STATES OF NON-ZERO DENSITY IN THE CHIRAL INVARIANT GROSS-NEVEU MODEL*

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States describing a non-zero number density of massive particles are investigated in the SU(2) chiral-invariant Gross-Neveu model. It is found that for a fixed positive density, the lowest energy state is color ferromagnetic, with all color spins aligned. For asymptotically large densities, the total energy and density are calculated as functions of the Fermi momentum. These quantities tend toward their counterparts in a non-interacting theory, with logarithmic corrections typical of an asymptotically free system.

1. Introduction

The chiral invariant Gross-Neveu (CGN) model, describing the interaction of fermion fields with SU(n) internal symmetry in (1+1)-dimensional space-time, has proved to be a fruitful theoretical laboratory in which a number of interesting results can be obtained exactly [1, 2]. Corresponding formally to the lagrangian density

$$\mathscr{L} = i\bar{\psi}^a \,\delta\psi_a + g[(\bar{\psi}^a\psi_a)^2 - (\bar{\psi}^a\gamma^5\psi_a)^2], \qquad (1.1)$$

the CGN model exhibits dynamical generation (via dimensional transmutation) of a non-zero mass m, accompanied by asymptotic freedom: in the limit of infinite ultraviolet cutoff K, the bare coupling constant g appearing in (1.1) vanishes according to

$$g(K) \sim \frac{1}{\ln\left(K/m\right)}.$$
 (1.2)

In contrast to what one expects of four-dimensional gauge theories, the asymptotic freedom is not accompanied by SU(n) color confinement; rather there exist n-1 multiplets of color-bearing particles of masses

$$m_r = m \sin \frac{r\pi}{n}, \qquad r = 1, 2, \dots, n-1.$$
 (1.3)

In addition, there is no spontaneous breaking of the chiral and SU(n) invariances (such breaking is, in fact, forbidden by Coleman's theorem [3]).

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Another interesting feature of the model is the absence of particle creation and annihilation in scattering processes, thanks to an infinite set of conservation laws [4]. The latter constrain the sets of incoming and outgoing momenta to be identical, and allow one to write the S-matrix as a product of two-body S-matrices. Because of the conservation of particle number, it is meaningful (here and below we restrict our attention to the SU(2) model, with only one species of particle) to speak of a gas of \mathcal{N} massive Gross-Neveu particles on a line segment of length L, and then take the limit $L \to \infty$ with $\rho = \mathcal{N}/L$ fixed.

It is our goal in this article to investigate the zero-temperature properties of the finite-density CGN gas. We shall primarily be interested in answering two questions:

(i) What is the nature of the ground state?

(ii) What are the physical manifestations of asymptotic freedom in the highdensity regime?

Although Coleman's theorem forces the vacuum of the local CGN field theory to be a color singlet, there is no such restriction on states of non-zero density, in which the massive particles are scattered by one another as a result of an effective non-local interaction. In sect. 3, we shall show that in fact the finite-density ground state is not a color singlet, but is rather the (degenerate) color ferromagnetic state with all color spins aligned. The approach to this ground state as the temperature tends to zero will be studied in sect. 4.

Turning to the second main item of interest, we note that in previous work on the CGN model, the notion of asymptotic freedom has referred primarily, if not exclusively, to the renormalization relation (1.2) expressing the cut-off dependence of the bare coupling constant. It is expected that asymptotic freedom in this constructive sense will be mirrored in the asymptotic short-distance behavior of the correlation functions of fields. Unfortunately, the correlation functions are not now accessible to calculation, and so one must look at simpler quantities, of a macroscopic or thermodynamic nature, to find a physical expression of asymptotic freedom. Such a program has been quite successful [5, 6] in a closely related model, the Kondo model of a magnetic impurity interacting with the conduction electrons of a non-magnetic metal. There one has total screening of the impurity's magnetic moment in the ground state, but for increasingly high temperature or magnetic field (relative to the so-called Kondo temperature), the contribution of the impurity to the magnetization approaches more and more that of a free moment, with logarithmic corrections typical of asymptotically free systems.

It is with the Kondo example in mind that we look for a manifestation of asymptotic freedom in the high-density regime of the CGN model. In sect. 5 of this article, we shall set up a perturbation scheme which allows us to determine, to arbitrary order in b^{-1} (m sinh b is the upper bound on the zero-temperature momentum distribution, i.e. the Fermi momentum) the rapidity (hence momentum) distribution of the density ground state. We then calculate explicitly, to order b^{-2} the relations among the density, the Fermi momentum and the total energy. We

shall find that these relations are all consistent with asymptotic freedom, with a renormalization of the effective mass very similar to the large magnetic field rescaling of the Kondo temperature found in ref. [6].

2. Brief survey of the chiral-invariant Gross-Neveu model

In this section we summarize the main results of ref. [1]. For a more detailed introduction to the chiral-invariant Gross-Neveu model, the reader is referred to ref. [7].

In a convenient basis with "fermionic bookkeeping" [7] each energy eigenstate (in the presence of volume and ultraviolet cut-offs, L and K, respectively) is labeled by

(a) sets of distinct integers, $n_1^{\pm}, n_2^{\pm}, \ldots, n_{N_{\pm}}^{\pm}$, where N_{\pm} are integers;

(b) a set of distinct complex "rapidity" parameters $\chi_1, \chi_2, \ldots, \chi_M$, with $M \le \frac{1}{2}(N_+ + N_-) = \frac{1}{2}N$, satisfying

$$\left[\frac{i(\pi/c - \chi_{\gamma}) + \frac{1}{2}\pi}{i(\pi/c - \chi_{\gamma}) - \frac{1}{2}\pi}\right]^{N_{+}} \left[\frac{i(-\pi/c - \chi_{\gamma}) + \frac{1}{2}\pi}{i(-\pi/c - \chi_{\gamma}) - \frac{1}{2}\pi}\right]^{N_{-}} = -\prod_{\beta=1}^{M} \left[\frac{i(\chi_{\beta} - \chi_{\gamma}) + \pi}{i(\chi_{\beta} - \chi_{\gamma}) - \pi}\right],$$
$$c = \frac{4g}{1 - g^{2}}, \gamma = 1, \dots, M, \qquad (2.1)$$

or, taking the logarithm,

$$2N_{+}\tan^{-1}\left(\frac{\pi/c-\chi_{\gamma}}{\frac{1}{2}\pi}\right) - 2N_{-}\tan^{-1}\left(\frac{\pi/c+\chi_{\gamma}}{\frac{1}{2}\pi}\right) = 2\sum_{\beta=1}^{M}\tan^{-1}\left(\frac{\chi_{\beta}-\chi_{\gamma}}{\pi}\right) - 2\pi J(\chi_{\gamma}),$$
(2.2)

where $J(\chi_{\gamma})$ is an integer (half odd integer) if N-M is odd (even). Each set of quantum numbers $\{\underline{n}^{\pm}, \underline{\chi}\}$ corresponds to a multiplet of states of color spin $\frac{1}{2}(N-2M)$.

The ground state, which becomes the physical vacuum in the infinite cut-off limit, corresponds to a filled Dirac sea with

$$n_j^{\pm} = n_{0j}^{\pm} = \pm n_{0j},$$

$$n_{0j} = -\frac{1}{2}N_0 + j - 1, \qquad j = 1, 2, \dots, N_0 \sim \frac{2KL}{\pi}.$$

It is a color singlet, with $M = \frac{1}{2}N_0$. All χ_{γ} are real, and the quantities $J(\chi_{\gamma})$ in (2.2) are consecutive integers (half odd integers) for M odd (even).

