of elasticity and factorization, and may be obtained explicitly [3-13]. Once the $S$-matrices are known it is possible to obtain information about the off-shell theory by considering the form factors, which are matrix elements of local fields $\mathscr{C}(x)$ between the asymptotic states. The correlation functions may then be constructed as sums over intermediate states. As a result of unitarity and CPT invariance, the form factors obey the Watson equations, which, in the case of factorizing, elastic $S$-matrices, become rather simple functional equations [6]. In addition, the form factors corresponding to multiparticle asymptotic states are related by LSZ reduction to those with fewer particles.

These two types of condition, together with analyticity requirements, have been used to determine some of the form factors in several theories [ $6,14,15$ ]. In certain cases the results may be tested against exact or perturbative solutions. However, in the calculations it is necessary to make certain "minimality" assumptions, roughly speaking that the form factors have the smallest number of zeros consistent with satisfying the above two conditions. The status of this assumption is unclear. In general, the Watson equations and the LSZ reduction formulae form a linear system of equations, whose solutions therefore span a linear space. In deriving the equations, no reference is made to whether the operator $\mathscr{O}$ is the non-critical deformation of a primary or a descendent operator. Thus one might suppose that the arbitrariness inherent in solving the equations reflects precisely this fact. One would then expect that the space of solutions to the Watson + LSZ system is isomorphic to the space of descendent operators, that is, a Virasoro irreducible representation.

In this paper we investigate this possibility in the simplest possible model, that of the $c=\frac{1}{2}$ Ising conformal field theory perturbed by the energy operator with scaling dimensions ( $\frac{1}{2}, \frac{1}{2}$ ). This theory is known to be integrable; in fact, it is equivalent to a massive free fermion [16-20]. For that reason, some of the form factors, for example those of the energy operator itself, are rather too trivial to test
the above ideas. However, the spin operator of the Ising model is non-local when expressed in terms of the fermions, and its form factors, as well as those of its descendents, are non-trivial.

We shall show that there exists a natural grading of the space of solutions to the equations for the form factors and that the dimension at the level ( $n, \bar{n}$ ) agrees with that of the corresponding Virasoro representation. We also show that when the form factors are used to construct the non-critical two-point functions, the analysis of the ultraviolet behavior leads to scaling dimensions which are precisely shifted by $n$ and $\bar{n}$ with respect to those of the primary operator.

Our results suggest that there is a pair of Virasoro algebras acting on the space of form factors in the non-critical theory. A similar, but distinct, observation has been made in the spectrum of the corner transfer matrix in non-critical exactly solvable lattice models [21]. We emphasize that our results concern the continuum theory in the scaling region.

## 2. Equations for the form factors

In this section we review the Watson equations satisfied by the form factors. We stay close to the notation and development of ref. [6]. Similar results may be found in ref. [14]. For convenience we consider a theory with only one type of particle. We have complete sets of in and out asymptotic states $\left|p_{1}, \ldots, p_{n}\right\rangle_{\text {in }}$ and $\left|p_{1}, \ldots, p_{n}\right\rangle_{\text {out }}$, and $S$-matrix elements defined by

$$
\begin{equation*}
S_{n}\left(p_{1}, \ldots, p_{n}\right)={ }_{\text {out }}\left\langle p_{1}, \ldots, p_{n} \mid p_{1}, \ldots, p_{n}\right\rangle_{\text {in }}, \tag{2}
\end{equation*}
$$

where the $n$-particle $S$-matrix has the factorizing form

$$
\begin{equation*}
S_{n}\left(p_{1}, \ldots, p_{n}\right)=\prod_{i<j} S_{2}\left(p_{i}, p_{j}\right) \tag{3}
\end{equation*}
$$

As usual, it is more convenient to label the momenta by the rapidities $\theta_{i}$, where $p_{i}=\left(\cosh \theta_{i}, \sinh \theta_{i}\right)$ (we use units where the mass of the particle is set equal to unity). Then $S_{2}$ depends only on the difference $\theta_{i j}=\left|\theta_{i}-\theta_{j}\right|$.

The form factors are matrix elements of local operators $\mathscr{C}(x)$ between out-states and in-states. We define the functions

$$
\begin{equation*}
F_{n}=\langle 0| \mathscr{C}(0)\left|p_{1}, \ldots, p_{n}\right\rangle_{\mathrm{in}} . \tag{4}
\end{equation*}
$$

If $O$ has spin $s$, Lorentz invariance implies that $F_{n}$ is of the form $\mathrm{e}^{s \theta_{1}}$ times a function depending only on the differences $\theta_{i j}$. The usual arguments imply that this function is the boundary value on the real axis of an analytic function of the
$\theta_{i j}$. The most general $n$-particle form factor is

$$
\begin{equation*}
\text { out }\left\langle p_{1}, \ldots, p_{m}\right| \mathscr{O}(0)\left|p_{m+1}, \ldots, p_{n}\right\rangle_{\text {in }} . \tag{5}
\end{equation*}
$$

