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ANALYSIS OF THE BETHE-ANSATZ EQUATIONS OF THE CHIRAL-INVARIANT GROSS-NEVEU MODEL*

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The Bethe-ansatz equations of the chiral-invariant Gross-Neveu model are reduced to a simple form in which the parameters of the vacuum solution have been eliminated. The resulting system of equations involves only the rapidities of physical particles and a minimal set of complex parameters needed to distinguish the various internal symmetry states of these particles. The analysis is performed without invoking the time-honored assumption that the solutions of the Bethe-ansatz equations, in the infinite-volume limit, are comprised entirely of strings ("bound states"). Surprisingly, it is found that the correct description of the n-particle states involves no strings of length greater than two (except for special values of the momenta).

1. Introduction

The problem of diagonalizing the hamiltonian of the SU(2) chiral-invariant Gross-Neveu model has been shown [1–3] to be equivalent to finding the self-consistent solutions $\chi_1,\ldots,\chi_M$ and "holes" $\chi_1^{\mathrm{h}},\ldots,\chi_{M^{\mathrm{h}}}^{\mathrm{h}}$ of the system of coupled equations

$$\prod_{j=1}^{2}\left(\frac{i(\alpha_j-\chi)+\frac{1}{2}\pi}{i(\alpha_j-\chi)-\frac{1}{2}\pi}\right)^{N_j}=-\prod_{\beta=1}^{M}\frac{i(\chi_\beta-\chi)+\pi}{i(\chi_\beta-\chi)-\pi},$$

$$\chi\in\{\chi_\gamma:\gamma=1,\ldots,M\}\cup\{\chi_\delta^{\mathrm{h}}:\delta=1,\ldots,M^{\mathrm{h}}\},\tag{1.1}$$

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where $\alpha_1 = \pi/c = -\alpha_2$. Here c is a bare coupling parameter, which vanishes logarithmically when the ultraviolet cut-off K is removed, and

$$N = N_1 + N_2, \qquad Q^5 = N_1 - N_2$$

are the bare charge and chiral charge, respectively.

In ref. [1], the vacuum state (with all real χ_γ and no real holes) and low-lying excited states were studied. It was found that a state describing n particles of color-spin one-half, non-zero mass m, and rapidities $\theta_1, \ldots, \theta_n$ corresponds to a solution of (1.1) possessing precisely n real holes, located at $\chi = \theta_i$, $i = 1, \ldots, n$, provided that the limit of infinite K is taken with fixed

$$m = (4K/\pi)\exp(-\pi/c).$$

The interparticle interactions are reflected in departures of the particle momenta from integer multiples of $2\pi/L$, where L is the length of the re-entrant line segment traversed by the particles. In fact, by studying the $O(1/L)$ momentum shifts of the two-body states, it has been possible to calculate [4] the factorizable S-matrix of the model.

Prior to the present investigation, it has been assumed that the generalization of the results of [1] to describe the scattering states of arbitrarily many massive particles would be completely straightforward, thanks to the so-called string hypothesis, according to which the solutions of (1.1) (and the closely related equations which arise in some models of one-dimensional magnetic systems) consist, in the limit of infinite N or infinite L, solely of strings, i.e. families

$$\psi + ik\pi, \qquad k = -\tfrac{1}{2}(r-1), -\tfrac{1}{2}(r-1)+1, \ldots, \tfrac{1}{2}(r-1), \qquad \psi \text{ real}.$$

This conjecture has a long history, going back fifty years to Bethe's treatment [5] of the one-dimensional Heisenberg spin chain. Although the hypothesis has never been rigorously proven, there do exist highly plausible consistency checks [6–11]. In addition, the string conjecture has been applied so many times (we shall not attempt to make a list), never encountering contradictions or physically implausible results, that few question its validity.

The main purpose of the present article is to study systematically the n-particle scattering states of the chiral-invariant Gross-Neveu model *without* assuming from the beginning the string hypothesis. One of our principal results is that strings of length greater than two are *not* typical of the solutions of (1.1) (although they may occur accidentally for special choices of the particle rapidities). They become a good approximation only when n tends to infinity. The latter assumption is appropriate to the calculations of thermodynamic quantities at non-zero temperature in magnetic systems (so that the use of the hypothesis in such cases remains valid), but not to the study of particle physics, nor to the strictly zero-temperature states of an antiferromagnetic magnetic system.

The paper is organized as follows. In sect. 2, it will be shown that in the infinite cut-off limit, states of n massive particles with given rapidities $\theta_1,\dots,\theta_n$ and total color spin $\frac{1}{2}n - \hat{M}$ may be parametrized by $\hat{M}$ complex parameters $\hat{\chi}_\gamma$ which satisfy a system of coupled equations formally identical to those which arise in the model [12, 13] of a one-dimensional gas of non-relativistic fermions with a delta-function interaction. The simplest non-string solutions will be discussed in sect. 3. Then, in sect. 4, we shall show that the momentum shifts due to the interaction of pairs of particles, as well as the equations for $\hat{\chi}_\gamma$ derived in sect. 2, may be elegantly summarized in a simple eigenvalue problem involving the 2-body S-matrix. At the end of sect. 4, we shall interpret our results in terms of a higher-level x-space Bethe ansatz.