All other states have strictly positive energy (there is a mass gap), and may have $N^{\pm} \neq \frac{1}{2}N_0$ or $M \neq \frac{1}{2}N_0$. In addition, some of the χ_{γ} may not be real, and interspersed among the real χ_{γ} there may be "holes", i.e. solutions $\chi_i^{\rm h}$ of (2.2) such that $\chi_i^{\rm h} \notin \{\chi_1 \cdots \chi_M\}$. The task of classifying, in general, the solutions of (2.1) is simplified considerably by the standard assumption, whose consistency has been thoroughly

checked [8], that all χ_{γ} in a solution set are members of *strings*. An *n*-string is defined as a family of $n \chi_{\gamma}$'s with the same real part, of the form

$$\chi^{s} + il\pi$$
, $l = -\frac{n-1}{2}, \dots, \frac{n-1}{2}, \quad (\chi^{s} \text{ real}).$ (2.3)

There is a further simplification if the number of 1-strings is macroscopic (i.e. $\sim L$), as is the case for physical states in our model. Then one may introduce a density $\sigma_1(\chi)$ such that for all 1-strings χ_i

$$\int_{\chi_i}^{\chi_{i+1}} \mathrm{d}\chi \sigma_1(\chi) = 1 \; .$$

Introducing as well a hole density $\sigma_{1,h}(\chi)$, which will be merely a sum of delta functions if the number of holes is fixed for $L \to \infty$, one sees that, in general,

 $\sigma(\chi) = \sigma_1(\chi) + \sigma_{1,h}(\chi)$

is a smooth, positive function.

We now rewrite (2.2), making use of the string hypothesis and the density $\sigma_1(\chi)$:

$$2N_{+} \tan^{-1}\left(\frac{\pi/c-\chi}{\frac{1}{2}\pi}\right) - 2N_{-} \tan^{-1}\left(\frac{\pi/c+\chi}{\frac{1}{2}\pi}\right)$$

= $2\sum_{n=2}^{\infty} \sum_{j}^{n-\text{strings}} \left[\tan^{-1}\left(\frac{\chi_{nj}-\chi}{\frac{1}{2}(n+1)\pi}\right) + \tan^{-1}\left(\frac{\chi_{nj}-\chi}{\frac{1}{2}(n-1)\pi}\right) \right]$
+ $2\int_{-\infty}^{+\infty} d\chi' \sigma_{1}(\chi') \tan^{-1}\left(\frac{\chi'-\chi}{\pi}\right) - 2\pi J(\chi),$ (2.4)

where

$$J(\chi) = \int_{\chi_0}^{\chi} d\chi' \sigma(\chi') , \qquad (2.5)$$

with χ_0 chosen so that

$$2\pi J(-\infty) = -\left(N - M_1 - 2\sum_n M_n\right)\pi = -2\pi J(+\infty),$$

where M_n is the number of *n*-strings. The 1-strings and 1-string holes will then be located at the points where $J(\chi)$ takes on the (half) integral values $J(-\infty) + \frac{1}{2}$, $J(-\infty) + \frac{3}{2}, \ldots, J(+\infty) - \frac{3}{2}, J(+\infty) - \frac{1}{2}$.

Differentiating (2.4) with respect to χ now gives the integral equation

$$\sigma_{1}(\chi) + \sigma_{1,h}(\chi) = \frac{\frac{1}{2}N_{+}}{(\frac{1}{2}\pi)^{2} + (\chi - \pi/c)^{2}} + \frac{\frac{1}{2}N_{-}}{(\frac{1}{2}\pi)^{2} + (\chi + \pi/c)^{2}} - \int_{-\infty}^{+\infty} d\chi' \sigma_{1}(\chi') \frac{1}{\pi^{2} + (\chi - \chi')^{2}} + \sum_{n=2}^{\infty} \sum_{j}^{M_{n}} \left[\frac{\frac{1}{(\frac{1}{2}(n+1)\pi)^{2} + (\chi - \chi_{nj})^{2}}}{(\frac{1}{(\frac{1}{2}(n+1)\pi)^{2} + (\chi - \chi_{nj})^{2}} + \frac{\frac{1}{(\frac{1}{2}(n-1)\pi)^{2} + (\chi - \chi_{nj})^{2}}}{(\frac{1}{(\frac{1}{2}(n-1)\pi)^{2} + (\chi - \chi_{nj})^{2}}} \right], (2.6)$$

which may be solved algebraically for the Fourier transform

$$\tilde{\sigma}_1(p) = \int_{-\infty}^{+\infty} \mathrm{d}\chi \,\,\mathrm{e}^{-ip\chi}\sigma_1(\chi)\,,$$

in terms of $\tilde{\sigma}_{1,h}(p)$ and string positions:

$$\tilde{\sigma}_{1}(p) = \frac{N_{+} e^{-ip\pi/c} + N_{-} e^{ip\pi/c}}{2\cosh\left(\frac{1}{2}\pi p\right)} - \frac{\tilde{\sigma}_{1,h}(p)}{1 + e^{-\pi|p|}} - \sum_{n=2}^{\infty} \sum_{j=1}^{M_{n}} e^{-(n-1)(\pi/2)|p|} e^{-ip\chi_{nj}}.$$
 (2.7)

For the vacuum, we have $N_{\pm} = \frac{1}{2}N_0$ and only 1-strings, so that the density is given by

$$\tilde{\sigma}_0(p) = \frac{N_0 \cos p\pi/c}{2 \cosh(\frac{1}{2}\pi p)}.$$
(2.8)

Once a solution of (2.1) has been found in a self-consistent way, the momentum and energy, relative to the vacuum, may be computed simply, as

$$P = n \frac{2\pi}{L} + \sum_{j=1}^{M_{\rm h}} \frac{N_0}{L} \tan^{-1} \left(\frac{\sinh \chi_j^{\rm h}}{\cosh \pi/c} \right), \qquad (2.9a)$$

$$E = |n| \frac{2\pi}{L} + \sum_{j=1}^{M_{\rm h}} \frac{N_0}{L} \tan^{-1} \left(\frac{\cosh \chi_j^{\rm h}}{\sinh \pi/c} \right), \qquad (2.9b)$$

 $(M_{\rm h} = \tilde{\sigma}_{1,\rm h}(0))$,

where *n* is an integer. In the limit $K \to \infty$, $c \to 0$ with the quantity $(N_0/L) \tan^{-1} (1/\sinh[\pi/c])$ held fixed at *m*, the renormalized mass, eqs. (2.9) become

$$P = n \frac{2\pi}{L} + \sum_{j=1}^{M_{\rm h}} m \sinh \chi_j^{\rm h}, \qquad (2.10a)$$

$$E = |n| \frac{2\pi}{L} + \sum_{j=1}^{M_{\rm h}} m \cosh \chi_j^{\rm h}.$$
 (2.10b)

We observe that there are two types of terms in the energy-momentum spectrum, corresponding to the sets of quantum numbers n_i^{\pm} and χ_{γ} , respectively. The former corresponds to a spectrum of massless, colorless, non-interacting particles and antiparticles of charge ± 1 and chirality ± 1 . The second class of terms provides a natural interpretation of the hole position $\chi_i^{\rm h}$: it is the rapidity of a particle of mass m and color spin one-half. The non-real χ_{γ} do not contribute directly to the energy-momentum [the r.h.s. of (2.10) depends only on the hole positions], although there is an indirect O(1/L) contribution, coming from the influence on hole positions the "string" terms in (2.7). The non-real χ_{γ} label the various states (degenerate in the limit $L \rightarrow \infty$) corresponding to the different ways in which $M_{\rm h}$ color spins $\frac{1}{2}$ can be combined to yield total color spin $\frac{1}{2}(N-2M)$. Henceforth we shall identify $M_{\rm h}$ with \mathcal{N} , the number of massive particles.