Crossing implies that this is obtained by analytic continuation of eq. (4), and is equal to

$$
\begin{equation*}
F_{n}\left(\theta_{i j}, i \pi-\theta_{r s}, \theta_{k l}\right) \tag{6}
\end{equation*}
$$

where $1 \leqslant i<j \leqslant m, 1 \leqslant r \leqslant m<s \leqslant n$, and $m<k<l \leqslant n$.
The Watson equations are derived by inserting a complete set of in-states before the operator $\mathscr{C}$, and of out-states after this operator, in eq. (5), and using eq. (2). The matrix element with in- and out-states interchanged is obtained by CPT invariance from eq. (5) by changing the signs of all the $\theta_{i j}$. Thus

$$
\begin{equation*}
F_{n}\left(\theta_{i j}, i \pi-\theta_{r s}, \theta_{k l}\right)=\left(\prod_{i<j} S\left(\theta_{i j}\right)\right) F_{n}\left(-\theta_{i j}, i \pi+\theta_{r s},-\theta_{k l}\right)\left(\prod_{k<l} S\left(\theta_{k l}\right)\right) . \tag{7}
\end{equation*}
$$

In the case $n=2$, these simplify to

$$
\begin{equation*}
F_{2}(\theta)=F_{2}(-\theta) S_{2}(\theta), \quad F_{2}(i \pi-\theta)=F_{2}(i \pi+\theta) \tag{8}
\end{equation*}
$$

It was shown in ref. [6] that the general solution to the Watson equations has the form

$$
\begin{equation*}
F_{n}=K_{n} \prod_{i<j} F_{\min }\left(\theta_{i j}\right), \tag{9}
\end{equation*}
$$

where $F_{\min }(\theta)$ has the properties that it satisfies eq. (8), is analytic in $0 \leqslant \operatorname{Im} \theta \leqslant 2 \pi$, and has no zeros in $0<\operatorname{Im} \theta<2 \pi$. These requirements uniquely determine this function. The remaining factor $K_{n}$ then satisfies the Watson equations with $S_{2}=1$, which implies that it is a completely symmetric, periodic function of the $\theta_{i}$.

The other constraint on the $K_{n}$ is that they contain all the physical poles expected in the form factor under consideration. This will of course depend on the operator $\mathscr{O}$, and its transformation properties under any global symmetries the theory may possess. However, all operators corresponding to states within the same Virasoro representation should have the same global symmetry, and therefore a priori the same pole structure in the $K_{n}$. The arbitrariness therefore resides in the numerator which multiplies these poles. However, this is further constrained by the requirement that the residues of the poles are proportional to form factors with fewer particles.

To proceed further, it is necessary to specify the particular theory. We therefore proceed to consider the main example of this paper. The massive quantum field theory corresponding to the zero-field Ising model perturbed by the energy
operator is known to be integrable, with a single $\mathbb{Z}_{2}$-odd particle and $S_{2}=-1$ [6]. This is of course related to the free fermion nature of the theory, but we shall not use this fact, since we wish to construct an example which may be generalized to other perturbed conformal field theories. In this case, the minimal solution of eq. (8) is simply

$$
\begin{equation*}
F_{\min }(\theta)=\sinh \frac{1}{2} \theta \tag{10}
\end{equation*}
$$

We consider the case where $\mathscr{O}$ is a $\mathbb{Z}_{2}$-odd operator, and is therefore the non-critical deformation of some operator in the conformal tower of the primary magnetization operator. In that case the $\mathbb{Z}_{2}$ symmetry implies that $F_{n}$ vanishes when $n$ is even. The pole structure of $K_{n}$ may be deduced as follows. There should be poles in every three-body channel. One may argue that no explicit poles should occur in $n$-body channels with $n>3$, because crossing would then imply the existence of inelastic processes. Using the fact that

$$
\begin{equation*}
\left(p_{i}+p_{j}+p_{k}\right)^{2}-1=8 \cosh \frac{1}{2} \theta_{i j} \cosh \frac{1}{2} \theta_{j k} \cosh \frac{1}{2} \theta_{k i} \tag{11}
\end{equation*}
$$

we see that all possible three-body poles may be taken into account by letting

$$
\begin{equation*}
K_{n}=\frac{R_{n}}{\Pi_{i<j} \cosh \frac{1}{2} \theta_{i j}}, \tag{12}
\end{equation*}
$$

where the function $R_{n}$ has no singularities. Note that when $n$ is odd, the denominator in eq. (12) is periodic in each rapidity variable $\theta_{i}$, and therefore so must be $R_{n}$. We may therefore consider it as having a Taylor expansion in the variables $\mathrm{e}^{\theta_{i}}$ and $\mathrm{e}^{-\theta_{i}}$. As will become clear in sect. 3, in order for the ultraviolet behavior of the two-point function to be power-law bounded, this expansion should in fact terminate, so that $R_{n}$ may be written in the form