2. Eliminating the vacuum parameters from the Bethe-ansatz equations

In order to study the solutions of (1.1) in detail, it is convenient to distinguish three main categories of χ_γ:

$$\begin{aligned}
&\text{real } \chi_\gamma: && \lambda_k, && k=1,2,\dots,M_\mathrm{r}, && \\
&\text{close pair:} && \xi_l \pm i\eta_l, && l=1,\dots,M_\mathrm{cp}, && 0<\eta_l<\pi, \\
&\text{wide pair:} && \zeta_r \pm i\omega_r, && r=1,\dots,M_\mathrm{wp}, && \omega_r \geqslant \pi.
\end{aligned} \tag{2.1}$$

With this notation (which we shall employ consistently throughout the following), eq. (1.1) assumes the form

$$\begin{aligned}
\prod_{j=1}^{2}\left(\frac{i(\alpha_j-\lambda)+\frac{1}{2}\pi}{i(\alpha_j-\lambda)-\frac{1}{2}\pi}\right)^{N_j} &= -\prod_{k=1}^{M_\mathrm{r}}\frac{i(\lambda_k-\lambda)+\pi}{i(\lambda_k-\lambda)-\pi}\prod_{l=1}^{M_\mathrm{cp}}\frac{i(\xi_l-\lambda)-\eta_l+\pi}{i(\xi_l-\lambda)-\eta_l-\pi} \\
&\quad\times\frac{i(\xi_l-\lambda)+\eta_l+\pi}{i(\xi_l-\lambda)+\eta_l-\pi}\prod_{r=1}^{M_\mathrm{wp}}\frac{i(\zeta_r-\lambda)-\omega_r+\pi}{i(\zeta_r-\lambda)-\omega_r-\pi}\,\frac{i(\zeta_r-\lambda)+\omega_r+\pi}{i(\zeta_r-\lambda)+\omega_r-\pi},
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\prod_{j=1}^{2}\left(\frac{i(\alpha_j-\sigma)+\tau+\frac{1}{2}\pi}{i(\alpha_j-\sigma)+\tau-\frac{1}{2}\pi}\right)^{N_j} &= -\prod_{k=1}^{M_\mathrm{r}}\frac{i(\lambda_k-\sigma)+\tau+\pi}{i(\lambda_k-\sigma)+\tau-\pi} \\
&\quad\times\prod_{l=1}^{M_\mathrm{cp}}\frac{i(\xi_l-\sigma)+\tau-\eta_l+\pi}{i(\xi_l-\sigma)+\tau-\eta_l-\pi}\,\frac{i(\xi_l-\sigma)+\tau+\eta_l+\pi}{i(\xi_l-\sigma)+\tau+\eta_l-\pi} \\
&\quad\times\prod_{r=1}^{M_\mathrm{wp}}\frac{i(\zeta_r-\sigma)+\tau-\omega_r+\pi}{i(\zeta_r-\sigma)+\tau-\omega_r-\pi}\,\frac{i(\zeta_r-\sigma)+\tau+\omega_r+\pi}{i(\zeta_r-\sigma)+\tau+\omega_r-\pi},
\end{aligned} \tag{2.3}$$

where $\sigma \pm i\tau$ denotes either a close pair $\xi \pm i\eta$ or a wide pair $\zeta \pm i\omega$, and λ denotes a real χ_γ or hole θ.

Our first task is to obtain an expression for the density of real χ_γ (that such a density is meaningful is due to the fact that, after the ultraviolet cut-off has been removed, the separation between real χ_γ is of order $1/mL$). This can be done by taking the logarithm of (2.2):

$$\sum_{j=1}^{2} N_j \tan^{-1}\left(\frac{\lambda-\alpha_j}{\frac{1}{2}\pi}\right) - \sum_{k=1}^{M_r} \tan^{-1}\left(\frac{\lambda-\lambda_k}{\pi}\right)$$
$$- \sum_{l=1}^{M_{cp}} \left[\tan^{-1}\left(\frac{\lambda-\xi_l}{\pi-\eta_l}\right) + \tan^{-1}\left(\frac{\lambda-\xi_l}{\pi+\eta_l}\right)\right]$$
$$- \sum_{r=1}^{M_{wp}} \left[\tan^{-1}\left(\frac{\lambda-\zeta_r}{\omega_r+\pi}\right) - \tan^{-1}\left(\frac{\lambda-\zeta_r}{\omega_r-\pi}\right)\right] = \pi J(\lambda), \tag{2.4}$$

where $J(\lambda)$ takes on integer (or half-odd-integer) values as one runs through the successive χ_γ and holes on the real axis. Thus $\mathrm{d}J/\mathrm{d}\lambda$ provides a reasonable definition for a smoothed out density of real χ_γ and holes:

$$\frac{\mathrm{d}J}{\mathrm{d}\lambda}(\lambda) = \sigma(\lambda) + \sum_{j=1}^{n} \delta(\lambda-\theta_j), \tag{2.5}$$

where $\sigma(\lambda)$ is the density of λ_j alone. If we differentiate (2.4), insert (2.5) and make the replacement

$$\sum_{i=1}^{M_r} F(\lambda_i) \to \int \mathrm{d}\lambda\, \sigma(\lambda) F(\lambda), \tag{2.6}$$

we obtain the following integral equation for $\sigma(\lambda)$:

$$\sigma(\lambda) + \sum_{j=1}^{n} \delta(\lambda-\theta_j) = \sum_{i=1}^{2} N_i K_{\pi/2}(\lambda-\alpha_i) - \int \mathrm{d}\lambda' \sigma(\lambda') K_\pi(\lambda-\lambda')$$
$$- \sum_{l=1}^{M_{cp}} \left[K_{\pi-\eta_l}(\lambda-\xi_l) + K_{\pi+\eta_l}(\lambda-\xi_l)\right]$$
$$- \sum_{r=1}^{M_{wp}} \left[K_{\omega_r+\pi}(\lambda-\zeta_r) - K_{\omega_r-\pi}(\lambda-\zeta_r)\right], \tag{2.7}$$

where

$$K_a(\lambda)=(a/\pi)(a^2+\lambda^2)^{-1}.$$

Eq. (2.7) may be solved by the method of Fourier transformation:

$$\sigma(\lambda)=\int\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{ip\lambda}\tilde{\sigma}(p),$$

$$\begin{aligned}\tilde{\sigma}(p)=&\sum_{j=1}^{2}\frac{N_j\exp(-i\alpha_j p)}{2\,\mathrm{ch}(\frac{1}{2}\pi p)}-\frac{\exp(\frac{1}{2}\pi|p|)}{2\,\mathrm{ch}(\frac{1}{2}\pi p)}\sum_{k=1}^{n}\exp(-i\theta_k p)\\ &-\sum_{l=1}^{M_{\mathrm{cp}}}\frac{\mathrm{ch}\,\eta_l p}{\mathrm{ch}(\frac{1}{2}\pi p)}\exp(-\tfrac{1}{2}\pi|p|-i\xi_l p)\\ &-\sum_{r=1}^{M_{\mathrm{wp}}}\left[\exp(-\omega_r|p|)-\exp(-(\omega_r-\pi)|p|)\right]\exp(-i\zeta_r p).\end{aligned}\tag{2.8}$$

This formula will be of key importance in what follows.