To illustrate the above concepts and formulas, consider a state without massless excitations (in particular, $N_+ = N_- = \frac{1}{2}N_0$) containing two mass-*m* particles. The color spins can add to form either a singlet or a triplet state. Both are characterized by two holes, χ_1^h and χ_2^h , but are distinguished by the fact that in the triplet state, all χ_{γ} are real, whereas in the singlet state, there is a 2-string at $\chi^s = \frac{1}{2}(\chi_1^h + \chi_2^h)$. From (2.7) we have for the triplet and singlet densities σ_1^t and σ_2^s ,

$$\tilde{\sigma}_{1}^{t}(p) = \frac{N_{0}\cos\left(\pi p/c\right)}{2\cosh\left(\frac{1}{2}\pi p\right)} - \frac{e^{-ip\chi_{1}^{n}} + e^{-ip\chi_{2}^{n}}}{1 + e^{-\pi |p|}},$$
(2.11a)

$$\tilde{\sigma}_{1}^{s}(p) = \tilde{\sigma}_{1}^{t}(p) - e^{-(\pi/2)|p|} e^{-ip(\chi_{1}^{h} + \chi_{2}^{h})/2}.$$
(2.11b)

Substitution into (2.4) then yields [1, 7], for $K \rightarrow \infty$,

$$m \sinh \chi_{i}^{h} = \frac{2\pi}{L} J(\chi_{i}^{h}) - \frac{\delta_{ij}^{\kappa}}{L},$$

$$i \neq j = 1, 2, \qquad \kappa = t, s,$$

$$\delta_{12}^{t} = -\delta_{21}^{t} = \int \frac{dp}{p} \frac{\sin (\chi_{1}^{h} - \chi_{2}^{h})p}{1 + e^{\pi |p|}},$$

$$\delta_{12}^{s} = -\delta_{21}^{s} = \delta_{12}^{t} - 2 \tan^{-1} \left(\frac{\chi_{1}^{h} - \chi_{2}^{h}}{\pi}\right).$$
(2.12)

If, say, N_0 is twice an odd integer before cutoff removal, then in the singlet state (with $N_0 = 2M$) $J(\chi_i^h)$ is an integer while in the triplet state (with $N_0 = 2(M+1)$) $J(\chi_i^h)$ is a half odd integer.

The terms in (2.12) which involve the rapidity difference $\chi_i^h - \chi_j^h$ represent a shift in momentum of each particle away from its non-interacting value (a multiple of $2\pi/L$). In refs. [1, 7] this momentum shift was interpreted as the two-particle scattering phase shift. Our main interest in this term is in its contribution to the energy of the two-particle state, as a clue to what we may expect when we put together \mathcal{N} particles in a state of finite density.

We calculate in the c.m. frame (which eliminates, in order L^{-1} , the spurious π/L momentum shifts due to the fact that $J(\chi_i^h)$ is a half-odd-integer in the triplet state). Then, to order L^{-1} ,

$$E = 2\sqrt{p^2 + m^2} + \frac{2p\delta p}{\sqrt{p^2 + m^2}},$$
 (2.13)

where p is an integer multiple of $2\pi/L$ (the same for both triplet and singlet states) and, assuming $\chi_1^h > 0$,

$$\delta p = \begin{cases} -L^{-1} \delta_{12}^{t}, & \text{(triplet)}, \\ -L^{-1} \delta_{12}^{t} + 2L^{-1} \tan^{-1} \left(\frac{2\chi_{1}^{h}}{\pi}\right), & \text{(singlet)} \end{cases}$$

We see that the singlet energy is higher than that of the triplet. The two-body interaction between massive physical particles is color ferromagnetic, in contrast to the color antiferromagnetic bare interaction between pseudoparticles in the Bethe ansatz construction [1, 7]. Combining this result with the absence of *n*-body interaction for N > 2 (the S-matrix is in fact factorizable, i.e. it may be written as a product of 2-body S-matrices), we are led to expect that for a finite density system, $\lim (N/L) = \rho > 0$, the ground state will have all color spins aligned.

3. Finite density states at zero temperature

The results of the preceding section suggest that the ground state of a system of \mathcal{N} massive Gross-Neveu particles is one (more precisely, a multiplet) in which the color spin is maximal, i.e. $\frac{1}{2}\mathcal{N}$. We now turn to the problem of constructing such a state and showing that it indeed minimizes the energy.

The states of maximal color spin are particularly simple, since in them all rapidity parameters χ_{γ} are 1-strings. This can be seen from the normalization condition for the density $\sigma_1(\chi)$ (see (2.7)),

$$M = \int_{-\infty}^{+\infty} d\chi \sigma_1(\chi) + \sum_{n \ge 2} n M_n$$

= $\frac{1}{2} N - \frac{1}{2} M_h + \sum_{n \ge 2} (n-1) M_n$. (3.1)

Expression (3.1) is a minimum for $M_n = 0$, n > 1. In addition, the energy (in the absence of massless excitations)

$$E = \sum_{j=1}^{M_{\rm h}=\mathcal{N}} E_{\rm h}(\chi_j^{\rm h}), \qquad E_{\rm h}(\chi) = m \cosh \chi,$$

will obviously be minimized by choosing a density of holes (i.e. of particle rapidities) of bounded support:

$$\sigma_{1,\mathbf{h}}(\chi) = \begin{cases} \sigma(\chi) , & |\chi| \le b ,\\ 0, & |\chi| > b , \end{cases}$$
(3.2)

where b, the rapidity bound, is related to the Fermi momentum and particle number density (in the c.m. frame) by

$$p_{\rm F} = m \sinh b$$

$$\rho L = \mathcal{N} = \int_{-\infty}^{+\infty} d\chi \sigma_{1,\rm h}(\chi) = \int_{-b}^{b} d\chi \sigma(\chi) . \qquad (3.3)$$

The density $\sigma(\chi)$ of solutions and holes must satisfy, according to (2.6), the integral equation

$$\sigma(\chi) = \frac{1}{4} N_0 \left[\frac{1}{(\frac{1}{2}\pi) + (\chi - \pi/c)^2} + \frac{1}{(\frac{1}{2}\pi)^2 + (\chi + \pi/c)^2} \right] - \int_{-\infty}^{-b} d\chi' \frac{\sigma(\chi')}{\pi^2 + (\chi - \chi')^2} - \int_{b}^{\infty} d\chi' \frac{\sigma(\chi')}{\pi^2 + (\chi - \chi')^2}.$$
(3.4)

By Fourier transformation, eq. (3.4) is easily transformed into the alternative form

$$\sigma(\chi) = \sigma_0(\chi) + \int_{-b}^{b} \mathrm{d}\chi' K(\chi - \chi')\sigma(\chi') , \qquad (3.5)$$

where

$$K(\chi) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{e}^{ip\chi}}{1 + \mathrm{e}^{\pi|p|}},$$
 (3.6)

and

$$\sigma_0(\chi) = \frac{N_0}{4\pi} \left[\frac{1}{\cosh(\chi - \pi/c)} + \frac{1}{\cosh(\chi + \pi/c)} \right] \xrightarrow{K \to \infty} \frac{mL}{2\pi} \cosh \chi$$

is the vacuum distribution [see (2.8)].

The integral equation (3.4) is very similar to that which arose in the treatment by Yang and Yang of the Heisenberg model [9]. Unfortunately, methods of exact solution have not been found for such equations, but approximation methods are possible for asymptotically large or small Fermi momenta. We shall postpone these calculations until sect. 5.

Let us now turn to the question of whether the color ferromagnetic state of lowest energy is in fact the ground state of the finite density system. We shall examine this question from two points of view. First, we shall show immediately that small perturbations of the alleged ground state (i.e. introduction of arbitrary, but non-macroscopic, numbers of non-real χ_{γ}) always raise the energy. Then, in sect. 4, we shall examine thermodynamic states with macroscopic distributions of non-real χ_{γ} . We shall see that the temperature-T equilibrium state approaches the proposed ground state in the limit $T \rightarrow 0$.

