# DERIVATION OF THE CHIRAL GROSS-NEVEU SPECTRUM FOR ARBITRARY SU( $N$ ) SYMMETRY ${ }^{\text {T }}$ 

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The $\operatorname{SU}(N)$ chiral invariant Gross-Neveu hamiltonian is diagonalized using iterated Bethe-ansätze. This extends to arbitrary $N$ the results previously obtained for $\operatorname{SU}(2)$, and reproduces the spectrum obtained by assuming factorizability of the $S$-matrix.

In a previous article [1] ${ }^{\neq 1}$, we used Yang's version [2] of the Bethe-ansatz [3] to determine exactly the spectrum of the $\mathrm{SU}(2)$ chiral invariant Gross-Neveu model, with lagrangian

$$
\begin{equation*}
\mathcal{L}=\mathrm{i} \bar{\psi}_{\mathrm{a}} \not \partial \psi_{\mathrm{a}}+g\left[\left(\bar{\psi}_{\mathrm{a}} \psi_{\mathrm{a}}\right)^{2}-\left(\bar{\psi}_{\mathrm{a}} \gamma^{5} \psi_{\mathrm{a}}\right)^{2}\right] . \tag{1}
\end{equation*}
$$

We now use Sutherland's generalization [4] of the Bethe-Yang technique to extend our results to arbitrary $\operatorname{SU}(N)$, obtaining the spectrum of massive particles found in refs. [5-7].

We begin by summarizing the $\operatorname{SU}(2)$ case. There, one diagonalizes a hamiltonian $H$ for a cut-off theory, in which the basic fields are canonically quantized on a line segment of length $L$ and with momentum cut-off $K$. The energy eigenvalue problem is found to be equivalent to that of an $\chi$-body hamiltonian describing $\chi^{ \pm}$colored pseudoparticles of chirality $\alpha= \pm 1$. Pairs of pseudoparticles of opposite chirality interact via a chirality exchanging deltafunction potential. Eigenstates of $H$ of chirality $\varkappa^{+}-\varkappa^{-}$and color spin $\frac{1}{2}\left(\varkappa^{+}+\varkappa^{-}-2 M\right)$ are labeled by sets of "momenta" $\left\{k_{j}^{+}, k_{l}^{-}: j=1, \ldots, \mathcal{K}^{+}, l=1, \ldots \mathcal{K}^{-}\right\}$and "rapidities" $\left\{\Lambda_{\gamma}: \gamma=1, \ldots, M\right\}$, satisfying
$\chi^{+} \theta\left(2 \Lambda_{\gamma}-2\right)+\chi^{-} \theta\left(2 \Lambda_{\gamma}+2\right)=-2 \pi J\left(\Lambda_{\gamma}\right)+\sum_{\delta=1}^{M} \theta\left(\Lambda_{\gamma}-\Lambda_{\delta}\right)$,
where $\theta(x)=-2 \tan ^{-1}(x / c),-\pi \leqslant \theta<\pi, c=4 g /\left(1--g^{2}\right)$, the $J\left(\Lambda_{\gamma}\right)$ are half-integers (integers) for $M$ even (odd), and

$$
\begin{equation*}
k_{j}^{\alpha_{j}}=L^{-1}\left\{2 \pi n_{j}^{\alpha_{j}}+\alpha_{j} \Re^{\left(-\alpha_{j}\right)} \varphi+\alpha_{j}\left(\Re^{\left(\alpha_{j}\right)}-1\right) \pi+\sum_{\gamma=1}^{M}\left[\theta\left(2 \Lambda_{\gamma}-2 \alpha_{j}\right)-\alpha_{j} \pi\right]\right\} \tag{3}
\end{equation*}
$$

[^0]where $n_{j}$ is an integer, $\alpha_{j}= \pm 1$, and $\varphi=\tan ^{-1} \frac{1}{2} c$. The energy and momentum of such a state are
$E=\sum_{j} k_{j}^{+}-\sum_{l} k_{l}^{-}, \quad P=\sum_{j} k_{j}^{+}+\sum_{l} k_{l}^{-}$.
The vacuum of the $\operatorname{SU}(2)$ model is characterized by (i) $\varkappa^{+}=\boldsymbol{K}^{-}=M$, (ii) all $\Lambda_{\gamma}$ are real, and (iii) the pseudoparticles occupy the lowest possible levels, forming the Dirac sea. Excitations consist of (i) massless excitations analogous to those of the massless Thirring model and (ii) massive, color doublet particles of zero chirality, corresponding to holes and strings in the set of $\Lambda_{\gamma}$. Holes occur where there is a jump in the sequence of $J\left(\Lambda_{\gamma}\right)$, i.e.
\[