From the value of $\tilde{\sigma}(p)$ at the origin, we obtain the following normalization conditions:

$$M_{\mathrm{r}}=\tilde{\sigma}(0)=\tfrac{1}{2}\sum_{i=1}^{2}N_i-\tfrac{1}{2}n-M_{\mathrm{cp}},$$

$$M=M_{\mathrm{r}}+2M_{\mathrm{cp}}+2M_{\mathrm{wp}}=\tfrac{1}{2}\sum_{i=1}^{2}N_i-\tfrac{1}{2}n+M_{\mathrm{cp}}+2M_{\mathrm{wp}}.\tag{2.9}$$

Moreover, from the behavior of $\tilde{\sigma}(p)$ at infinity, we see that $\sigma(\lambda)$ will have negative delta-function singularities at the hole positions [as is obvious from (2.5)]. There are no other singular points, except in the special case of a wide pair $\zeta\pm i\pi$, where there is a positive delta function at $\lambda=\zeta$. There would then be an "extra" real χ_γ at that location: in other words there would be a 3-string at ζ. As we shall soon verify, this is a borderline case which arises only for special values of the particle rapidities (i.e. holes), or in the limit of infinitely many holes.

We now turn our attention to the wide-pair version of (2.3). We evaluate the product over λ_k by writing

$$\prod_{k}^{M_{\mathrm{r}}}\frac{i(\lambda_k-\zeta)+\omega+\pi}{i(\lambda_k-\zeta)+\omega-\pi}=\exp\int\mathrm{d}\lambda\,\sigma(\lambda)\ln\frac{i(\lambda-\zeta)+\omega+\pi}{i(\lambda-\zeta)+\omega-\pi},\tag{2.10}$$

computing the integral by means of the formula

$$\int \mathrm{d}\lambda\, f(\lambda) \ln \frac{i(\lambda-\zeta)+a}{i(\lambda-\zeta)+b} = \int_0^\infty \frac{\mathrm{d}p}{p} \tilde{f}(p) \mathrm{e}^{i\zeta p} (\mathrm{e}^{-bp} - \mathrm{e}^{-ap}), \tag{2.11}$$

which is valid for real-valued $f(\lambda)$ and $a, b > 0$. Substituting (2.8) into (2.10), we obtain

$$\prod_{k=1}^{M_r} \frac{i(\lambda_k - \zeta) + \omega + \pi}{i(\lambda_k - \zeta) + \omega - \pi} = \exp\left[\sum_{i=1}^{2} I_i + \sum_{j=1}^{n} I_j^{\mathrm{h}} + \sum_{l=1}^{M_{\mathrm{cp}}} I_l^{\mathrm{cp}} + \sum_{r=1}^{M_{\mathrm{wp}}} I_r^{\mathrm{wp}} \right],$$

where

$$I_i = \int_0^\infty \frac{\mathrm{d}p}{p} (\mathrm{e}^{-(\omega-\pi/2)p} - \mathrm{e}^{-(\omega+\pi/2)p}) \mathrm{e}^{i(\zeta-\alpha_i)p},$$

$$I_j^{\mathrm{h}} = \int_0^\infty \frac{\mathrm{d}p}{p} (\mathrm{e}^{-\omega p} - \mathrm{e}^{-(\omega-\pi)p}) \mathrm{e}^{i(\zeta-\theta_j)p},$$

$$I_l^{\mathrm{cp}} = \int_0^\infty \frac{\mathrm{d}p}{p} (\mathrm{e}^{-(\omega-\eta_l+\pi)p} + \mathrm{e}^{-(\omega+\eta_l+\pi)p} - \mathrm{e}^{-(\omega-\eta_l)p} - \mathrm{e}^{-(\omega+\eta_l)p}) \mathrm{e}^{i(\zeta-\xi_l)p},$$

$$I_r^{\mathrm{wp}} = \int_0^\infty \frac{\mathrm{d}p}{p} (\mathrm{e}^{-(\omega+\omega_r+\pi)p} + \mathrm{e}^{-(\omega+\omega_r-2\pi)p}$$
$$- \mathrm{e}^{-(\omega+\omega_r)p} - \mathrm{e}^{-(\omega+\omega_r-\pi)p}) \mathrm{e}^{i(\zeta-\zeta_r)p}. \tag{2.12}$$

Applying (2.11) once more (with $\tilde{f}$ set equal to one) to evaluate the integrals in (2.12), and then inserting the result in (2.3), we get

$$\prod_{j=1}^{n} \frac{i(\theta_j - \zeta) + \omega}{i(\theta_j - \zeta) + \omega - \pi} = - \prod_{l=1}^{M_{\mathrm{cp}}} \frac{i(\xi_l - \zeta) + \omega - \eta_l}{i(\xi_l - \zeta) + \omega - \eta_l - \pi} \frac{i(\xi_l - \zeta) + \omega + \eta_l}{i(\xi_l - \zeta) + \omega + \eta_l - \pi}$$
$$\times \prod_{r=1}^{M_{\mathrm{wp}}} \frac{i(\zeta_r - \zeta) + \omega - \omega_r + \pi}{i(\zeta_r - \zeta) + \omega - \omega_r - \pi} \frac{i(\zeta_r - \zeta) + \omega + \omega_r}{i(\zeta_r - \zeta) + \omega + \omega_r - 2\pi}. \tag{2.13}$$

Note that this expression involves only quantities which remain finite in the infinite cut-off limit, $K \to \infty$, $L \to \infty$.