Directly from eq. (2.7), we obtain as a generalization of (2.12)

$$\frac{2\pi}{L} J_{f}(\chi) = m \sinh \chi + \frac{1}{L} \sum_{j=1}^{N} \int_{-\infty}^{+\infty} \frac{dp}{p} \frac{\sin (\chi - \chi_{j}^{h})p}{1 + e^{\pi |p|}} - f(\chi) , \qquad (3.7)$$
$$f(\chi) = \frac{2}{L} \sum_{j=1}^{\text{strings}} \tan^{-1} \left(\frac{\chi - \chi_{j}^{s}}{\frac{1}{2}(n_{j} - 1)\pi}\right) .$$

For a fixed number of strings and $L \to \infty$, the last term in (3.7) is $O(L^{-1})$, and we may use (3.7) to calculate the first order shifts $\varepsilon(\chi_{0i}^{h})$ in hole positions when our

proposed ground state (with holes χ_{oi}^{h} between -b and b) is perturbed slightly. Assuming that the (half-) integers $J_{f}(\chi_{fi}^{h})$, $\chi_{fi}^{h} = \chi_{0i}^{h} + \varepsilon(\chi_{oi}^{h})$, are in sequence (otherwise they could be relocated with decrease of energy), we have, to first order in L^{-1} ,

$$J_{\rm f}(\chi + \varepsilon(\chi)) - J_0(\chi) = ({\rm half}) {\rm integer}$$

Inserting (3.7) for both J_t and J_0 , and simplifying with the aid of (3.5), (3.6), we obtain for $-b \leq \chi \leq b$,

$$\phi(\chi) = \int_{-b}^{b} d\chi' K(\chi - \chi') \phi(\chi') + f(\chi) + (\text{half}) \text{ integer}, \qquad (3.8)$$

where

$$\phi(\chi) = \frac{\sigma(\chi)\varepsilon(\chi)}{mL}, \qquad \sigma(\chi) = \text{solution of } (3.5)$$

In terms of $\phi(\chi)$, the energy difference (to order L^0) relative to our proposed ground state is

$$\delta E = m \int_{-b}^{b} d\chi \sinh \chi \sigma(\chi) \varepsilon(\chi)$$
$$= 2m^{2}L \int_{0}^{b} d\chi \sinh \chi \phi^{a}(\chi), \qquad (3.9)$$

where $\phi^{a}(\chi) = \frac{1}{2}[\phi(\chi) - \phi(-\chi)]$, the antisymmetric part of $\phi(\chi)$, satisfies

$$\phi^{a}(\chi) = \int_{-b}^{b} d\chi' K(\chi - \chi') \phi^{a}(\chi') + f^{a}(\chi)$$

=
$$\int_{0}^{b} d\chi' [K(\chi - \chi') - K(\chi + \chi')] \phi^{a}(\chi') + f^{a}(\chi) \qquad (3.10)$$

where

$$f^{a}(\chi) = \frac{1}{2}[f(\chi) - f(-\chi)].$$

Now since $f(\chi)$ is monotonically increasing, we have

$$f^{\mathrm{a}}(\chi) > 0$$
, for $\chi > 0$,

and since

$$K(\chi-\chi')-K(\chi+\chi')>0, \quad \text{for } \chi, \chi'>0,$$

we also have

$$\int_0^b d\chi' [K(\chi - \chi') - K(\chi + \chi')] f^a(\chi') > 0, \quad \text{for } \chi > 0,$$

and, by iteration,

$$\phi^{a}(\chi) > 0$$
, for $\chi > 0$. (3.11)

This establishes the positivity of δE .

4. The zero-temperature limit

To further explore the color ferromagnetism of a gas of massive Gross-Neveu particles, we consider the behavior of the rapidity distribution as the temperature approaches zero. To relieve the anticipated vacuum degeneracy, we assume the presence of a constant color-magnetic field H, pointing in the z-direction in color space and having a non-zero, but arbitrarily small magnitude. The techniques for setting up the finite-temperature formalism are standard [6, 8, 10], so that we shall omit details except where special features of the model appear.

For non-zero temperatures, the state of the system will be a statistical one labeled by densities $\sigma_n(\chi)$, $\sigma_{n,h}(\chi)$ of strings of rank *n* and their holes (in contrast to zero temperature, the numbers of such strings and holes will typically be ∞L). These satisfy resursive equations derived from (2.2),

$$\sigma_n + \sigma_{n,h} = G(\sigma_{n-1,h} + \sigma_{n+1,h}), \qquad n = 1, 2, \dots,$$
 (4.1)

where G is the convolution operator

$$(Gf)(\chi) = \int_{-\infty}^{+\infty} \mathrm{d}\chi' G(\chi - \chi') f(\chi') \,,$$

with kernel

$$G(\chi) = 1/2\pi \cosh \chi, \qquad (4.2)$$

and

$$G\sigma_{0,h} = G\left(\frac{1}{2}N_0\delta\left(\chi - \frac{\pi}{c}\right) + \frac{1}{2}N_0\delta\left(\chi + \frac{\pi}{c}\right)\right)$$
$$= \sigma_0(\chi) .$$

The equilibrium distributions at temperature T can then be obtained by minimizing the thermodynamic functional

$$\Phi = E - TS - \mu \mathcal{N} + \frac{1}{2}H(N - 2M), \qquad (4.3)$$

where

$$E = \int_{-\infty}^{+\infty} \mathrm{d}\chi \sigma_{1,\mathrm{h}}(\chi) E_{\mathrm{h}}(\chi)$$

is the energy functional and

$$S = \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} d\chi [(\sigma_n(\chi) + \sigma_{n,h}(\chi)) \ln (\sigma_n(\chi) + \sigma_{n,h}(\chi)) - \sigma_n(\chi) \ln \sigma_n(\chi) - \sigma_{n,h}(\chi) \ln \sigma_{n,h}(\chi)]$$
(4.4)

is the entropy. The chemical potential μ has been introduced as a Lagrange multiplier, and is to be fixed by the constraint that the mean particle number (integral of $\sigma_{1,h}$) is \mathcal{N} . In varying Φ , the σ_n , $\sigma_{n,h}$ are to be constrained by the linear relations (4.1). The result (see [8, 10]) is a set of recursion relations for

$$\phi_n(\chi) \equiv \ln \eta_n(\chi) \equiv \ln \frac{\sigma_{n,h}(\chi)}{\sigma_n(\chi)}, \qquad (4.5)$$

namely,

$$\phi_1 = \frac{1}{T} \left(\mu - E_{\rm h} \right) + G \ln \left(1 + \eta_2 \right), \qquad (4.6a)$$

$$\phi_n = G \ln (1 + \eta_{n-1})(1 + \eta_{n+1}), \qquad (4.6b)$$

together with the boundary condition for $n \to \infty$,

$$\lim_{n \to \infty} \left[D_n \ln \left(1 + \eta_{n+1} \right) - D_{n+1} \ln \left(1 + \eta_n \right) \right] = \frac{2H}{T}, \qquad (4.6c)$$

with

$$D_n(\chi) = \frac{\frac{1}{2}n}{\left(\frac{1}{2}n\pi\right)^2 + \chi^2}.$$
(4.7)

We observe that the equilibrium free energy $F = E - TS - \mu N$ may be calculated by making use of (4.6). In the limit of infinite ultraviolet cutoff, we obtain the simple and suggestive result,

$$F = -T \frac{mL}{2\pi} \int_{-\infty}^{+\infty} d\chi \cosh \chi \ln (1 + \eta_1(\chi))$$

= $-T \frac{L}{2\pi} \int_{-\infty}^{+\infty} dk \ln (1 + e^{\hat{\phi}_1(k)}),$ (4.8)

where $\hat{\phi}_1(m \sinh \chi) = \phi_1(\chi)$. If the second term on the r.h.s. of (4.6a) were absent, the expression (4.8) would be precisely the free energy of a system of free, spinless, relativistic particles of mass *m* in one dimension.

