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# The algebraic Bethe ansatz for rational braid–monoid lattice models

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## Abstract

In this paper we study isotropic integrable systems based on the braid–monoid algebra. These systems constitute a large family of rational multistate vertex models and are realized in terms of the  $B_n$ ,  $C_n$  and  $D_n$  Lie algebra and by the superalgebra  $Osp(n|2m)$ . We present a unified formulation of the quantum inverse scattering method for many of these lattice models. The appropriate fundamental commutation rules are found, allowing us to construct the eigenvectors and the eigenvalues of the transfer matrix associated to the  $B_n$ ,  $C_n$ ,  $D_n$ ,  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$  models. The corresponding Bethe ansatz equations can be formulated in terms of the root structure of the underlying algebra. © 1997 Elsevier Science B.V.

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## 1. Introduction

In this paper we look at the problem of diagonalization of the transfer matrix of a certain class of integrable two-dimensional lattice models. Their Boltzmann weights are intimately connected with a rational Baxterization of the braid–monoid algebra (see e.g. Ref. [1] for a review). In particular, we are going to analyse multistate vertex models which are based on the symmetries  $B_n$ ,  $C_n$ ,  $D_n$  and  $Osp(n|2m)$ .

One possible method of finding the eigenvalues of a given transfer matrix is by using the so-called analytical Bethe ansatz [2]. This technique relies on the unitarity, crossing

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and analyticity properties of the transfer matrix and, in some cases, an extra amount of phenomenological input is also required. This method has been applied to some of the models which we are going to consider in this paper, more precisely for the systems  $B_n$ ,  $C_n$ ,  $D_n$  [3,4] and  $Osp(1|2n)$  [5,6]. Unfortunately, the explicit construction of eigenvectors of the transfer matrix is beyond the scope of the analytical Bethe ansatz. The construction of exact eigenvectors, besides being an interesting problem on its own, is certainly an important step in the program of solving integrable systems. Thus, another route has to be taken if one wants to benefit from the knowledge of the eigenvectors. This would be the case of computing lattice correlation functions in the framework developed by Izergin and Korepin [8,9].

A more powerful mathematical method, based on first principles, is the quantum inverse scattering method [7]. This technique, together with the Yang–Baxter relation, offers us a unified viewpoint for studying the properties of integrability of two-dimensional solvable models [7,9–11]. One important feature of this method is that it permits us to present an algebraic formulation of the Bethe states. This step, however, depends much on our ability to disentangle the Yang–Baxter algebra in terms of appropriate commutation relations. The simplest structure of commutation rules has been discovered in the context of the 6-vertex model [7] and its multistate generalizations [12–14].

In this paper, we shall deal with the diagonalization problem of the transfer matrix of certain rational braid–monoid vertex models by means of the quantum inverse scattering method. We shall see that this program can be carried out in a universal way for a quite general class of systems: the  $B_n$ ,  $C_n$ ,  $D_n$ ,<sup>2</sup>  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$  vertex models. We show that the fundamental commutation rules have a common form in terms of the corresponding Boltzmann weights. As a consequence, the derivation of the eigenvectors, the eigenvalues and the associated Bethe ansatz equations also have a quite general character for these vertex models. We believe that the unified picture proposed in this paper is a new result in the literature as well as the Bethe ansatz results for the superalgebra  $Osp(n|2m)$ . We remark that much of our motivation, concerning such a general picture, was prompted by our previous effort of presenting the algebraic Bethe ansatz solution [17] for the four-dimensional representation of the supersymmetric  $spl(2|1)$  vertex model [18–21].

We have organized this papers as follows. In the next section, in order to make this paper self-contained, we review the basic properties of the braid–monoid algebra and its rational Baxterization. A convenient representation for Bethe ansatz analysis is then presented for the  $Osp(n|2m)$  symmetry. In Section 3 we formulate the eigenvalue problem for the transfer matrix in terms of the quantum inverse scattering method. The fundamental commutation rules are explicitly exhibited. In Section 4 we elaborate on the construction of the eigenvectors and eigenvalues of the  $B_n$ ,  $C_n$ ,  $D_n$ ,  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$  vertex models and in Section 5 the

<sup>2</sup> We remark that the algebraic Bethe ansatz solution for the  $D_n$  vertex model has been previously discussed by de Vega and Karowski in Ref. [15] (see also Refs. [2,16]) by using a different construction than the one pursued here (see further discussion in Section 5).

corresponding nested Bethe ansatz equations are derived. These results allow us to conjecture the Bethe ansatz equations of the general  $Osp(n|2m)$  chain. Section 6 is reserved for our conclusion and remarks on the universal picture we have found for braid–monoid vertex systems. In Appendices A and B we present details concerning the two- and the three-particle state, respectively. In Appendix C we collect some useful relations for the supersymmetric formulation of the quantum inverse scattering method.