The corresponding reduction of the close-pair version of (2.3) is not quite as tidy. We can again apply (2.10) to evaluate the product over λ_k, but now we must apply

$$\int \mathrm{d}\lambda\, f(\lambda) \ln \frac{i(\lambda-\xi)+a}{i(\lambda-\xi)-b} = \int_0^\infty \frac{\mathrm{d}p}{p} (\tilde{f}(-p) \mathrm{e}^{-bp-i\xi p} - \tilde{f}(p) \mathrm{e}^{-ap+i\xi p}) - i\pi \tilde{f}(0). \tag{2.14}$$

This gives

$$\prod_{k=1}^{M_r} \frac{i(\lambda_k-\xi)+\eta+\pi}{i(\lambda_k-\xi)+\eta-\pi} = (-1)^{\frac{1}{2}\Sigma N_i - \frac{1}{2}n - M_{cp}}$$

$$\times \exp\left[\sum_{i=1}^{2} J_i + \sum_{j=1}^{n} J_j^{h} + \sum_{l=1}^{M_{cp}} J_l^{cp} + \sum_{r=1}^{M_{wp}} J_r^{wp}\right], \tag{2.15}$$

where

$$J_i = J_i' + J_i'', \tag{2.16}$$

$$J_i' = \begin{cases} N_i \displaystyle\int_0^\infty \frac{dp}{p} e^{-i(\alpha_i-\xi)p}\left(e^{-(\eta-\pi/2)p} - e^{-(\eta+\pi/2)p}\right), & \eta > \frac{1}{2}\pi, \\ N_i \displaystyle\int_0^\infty \frac{dp}{p}\left(e^{i(\alpha_i-\xi)p-(\pi/2-\eta)p} - e^{-i(\alpha_i-\xi)p-(\pi/2+\eta)p}\right), & \eta < \frac{1}{2}\pi, \end{cases}$$

$$J_i'' = \begin{cases} -N_i \displaystyle\int_0^\infty \frac{dp}{p\,\mathrm{ch}(\frac{1}{2}\pi p)} \mathrm{sh}\left(\pi-\eta-i(\alpha_i-\xi)\right)p, & \eta > \frac{1}{2}\pi, \\ -N_i \displaystyle\int_0^\infty \frac{dp}{p\,\mathrm{ch}(\frac{1}{2}\pi p)} \mathrm{sh}\left(\eta+i(\alpha_i-\xi)\right)p, & \eta < \frac{1}{2}\pi, \end{cases}$$

$$J_j^{h} = J_j^{h\prime} + J_j^{h\prime\prime}, \tag{2.17}$$

$$J_j^{h\prime} = \int_0^\infty \frac{dp}{p}\left(e^{-\eta p}-1\right)e^{-i(\theta_j-\xi)p} - \tfrac{1}{2}i\pi\varepsilon(\theta_j-\xi),$$

$$J_j^{h\prime\prime} = -\int_0^\infty \frac{dp}{p\,\mathrm{ch}(\frac{1}{2}\pi p)}\left[\mathrm{ch}\left(\eta-\tfrac{1}{2}\pi+i(\theta_j-\xi)\right)p - \mathrm{ch}(\tfrac{1}{2}\pi p)\cos(\theta_j-\xi)p\right],$$

$$J_l^{cp} = J_l^{cp\prime} + J_l^{cp\prime\prime}, \tag{2.18}$$

$$J_l^{cp\prime} = \int_0^\infty \frac{dp}{p} e^{-i(\xi_l-\xi)p}\left[e^{-(\eta+\eta_l+\pi)p} + e^{-(\eta-\eta_l+\pi)p} - 2e^{-\pi p}\right] - 2i\tan^{-1}\left(\frac{\xi_l-\xi}{\pi}\right),$$

$$J_l^{cp\prime\prime} = -2\int_0^\infty \frac{dp\, e^{-\pi p}}{p\,\mathrm{ch}(\frac{1}{2}\pi p)}\left[\mathrm{ch}\,\eta_l p\,\mathrm{ch}\left(\eta-\tfrac{1}{2}\pi+i(\xi_l-\xi)\right)p - \mathrm{ch}(\tfrac{1}{2}\pi p)\cos(\xi_l-\xi)p\right],$$

$$J_r^{wp} = \int_0^\infty \frac{dp}{p}\left[e^{-i(\zeta_r-\xi)p-(\eta+\omega_r+\pi)p} - e^{i(\zeta_r-\xi)p-(\omega_r-\eta+\pi)p}\right.$$

$$\left. + e^{i(\zeta_r-\xi)p-(\omega_r-\eta)p} - e^{-i(\zeta_r-\xi)p-(\omega_r+\eta)p}\right]. \tag{2.19}$$