$$
\begin{equation*}
R_{n}=P_{n}\left(p_{1}, \ldots, p_{n}\right) \exp \left(-N \sum_{i} \theta_{i}\right) \tag{13}
\end{equation*}
$$

for some integer $N$. Here $P_{n}$ is a totally symmetric polynomial in the variables $p_{i}=\mathrm{e}^{\theta_{i}}$. From this we may read off the transformation properties under a Lorentz boost $\theta_{i} \rightarrow \theta_{i}+\alpha$. If the spin of $\theta$ is $s$, we see that $P_{n}$ must in fact be homogeneous of degree $s+N$.

In the conformal theory, the scaling dimension and spin of an operator at level ( $n, \bar{n}$ ) are equal to $h+\bar{h}+n+\bar{n}$ and $h-\bar{h}+n-\bar{n}$ respectively. Initially we shall consider only those operators corresponding to states with $\bar{n}=0$, that is, given by only the generators $L_{-k}$ (rather than the $\bar{L}_{-k}$ ) acting on the highest weight state. These operators are located along the diagonal of the fig. 1. For a given $s$, these are the operators with the lowest scaling dimension. Thus we should expect their


Fig. 1. Tower of operators in the conformal family of the primary field $\Phi$.
form factors to be those with the mildest possible ultraviolet behavior, that is, the smallest possible value of $\operatorname{deg} P_{n}+N$. Thus we choose $N=0$. This argument will be justified more completely in sect. 3 when we consider the ultraviolet behavior in more detail. We are therefore faced with the problem of constructing symmetric homogeneous polynomials of degree $s$ in $n$ variables. A well-known theorem (see, for instance, ref. [22]) states that any such polynomial may be expressed as a sum of products of the form $\sigma_{k_{1}} \sigma_{k_{2}} \ldots$, with $\sum_{i} k_{i}=s$, where the $\sigma_{k}$ are the elementary symmetric polynomials

$$
\begin{equation*}
\sigma_{1}=p_{1}+p_{2}+\ldots, \quad \sigma_{2}=p_{1} p_{2}+p_{1} p_{3}+\ldots, \quad \sigma_{3}=p_{1} p_{2} p_{3}+\ldots, \tag{14}
\end{equation*}
$$

and so on. Then, before the LSZ reduction formulae are used, the dimension of the space of solutions to the Watson equations at this level is given by the number $P(s)$ of partitions of $s$. (We here assume that $n \geqslant s$, since we are interested ultimately in large $n$. The $\sigma_{k}$ are defined to vanish when $k>n$.)

However, not all linear combinations behave correctly when we go to one of the three-particle poles (see fig. 2). We next investigate this constraint. We have

$$
\begin{equation*}
F_{n}=P_{n}\left(p_{1}, p_{2}, \ldots, p_{n}\right) \prod_{i<j} \tanh \frac{1}{2} \theta_{i j} \tag{15}
\end{equation*}
$$



Fig. 2. Bootstrap equation for the form factor $\mathscr{F}_{n}$.

From eq. (11) we see that the pole at $\left(p_{1}+p_{2}+p_{3}\right)^{2}=1$ may be reached by, for example, setting $\theta_{2}=i \pi+\theta_{1}$. Note that this constraint is independent of $\theta_{3}$, a remarkable peculiarity of $(1+1)$-dimensional kinematics. The residue of $F_{n}$ at this pole is

$$
\begin{align*}
& P_{n}\left(p_{1},-p_{1}, p_{3}, \ldots, p_{n}\right) \cdot 8 \sinh \left(-\frac{1}{2} i \pi\right) \sinh \frac{1}{2}\left(i \pi+\theta_{13}\right) \sinh \frac{1}{2} \theta_{13} \\
& \quad \times \prod_{k \geqslant 4} \tanh \frac{1}{2} \theta_{1 k} \tanh \frac{1}{2}\left(i \pi+\theta_{1 k}\right) \prod_{3 \leqslant k<l} \tanh \frac{1}{2} \theta_{k l} \\
& \quad=P_{n}\left(p_{1},-p_{1}, p_{3}, \ldots, p_{n}\right) \cdot 4 \sinh \theta_{13} \prod_{3 \leqslant k<l} \tanh \frac{1}{2} \theta_{k l} . \tag{16}
\end{align*}
$$

On the other hand, by LSZ reduction, this residue should be equal to

$$
\begin{equation*}
i_{\text {out }}\left\langle\theta_{3} \mid \theta_{1}, i \pi+\theta_{1}, \theta_{3}\right\rangle_{\text {in }} F_{n-2}\left(\theta_{3}, \ldots, \theta_{n}\right) . \tag{17}
\end{equation*}
$$

By crossing, the amplitude appearing in eq. (17) is, apart from a factor of $i$, just the two-body $T$-matrix element, related to the $S$-matrix by

$$
\begin{equation*}
i T_{2}=4 \sinh \theta_{13}\left[S\left(\theta_{13}\right)-1\right] . \tag{18}
\end{equation*}
$$

The kinematic factor of $4 \sinh \theta_{13}$ derives from relating the conventional $T$-matrix obtained by LSZ reduction (in which an overall energy-momentum conserving delta-function is taken out) to the quantity $S$, in which rapidity-conserving deltafunctions have been factored out.