We are unable to solve eqs. (4.6) exactly. However, we are here only interested in the limit $T \rightarrow 0$, and for this purpose we can satisfy ourselves with some general properties of the solutions of (4.6): (i) $\phi_n(\chi) = \phi_n(-\chi)$ for all *n*, since *G*, *D_n* and *E_h* are all symmetric.

(ii) $\phi_n(\chi) \ge 0$ for $n \ge 2$, since G is positive definite.

(iii) $\phi_n(\chi)$, for all *n*, is a monotonically decreasing function of $|\chi|$, since $-E_h$ is such a function, $dG/d\chi < 0$ for $\chi > 0$ and property (i) is satisfied.

Now, when $|\chi| \to \infty$, $T^{-1}(\mu - E_h) \to -\infty$, $\eta_1(\chi) \to 0$ and all higher η_n approach finite asymptotic values. Since the operators D_n and 2G behave like the identity on constant functions, the asymptotic values of η_n , $\eta_n(\infty)$, satisfy the algebraic system

$$\eta_1(\infty) = 0 , \qquad (4.9a)$$

$$\eta_n^2(\infty) = (1 + \eta_{n+1}(\infty))(1 + \eta_{n-1}(\infty)), \qquad (4.9b)$$

$$\lim_{n \to \infty} \frac{1 + \eta_n(\infty)}{1 + \eta_{n+1}(\infty)} = e^{-2H/T},$$
(4.9c)

whose solution is

$$\eta_n(\infty) = \left[\frac{\sinh(nH/T)}{\sinh(H/T)}\right]^2 - 1$$

~ $e^{2(n-1)H/T}$, for $T \to 0, n \ge 2$. (4.10)

From property (iii) we have

$$\eta_n(\chi) \geq \eta_n(\infty)$$

for all T, and so

$$\eta_n(\chi) \xrightarrow[T \to 0]{} \infty, \qquad n \ge 2,$$
(4.11)

if $H \neq 0$. Comparing with (4.5), we find

$$\sigma_n(\chi) \xrightarrow[T \to 0]{} 0, \qquad n \ge 2, \qquad (4.12)$$

i.e. all complex χ_{γ} disappear in the limit, leaving the gas in a state of maximal color spin.

Result (4.11) allows us to scale out of ϕ_n a trivial temperature dependence. Writing $\phi_n = \psi_n/T$, eqs. (4.6) are transformed into

$$\psi_{1} = \mu - E_{h} + G\psi_{2},$$

$$\psi_{2} = G(\psi_{1}^{(+)} + \psi_{3}),$$

$$\psi_{n} = G(\psi_{n+1} + \psi_{n-1}), \qquad n \ge 3,$$

$$\lim_{n \to \infty} (D_{n}\psi_{n+1} - D_{n+1}\psi_{n}) = 2H,$$

(4.13)

where

$$\psi_1^{(+)}(\chi) = \begin{cases} \psi_1(\chi), & \text{if } \psi_1(\chi) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

These may be solved by Fourier transformation:

$$\tilde{\psi}_{n} = \tilde{\psi}_{1}^{(+)} e^{-(n-1)\pi|p|/2} + 4\pi nH\delta(p), \qquad n \ge 2,
\tilde{\psi}_{1} = 2\pi(\mu+H)\delta(p) - \tilde{E}_{h} + (1+e^{\pi|p|})^{-1}\tilde{\psi}_{1}^{(+)}.$$
(4.14)

Going back to χ -space, and using property (iii) for a last time, we obtain for $\psi_1(\chi)$

$$\psi_1(\chi) = \mu + H - E_h(\chi) + \int_{-b}^{b} d\chi' K(\chi - \chi') \psi_1(\chi'), \qquad (4.15)$$

where [-b, b] is the interval over which ψ_1 is positive. Notice that if $\mu + H < E_h(0) = m$, the solution of (4.15) is

$$\psi_1(\chi) = \mu + H - E_{\rm h}(\chi) < 0,$$

for all χ . Then $\lim_{T\to 0} \sigma_{1,h}(\chi) = 0$, i.e. there are no massive particles present, inconsistent with hypothesis. Thus, $\mu + H > m$. Substituting (4.12) into (4.1), one easily finds

$$\tilde{\sigma}_{n,h} = \tilde{\sigma}_{1,h} e^{-(n-1)\pi|p|/2}, \quad n \ge 2,$$

$$\tilde{\sigma}_{1,h} + \tilde{\sigma}_1 = \tilde{\sigma}_0 + \tilde{G}\tilde{\sigma}_{2,h} = \frac{\tilde{\sigma}_{1,h} e^{-\pi|p|/2}}{2\cosh\frac{1}{2}\pi p}.$$
(4.16)

From (4.15) we have

$$\sigma_{1,h}(\chi) = 0, \quad \text{for } \chi^2 > b^2,$$

$$\sigma_1(\chi) = 0, \quad \text{for } \chi^2 < b^2,$$

and so in χ -space, (4.16) may be written as an integral equation for $\sigma = \sigma_1 + \sigma_{1,h}$:

$$\sigma(\chi) = \sigma_0(\chi) + \int_{-b}^{b} d\chi' K(\chi - \chi') \sigma(\chi'), \qquad (4.17)$$

which we recognize as nothing but (3.5).

5. Asymptotic freedom at high density

Eq. (3.5) is a Fredholm equation of the second kind with symmetric kernel; since $K(\chi)$ is positive and

$$\int_{-\infty}^{\infty} \mathrm{d}\chi K(\chi) = \frac{1}{2} < 1 ,$$

there exists a unique solution which may be computed by interation (for $b \ll 1$, this gives a perturbative series in increasing powers of b). In this section we shall instead be interested in the high density regime (where $\rho \gg m$ and asymptotic freedom is expected to set in), and there $b \gg 1$. In this case it is possible to obtain an asymptotic expansion by adapting to our needs a method developed by Yang and Yang [9].

A direct application of the method of ref. [9] is not possible, because of the large- χ growth of the inhomogeneous term (for $K \to \infty$)

$$\sigma_0(\chi) = \frac{mL}{2\pi} \cosh \chi \,.$$

We therefore rewrite (3.5) as an equation for

$$\psi(\chi) \equiv \frac{2\pi}{mL} \sigma(\chi) - \cosh \chi , \qquad (5.1)$$

so that

$$\psi(\chi) = \int_{-b}^{b} d\chi' K(\chi - \chi') [\cosh \chi' + \psi(\chi')]. \qquad (5.2)$$

Setting

$$\psi_{\rm in}(\chi) = \begin{cases} \psi(\chi) , & \chi^2 < b^2 , \\ 0 , & \chi^2 > b^2 ; \end{cases}$$
$$\psi_{\rm out}(\chi) = \begin{cases} 0 , & \chi^2 < b^2 , \\ \psi(\chi) , & \chi^2 > b^2 ; \end{cases}$$
$$h(\chi) = \begin{cases} \cosh \chi , & \chi^2 < b^2 , \\ 0 , & \chi^2 > b^2 ; \end{cases}$$

and Fourier transforming, (5.2) becomes

$$\tilde{\psi}_{\rm in}(p) = e^{-\pi |p|} \tilde{h}(p) - (1 + e^{-\pi |p|}) \tilde{\psi}_{\rm out}(p) , \qquad (5.3)$$

or, back in χ -space [using $\psi(\chi) = \psi(-\chi)$],

$$\psi(\chi) = \int_{-b}^{b} d\chi' \frac{\cosh \chi'}{\pi^2 + (\chi - \chi')^2} - \int_{0}^{\infty} d\chi' \psi(\chi') \left[\frac{1}{\pi^2 + (\chi - \chi')^2} + \frac{1}{\pi^2 + (\chi + \chi')^2} \right].$$
(5.4)