These integrals may be evaluated with the aid of (2.11) and (2.14). After inserting the results into (2.3), we obtain

$$F(\xi,\eta)\equiv\prod_{j=1}^{n}\frac{i(\theta_j-\xi)+\eta}{i(\theta_j-\xi)}$$

$$=-\prod_{l=1}^{M_{\rm cp}}\frac{i(\xi_l-\xi)+\pi}{i(\xi_l-\xi)+(\eta-\eta_l-\pi)}\,\frac{-i(\xi_l-\xi)+\pi}{i(\xi_l-\xi)+(\eta+\eta_l-\pi)}$$

$$\times\prod_{r=1}^{M_{\rm wp}}\frac{i(\zeta_r-\xi)+(\eta-\omega_r+\pi)}{i(\zeta_r-\xi)+(\eta-\omega_r)}\,\frac{i(\zeta_r-\xi)+(\eta+\omega_r)}{i(\zeta_r-\xi)+(\eta+\omega_r-\pi)}\exp J''(\xi,\eta), \tag{2.20}$$

where

$$J''(\xi,\eta)=\sum_{i=1}^{2}\left(J_i''+\tfrac{1}{2}i\pi N_i\varepsilon\left(\eta-\tfrac{1}{2}\pi\right)\right)+\sum_{j=1}^{n}J_j^{\mathrm{h}\prime\prime}+\sum_{l=1}^{M_{\rm cp}}J_l^{\mathrm{cp}\prime\prime}+\sum_{r=1}^{M_{\rm wp}}J_r^{\mathrm{wp}\prime\prime}$$

$$=J''(\xi,\pi-\eta)^*.$$

Now for $K\to\infty$, $L\to\infty$,

$$\operatorname{Re}J''(\xi,\eta)\sim-mL\operatorname{ch}\xi\sin\eta,$$

which forces $\eta+\eta_l-\pi$ to vanish [with corrections of order $\exp(-mL)$] for some l. There are two ways this can occur: either the close pair $\xi\pm i\eta$ is a *2-string*

$$\xi=\xi_l,\qquad \eta=\eta_l=\tfrac{1}{2}\pi,$$

or else it has a partner $\xi_l\pm i\eta_l$ such that

$$\xi=\xi_l,\quad \eta=\pi-\eta_l\neq\eta_l.$$

In the latter case, we have a *quartet* (no relation to a 4-string!) of complex χ_γ with the same real part. Whichever is the case, ξ and η may be determined by calculating

$$F(\xi,\eta)/F(\xi,\pi-\eta)^*,$$

where $F(\xi, \eta)$ is defined by (2.20). The result is

$$\prod_{j=1}^{n} \frac{i(\theta_j - \xi) + \eta}{i(\theta_j - \xi) + \eta - \pi} = -\prod_{l=1}^{M_{\mathrm{cp}}} \frac{i(\xi_l - \xi) + \eta + \eta_l}{i(\xi_l - \xi) + \eta + \eta_l - \pi} \frac{i(\xi_l - \xi) + \eta - \eta_l}{i(\xi_l - \xi) + \eta - \eta_l - \pi}$$
$$\times \prod_{r=1}^{M_{\mathrm{wp}}} \frac{i(\zeta_r - \xi) + \eta + \omega_r}{i(\zeta_r - \xi) + \eta + \omega_r - 2\pi} \frac{i(\zeta_r - \xi) + \eta - \omega_r + \pi}{i(\zeta_r - \xi) + \eta - \omega_r - \pi}, \tag{2.21}$$

which is the exact analogue of (2.13).

As a final step in the analysis of this section, we re-express the product of close-pair factors in (2.13) and (2.21) in terms of 2-strings and quartets:

$$\prod_{j=1}^{n} \frac{i(\theta_j - \xi) + \eta}{i(\theta_j - \xi) + \eta - \pi} = -\prod_{l=1}^{M_2} \frac{i(\psi_l - \xi) + \eta + \frac{1}{2}\pi}{i(\psi_l - \xi) + \eta - \frac{3}{2}\pi}$$
$$\times \prod_{s=1}^{M_{\mathrm{q}}} \frac{i(\xi_s - \xi) + \eta + \eta_s}{i(\xi_s - \xi) + \eta + \eta_s - 2\pi} \frac{i(\xi_s - \xi) + \eta - \eta_s + \pi}{i(\xi_s - \xi) + \eta - \eta_s - \pi}$$
$$\times \prod_{r=1}^{M_{\mathrm{wp}}} \frac{i(\zeta_r - \xi) + \eta + \omega_r}{i(\zeta_r - \xi) + \eta + \omega_r - 2\pi} \frac{i(\zeta_r - \xi) + \eta - \omega_r + \pi}{i(\zeta_r - \xi) + \eta - \omega_r - \pi}, \tag{2.22}$$

where

$$\psi_l \pm \tfrac{1}{2} i\pi = l\text{th two-string}, \qquad l = 1, 2, \dots, M_2;$$
$$\xi_s + i\eta_s = \text{top member of } s\text{th quartet}, \qquad s = 1, 2, \dots, M_{\mathrm{q}};$$
$$\xi + i\eta = \text{any complex } \chi_\gamma \text{ with } \eta > 0.$$

If we now define new quantities $\hat{\chi}_1, \dots, \hat{\chi}_{\hat{M}}$, $\hat{M} = M_2 + 2M_{\mathrm{q}} + 2M_{\mathrm{wp}}$, so that

$$\{\hat{\chi}_1, \dots, \hat{\chi}_{\hat{M}}\} = \{\psi_l : l = 1, \dots, M_2\} \cup \left\{\xi_s \pm i\left(\eta_s - \tfrac{1}{2}\pi\right) : s = 1, \dots, M_{\mathrm{q}}\right\}$$
$$\cup \left\{\zeta_r \pm i\left(\omega_r - \tfrac{1}{2}\pi\right) : r = 1, \dots, M_{\mathrm{wp}}\right\},$$

then eq. (2.22) assumes the particularly simple form

$$\prod_{j=1}^{n} \frac{i(\theta_j - \hat{\chi}_\gamma) + \frac{1}{2}\pi}{i(\theta_j - \hat{\chi}_\gamma) - \frac{1}{2}\pi} = -\prod_{\beta=1}^{\hat{M}} \frac{i(\hat{\chi}_\beta - \hat{\chi}_\gamma) + \pi}{i(\hat{\chi}_\beta - \hat{\chi}_\gamma) - \pi}. \qquad (2.23)$$

We note that these equations are formally the same as those of the non-relativistic gas of fermions with repulsive delta-function interaction [12, 13]. In the latter equations, the particle momenta (divided by the strength of the potential) played the role of our rapidities.