Using $S=-1$, we see that eq. (16) may be satisfied, as long as

$$
\begin{equation*}
P_{n}\left(p_{1},-p_{1}, p_{3}, \ldots, p_{n}\right)=2 i P_{n-2}\left(p_{3}, \ldots, p_{n}\right) \tag{19}
\end{equation*}
$$

Note that $\operatorname{deg} P_{n}=\operatorname{deg} P_{n-2}$, which shows that it is possible to find a solution with $N=0$, that is $\operatorname{deg} P_{n}=s$, independent of $n$. In other theories, the situation is not quite so simple, and the degree of $P_{n}$ increases with $n$. In the case $s=0$, which corresponds to the primary magnetization operator, we see that $P_{n}$ is just a constant, proportional to $(2 i)^{n / 2}$. This is in agreement with the result of ref. [6]. In the general case, it is convenient to absorb this factor, and to define $\tilde{P}_{n}=(2 i)^{n / 2} P_{n}$, so that

$$
\begin{equation*}
\tilde{P}_{n}\left(p_{1},-p_{1}, p_{3}, \ldots, p_{n}\right)=\tilde{P}_{n-2}\left(p_{3}, \ldots, p_{n}\right) \tag{20}
\end{equation*}
$$

The important feature of the above equation is that the right-hand side is independent of $p_{1}$. This places a severe constraint on the allowed polynomials. Since these polynomials are to be expressed in terms of the $\sigma_{k}$, we must see how
the latter behave when we set $p_{2}=-p_{1}$. It is straightforward to show that

$$
\begin{equation*}
\sigma_{k}\left(p_{1},-p_{1}, p_{3}, \ldots, p_{n}\right)=\sigma_{k}\left(p_{3}, \ldots, p_{n}\right)-p_{1}^{2} \sigma_{k-2}\left(p_{3}, \ldots, p_{n}\right) \tag{21}
\end{equation*}
$$

where we adopt the convention that $\sigma_{0}=1$, and that $\sigma_{k}=0$ if $k<0$. It is conceptually simpler to suppress the explicit dependence of the $\sigma_{k}$ on the $p_{i}$, and to regard the $P_{n}$ as depending on the variables $\sigma_{k}$, which transform according to

$$
\begin{equation*}
\sigma_{k} \rightarrow \sigma_{k}+\beta_{2} \sigma_{k-2} \tag{22}
\end{equation*}
$$

where $\beta_{2}=-p_{1}^{2}$ is an arbitrary parameter. The above linear transformations, when iterated with parameters $\beta_{2}^{(1)}, \beta_{2}^{(2)}, \ldots, \beta_{2}^{(M)}$, generate a group whose most general element is

$$
\begin{equation*}
\sigma_{k} \rightarrow \sigma_{k}+\beta_{2} \sigma_{k-2}+\beta_{4} \sigma_{k-4}+\ldots+\beta_{2[k / 2]} \sigma_{k-2[k / 2]} \tag{23}
\end{equation*}
$$

where $\beta_{2}=\sum_{i} \beta_{2}^{(i)}, \beta_{4}=\sum_{i \neq j} \beta_{2}^{(i)} \beta_{2}^{(j)}$, and so on. We may always choose $M$ sufficiently large so that the parameters $\beta_{j}$ are independent. Thus, the general transformation is of the form

$$
\begin{equation*}
\sigma_{k} \rightarrow \sum_{l} \beta_{k-l} \sigma_{l} \tag{24}
\end{equation*}
$$

where $\beta_{j}=0$ if $j$ is odd, or if $j<0$, and $\beta_{0}=1$. The condition eq. (19) may therefore be generalized to

$$
\begin{equation*}
\tilde{P}_{n}\left(\left\{\sigma_{k}\right\}\right)=\tilde{P}_{n-2}\left(\sum_{l} \beta_{k-l} \sigma_{l}\right) \tag{25}
\end{equation*}
$$

Now the polynomials $P_{n}$ belong to the space $\mathscr{P}_{s}$ spanned by the $\sigma_{k_{1}} \sigma_{k_{2}} \ldots$ with $\sum_{i} k_{i}=s$. In general, a polynomial in this space, will, under the above transformation, be mapped out of this space, in fact, into a linear combination of polynomials in $\mathscr{P}_{s}, \mathscr{P}_{s-2}$, and so on. For a polynomial to satisfy eq. (25) it must not be mapped out of the space. The structure of the transformation eq. (24) then implies that this polynomial will be invariant under the group of transformations eq. (24), and that the polynomial $P_{n-2}$ will have, as a function of the $\sigma_{k}$, the same form as $P_{n}$.