We now introduce the translated function

$$\rho(\chi) = \psi(\chi + b), \qquad (5.5)$$

as well as

$$\rho_{+}(\chi) = \begin{cases} \rho(\chi), & \chi > 0, \\ 0, & \chi < 0, \end{cases} \qquad \rho_{-}(\chi) = \begin{cases} 0, & \chi > 0, \\ \rho(\chi), & \chi < 0, \end{cases}$$
(5.6)

so that

$$\psi_{\text{out}}(\chi) = \begin{cases} \rho_{+}(\chi - b), & \chi > 0, \\ \rho_{+}(-\chi - b), & \chi < 0. \end{cases}$$
(5.7)

Since ψ_{in} , by (5.3), may be written in terms of ψ_{out} , we see that only $\rho_+(\chi)$ need be calculated. The function $\rho(\chi)$ satisfies

$$\rho(\chi) = \int_{-\infty}^{\infty} d\chi' \frac{h(\chi'+b)}{\pi^2 + (\chi-\chi')^2} - \int_{0}^{\infty} d\chi' \frac{\rho(\chi')}{\pi^2 + (\chi-\chi')^2} + \int_{0}^{\infty} d\chi' \frac{\rho(\chi')}{\pi^2 + (\chi+\chi'+2b)^2}.$$
(5.8)

It is to (5.8) that we apply the Yang-Yang approach, writing

$$\rho = \rho_0 + \rho_1 + \rho_2 + \cdots,$$

$$\rho_0(\chi) = \int_{-\infty}^{+\infty} d\chi' \frac{h(\chi' + b)}{\pi^2 + (\chi - \chi')^2} - \int_0^\infty d\chi' \frac{\rho_0(\chi')}{\pi^2 + (\chi - \chi')^2},$$
(5.9)

$$\rho_n(\chi) = -\int_0^\infty d\chi' \frac{\rho_{n-1}(\chi')}{\pi^2 + (\chi + \chi' + 2b)^2} - \int_0^\infty d\chi' \frac{\rho_n(\chi')}{\pi^2 + (\chi - \chi')^2}, \qquad n \ge 1,$$
(5.10)

with

$$\frac{\rho_{n+1,+}}{\rho_{n,+}} \sim \mathcal{O}\left(\frac{1}{b}\right).$$

In appendix A, the Wiener-Hopf factorization technique [11] is used to solve the system (5.9), (5.10). We find, collecting (5.3), (5.7) and (A.9a, b),

$$\psi_{\rm in}(\chi) = \int_{-b}^{b} \mathrm{d}\chi' \frac{\cosh\chi'}{\pi^2 + (\chi - \chi')^2} - \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}p \, \cos p\chi \, K_+(-p) \, \mathrm{e}^{-ipb} F(p) \,, \quad (5.11)$$

where

$$F(p) = \sum_{n=0}^{\infty} F_n(p) ,$$

$$F_0(p) = -\frac{i}{\pi} \int_0^{\infty} dt \sin \pi t K_-(it) e^{-bt} \frac{\tilde{h}(it)}{p - it}, \qquad (5.12a)$$

$$F_n(p) = \frac{i}{\pi} \int_0^\infty dt \sin \pi t \left[K_{-}(it) e^{-bt} \right]^2 \frac{F_{n-1}(-it)}{p-it}, \qquad n \ge 1, \qquad (5.12b)$$

and K_{\pm} are defined in appendix A.

One can check explicitly that the *n*th approximation in (5.11) produces a function $\psi_{in}^{(n)}$ which vanishes outside of [-b, b] such that

$$\frac{\psi_{\rm in}^{(n)}}{\psi_{\rm out}^{(n)}} = \mathcal{O}(b^{-(n+1)}), \qquad \chi^2 > b^2.$$

.

Further progress in evaluating (5.11) can be obtained by throwing away a piece of $e^{-bt}\tilde{h}(it)$ which vanishes like e^{-b} and hence is completely negligible for $b \gg 1$. In particular,

$$e^{-bt}\tilde{h}(it) = \frac{1}{2}e^{b}\left(\frac{1}{1+t} + \frac{e^{-2bt}}{1-t}\right) + O(e^{-b}).$$
 (5.13)

Inserting (5.13) in (5.12) and using some tricks explained in appendix B, one obtains, finally,

$$\sigma(\chi) = \frac{mL}{2\pi} g(\chi), \qquad \chi^2 < b^2,$$

$$g(\chi) = \frac{e^b}{\sqrt{\pi e}} \int_0^\infty dt \sin \pi t \, K_-(it) \, e^{-bt} \cosh \chi t \, \sum_{n=0}^\infty Z_n(t),$$

$$Z_0(t) = \frac{1}{1-t},$$
(5.15a)

$$Z_n(t) = -\frac{1}{\pi} \int_0^\infty dt' \sin \pi t' \left[K_-(it') e^{-bt'} \right]^2 \frac{Z_{n-1}(t')}{t+t'}, \qquad n \ge 1, \qquad (5.15b)$$

We are now in a position to calculate explicitly the asymptotic expansions, in powers of b^{-1} and $\ln b$, of the density and energy,

$$\rho = \frac{\mathcal{N}}{L} = \frac{1}{L} \int_{-b}^{b} d\chi \sigma(\chi) ,$$

$$E = m \int_{-b}^{b} d\chi \sigma(\chi) \cosh \chi .$$
(5.16)

Fortunately each of the integrals which we need to evaluate for $b \gg 1$ is either (a) a Laplace transform whose asymptotic behavior may be determined by expanding the integrand near t = 0, or (b) an integral which may be calculated exactly by contour integration, as illustrated in appendix B. The resursive structure (5.15) makes the bookkeeping of all corrections quite transparent. Our results are:

$$\rho = \frac{m}{2\pi} \sqrt{\frac{2\pi}{e}} e^{b} \left\{ 1 - \frac{1}{4b} - \frac{3}{32b^2} - \frac{\ln b}{8b^2} + O(b^{-3} \ln^2 b) \right\}$$
(5.17)

$$E = \frac{m^2 L}{8\pi} \left(\frac{2\pi}{e}\right) e^{2b} \left\{ 1 + \frac{1}{2b^2} + O(b^{-3}) \right\}.$$
 (5.18)

To obtain the relation between energy and density, we may solve (5.17) for b,

$$b = \frac{1}{2} \ln 2\pi + \frac{1}{2} + \ln \frac{\rho}{m} + \frac{1}{4 \ln (\rho/m)} - \frac{\ln 2\pi}{8 \ln^2 (\rho/m)} \frac{\ln \ln (\rho/m)}{8 \ln^2 (\rho/m)} + O\left(\frac{\ln^2 \ln (\rho/m)}{\ln^3 (\rho/m)}\right),$$
(5.19)

and substitute into (5.18) to obtain

$$E = \frac{1}{4}\pi\rho^{2}L\left\{1 + \frac{1}{4\ln(\rho/m)} + \frac{5-2\ln 2\pi}{8\ln^{2}(\rho/m)} + \frac{\ln\ln(\rho/m)}{8\ln^{2}(\rho/m)} + O\left(\frac{\ln^{2}\ln(\rho/m)}{\ln^{3}\rho/m}\right)\right\}.$$
 (5.20)

Recalling that for large b, the Fermi momentum p_F is proportional to e^b , we note that in (5.17) the density ρ is proportional, asymptotically, to p_F , whereas in (5.18) E is quadratic in p_F . These relations suggest an approach to a system of non-interacting particles. In addition, there appears to be a renormalization of the effective mass, from m to $m\sqrt{2\pi/e}$, which is reminiscent of the relation between low and high magnetic field dimensional scales in the Kondo model [6].