3. Illustrative examples

In this section, we present some simple examples of solutions of eqs. (2.23). Specifically, we discuss the simplest cases (4- and 6-particle singlet states) in which a wide pair or quartet of complex χ_γ arises.

3.1. FOUR-PARTICLE SINGLET STATES

With $n=4$ and $\hat{M}=2$, we expect two distinct solutions of the equations. If we choose reflection-symmetric rapidities,

$$\theta_1 = \tfrac{1}{2}a_1\pi = -\theta_3, \qquad \theta_2 = \tfrac{1}{2}a_2\pi = -\theta_4,$$

the solutions (if they exist) will also possess this symmetry, i.e.

$$\hat{\chi}_1 = \tfrac{1}{2}z\pi = -\hat{\chi}_2,$$

where z satisfies

$$\prod_{i=1}^{2} \frac{(z+i)^2 - a_i^2}{(z-i)^2 - a_i^2} = \frac{z+i}{z-i}. \qquad (3.1)$$

This equation can be written more simply as a quadratic equation for z^2:

$$3z^4 + (2 - a_1^2 - a_2^2)z^2 - (1 + a_1^2)(1 + a_2^2) = 0, \qquad (3.2)$$

so that

$$z^2 = \tfrac{1}{6}(a_1^2 + a_2^2 - 2) \pm \tfrac{1}{6}(a_1^4 + 14a_1^2a_2^2 + a_2^4 + 8a_1^2 + 8a_2^2 + 16)^{1/2}. \qquad (3.3)$$

We see that, regardless of what rapidities are selected, there will always be one positive root $z_+^2 \geqslant \frac{1}{3}$, corresponding to 2-strings

$$\xi \pm \tfrac{1}{2} i\pi, \qquad -\xi \pm \tfrac{1}{2} i\pi, \qquad \xi \geqslant \pi/2\sqrt{3},$$

and one negative root, $z_-^2 \leqslant -1$, corresponding to a wide pair

$$\zeta \pm i\omega, \qquad \zeta = 0, \qquad \omega \geqslant \pi.$$

From (3.3) we see that ω is always greater than π, except in the limiting case where two of the particles are at rest ($a_1 = 0$ or $a_2 = 0$). In that limit, the wide pair becomes a 3-string.

3.2. SIX PARTICLE SINGLET STATES

Here, with $n = 6$ and $\hat{M} = 3$, we must find

$$\binom{6}{3} - \binom{6}{2} = 5$$

solutions of (2.23) to establish completeness of the basis in this subspace. As in the previous example, the problem simplifies considerably if we choose symmetrically placed rapidities

$$\theta_i = \tfrac{1}{2} a_i \pi = -\theta_{3+i}, \qquad i = 1, 2, 3.$$

We would expect that at least some of the five solutions will be reflection symmetric, i.e. of the form

$$\hat{\chi}_1 = \tfrac{1}{2} z\pi = -\hat{\chi}_3, \qquad \hat{\chi}_2 = 0,$$

with z a solution of

$$\prod_{i=1}^{3} \frac{z - a_i + i}{z - a_i - i} \frac{z + a_i + i}{z + a_i - i} = \frac{z + 2i}{z - 2i} \frac{z + i}{z - i}. \tag{3.4}$$

Eq. (3.4) may be written as

$$zP(z^2) = 0, \tag{3.5}$$

where

$$P(y) = 3y^3 + By^2 + Cy + D,$$

and B, C, D are polynomials in a_1^2, a_2^2, a_3^2, with integer coefficients. In addition to the trivial solution $z = 0$, eq. (3.5) may be solved in closed form for z^2; here we shall content ourselves with a qualitative discussion.

First of all, we observe that $P(y)=0$ has at least one positive root. To see this, we consider (3.4) restricted to real z. Taking the logarithm gives

$$\begin{aligned}\pi J(z) &\equiv \sum_{i=1}^{3}\left[\tan^{-1}(z-a_i)+\tan^{-1}(z+a_i)\right]-\tan^{-1}(\tfrac{1}{2}z)-\tan^{-1}z\\ &=\pi\times\text{integer}.\end{aligned} \tag{3.6}$$

We see immediately that since $J(z)$ is a continuous function on the entire real line with

$$\lim_{z\to\pm\infty} J(z)=\pm 2,$$

there will always be at least one solution with $z>0$, $|J|=1$.

The existence of a negative root is even easier to establish. Since by direct evaluation

$$P(-4)=3\prod_{i=1}^{3}\left(1+a_i^2\right),\qquad P(-\infty)=-\infty,$$

we conclude that $P(y)$ must vanish for some $y<-4$. Moreover, since $P(y)$ is a cubic polynomial with real coefficients, there must also be a *third* root, whose sign is just the sign of

$$P(0)=D=\prod_{i=1}^{3}\left(1+a_i^2\right)\left[3-4\left(\frac{1}{1+a_1^2}+\frac{1}{1+a_2^2}+\frac{1}{1+a_3^2}\right)\right]. \tag{3.7}$$

If any one of the a_i is sufficiently small, the sign will be negative and we will have, in addition to the two-string $\pm\frac{1}{2}i\pi$ already present, a quartet of complex χ_γ,

$$i\left(\tfrac{1}{2}\pi\pm\varepsilon\right),\qquad -i\left(\tfrac{1}{2}\pi\pm\varepsilon\right),\qquad 0<\varepsilon<\tfrac{1}{2}\pi$$

(or a 3-string if some a_i vanishes). Note that the possibility of a second wide pair $\pm i\omega$ is eliminated by the inequality

$$P(-1)=3\prod_{i=1}^{3}a_i^2\geqslant 0.$$

If none of the a_i is sufficiently small, we will have $P(0)>0$ and there will be a second solution consisting of three 2-strings. Such a solution satisfies (3.6) with $J(z)=0$ for $z=0$ *and* for $z=\pm\psi\neq 0$.