The structure of the transformations eq. (24) suggests that we define the generating function

$$
\begin{equation*}
f\left(x_{1}, x_{2}, \ldots, x_{n} ; t\right) \equiv \sum_{k=0}^{\infty} \sigma_{k} t^{k} \tag{26}
\end{equation*}
$$

whose transformation properties are simple:

$$
\begin{equation*}
f(t) \rightarrow\left(1+\beta_{1} t^{2}+\beta_{2} t^{4}+\ldots\right) f(t) \tag{27}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
f(t) \rightarrow \beta(t) f(t) \tag{28}
\end{equation*}
$$

where $\beta(t)$ is any even function of $t$, satisfying $\beta(0)=1$.
In order to construct polynomial invariants of degree $s$ we therefore need to construct functions of $f(t)$ invariant under eq. (28), and expand them in powers of $t$. For example, consider

$$
\begin{equation*}
g(t) \equiv \frac{f(t)-f(-t)}{f(t)+f(-t)} \tag{29}
\end{equation*}
$$

This generates an important class of invariants $I_{s}$, of degree $s$, with $s$ odd, of the form

$$
\begin{equation*}
I_{s}=\sigma_{s}+\ldots, \tag{30}
\end{equation*}
$$

where the omitted terms each contain at least two factors of the $\sigma_{i}$. The first few examples are

$$
\begin{align*}
& I_{1}=\sigma_{1} \\
& I_{3}=\sigma_{3}-\sigma_{2} \sigma_{1} \\
& I_{5}=\sigma_{5}-\sigma_{4} \sigma_{1}-\sigma_{3} \sigma_{2}+\sigma_{2}^{2} \sigma_{1} \tag{31}
\end{align*}
$$

They obey the recursion relations

$$
\begin{equation*}
\sigma_{2 k+1}=I_{2 k+1}+\sigma_{2} I_{2 k-1}+\sigma_{4} I_{2 k-3}+\ldots+\sigma_{2 k} I_{1} \tag{32}
\end{equation*}
$$

We now show that any invariant $\mathscr{I}$ of finite degree may be expressed as a polynomial in the $I_{\text {odd }}$. For it is certainly a polynomial in the $\sigma_{\text {odd }}$ and the $\sigma_{\text {even }}$. Using eq. (32) repeatedly, it may then be expressed as a polynomial in the $I_{\text {odd }}$ and the $\sigma_{\text {even }}$. Let $k$ be the largest integer such that $\sigma_{2 k}$ appears in the expression. Consider the particular transformation eq. (24) with $\beta_{2}=\beta_{4}=\ldots=\beta_{2 k-2}=0$, and $\beta_{2 k} \neq 0$, so that $\sigma_{2 k} \rightarrow \sigma_{2 k}+\beta_{2 k}$, while all the other $\sigma_{\text {even }}$, and of course the $I_{\text {odd }}$, remain unshifted. Since $\mathscr{I}$ is invariant, it must therefore be independent of $\sigma_{2 k}$. Now consider the transformation with all the $\beta_{j}=0$ except for $\beta_{2 k-2}$. Since $\sigma_{2 k}$ has already been eliminated, the only argument of $\mathscr{I}$ which shifts is $\sigma_{2 k-2}$. Thus $\mathscr{I}$ must be independent of $\sigma_{2 k-2}$. Repeating this argument, it is clear that in fact $\mathscr{I}$ cannot depend on any of the $\sigma_{\text {even }}$, and is therefore a polynomial in the $I_{\text {odd }}$.
Q.E.D.

It follows that the dimension of the space of invariants of degree $s$ is equal to the number $q(s)$ of partitions of $s$ into odd integers. It is well known that this is equal to the number of partitions of $s$ into distinct integers [22], whose generating function is

$$
\begin{equation*}
\sum_{s=0}^{\infty} q(s) x^{s}=\prod_{r=1}^{\infty}\left(1+x^{r}\right) . \tag{33}
\end{equation*}
$$

Apart from the factor of $x^{1 / 24}$, this is the character for the $h=\frac{1}{16}$ representation of the Virasoro algebra with $c=\frac{1}{2}$. Thus we establish our main result, that the dimension of the space of solutions to the equations for the off-critical form factors of the most relevant operators with spin $s$ is equal to that of the space of such operators in the conformal field theory.