## 2. The rational braid–monoid solution

The braid–monoid algebra [1] is generated by the identity  $I$ , the braid operator  $b_i$  and the monoid operator  $E_i$  acting on the sites  $i$  of a chain of length  $L$ . In general, these operators satisfy a set of relations which goes by the name of Birman–Wenzl–Murakami algebra [22,1]. We recall that for a braid operator  $b_i$  we mean a object satisfying the braiding relation  $(b_i b_{i\pm 1} b_i = b_{i\pm 1} b_i b_{i\pm 1})$  such that  $b_i b_j = b_j b_i$  for  $|i - j| \geq 2$ . In this paper, we are interested in a degenerated point of this algebra, when the braid operator  $b_i$  and its inverse  $b_i^{-1}$  are identical, i.e.  $b_i = b_i^{-1}$ . Here we choose the braid operator as the graded permutation operator  $b_i \equiv P_i^g$ , defined by the following matrix elements [11]:

$$(P_i^g)_{ab}^{cd} = (-1)^{p(a)p(b)} \delta_{ad} \delta_{bc}, \quad (1)$$

where  $p(a)$  is the Grassmann parity of the  $a$ th degree of freedom, assuming values  $p(a) = 0, 1$ . This is a generalization of the standard operation of permutation [11], which distinguishes the “bosonic” ( $p(a) = 0$ ) and the “fermionic” ( $p(a) = 1$ ) degrees of freedom. The monoid  $E_i$  is a Temperley–Lieb operator [23] and satisfies the relations

$$\begin{aligned} E_i^2 &= \mathcal{K} E_i, \\ E_i E_{i\pm 1} E_i &= E_i, \\ E_i E_j &= E_j E_i, \quad |i - j| \geq 2, \end{aligned} \quad (2)$$

where  $\mathcal{K}$  is a  $c$ -number. The choice we have made for the braid operator (see Eq. (1)) greatly simplifies the constraints between the braid and the monoid. It is possible to show that the constraints closing the braid–monoid algebra on such a degenerated point are given by (see e.g. Refs. [24,25])

$$\begin{aligned} P_i^g E_i &= E_i P_i^g = \hat{t} E_i, \\ E_i P_{i\pm 1}^g P_i^g &= P_{i\pm 1}^g P_i^g E_{i\pm 1} = E_i E_{i\pm 1}, \end{aligned} \quad (3)$$

where the constant  $\hat{t}$  assumes only the values  $\pm 1$ . Any other constraint coming from the Birman–Wenzl–Murakami algebra [1,22] can be derived from (3), from the braiding properties of  $P_i^g$ , the fact that  $(P_i^g)^2 = I$ , and the Temperley–Lieb relations (2). Lastly, we note that these set of relations are invariant by the transformation  $\hat{t} \rightarrow -\hat{t}$  and  $P_i^g \rightarrow -P_i^g$ .

It turns out that the algebraic relations (2) and (3) can be “Baxterized” in terms of rational functions. In other words, it is possible to find a solution  $R(\lambda)$  of the

Yang–Baxter equation in terms of a certain combination of the identity,  $P_i^g$  and  $E_i$ . The solution comes in terms of rational functions [24,25] and is given by<sup>3</sup>

$$R(\lambda) = I + \lambda P^g - \frac{\lambda}{\lambda - \Delta} E, \quad (4)$$

where  $\Delta = (2 - \mathcal{K})\hat{t}/2$ . This solution has a quasi-classical analog,<sup>4</sup> it is regular at  $\lambda = 0$  and also has a crossing point at  $\lambda = \Delta$ . The next step is to search for explicit representations of the Temperley–Lieb operator  $E_i$ . In order to satisfy the braid–monoid restrictions (2) and (3), one possible choice is to set the following ansatz for the monoid  $E_i$  [24]:

$$(E_i)_{ab}^{cd} = \alpha_{ab} \alpha_{cd}^{-1}, \quad (5)$$

where  $\alpha_{ab}$  are the elements of an invertible matrix. A quite general representation for the matrix  $\alpha$  can be found in the context of the  $Osp(n|2m)$  symmetry. The integers  $n$  and  $2m$  stand for the number of bosonic and fermionic degrees of freedom, respectively. The superalgebra  $Osp(n|2m)$  combines the orthogonal  $O(n)$  and the symplectic  $Sp(2m)$  Lie algebras (see e.g. Ref. [27]), and its element  $Z$  satisfies

$$Z + M_{Osp} Z^{st} M_{Osp}^{-1} = 0, \quad (6)$$

where the symbol st denotes the supertranspose operation and the matrix  $M_{Osp}$  is given by

$$M_{Osp} = \begin{pmatrix} I_{n \times n} & O_{n \times 2m} \\ O_{2m \times n} & \begin{pmatrix} O_{m \times m} & I_{m \times m} \\ -I_{m \times m} & O_{m \times m} \end{pmatrix} \end{pmatrix}, \quad (7)$$

where  $I_{k \times k}$  and  $O_{k \times k}$  are the identity and the null  $k \times k$  matrices, respectively. The elements of matrix (7) can be used as an explicit representation [24] for  $\alpha_{ab}$ , i.e.  $\alpha_{ab} = [M_{Osp}]_{ab}$ , and the Temperley–Lieb parameter  $\mathcal{K}$  is then fixed by

$$\mathcal{K} = n - 2m. \quad (8)$$

For general values of  $n$  and  $m$ , however, such a representation breaks the  $U(1)$  invariance of the monoid  $E_i$  [26],<sup>5</sup> and it is not appropriate for Bethe ansatz analysis. Usually, the lack of  $U(1)$  invariance induces extra difficulties on the formulation of a Bethe ansatz, and therefore we should look for other alternatives. This problem can be resolved by the following construction. The monoid preserving the symmetry  $U(1)$  is built in terms of an anti-diagonal matrix  $\alpha$ , whose elements are either +1 or –1. The integer  $m$  is the number of minus signs (–1) and  $n$  is the anti-trace of  $\alpha$ . Furthermore, concerning the

<sup>3</sup> In this paper the  $R$ -matrix is read as  $R(\lambda) = \sum_{abcd} R(\lambda)_{ab}^{cd} e_{ac} \otimes e_{bd}$ , where the matrix elements of  $e_{ab}$  are  $[e_{ab}]_{ij} = \delta_{ai} \delta_{bj}$ .

<sup>4</sup> The quasi-classical  $r$ -matrix can be obtained by redefining  $\lambda \rightarrow \lambda/\eta$  and by expanding it around  $\eta = 0$ .

<sup>5</sup> We remark that the  $Osp(1|2m)$  is an exception to this rule. In this case, a canonical transformation can bring (7) in a  $U(1)$  invariant form.

Table 1

The number of possible states per link  $q$  and the parameters  $\hat{t}$  and  $\mathcal{K}$  for the models  $B_n$ ,  $C_n$ ,  $D_n$ ,  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$

	$B_n$	$C_n$	$D_n$	$Osp(2n-1 2)$	$Osp(2 2n-2)$	$Osp(2n-2 2)$	$Osp(1 2n)$
$q$	$2n+1$	$2n$	$2n$	$2n+1$	$2n$	$2n$	$2n+1$
$\mathcal{K}$	$2n+1$	$-2n$	$2n$	$2n-3$	$4-2n$	$2n-4$	$1-2n$
$\hat{t}$	1	-1	1	1	-1	1	-1

grading structure, the elements  $\alpha_{ab}$  only link degrees of freedom of the same specie, namely  $p(a) = p(b)$ . The possible ways to distribute  $\pm 1$  in the anti-diagonal, for a fixed  $\mathcal{K} = n - 2m$ , are then related by permutation of the grading indices (canonical transformations). However, one needs to make sure that  $\hat{t}$ , the elements  $\alpha_{ab}$  and the parities  $p(a)$  satisfy the braid-monoid relation (3), that is  $\hat{t}\alpha_{ab} = (-1)^{p(a)}\alpha_{ba}$ . For example, one possible representation for the matrix  $\alpha$  is the following block anti-diagonal structure:

$$\alpha_{Osp(n|2m)} = \begin{pmatrix} O_{m \times m} & O_{m \times m} & \mathcal{I}_{n \times n} \\ O_{m \times m} & \mathcal{I}_{m \times m} & O_{n \times n} \\ -\mathcal{I}_{m \times m} & O_{m \times m} & O_{n \times n} \end{pmatrix}, \quad (9)$$

where  $\mathcal{I}_{k \times k}$  is a  $k \times k$  anti-diagonal matrix. In this case, the two compatible sequences of grading are  $f_1 \dots f_m b_1 \dots b_n f_{m+1} \dots f_{2m}$  for  $\hat{t} = 1$  and  $b_1 \dots b_m f_1 \dots f_n b_{m+1} \dots b_{2m}$  for  $\hat{t} = -1$ . One extra advantage of our construction is that the vertex models  $B_n$ ,  $D_n$  and  $C_m$  can be nicely represented in terms of the limits  $m \rightarrow 0$  and  $n \rightarrow 0$ , respectively. More precisely, we have

$$\alpha_{B_n} = \mathcal{I}_{2n+1 \times 2n+1}, \quad \alpha_{D_n} = \mathcal{I}_{2n \times 2n}, \quad \alpha_{C_m} = \begin{pmatrix} O_{m \times m} & \mathcal{I}_{m \times m} \\ -\mathcal{I}_{m \times m} & O_{m \times m} \end{pmatrix}. \quad (10)$$

In this paper, besides the  $B_n$ ,  $C_n$  and  $D_n$  models, we are primarily interested in the supersymmetric models related to the  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$  symmetries. The first reason is because, as we shall see below, the nested Bethe ansatz formulation for the  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$  and  $Osp(2n-2|2)$  models goes fairly parallel to that of the  $B_n$ ,  $C_n$  and  $D_n$  vertex models, respectively. Secondly, because they exhaust the main basic symmetries present in the general  $Osp(n|2m)$  superalgebra. In order to see these relations, we show in Fig. 1 the Dynkin diagrams of the superalgebra  $Osp(n|2m)$  as well as those of the Lie algebras  $B_n$ ,  $C_n$  and  $D_n$ . We note that the  $Osp(1|2n)$  superalgebra has a special structure of roots. In fact, such a special character will be present in many points of our Bethe ansatz analysis of the  $Osp(1|2n)$  vertex models.

We turn now to the analysis of the Boltzmann weights of these vertex models. In general, these are multistate vertex models having one of  $q$  possible states on each bond of the two-dimensional square lattice. The functional form of the Boltzmann weights depends directly on the values of  $\hat{t}$ ,  $\mathcal{K}$ . For the  $Osp(n|2m)$  vertex models, the weights also depend on the sequence of grading that has been chosen. In Table 1 we