$$
\begin{align*}
J\left(\Lambda_{\gamma+1}\right) & =J\left(\Lambda_{\gamma}\right)+1, & & \Lambda_{\gamma} \neq \Lambda^{\mathrm{h}}  \tag{5}\\
& =J\left(\Lambda_{\gamma}\right)+2, & & \Lambda_{\gamma}=\Lambda^{\mathrm{h}}
\end{align*}
$$
\]

Strings are families of complex $\Lambda_{\gamma}$ which, for $\mathcal{K}$ tending to infinity, have the same real part. For an $n$-string, these are located at
$\Lambda_{\gamma}^{(n) j}=\Lambda_{\gamma}^{(n)}+\frac{1}{2} \mathrm{i} c(n+1-2 j), \quad j=1, \ldots, n$.
To compute the contribution to the energy of holes at $\Lambda_{i}^{\mathrm{h}}$ and $n$-strings at $\Lambda_{j}^{\mathrm{s}}$, one makes the replacement (for $n \rightarrow \infty$ )
$\sum_{\Lambda_{\gamma}} \ldots \rightarrow \int \mathrm{d} \Lambda \sigma(\Lambda) \ldots+\sum_{\substack{n \text {-strings } \\ n>1}} \ldots$,
in eqs. (2) and (3). One can solve eq. (2) for the Fourier transform of $\sigma(\Lambda)$, obtaining

$$
\begin{align*}
& \sigma(p)=\sigma^{\text {sea }}(p)+\sum_{i}^{\text {holes }} \sigma_{i}^{\mathrm{h}}(p)+\sum_{j}^{\text {strings }} \sigma_{j}^{\mathrm{s}}(p)  \tag{7}\\
& \quad \equiv \frac{\Upsilon^{+} \exp (-\mathrm{i} p)+\Upsilon^{-} \exp (\mathrm{i} p)}{2 \cosh c^{\prime} p} \frac{\exp c^{\prime} p}{2 \cosh c^{\prime} p} \sum_{i}^{\text {holes }} \exp \left(-\mathrm{i} \Lambda_{i}^{\mathrm{h}} p\right)-\sum_{j}^{\text {strings }} \exp \left[-\left(n_{j}-1\right) c^{\prime}|p|\right] \exp \left(-\mathrm{i} \Lambda_{j}^{\mathrm{s}} p\right), \quad c^{\prime}=\frac{1}{2} c .
\end{align*}
$$