So far, we have considered only the case $N=0$ in eq. (13), which we have argued gives the form factors of operators corresponding to states obtained by the action of a single Virasoro algebra on the highest weight state. In the general case, we would like to argue that the function $R_{n}$ may be written as a sum of products of the invariants $I_{k}\left(p_{i}\right)$ and the invariants $I_{\bar{k}}\left(\bar{p}_{i}\right)$, where $\bar{p}_{i}=p_{i}^{-1}$. This would then correspond to the usual commuting left-moving and right-moving Virasoro algebras. We shall argue that this is indeed the case.

First, note that the function $R_{n}$ may always be written in a unique way as sums of products of the $\sigma_{k}$ and the $\bar{\sigma}_{k}$, where $\bar{\sigma}_{k} \equiv \sigma_{k}\left(\bar{p}_{i}\right)$. This is because the polynomial $P_{n}$ may certainly be written in terms of the $\sigma_{k}$, as a sum of terms of the form

$$
\begin{equation*}
\sigma_{k_{1}} \sigma_{k_{2}} \ldots \sigma_{k_{m}} \tag{34}
\end{equation*}
$$

with $k_{1} \leqslant k_{2} \leqslant \ldots \leqslant k_{m}$, and $\sum_{j} k_{j}=\operatorname{deg} P_{n}=s+N n$. Now apply the identity

$$
\begin{equation*}
\sigma_{n-k}=\left(p_{1} \ldots p_{n}\right) \bar{\sigma}_{k} \tag{35}
\end{equation*}
$$

to absorb the factors of $\left(p_{1} \ldots p_{n}\right)^{-1}$. We choose to apply this beginning with $k_{m}$, in decreasing order of the $k_{j}$ in eq. (34), until all the factors are exhausted. If there are such factors remaining at the end, they should simply be renamed as $\bar{\sigma}_{n}$.

We therefore have written the function $R_{n}$ as sums of terms of the form

$$
\begin{equation*}
\sigma_{k_{1}} \ldots \sigma_{k_{p}} \bar{\sigma}_{\bar{k}_{1}} \ldots \bar{\sigma}_{\bar{k}_{q}} \tag{36}
\end{equation*}
$$

Note that, because of eq. (35), not all such terms are independent. However, we now declare that we only consider such expressions in which the degrees in the $p_{i}$ and the $\bar{p}_{i}$, namely $\sum_{j} k_{j}$ and $\sum_{j} \bar{k}_{j}$ are finite as the number of particles $n$ is allowed to grow. This is because, as will be discussed in sect. 3, only such form factors will lead to acceptable power-law ultraviolet behavior for the two-point functions. In
that case, for large enough $n$, all the above expressions will be independent, and thus form a useful basis for the function $R_{n}$.

Now consider the behavior of such functions under the restriction $p_{2}=-p_{1}$ required when we study the residue of the three-particle poles. In analogy to eq. (21) we have

$$
\begin{equation*}
\sigma_{k} \rightarrow \sigma_{k}-p_{1}^{2} \sigma_{k-2}, \quad \bar{\sigma}_{k} \rightarrow \bar{\sigma}_{k}-p_{1}^{-2} \bar{\sigma}_{k-2} \tag{37}
\end{equation*}
$$

The apparent problem is that the two transformations are not independent, and it might therefore seem possible to construct invariants which are not invariant under the transformations separately. This is not the case, however, since we may iterate the transformations eq. (37) as many times as we choose, and, in analogy to eq. (24) find that they close only on the group of transformations

$$
\begin{equation*}
\sigma_{k} \rightarrow \sum_{l} \beta_{k-l} \sigma_{l}, \quad \bar{\sigma}_{k} \rightarrow \sum_{l} \bar{\beta}_{k-l} \bar{\sigma}_{l} \tag{38}
\end{equation*}
$$

where the coefficients $\beta_{j}$ are as before, and $\bar{\beta}_{2}=\sum_{i}\left(\underline{\beta}_{2}^{(i)}\right)^{-1}$, and so on. Once again, by choosing $M$ sufficiently large, the parameters $\bar{\beta}_{j}$ may be taken independent of the $\beta_{j}$. Thus the two groups of transformations may be taken to act independently on the $\sigma_{k}$ and the $\bar{\sigma}_{k}$.

We conclude, by our previous analysis, that the only functions $R_{n}$ satisfying the LSZ reduction formulae are expressible as sums of products of invariant polynomials of the $p_{i}$, multiplied by similar invariant polynomials of the $\bar{p}_{i}$. Note that, for this argument to work, we had to insist that the number of particles $n$ was larger than the degrees of the polynomials. However, we may always start with this case. The lower values of $n$ will then follow by LSZ reduction, and by the general structure, must also be expressible as invariants. However, it may well occur that two operators which have distinct form factors when coupling to a sufficiently large number of particles, have the same coupling to fewer particles. It should therefore be possible to form linear combinations of operators which do not couple to less than a certain number of particles. It would be interesting to investigate this structure in more detail.