We have thus succeeded in finding three six-particle color singlet states. For values of the rapidities in the vicinity of $\pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi$, we have solved the equations numerically and have found two additional solutions (mirror images of one another), each containing a wide pair and a 2-string. Thus far we have not been able to find a general existence proof for this type of solution.

4. Equations for the rapidities and indications of a higher level Bethe ansatz

The remarkable equations (2.23) do not exhaust the useful information which can be extracted from the original system (1.1). In particular, we must still obtain equations for determining the quantized values of the particle momenta, $p_i = m\,\mathrm{sh}\,\theta_i$ to an accuracy of order $1/L$. Such knowledge is crucial for obtaining the S-matrix and densities of states in the continuum limit. The relevant equations are derived from (2.2) by evaluating the product over λ_k with the aid of the density $\sigma(\lambda)$, whose Fourier transform may now be written

$$\begin{aligned}\tilde{\sigma}(p) = &\sum_{j=1}^{2} \frac{N_j \exp(-i\alpha_j p)}{2\,\mathrm{ch}(\frac{1}{2}\pi p)} - \frac{\exp(\frac{1}{2}\pi|p|)}{2\,\mathrm{ch}(\frac{1}{2}\pi p)} \sum_{k=1}^{n} \exp(-i\theta_k p)\\ &- \sum_{l=1}^{M_2} \exp(-\tfrac{1}{2}\pi|p|)\exp(-i\psi_l p)\\ &- \sum_{s=1}^{M_\mathrm{q}} \left[\exp(-\eta_s|p|) + \exp(-(\pi-\eta_s)|p|)\right]\exp(-i\xi_s p)\\ &- \sum_{r=1}^{M_\mathrm{wp}} \left[\exp(-\omega_r|p|) - \exp(-(\omega_r-\pi)|p|)\right]\exp(-i\zeta_r p).\end{aligned} \tag{4.1}$$

Thus, from (2.14)

$$\prod_{k=1}^{M_\mathrm{r}} \frac{i(\lambda_k-\lambda)+\pi}{i(\lambda_k-\lambda)-\pi} = (-1)^{M_\mathrm{r}} \exp\int_0^\infty \frac{\mathrm{d}p}{p} \mathrm{e}^{-\pi p}\left[\tilde{\sigma}(-p)\mathrm{e}^{-i\lambda p} - \tilde{\sigma}(p)\mathrm{e}^{i\lambda p}\right], \tag{4.2}$$

where

$$M_\mathrm{r} = \tfrac{1}{2}\sum_{j=1}^{2} N_j - \tfrac{1}{2}n - M_2 - 2M_\mathrm{q}.$$

With the aid of (2.14) and formulas 3.411(28) and 4.111(7) of [14], the integral in (4.2) is readily evaluated. The result may be substituted into (2.2) to yield

$$\exp(iLm\,\mathrm{sh}\,\lambda)=\pm\prod_{k=1}^{n}\hat{S}_{\mathrm{t}}^{-1}(\lambda-\theta_k)\prod_{\gamma=1}^{\hat{M}}\frac{i(\lambda-\hat{\chi}_\gamma)+\frac{1}{2}\pi}{i(\lambda-\hat{\chi}_\gamma)-\frac{1}{2}\pi}\,, \tag{4.3}$$

where the overall sign is positive (our choice from now on) or negative depending on whether one-half the sum of the bare charge and chiral charge is odd or even, and

$$\hat{S}_{\mathrm{t}}(\theta)=i\frac{\Gamma(1+i\theta/2\pi)\Gamma(\frac{1}{2}-i\theta/2\pi)}{\Gamma(1-i\theta/2\pi)\Gamma(\frac{1}{2}+i\theta/2\pi)}\,.$$

The quantity $\hat{S}_{\mathrm{t}}(\theta)$ may be interpreted as the eigenvalue of the S-matrix in the color-triplet scattering state of two particles with rapidity difference θ. To see this, consider a color-triplet state of particles with rapidities $\theta_1=\theta>0$ and $\theta_2=0$ (rest frame of the second particle). An "in" state $|\theta,0,\mathrm{triplet}\rangle^{\mathrm{in}}$ will be described by a wave function $\exp(ix_1m\,\mathrm{sh}\,\theta)$ for x_1 large and negative, and by $\exp(i\hat{\delta}_{\mathrm{t}}+ix_1m\,\mathrm{sh}\,\theta)$ for x_1 large and positive. Periodicity now requires

$$\exp(i\hat{\delta}_{\mathrm{t}}+imL\,\mathrm{sh}\,\theta)=1\,. \tag{4.4}$$

On the other hand, eq. (4.3) with $\hat{M}=0$, together with (4.4), gives

$$\exp(iLm\,\mathrm{sh}\,\theta)=\hat{S}_{\mathrm{t}}^{-1}(\theta)=\exp(-i\hat{\delta}_{\mathrm{t}}(\theta))\,. \tag{4.5}$$

The scattering phase for the singlet state (which has a 2-string with real part $\frac{1}{2}(\theta_1+\theta_2)$, may be calculated in similar fashion [4]. The full two-particle S-matrix can then be written in the compact form [3]

$$\hat{S}_{12}=\hat{S}_{\mathrm{t}}(\theta_1-\theta_2)\frac{i(\theta_1-\theta_2)+\pi P_{12}}{i(\theta_1-\theta_2)+\pi}\,, \tag{4.6}$$

where P_{12} is the color-exchange operator. Not that the S-matrix has a form analogous to that of the pseudoparticle S-matrix which arose in the construction of the model [1]:

$$S_{12}=S_{\mathrm{t}}(\alpha_1-\alpha_2)\frac{i(\alpha_1-\alpha_2)+\pi P_{12}}{i(\alpha_1-\alpha_2)+\pi}\,,\qquad \alpha_i=\pm\pi/c\,,$$

$$S_{\mathrm{t}}(\alpha)=\exp(-\tfrac{1}{2}i\alpha\eta(c))\,, \tag{4.7}$$

where η is independent of α.