Computing $k_{j}^{ \pm}$using eq. (3) and substituting into eq: (4) yields
$E_{i}^{\text {hole }}=\Pi L^{-1}\left[\int \mathrm{~d} p \frac{\sin p}{p} \exp \left(-c^{\prime}|p|\right) \sigma_{i}^{\mathrm{h}}(p)+\pi / 2\right]=\Upsilon L^{-1} \tan ^{-1}\left[\cosh \left(\Lambda_{i}^{\mathrm{h}} \pi / c\right) / \sinh (\pi / c)\right]$,
$E_{j}^{\text {string }}=\Pi L^{-1}\left[\int \mathrm{~d} p \frac{\sin p}{p} \exp \left(-c^{\prime}|p|\right) \sigma_{j}^{\mathrm{s}}(p)\right]+\frac{1}{2} \varkappa L^{-1}\left[\theta\left(\frac{2 \Lambda_{j}^{\mathrm{s}}-2}{n_{j}}\right)-\theta\left(\frac{2 \Lambda_{j}^{\mathrm{s}}+2}{n_{j}}\right)\right]=0$.
Taking the limit $K \rightarrow \infty, c(K) \rightarrow 0$, with $m=(4 K / \pi) \mathrm{e}^{-\pi / c}$ and $\chi_{i}=\Lambda_{i}^{\mathrm{h}} \pi / c$ fixed then yields for $E_{i}^{\text {hole }}$ the spectrum of a mass $m$ particle:
$E_{i}^{\text {hole }} \rightarrow m \cosh \chi_{i}$.
Sutherland's method [4] of iterative Bethe-Yang ansätze makes possible a straightforward extension of the above results to arbitrary $\operatorname{SU}(N)$. The color symmetry is now described by a Young diagram with rows of lengths $M_{i-1}-M_{i}$, with $M_{0}=\chi$ and $M_{N}=0$. In addition, the state is labeled by "momenta" $k_{i}^{ \pm}, i=1,2, \ldots \mathcal{K}^{ \pm}$, and "rapidities" $\Lambda_{r \gamma}, r=0,1, \ldots, N-1, \gamma=1,2, \ldots, M_{r}$, where, by convention, $\Lambda_{0 \gamma} \equiv \alpha_{\gamma} \equiv$ chirality $= \pm 1, \gamma=1,2$, $\ldots \boldsymbol{\sim}$. According to ref. [4], these parameters satisfy a chain of coupled equations, analogous to eq. (2) :
$\sum_{\delta=1}^{M_{r+1}} \theta\left(2 \Lambda_{r \gamma}-2 \Lambda_{r+1, \delta}\right)+\sum_{\epsilon=1}^{M_{r-1}} \theta\left(2 \Lambda_{r \gamma}-2 \Lambda_{r-1, \epsilon}\right)-\sum_{\beta=1}^{M_{r}} \theta\left(\Lambda_{r \gamma}-\Lambda_{r \beta}\right)=-2 \pi J_{r}\left(\Lambda_{r \gamma}\right)$.
For $\Pi \rightarrow \infty$, we introduce densities $\sigma_{r}\left(\Lambda_{r}\right), r=0,1, \ldots, N-1$. Moreover, for each $r$ there may be holes, at $\Lambda_{r}=\Lambda_{r j}^{\mathrm{h}}$, and $n$-strings of complex $\Lambda_{r}$ with real parts $\Lambda_{r j}^{s}$. Making the substitution (6), eq. (11) becomes

$$
\begin{align*}
& {\left[\int \mathrm{d} \Lambda \sigma_{r+1}(\Lambda) \theta\left(2 \Lambda_{r}-2 \Lambda\right)+\sum_{j}^{\text {strings }} \theta\left(\frac{2 \Lambda_{r}-2 \Lambda_{r+1, j}^{\mathrm{s}}}{n_{r+1, j}}\right)\right]} \\
& +\left[\int \mathrm{d} \Lambda \sigma_{r-1}(\Lambda) \theta\left(2 \Lambda_{r}-2 \Lambda\right)+\sum_{j}^{\text {strings }} \theta\left(\frac{2 \Lambda_{r}-2 \Lambda_{r-1, j}^{s}}{n_{r-1, j}}\right)\right] \\
& \quad-\left[\int \mathrm{d} \Lambda \sigma_{r}(\Lambda) \theta\left(2 \Lambda_{r}-\Lambda\right)+\sum_{j}^{\text {strings }} \theta\left(\frac{2 \Lambda_{r}-2 \Lambda_{r j}^{\mathrm{s}}}{n_{r j}+1}\right)+\theta\left(\frac{2 \Lambda_{r}-2 \Lambda_{r j}^{\mathrm{s}}}{n_{r j}-1}\right)\right]=-2 \pi J_{r}\left(\Lambda_{r}\right) \tag{12}
\end{align*}
$$

Differentiating eq. (12) with respect to $\Lambda_{r}$ and taking the Fourier transform leads to a simple recursive formula:
$2 \cosh c^{\prime} p \sigma_{r}^{\prime}(p)=\sigma_{r-1}^{\prime}(p)+\sigma_{r+1}^{\prime}(p)-h_{r}(p)$,
where
$\sigma_{r}^{\prime}(p)=\sigma_{r}(p)+\sum_{j} \exp \left(-\mathrm{i} \Lambda_{r j}^{\mathrm{s}} p\right) \exp \left[-\left(n_{r j}-1\right) c^{\prime}|p|\right], h_{r}(p)=\exp \left(c^{\prime}|p|\right) \sum_{j} \exp \left(-\mathrm{i} \Lambda_{r j}^{\mathrm{h}} p\right)$,
$\sigma_{N}^{\prime}(p) \equiv 0, \quad \sigma_{0}^{\prime}(p) \equiv \Upsilon^{+} \exp (-\mathrm{i} p)+\Upsilon^{-} \exp (\mathrm{i} p)$.