## 3. Ultraviolet behavior

In the last section, we showed that there is a way of classifying the solutions to the Watson + LSZ system of equations according to a set of integers ( $n, \bar{n}$ ) such that the dimensions of the vector spaces at level $(n, \bar{n})$ match with those at the same level in the representations of the right- and left-moving Virasoro algebras. To complete the identification one should show that the scaling dimensions of the operators whose form factors are thus constructed come out correctly when the ultraviolet limit of their two-point functions is taken.

The two-point function of an operator in momentum space is

$$
\begin{equation*}
\tilde{G}(k)=\int \frac{\rho\left(\kappa^{2}\right) \mathrm{d} \kappa^{2}}{k^{2}-\kappa^{2}+i \epsilon} \tag{39}
\end{equation*}
$$

where the spectral function is given in terms of the form factors by

$$
\begin{equation*}
\rho\left(\kappa^{2}\right)=\sum_{n} \frac{1}{n!} \int \prod_{i} \frac{\mathrm{~d} \theta_{i}}{4 \pi} \delta\left(\kappa_{0}-\sum_{i} \cosh \theta_{i}\right) \delta\left(\kappa_{1}-\sum_{i} \sinh \theta_{i}\right)\left|F_{n}\right|^{2}, \tag{40}
\end{equation*}
$$

where $\kappa=\left(\kappa_{0}, \kappa_{1}\right)$ and $\kappa^{2}=\kappa_{0}^{2}-\kappa_{1}^{2}$. A less familiar, but simpler result follows for the two-point function in real euclidean space:

$$
\begin{equation*}
G(r)=\sum_{n} \frac{1}{n!} \int \prod_{i} \frac{\mathrm{~d} \theta_{i}}{4 \pi} \exp \left(-|r| \sum_{i} \cosh \theta_{i}\right)\left|F_{n}\right|^{2} \tag{41}
\end{equation*}
$$

First consider the case of the primary magnetization operator, where $F_{n}$ is given by eq. (15) with $P_{n}=(2 i)^{n / 2}$. Then if we define

$$
\begin{equation*}
V(y)=-\ln \tanh ^{2} \frac{1}{2} y, \quad U(y)=\mathrm{e}^{-y}, \quad l=\ln (2 / r) \tag{42}
\end{equation*}
$$

the above expression may be rewritten in the very suggestive form

$$
\begin{equation*}
G(r)=\frac{1}{2} \Xi(1 / 2 \pi, l)+\frac{1}{2} \Xi(-1 / 2 \pi, l), \tag{43}
\end{equation*}
$$

where

$$
\begin{align*}
\Xi(z, l)= & \sum_{n=0}^{\infty} \frac{z^{n}}{n!} \int \prod_{i} \mathrm{~d} \theta_{i} \exp \left(-\sum_{i} U\left(\theta_{i}+l\right)+U\left(l-\theta_{i}\right)\right) \\
& \times \exp \left(-\sum_{i<j} V\left(\theta_{i}-\theta_{j}\right)\right) \tag{44}
\end{align*}
$$

This is nothing other than the grand partition function for a one-dimensional gas of particles, of fugacity $z$, interacting with each other via a two-body potential $V$, and with two walls, located at $\pm l$, with a potential $U$. Note that both of these potentials are repulsive and short range. Thus, we expect, on the basis of the usual arguments about the thermodynamic limit of such a system, that as $l \rightarrow \infty$, we may write the thermodynamic potential as

$$
\begin{equation*}
-\ln \Xi \sim-2 p l+2 f_{\mathrm{s}}+\mathrm{O}\left(\mathrm{e}^{-l / \xi}\right) \tag{45}
\end{equation*}
$$

where $p$ is the pressure, $f_{\mathrm{s}}$ is the surface or boundary contribution to the free
energy, and $\xi$ is of the order of the correlation length. We see therefore that each term in eq. (43) should have a power law dependence on $r$, with an exponent proportional to the pressure $p$. Since we expect the gas with positive fugacity [corresponding to the first term in eq. (43)] to have the larger pressure (a result which may be checked within the virial expansion), we have the prediction that, as $r \rightarrow 0$,

$$
\begin{equation*}
G(r) \sim \frac{\text { const. }}{r^{2 p}} \tag{46}
\end{equation*}
$$

Hence the overall scaling dimension of the operator is nothing other than the pressure of this fictitious gas, with fugacity $z=1 / 2 \pi$. In fact this argument is very similar to that used to derive Regge behavior of elastic scattering amplitudes when expressed as unitarity sums over multiparticle production amplitudes. In that context, it is often called the Feynman gas [23].

Note that this argument does not immediately give the value of $p$. In fact it is rather surprising that it should turn out to have the rational value of $\frac{1}{8}$. We expect, of course, the pressure to depend smoothly on the fugacity, so it would seem that for this particular value this one-dimensional gas should be exactly solvable^. We have solved a simpler problem, in which the particles interact only with their nearest neighbors. Since the interaction is repulsive, and the density turns out to be small, this gives a remarkably good approximation to the exact answer, namely $p \approx 0.12529$.