We shall presently exploit the formal similarity of (4.6) and (4.7). Before doing so, it is worthwhile to point out a significant physical difference between the two scattering matrices: for the pseudoparticles, the triplet interaction is repulsive

($\delta_t < 0$) and the singlet is attractive ($\delta_s > 0$), whereas for the massive physical particles the signs are reversed. Thus, whereas the ground state of a large collection of pseudoparticles has a minimum possible total color spin (i.e. is antiferromagnetic), that of a large collection of massive particles is ferromagnetic. And so, whereas the physical vacuum, consisting of right- and left-moving Dirac seas of pseudoparticles, is a color singlet, the lowest-energy state of n physical particles has all of the color spins aligned (see [15]).

Observing the formal similarity of (4.6) and (4.7), we are led to ask the following: given that eqs. (1.1), together with the formula for the pseudoparticle momenta k_j,

$$e^{ik_jL} = \prod_{k=1}^{N} S_t^{-1}(\alpha_j - \alpha_k) \prod_{\gamma=1}^{M} \frac{i(\alpha_j - \chi_\gamma) + \frac{1}{2}\pi}{i(\alpha_j - \chi_\gamma) - \frac{1}{2}\pi}, \tag{4.8}$$

arise [1] from a discrete eigenvalue problem of the form

$$Z_j \Phi \equiv S_{(j+1)j} \cdots S_{N_j} S_{1j} \cdots S_{(j-1)j} \Phi = \exp(ik_j L)\Phi, \tag{4.9}$$

might it not be possible to write down an analogous eigenvalue problem which would lead to (2.23) and (4.3)?

To answer this question, we have reviewed the modified Bethe-ansatz solution of the discrete eigenvalue problem, and we find neither dependence on the precise functional form of $S_t(\alpha)$, nor on the number of allowed values of α_i. We may therefore write down immediately that the eigenvalue problem

$$\hat{Z}_j \hat{\Phi} = \exp(ip_j L)\hat{\Phi},$$

$$\hat{Z}_j = \hat{S}_{(j+1)j} \cdots \hat{S}_{N_j} \hat{S}_{1j} \cdots \hat{S}_{(j-1)j}, \qquad j = 1, 2, \ldots, n, \tag{4.10}$$

has the solution

$$\hat{\Phi} = \sum_{1 \leq y_1 < y_2 < \cdots < y_{\hat{M}} \leq n} \hat{\Phi}(y_1, \ldots, y_{\hat{M}}) \tau_{y_1}^- \cdots \tau_{y_{\hat{M}}}^- \hat{\Psi}_0,$$

$$\hat{\Psi}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n,$$

$$\tau_y^- = \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \cdots \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_y \otimes \cdots \otimes \mathbb{1}_n,$$

$$\hat{\Phi}(y_1, \ldots, y_{\hat{M}}) = \sum_{p \in S_{\hat{M}}} A_p \prod_{\gamma=1}^{\hat{M}} \hat{f}(\hat{\chi}_{p\gamma}, y_\gamma),$$

$$\hat{f}(\hat{\chi}, y) = \prod_{j=1}^{y-1} \frac{i(\theta_j - \hat{\chi}) + \frac{1}{2}\pi}{i(\theta_{j+1} - \hat{\chi}) - \frac{1}{2}\pi}, \tag{4.11}$$

where $\hat{\chi}_\gamma$ *must satisfy* (2.23) *and the eigenvalue is given by* (4.3).

Eq. (4.10) thus summarizes in a convenient form the equations derived in this and previous sections. It is tempting to interpret (4.10) in terms of a higher level x-space Bethe ansatz. In particular, without assuming a strictly local interaction between particles, one might associate with each of our n-particle energy eigenstates a set of asymptotic wave functions

$$\exp[i(p_1x_1 + \cdots + p_nx_n)]\hat{\Phi}^{Q}_{a_1\cdots a_n}, \tag{4.12}$$

describing, for each permutation Q of $123\cdots n$, free propagation in the region

$$x_{Q1} \ll x_{Q2} \ll \cdots \ll x_{Qn}.$$

The spinorial tensors Φ^Q for neighboring regions Q and Q' [with $Q'i = Q(i+1)$, $Qi = Q'(i+1)$] would then be linked by the two-particle S-matrix,

$$\hat{\Phi}^{Q'} = \hat{S}_{i(i+1)}\hat{\Phi}^{Q}. \tag{4.13}$$

Single valuedness on our "ring" of length L would then require

$$\exp(ip_jL)\hat{\Phi}^{12\cdots(j-1)(j+1)\cdots nj} = \hat{\Phi}^{j12\cdots(j-1)(j+1)\cdots n}. \tag{4.14}$$

From (4.13) it follows,

$$\begin{aligned}\hat{\Phi}^{j12\cdots(j-1)(j+1)\cdots n} &= \hat{S}_{1j}\hat{\Phi}^{1j2\cdots(j-1)(j+1)\cdots n}\\ &= \cdots = \hat{S}_{1j}\hat{S}_{2j}\cdots\hat{S}_{(j-1)j}\hat{\Phi}^{12\cdots n},\\ \hat{\Phi}^{12\cdots(j-1)(j+1)\cdots nj} &= \hat{S}^{-1}_{nj}\hat{S}^{-1}_{(n-1)j}\cdots\hat{S}^{-1}_{(j+1)j}\hat{\Phi}^{12\cdots n}.\end{aligned} \tag{4.15}$$

Identifying $\hat{\Phi}$ with $\hat{\Phi}^{12\cdots n}$, we are led to (4.10).

From the above discussion, we see that the notion of a higher-level x-space Bethe ansatz (which still must be placed on a more solid footing) provides a simple and satisfying interpretation of our principal results.

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