One must be careful to interpret properly the asymptotic freedom which is incorporated in (5.17), (5.18) and (5.20). For example, one would not expect to find nearly free protons and neutrons in hadronic matter of sufficiently high density; rather one would look for the asymptotically free quark and gluon constituents of the nucleons. By the same token, we should not expect that the massive composite particles of the CGN model behave for high densities like free particles. And in fact they do not: the distribution of rapidities $\sigma(\chi)$ given by (5.14) may be investigated using asymptotic analysis for $\chi \ll b$ and $\chi \approx b$, and numerical integration for all $\chi \in [-b, b]$, and one finds little resemblance of this function (even near the Fermi surface) to the free-particle distribution proportional to $\cosh \chi$. One consequence of this is that the numerical coefficient in the expressions (5.18) and (5.20) is only one-half what it should be for a gas of $\mathcal{N} = \rho L$ free relativistic fermions, all with the same color.

In fact, what one is seeing in the formulas for $\sigma(\chi)$, $E(\rho)$, etc. is evidence for the pseudoparticle structure of the massive CGN particles; i.e. the system behaves, for asymptotically large densities, as if it consisted of N_{eff} massless fermions, in a state of total color spin $\frac{1}{2}\rho L$, interacting via the bare CGN interaction with an effective coupling parameter c_{eff} , which vanishes proportional to the inverse off the rapidity bound b. More precisely, we may set

$$\frac{\pi}{c_{\rm eff}} = \lambda b ,$$

with $\lambda > 1$. The value of λ is not crucial; once it is fixed, however, the number of pseudoparticles is determined by the renormalization relation

$$N_{\rm eff} = \frac{1}{2}mL \ {\rm e}^{\pi/c_{\rm eff}}$$

One can now compute $\sigma(\chi)$ using (3.5) and (3.6), replacing N_0 and c with N_{eff} and c_{eff} , respectively. The analysis leading to (5.14) is now repeated, except that $h(\chi)$ in (5.2) is replaced, for $|\chi| < b$, by

$$\frac{N_{\rm eff}}{2mL} \left[\frac{1}{\cosh\left(\chi - \pi/c_{\rm eff}\right)} + \frac{1}{\cosh\left(\chi + \pi/c_{\rm eff}\right)} \right].$$

The replacement simply leads to additional correction terms in (5.13) of order $e^{-(\lambda-1)b}$, which are negligible for sufficiently large b. Hence, expression (5.14) for $\sigma(\chi)$, as well as (5.17)-(5.20) for ρ and E, will be unchanged.

As an extra check of the initial term in (5.20), we may use the simpler language of many-body theory for N_{eff} non-interacting massless fermions. The color-singlet ground state of the system will have right- and left-moving Dirac seas of doubly occupied levels, extending in energy from zero down to $-K_{\text{eff}} = -\pi N_{\text{eff}}/2L$, with the separation between successive levels exactly equal to $2\pi/L$. The lowest energy state of color spin $\frac{1}{2}\rho L$ can then be obtained by converting, for each sign of the chirality, $\frac{1}{4}\rho L$ negative color spin pseudoparticles below sea level into an equal number of positive color spin pseudoparticles above sea level, with a net increase of energy

$$E = 4(1+2+\cdots+\frac{1}{4}\rho L)\frac{2\pi}{L} = \frac{1}{4}\rho^2 L,$$

in agreement with (5.20).

Appendix A

SOLUTION OF EQS. (5.9) AND (5.10)

Fourier transformation of (5.9), (5.10) yields

$$(1 + e^{-\pi |p|})\tilde{\rho}_{0,+}(p) + \tilde{\rho}_{0,-}(p) = e^{ipb} e^{-\pi |p|}\tilde{h}(p), \qquad (A.1a)$$

$$(1 + e^{-\pi |p|})\tilde{\rho}_{n,+}(p) + \tilde{\rho}_{n,-}(p) = -e^{2ipb} e^{-\pi |p|}\tilde{\rho}_{n-1,+}(-p), \qquad n \ge 1.$$
 (A.1b)

By definition, $\tilde{\rho}_{n,+}(p)$ ($\tilde{\rho}_{n,-}(p)$) is analytic in the half-plane Im p < 0 (Im p > 0). In addition, we do not expect $\rho_{n,\pm}(\chi)$ to become singular for any real χ , and hence we assume $\tilde{\rho}_{n,\pm}(p) \rightarrow 0$ for $|p| \rightarrow \infty$ on the real axis.

Guided by the formula

$$e^{-\pi |p|} = \frac{1}{\pi} \int_0^\infty dt \sin \pi t \left[\frac{i}{p+it} - \frac{i}{p-it} \right],$$

we write

$$\tilde{\rho}_{n,\pm}(p) = \frac{1}{\pi} \int_0^\infty dt \sin \pi t \phi_{n,\pm}(p,t) , \qquad (A.2)$$

where $\phi_{n,\pm}$ (not uniquely defined yet) may be assumed to have the same analyticity properties in p as $\tilde{\rho}_{n,\pm}$. Eqs. (A.1a, b) are satisfied by $\tilde{\rho}_{n,\pm}$ if $\phi_{n,\pm}$ satisfies

$$(1+e^{-\pi|p|})\phi_{0,+}(p,t)+\phi_{0,-}(p,t)=\left[\frac{i}{p+it}-\frac{i}{p-it}\right]e^{ipb}\tilde{h}(p), \qquad (A.3a)$$

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$$(1 + e^{-\pi |p|})\phi_{n,+}(p,t) + \phi_{n,-}(p,t) = -\frac{1}{\pi} \left[\frac{i}{p+it} - \frac{i}{p-it} \right] e^{2ipb} \int_0^\infty dt \sin \pi t \, \phi_{n-1,+}(-p,t) \,.$$
(A.3b)

Following the Wiener-Hopf technique [11] we factorize the kernel $1 + e^{-\pi |p|}$:

$$1 + e^{-\pi |p|} = \frac{K_{+}(p)}{K_{-}(p)}$$
(A.4)

$$K_{+}(p) = \sqrt{2\pi} \frac{\exp\left\{-\frac{1}{2}ip\left[1 + \frac{1}{2}i\pi - \ln\left(-\frac{1}{2}p + i0\right)\right]\right\}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}ip\right)},$$
(A.5)

$$K_{-}(p) = \frac{1}{K_{+}(-p)}.$$
 (A.6)

The function $K_+(p)(K_-(p))$ is analytic and free of zeros for Im p < 0 (Im p > 0) and both tend to constants for $|p| \rightarrow \infty$. We also use the following decompositions:

r.h.s. of (A.3a) =
$$q_{0,+}(p, t) + q_{0,-}(p, t)$$
,
r.h.s. of (A.3b) = $q_{n,+}(p, t) + q_{n,-}(p, t)$, $n \ge 1$,

where

$$q_{0,+}(p,t) = -iK_{-}(it) e^{-bt}\tilde{h}(it)\frac{1}{p-it},$$

$$q_{0,-}(p,t) = iK_{-}(p) e^{ipb}\tilde{h}(p)\frac{1}{p+it} - i[K_{-}(p) e^{ipb}\tilde{h}(p) - K_{-}(it) e^{-bt}h(it)]\frac{1}{p-it},$$

$$q_{n,+}(p,t) = iK_{-}(it) e^{-2bt}\tilde{\rho}_{n-1,+}(-it)\frac{1}{p-it},$$

$$q_{n,-}(p,t) = -iK_{-}(p) e^{2ipb}\tilde{\rho}_{n-1,+}(-p)$$

$$+ i[K_{-}(p) e^{2ipb}\tilde{\rho}_{n-1,+}(-p) - K_{-}(it) e^{-2bt}\tilde{\rho}_{n-1,+}(-it)]\frac{1}{p-it}.$$

By construction, all $q_{n,+}(p, t)$ $(q_{n,-}(p, t))$ are analytic in the half-plane Im p < 0 (Im p > 0) and $q_{n,+}(p, t) \rightarrow 0$ for $|p| \rightarrow \infty$ along the real axis. Eqs. (A.3a, b) can now be written, for all n, as

$$K_{+}(p)\phi_{n,+}(p,t) - q_{n,+}(p,t) = -K_{-}(p)\phi_{n,-}(p,t) + q_{n,-}(p,t).$$
(A.7)