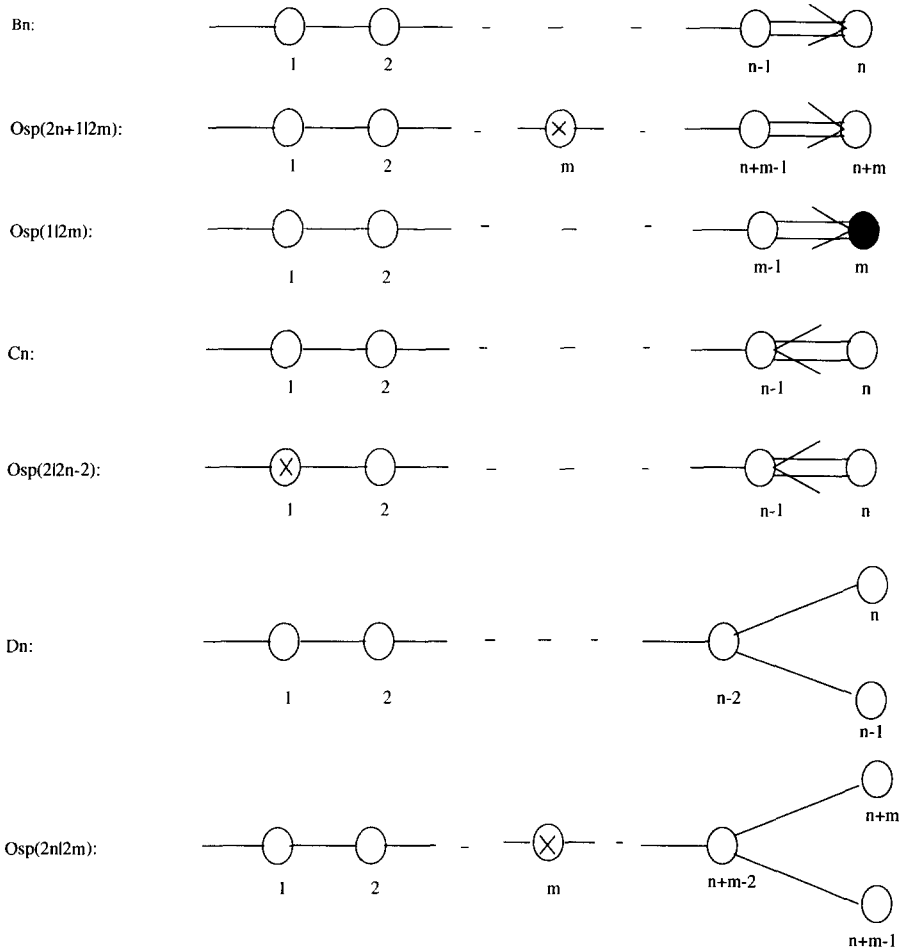


Fig. 1. The Dynkin diagrams of the Lie algebras  $B_n$ ,  $C_n$  and  $D_n$  and of the superalgebra  $Osp(n|2m)$  (see e.g. Ref. [27]). The symbols  $\circ$ ,  $\otimes$  and  $\bullet$  stand for the simple roots of the algebras  $Sl(2)$ ,  $Sl(1|1)$  and  $Osp(1|2)$ , respectively.

have collected the values of  $q$ ,  $\hat{t}$  and  $\mathcal{K}$  for the vertex models  $B_n$ ,  $C_n$ ,  $D_n$ ,  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$ . Moreover, for the first three supersymmetric models we have used the grading  $fb \dots bf$  and for  $Osp(1|2n)$  we have taken the sequence  $b \dots bfb \dots b$ . The basic reason for this choice is because it uses the minimum number of fermionic degrees of freedom in the grading sequence compatible with number  $\mathcal{K}$  and our choice of  $\hat{t}$  (see Table 1), and therefore saving the commutation rules of the presence of many extra minus signs. We remark that the structure of Eq. (9) is compatible with the above-mentioned grading sequences for the  $Osp(2n-1|2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$  vertex models. For the  $Osp(2|2n-2)$  system, however, we still have to perform an extra transformation in matrix (9), by exchanging the least  $-1$  and the  $(n+1)$ th  $+1$  degrees of freedom. More specifically, we have that  $\alpha_{Osp(2|2n-2)} = \text{anti-diagonal}(1_1, \dots, 1_n, -1_{n+1}, -1_{n+2}, \dots, -1_{2n-1}, 1_{2n})$ .

Table 2

The main five Boltzmann weights for the vertex models  $B_n$ ,  $C_n$ ,  $D_n$ ,  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$