Leaving aside the question of how to compute the scaling dimension exactly from eq. (44), we now consider correlation functions of descendent operators, whose form factors we have argued differ from those of the primary operator by invariant polynomials in the variables $\mathrm{e}^{\theta_{i}}$ and $\mathrm{e}^{-\theta_{i}}$, of respective degrees $n$ and $\bar{n}$. Their correlation functions are therefore related to expectation values of these polynomials in the fictitious gas ensemble, and will in general have the form

$$
\begin{equation*}
\Xi(z, l)\left\langle P\left(\left\{\mathrm{e}^{\theta_{i}}\right\},\left\{\mathrm{e}^{-\theta_{i}}\right\}\right)\right\rangle, \tag{47}
\end{equation*}
$$

where the partition function is, as before, essentially the two-point function of the primary operator, and $P$ is homogeneous of degrees $2 n, 2 \bar{n}$, respectively, in the two sets of variables. Recalling the definition of the interaction $U(\theta)$, we may write the above expectation value as

$$
\begin{equation*}
\mathrm{e}^{2(n+\bar{n})!}\left\langle P\left(U\left(l-\theta_{i}\right), U\left(l+\theta_{i}\right)\right)\right\rangle . \tag{48}
\end{equation*}
$$

[^1]Now we would expect the latter expectation value, which is related to the probability for finding a given number of particles close to one or other of the walls, to be finite in the thermodynamic limit $l \rightarrow \infty$. Thus, in that limit, the scaling dimension of an operator whose form factors contain polynomial of degree ( $n, \bar{n}$ ) has a scaling dimension shifted by $(n+\bar{n})$ from that of the primary field. Of course, it is clear from the Lorentz transformation properties of its form factors that it corresponds to spin $n-\bar{n}$.

## 4. Discussion

We have chosen a particularly simple, albeit non-trivial, case, that of the magnetization operator in the thermally perturbed Ising model, to illustrate our general thesis, which is that one may uncover the full structure of primary and descendent operators expected on the basis of conformal invariance by studying the equations determining the form factors of the theory away from criticality. However, the Ising model is in some ways a deceptively simple case, since the order of the polynomial multiplying the minimal solution turned out, for a given operator, to be independent of the number of particles $n$. In other models we have studied, for example the Yang-Lee or the three-state Potts model, the situation is already more complicated. In addition, in theories with a larger number of different types of particles, but no obvious selection rules, it is much harder to see how to proceed.

It is interesting to see how the infinite number of conserved charges, which makes the theory integrable, manifests itself in the computation of the form factors of the descendent operators. For the Ising model with a thermal perturbation, conserved charges $Q_{s}$ of spin $s$ exist for all odd $s$. When these charges act on an asymptotic state $\left|p_{1}, \ldots, p_{n}\right\rangle$ they give a factor of $\Sigma_{j} p_{j}^{s}$. Thus if we commute the primary operator $S$ with a conserved charge $Q_{s}$, we obtain an operator with form factor

$$
\begin{equation*}
\langle 0|\left[Q_{s}, S\right]\left|p_{1}, \ldots, p_{n}\right\rangle \propto \sum_{j} p_{j}^{s}\langle 0| S\left|p_{1}, \ldots, p_{n}\right\rangle \tag{49}
\end{equation*}
$$

It is straightforward to show that the polynomial $\Sigma_{j} p_{j}^{s}$ may be written in terms of the invariant polynomials $I_{k}$, with $k \leqslant s$, defined in sect. 3. By continuing the process, forming multiple commutators,

$$
\begin{equation*}
\left[Q_{s_{1}}, \ldots,\left[Q_{s_{m}}, S\right] \ldots\right] \tag{50}
\end{equation*}
$$

we may construct operators of different spins, whose form factors will all differ from those of the primary operator $S$ by invariant polynomials. Moreover, since the charges $Q_{s}$ all commute with each other, the order of the $Q$ 's is not important
in eq. (50). Thus, the number of different operators of total spin $s$ whose form factors we may obtain in this way is equal to the number of partitions of $s$ into odd integers. As we pointed out before, in this case this is just the dimension of the Virasoro representation at this level. This feature, that we generate all the operators in the representation by commutation with the conserved charges, is a particular feature of the thermally perturbed Ising model. In other cases, the number of conserved charges is smaller, and not all operators are obtained in this way.

A so far unsolved problem in this approach is the exact computation of the pressure of the Feynman gas which gives the scaling dimension of the primary operator. Our results suggest that there is some Virasoro algebra underlying this gas. If this could be identified explicitly, it should be possible to understand why such a quantity should be quantized to specific rational values.

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[^1]:    * Schroer and Truong [20] have shown how a slightly different expression for $G(r)$ may be expressed in terms of a free bosonic theory and hence evaluated in the UV limit. This method appears to be special to the Ising model.