The complementary analyticity domains and behavior for $|p| \rightarrow \infty$ imply that both sides of (A.7) vanish. Thus

$$\phi_{n,\pm}(p,t) = \frac{q_{n,\pm}(p,t)}{K_{\pm}(p)},$$
(A.8)

and so for $\tilde{\rho}_+(p) = \sum_{n=0}^{\infty} \tilde{\rho}_{n,+}(p)$,

$$\tilde{\rho}_{0,+}(p) = \frac{-i}{\pi K_{+}(p)} \int_{0}^{\infty} dt \sin \pi t \, K_{-}(it) \, e^{-bt} \frac{\tilde{h}(it)}{p-it}, \qquad (A.9a)$$

$$\tilde{\rho}_{n,+}(p) = \frac{i}{\pi K_{+}(p)} \int_{0}^{\infty} dt \sin \pi t \, K_{-}(it) \, e^{-2bt} \frac{\tilde{\rho}_{n-1,+}(-it)}{p-it} \,. \tag{A.9b}$$

Appendix B

DERIVATION OF EQ. (5.14)

In (5.11), we must evaluate the integral

$$I(t,\chi) = \int_{-\infty}^{+\infty} \mathrm{d}p \, \cos p\chi \, K_+(-p) \, \frac{\mathrm{e}^{-ipb}}{p-it} \tag{B.1}$$

for t > 0 and $|\chi| < b$. The integrand, as an analytic function of p, has a pole at *it*, a cut (say) along the negative imaginary axis, and exponential fall-off in the lower half-plane. Deforming the integration contour so that it comes from and goes to infinity along opposite sides of the cut, we have

$$I(t, \chi) = \int_{0}^{\infty} dx \ e^{-bx} \frac{\cosh \chi x}{x+t} [K_{+}(ix-\varepsilon) + K_{+}(ix+\varepsilon)]$$

$$= \sqrt{2\pi} \int_{0}^{\infty} dx \ e^{-bx} \frac{\cosh \chi x}{x+t} \cdot \frac{e^{(x+i\pi)/2}}{\Gamma(\frac{1}{2}+\frac{1}{2}x)} (e^{-x/2\ln(x/2+i\varepsilon)} - e^{-x/2\ln(x/2-i\varepsilon)})$$

$$= 2i\sqrt{2\pi} \int_{0}^{\infty} dx \ e^{-bx} \frac{\cosh \chi x}{x+t} \cdot \frac{e^{x/2(1-\ln x/2)}}{\Gamma(\frac{1}{2}+\frac{1}{2}x)} \sin \frac{1}{2}\pi x$$

$$= i\frac{\sqrt{2\pi}}{\pi} \int_{0}^{\infty} dx \ e^{-bx} \frac{\cosh \chi x}{x+t} e^{x/2(1-\ln x/2)} \Gamma(\frac{1}{2}+\frac{1}{2}x) \sin \pi x$$

$$= 2i \int_{0}^{\infty} dx \sin \pi x \ e^{-bx} \cosh \chi x \frac{K_{-}(ix)}{x+t}.$$
(B.2)

Using this result, eq. (5.11) can be transformed into

$$\psi_{\rm in}(\chi) = \int_{-b}^{b} d\chi' \frac{\cosh \chi'}{\pi^2 + (\chi - \chi')^2} - \frac{2}{\pi} \int_{0}^{\infty} dt \sin \pi t \, e^{-bt} \cosh \chi t \, K_{-}(it) \sum_{n=0}^{\infty} Q_n(t) \,,$$

re (B.3)

where

$$Q_{0}(t) = \frac{1}{\pi} \int_{0}^{\infty} dt' \sin \pi t' K_{-}(it') e^{-bt'} \frac{\tilde{h}(it')}{t+t'},$$
(B.4)

$$Q_n(t) = -\frac{1}{\pi} \int_0^\infty dt' \sin \pi t' \left[K_{-}(it') e^{-bt'} \right]^2 \frac{Q_{n-1}(t')}{t+t'}, \qquad n \ge 1.$$

It is now convenient to define new functions $Z_n(t)$ and $W_n(t)$ by

$$Z_{0} = \frac{1}{1-t},$$

$$Z_{n}(t) = -\frac{1}{\pi} \int_{0}^{\infty} dt' \sin \pi t' \left[K_{-}(it') e^{-bt'} \right]^{2} \frac{Z_{n-1}(t')}{t+t'}, \quad n \ge 1,$$
(B.5)

$$W_{0}(t) = \frac{K + (-t)}{1 - t} = \frac{1}{(1 - t)K_{-}(it)},$$

$$W_{n}(t) = -\frac{1}{\pi} \int_{0}^{\infty} dt' \sin \pi t' \left[K_{-}(it') e^{-bt'} \right]^{2} \frac{W_{n-1}(t')}{t + t'}, \quad n \ge 1,$$
(B.6)

We now evaluate $Q_0(t)$ using approximation (5.13).

.

$$Q_0(t) = \frac{e^b}{2\pi} \int_0^\infty dt' \sin \pi t' \frac{K_{-}(it)}{t+t'} \left(\frac{1}{1+t'} + \frac{e^{-2bt'}}{1-t'}\right).$$
(B.7)

For the first of the two terms in (B.7), we can perform the converse of the contour deformation that led to (B.2), to obtain

$$\int_{0}^{\infty} dt \sin \pi t' \frac{K_{-}(it')}{(t+t')(1+t')} = \frac{1}{2i} \int_{-\infty}^{+\infty} dp \frac{K_{+}(-p)}{(p-it)(1+ip)}.$$
 (B.8)

But now the contour can be closed in the upper half-plane giving us two pole contributions:

$$(B.8) = \frac{\pi}{1-t} \left(K_+(-it) - K_+(-i) \right) = \pi \left(W_0(t) - \sqrt{\frac{\pi}{e}} Z_0(t) \right). \tag{B.9}$$

Comparing (B.7), (B.9) and (B.6), we now get

$$Q_0(t) = \frac{1}{2} e^b \left[W_0(t) - W_1(t) - \sqrt{\frac{\pi}{e}} Z_0(t) \right]$$
(B.10)

and, since Q_n , W_n and Z_n all have the same recursive definition,

$$Q_n(t) = \frac{1}{2} e^b \left[W_n(t) - W_{n+1}(t) - \sqrt{\frac{\pi}{e}} Z_n(t) \right].$$
(B.11)

Therefore

$$\sum_{n=0}^{\infty} Q_n(t) = \frac{1}{2} e^b \left[W_0(t) - \sqrt{\frac{\pi}{e}} \sum_{n=0}^{\infty} Z_n(t) \right].$$
(B.12)

For large b, the finite integral over χ' in (B.3) can be approximated by

$$\int_{-b}^{b} d\chi' \frac{\cosh \chi'}{\pi^{2} + (\chi - \chi')^{2}} = \frac{1}{\pi} \int_{0}^{\infty} dt \sin \pi t \int_{-b}^{b} d\chi' e^{-|\chi - \chi'|t} \cosh \chi'$$
$$\cong -\cosh \chi + \frac{e^{b}}{\pi} \int_{0}^{\infty} dt \sin \pi t e^{-bt} \frac{\cosh \chi t}{1 - t} + O(e^{-b}) .$$

Inserting this and (B.12) into (B.3), one finally gets

$$\psi_{\rm in}(\chi) \cong -\cosh \chi + \frac{e^b}{\sqrt{\pi e}} \int_0^\infty {\rm d}t \sin \pi t \ e^{-bt} \cosh \chi t \ K_-(it) \sum_{n=0}^\infty Z_n(t) \ ,$$

which, together with definition (5.1), reproduces the result (5.14).

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