	$B_n$	$C_n$	$D_n$	$Osp(2n-1 2)$	$Osp(2 2n-2)$	$Osp(2n-2 2)$	$Osp(1 2n)$
$a(\lambda)$	$\lambda + 1$	$\lambda + 1$	$\lambda + 1$	$1 - \lambda$	$1 - \lambda$	$1 - \lambda$	$\lambda + 1$
$b(\lambda)$	$\lambda$	$\lambda$	$\lambda$	$\lambda$	$\lambda$	$\lambda$	$\lambda$
$c_n(\lambda)$	$\frac{n-1/2}{\lambda+n-1/2}$	$\frac{2\lambda+n+1}{\lambda+n+1}$	$\frac{n-1}{\lambda+n-1}$	$\frac{2\lambda+n-5/2}{\lambda+n-5/2}$	$\frac{n-1}{\lambda+n-1}$	$\frac{2\lambda+n-3}{\lambda+n-3}$	$\frac{2\lambda+n+1/2}{\lambda+n+1/2}$
$d_n(\lambda)$	$\frac{-\lambda}{\lambda+n-1/2}$	$\frac{\lambda}{\lambda+n+1}$	$-\frac{\lambda}{\lambda+n-1}$	$\frac{-\lambda}{\lambda+n-5/2}$	$\frac{\lambda}{\lambda+n-1}$	$\frac{-\lambda}{\lambda+n-3}$	$-\frac{\lambda}{\lambda+n+1/2}$
$e_n(\lambda)$	$\frac{\lambda(\lambda+n-3/2)}{\lambda+n-1/2}$	$\frac{\lambda(\lambda+n)}{\lambda+n+1}$	$\frac{\lambda(\lambda+n-2)}{\lambda+n-1}$	$\frac{-\lambda(\lambda+n-3/2)}{\lambda+n-5/2}$	$-\frac{\lambda(\lambda+n)}{\lambda+n-1}$	$\frac{-\lambda(\lambda+n-2)}{\lambda+n-3}$	$\frac{\lambda(\lambda+n-1/2)}{\lambda+n+1/2}$

This transformation is quite helpful because it allows us to treat the nesting problem for the  $C_n$  and  $Osp(2|2n-2)$  models in a common way.

Taking into account these considerations, we find that there are only a few possible distinct functional forms for the Boltzmann weights. In fact, as we shall see in the next section, our algebraic formulation will require at most five distinct functional dependences. We name these Boltzmann weights by  $a(\lambda)$ ,  $b(\lambda)$ ,  $c_n(\lambda)$ ,  $d_n(\lambda)$ ,  $e_n(\lambda)$  and they are summarized in Table 2. To some extent, this is the miraculous fact of integrability, which in our approach is encoded in the commutation rules to be presented in next section.

### 3. The eigenvalue problem and the commutation rules

We start this section by describing the eigenvalue problem for the transfer matrix  $T(\lambda)$  associated to the vertex models defined in Section 2. We are interested to determine the eigenvalues and the eigenvectors of  $T(\lambda)$  on a square lattice of size  $L \times L$ . The diagonalization problem is defined by

$$T(\lambda) |\Phi\rangle = A(\lambda) |\Phi\rangle. \quad (11)$$

An important object in the quantum inverse scattering method [7,9–11] is the Yang–Baxter algebra of the monodromy matrices. The monodromy matrix  $\mathcal{T}(\lambda)$  acts on the tensor product of an auxiliary space  $\mathcal{A} \equiv \mathbb{C}^q$  and on the quantum Hilbert space  $\mathcal{C}^{Lq}$ . The transfer matrix  $T(\lambda)$  is the trace of the monodromy matrix  $\mathcal{T}(\lambda)$  over the auxiliary space  $\mathcal{A}$ , i.e.  $T(\lambda) = \text{Tr}_{\mathcal{A}} \mathcal{T}(\lambda)$ . A convenient way of writing the monodromy matrix is in terms of a product of operators  $\mathcal{L}_{Ai}(\lambda)$  as

$$T(\lambda) = \mathcal{L}_{AL}(\lambda) \mathcal{L}_{AL-1}(\lambda) \dots \mathcal{L}_{A1}(\lambda), \quad (12)$$

where  $\mathcal{L}_{Ai}(\lambda)$  are  $q \times q$  matrices acting on the lattice sites  $i = 1, \dots, L$  whose elements are operators on the Hilbert space  $\mathcal{C}^{Lq}$ . The elements of the vertex operator  $\mathcal{L}_{ab}^{cd}(\lambda)$  are related to those of the  $R$ -matrix (4) by a permutation on the  $\mathbb{C}^q \times \mathbb{C}^q$  tensor space as

$$\mathcal{L}_{ab}^{cd}(\lambda) = R_{ba}^{cd}(\lambda) \quad (13)$$

and the Yang–Baxter algebra for two monodromy matrices with distinct spectral parameters reads

$$R(\lambda - \mu) T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) R(\lambda - \mu). \quad (14)$$