System (13) may be solved for any $\sigma_{r}$ :
$\sigma_{r}(p)=\frac{\sinh (N-r) c^{\prime} p}{\sinh N c^{\prime} p} \sigma_{0}(p)-\left[\sinh \left(N c^{\prime} p\right) \sinh \left(c^{\prime} p\right)\right]^{-1}\left\{\sum_{s=1}^{r} h_{s}(p) \sinh \left(s c^{\prime} p\right) \sinh (N-r) c^{\prime} p\right)$

$$
\begin{equation*}
\left.+\sum_{s=r+1}^{N-1} h_{s}(p) \sinh \left[(N-s) c^{\prime} p\right] \sinh \left(r c^{\prime} p\right)\right\}-\sum_{j} \exp \left(-i \Lambda_{r j}^{\mathrm{s}} p\right) \exp \left[-\left(n_{r j}-1\right) c^{\prime}|p|\right] \tag{14}
\end{equation*}
$$

Eq. (14) is the generalization to $\operatorname{SU}(N)$ of eq. (7). We observe that there are now contributions to $\sigma_{r}$ from holes of all ranks, whereas the explicit string contributions are limited to rank $r$. Having determined $\sigma_{1}$, the energy may be calculated from eqs. (4) and (3), with $\Lambda_{\gamma}$ replaced by $\Lambda_{1 \gamma}$.

We are now in a position to discuss the vacuum and low-lying excited states of the model. The vacuum is characterized by (i)
$\varkappa^{+}=\varkappa^{-}=\frac{1}{2} \varkappa, \quad M_{r}-M_{r+1}=\varkappa / N, \quad r=0,1, \ldots, N-1$
(assumę $\mathcal{X}$ is an even number divisible by $N$ ), (ii) the $\Lambda_{r \gamma}$ are real, and (iii) the pseudoparticles occupy the minimum energy levels, with $2 \pi_{j}^{\alpha_{j}} / L=-\alpha_{j} K$. The vacuum energy may be computed from eqs. (3), (4) and (14):
$E_{0}=-(K+\pi / L) \varkappa+\frac{1}{2} \varkappa^{2}(\hat{\varphi}+\pi / N) / L$,
where
$\hat{\varphi}=\varphi+\varkappa^{-1} \int \mathrm{~d} \Lambda \sigma_{1}^{\mathrm{vac}}(\Lambda)[\theta(2 \Lambda-2)-\theta(2 \Lambda+2)]-\frac{N-1}{N} \pi$
and, from eq. (14)
$\sigma_{1}^{\mathrm{vac}}(p)=\Upsilon\left\{\sinh \left[(N-1) c^{\prime} p\right] / \sinh \left(N c^{\prime} p\right)\right\} \cos p$.
The pseudoparticle number $\bigcap_{0}$ is obtained by minimizing $E_{0}$ (dropping terms of order $L^{0}$ )
$\chi_{0}=N K L /(\pi+N \hat{\varphi})$.
We note that for $c \rightarrow 0, \hat{\varphi} \sim c / 2 N, \bigcap_{0} / L \sim N K / \pi$.
The simplest excitation, as in the $\mathrm{SU}(2)$ case, consists of raising one pseudoparticle from its vacuum level to a higher level, without changing the $\Lambda_{r \gamma}$. This produces a massless boson spectrum analogous to that occurring in the Thirring model. A more interesting excited state is obtained by adding $r<N$ pseudoparticles to the vacuum, such that $\chi^{ \pm}=\frac{1}{2} \chi_{0}+r^{ \pm}$and $M_{s}=M_{\mathrm{s}}^{0}+r-s$. To the $\chi_{0} / M \times N$ rectangular Young diagram describing the color symmetry of the vacuum has been appended a column of length $r$. This generates a hole of rank $r$, say at $\Lambda^{0}$, and we have
$\Delta \sigma_{1}(p)=\frac{\sinh (N-1) c^{\prime} p}{\sinh N c^{\prime} p}\left(r^{+} \mathrm{e}^{-\mathrm{i} p}+r^{-} \mathrm{e}^{\mathrm{i} p}\right)-\frac{\sinh (N-r) c^{\prime} p}{\sinh N c^{\prime} p} \mathrm{e}^{c|p|} \mathrm{e}^{-\mathrm{i} \Lambda^{0} p}$.
The change in energy is readily calculated, yielding for $K \rightarrow \infty, c(K) \rightarrow 0$,
$\Delta E \sim \sum_{i=1}^{r^{+}} q_{i}^{+}+\sum_{i=1}^{r^{-}} q_{i}^{-}+\varkappa_{0} L^{-1}\left(\sin \frac{r \pi}{N}\right) \frac{\cosh \left(\Lambda^{0} \pi / N c^{\prime}\right)}{\sinh \left(\pi / N c^{\prime}\right)} \rightarrow \sum_{i=1}^{r^{+}} q_{i}^{+}+\sum_{i=1}^{r^{-}} q_{i}^{-}+m \sin (r \pi / N) \cosh \chi$,
where $m=(2 N K / \pi) \mathrm{e}^{-\pi / N c^{\prime}}$ and $\chi=\Lambda^{0} \pi / N c^{\prime}$ are taken to be cut-off independent. The state thus describes; in addition to $r$ Thirring-like excitation carrying the chirality and charge ( $\mathcal{\chi}-\mathcal{K}_{0}$ ), a color-bearing particle of nonzero mass
$m_{r}=m \sin (r \pi / N)$.
The massive particles of rank $r$ and $N-r$ have the same mass and transform contragrediently to one another under $\operatorname{SU}(N)$; they may thus be considered antiparticles of one another. Moreover, it is clear from our construction that an appropriate interpolating field for a particle of rank $r$ is an antisymmetrized normal product of $r \psi_{a \pm}^{*}$ fields, and thus, in a field-theoretic sense, the particle of rank $r$ is a bound state of $r$ fundamental particles of rank one. In particular, the antiparticle of the fundamental particle is a bound state of $N-1$ such particles. This interpretation was used recently, in refs [6,7] to derive the $S$-matrix of the model (assuming factorizability), including the mass formula (20).