The intertwining equation (14) is the corner-stone of the quantum inverse scattering approach, allowing us to derive the fundamental commutation relations. Unfortunately, there is no recipe to immediately find the appropriate commutation rules from the Yang–Baxter algebra. Much of the insight comes from the properties of the vertex operator  $\mathcal{L}_{Ai}(\lambda)$  itself and from the reference state we choose to begin the construction of the Hilbert space. As a reference state we take the standard ferromagnetic pseudovacuum given by

$$|0\rangle = \prod_{i=1}^L \otimes |0\rangle_i, \quad |0\rangle_i = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_q, \quad (15)$$

where  $q$  stands for the length of the vectors  $|0\rangle_i$ . The vertex operator  $\mathcal{L}_{Ai}(\lambda)$ , when acting on state  $|0\rangle_i$ , has the following triangular property:

$$\mathcal{L}_{Ai}(\lambda) |0\rangle_i = \begin{pmatrix} a(\lambda) |0\rangle_i & * & * & * & \dots & * & * \\ 0 & b(\lambda) |0\rangle_i & 0 & 0 & \dots & 0 & * \\ 0 & 0 & b(\lambda) |0\rangle_i & 0 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & b(\lambda) |0\rangle_i & * \\ 0 & 0 & 0 & 0 & \dots & 0 & e_n(\lambda) |0\rangle_i \end{pmatrix}_{q \times q}, \quad (16)$$

where the symbol  $*$  represents some values that are not necessary to evaluate explicitly for further discussion. The next step is to write an appropriate ansatz for the matrix representation of  $T(\lambda)$  on the auxiliary space  $\mathcal{A}$ . The triangular property (16) suggests us to seek for the following structure:

$$T(\lambda) = \begin{pmatrix} B(\lambda) & \mathbf{B}(\lambda) & F(\lambda) \\ \mathbf{C}(\lambda) & \hat{A}(\lambda) & \mathbf{B}^*(\lambda) \\ C(\lambda) & \mathbf{C}^*(\lambda) & D(\lambda) \end{pmatrix}_{q \times q}, \quad (17)$$

where  $\mathbf{B}(\lambda)$ ,  $\mathbf{B}^*(\lambda)$ ,  $\mathbf{C}(\lambda)$  and  $\mathbf{C}^*(\lambda)$  are two-component vectors with dimensions  $1 \times (q-2)$ ,  $(q-2) \times 1$ ,  $(q-2) \times 1$  and  $1 \times (q-2)$ , respectively. The operator  $\hat{A}(\lambda)$  is a  $(q-2) \times (q-2)$  matrix and we denote its elements by  $A_{ab}(\lambda)$ . The other remaining operators  $B(\lambda)$ ,  $C(\lambda)$ ,  $F(\lambda)$  and  $D(\lambda)$  are scalars. Putting them all together, we then



have a  $(q \times q)$  matrix representation for the monodromy matrix  $\mathcal{T}(\lambda)$ . Taking into account these definitions, the diagonalization problem (11) for the transfer matrix  $T(\lambda)$  becomes

$$\left[ B(\lambda) + \sum_{a=1}^{q-2} A_{aa}(\lambda) + D(\lambda) \right] |\Phi\rangle = A(\lambda) |\Phi\rangle. \quad (18)$$

As a consequence of definitions (12) and (17) and the triangular property (16), we find that the diagonal operators of  $\mathcal{T}(\lambda)$  satisfy the following relations:

$$\begin{aligned} B(\lambda) |0\rangle &= [a(\lambda)]^L |0\rangle, & D(\lambda) |0\rangle &= [e_n(\lambda)]^L |0\rangle, \\ A_{aa}(\lambda) |0\rangle &= [b(\lambda)]^L |0\rangle, & a &= 1, \dots, q-2, \end{aligned} \quad (19)$$

as well as the annihilation properties

$$\begin{aligned} C_a(\lambda) |0\rangle &= 0, & C_a^*(\lambda) |0\rangle &= 0, & C(\lambda) |0\rangle &= 0, \\ A_{ab}(\lambda) |0\rangle &= 0, & a, b &= 1, \dots, q-2, & a \neq b. \end{aligned} \quad (20)$$