Another low-lying excited state has $\mathfrak{K}^{ \pm}=\frac{1}{2} \bigcap_{0}$ and $M_{r}=(N-r)\left(\mathfrak{K}_{0} / N-1\right), r-1, \ldots, N-1$. It has $N$ holes of rank one, at $\Lambda^{l}, l=1, \ldots, N$. Thus
$\Delta \sigma_{1}=-\frac{\sinh (N-1) c^{\prime} p}{\sinh N c^{\prime} p} \mathrm{e}^{c^{\prime}|p|} \sum_{l=1}^{N} \mathrm{e}^{-\mathrm{i} \Lambda l p}$
and, in the infinite cut-off limit,

$$
\begin{equation*}
\Delta E=\sum_{l=1}^{N} m_{1} \cosh \left(\Lambda^{l} \pi / N c^{\prime}\right) . \tag{22}
\end{equation*}
$$

The state is thus one of $N$ unbound fundamental particles, transforming as a symmetric tensor of rank $N$ under $\mathrm{SU}(N)$.

There are, of course, hosts of other excitations. They all may be classified, however, in terms of Thirring-like massless excitations and the family of massive particles described above.

Conclusion. In refs. $[6,7]$, the spectrum of the chiral invariant Gross-Neveu model was obtained assuming the factorizability of the $S$-matrix. In the present note we have given a constructive, non-perturbative derivation of the spectrum without making assumptions about the $S$-matrix. In fact, we intend to show, in a forthcoming publication, that the factorizable $S$-matrix of refs. $[6,7]$ is a straightforward consequence of our construction.

## References

[1] N. Andrei and J.H. Lowenstein, Diagonalization of the chiral invariant Gross-Neveu hamiltonian, N.Y.U. preprint NYU/TR6/ 79, submitted to Phys. Rev. Lett.
[2] C.N. Yang, Phys. Rev. Lett. 19 (1967) 1312.
[3] H. Bethe, Z. Phys. 71 (1931) 205.
[4] B. Sutherland, Phys. Rev. Lett. 20 (1967) 98.
[5] S.S. Shei, Phys. Rev. D14 (1976) 535.
[6] B. Berg and P. Weisz, Exact $S$-matrix of the chiral invariant $\mathrm{SU}(N)$ Thirring model, F.U. Berlin preprint (Aug. 1978).
[7] V. Kurak and J.A. Swieca, Phys. Lett. 82B (1979) 289.


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    $\neq 1$ Similar results in a related model were obtained independently by A. Belavin (communicated to us by L.D. Fadeev).