This means that the reference state is an exact and trivial eigenvector with eigenvalue

$$A(\lambda) = [a(\lambda)]^L + (q-2)[b(\lambda)]^L + [e_n(\lambda)]^L \quad (21)$$

and also that the fields  $B(\lambda)$ ,  $B^*(\lambda)$  and  $F(\lambda)$  should play the role of the creation operators on the reference state. In order to construct the full Hilbert space we need to find the commutation relations between the creation, diagonal and annihilation fields. In principle, all the information concerning the commutation rules are encoded in the integrability condition (14). The basic problem is to collect them in a convenient form. For instance, for the 6-vertex model [7,9] and its multistate generalizations [10,12,13] they come almost directly, after substituting the appropriate form for  $\mathcal{T}(\lambda)$  and the associated  $R$ -matrix on the Yang–Baxter algebra (14). For the models we intend to analyse in this paper, that is the rational braid–monoid vertex models (4), this is not the case and some additional work is necessary. For example, in order to get the “nice” commutation rule between the creation operator  $B(\lambda)$  and the diagonal field  $\hat{A}(\lambda)$  we also have to disentangle the commutation relations between  $B(\mu)$  and  $B^*(\lambda)$ . The trick goes much along the lines we have already explained in detail for the  $spl(2|1)$  supersymmetric vertex model [17]. It turns out that these ideas can be generalized for the  $B_n$ ,  $C_n$ ,  $D_n$ ,  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$  multistate vertex models, and therefore we have been able to derive their fundamental commutation rules. As stressed below, the main procedure is quite cumbersome, and here we just list our final results for the most important commutation rules. Between the diagonal operators  $\hat{A}(\lambda)$ ,  $B(\lambda)$ ,  $D(\lambda)$  and the creation field  $B(\lambda)$  we have

Table 3

The main five Boltzmann weights of the “nested” matrix  $X^{(k)}$ . They are the same for the pairs  $\{B_n, Osp(2n-1|2)\}$ ,  $\{C_n, Osp(2|2n-2)\}$  and  $\{D_n, Osp(2n-2|2)\}$ . The corresponding crossing parameter  $\Delta^{(k)}$  is also listed

$X^{(k)}(\lambda)$	$\{B_n, Osp(2n-1 2)\}$	$\{C_n, Osp(2 2n-2)\}$	$\{D_n, Osp(2n-2 2)\}$	$Osp(1 2n)$
$a(\lambda)$	$\lambda + 1$	$\lambda + 1$	$\lambda + 1$	$\lambda + 1$
$b(\lambda)$	$\lambda$	$\lambda$	$\lambda$	$\lambda$
$c_{n-k}(\lambda)$	$\frac{n-k-1/2}{\lambda+n-k-1/2}$	$\frac{2\lambda+n-k+1}{\lambda+n-k+1}$	$\frac{n-k-1}{\lambda+n-k-1}$	$\frac{2\lambda+n-k+1/2}{\lambda+n-k+1/2}$
$d_{n-k}(\lambda)$	$\frac{-\lambda}{\lambda+n-k-1/2}$	$\frac{\lambda}{\lambda+n-k+1}$	$\frac{-\lambda}{\lambda+n-k-1}$	$-\frac{\lambda}{\lambda+n-k+1/2}$
$e_{n-k}(\lambda)$	$\frac{\lambda(\lambda+n-k-3/2)}{\lambda+n-k-1/2}$	$\frac{\lambda(\lambda+n-k)}{\lambda+n-k+1}$	$\frac{\lambda(\lambda+n-k-2)}{\lambda+n-k-1}$	$\frac{\lambda(\lambda+n-k-1/2)}{\lambda+n-k+1/2}$
$\Delta^{(k)}$	$-n+k+1/2$	$-n+k-1$	$-n+k+1$	$-n+k-1/2$

$$\begin{aligned} \hat{A}(\lambda) \otimes B(\mu) = & \frac{1}{b(\lambda - \mu)} [B(\mu) \otimes \hat{A}(\lambda)] \cdot \tilde{X}^{(1)}(\lambda - \mu) \\ & - \frac{1}{b(\lambda - \mu)} B(\lambda) \otimes \hat{A}(\mu) + \frac{d_n(\lambda - \mu)}{e_n(\lambda - \mu)} \left[ B^*(\lambda) B(\mu) \right. \\ & \left. + \frac{1}{b(\lambda - \mu)} F(\lambda) C(\mu) - \frac{\{1 - \hat{b}(\lambda - \mu)\}}{b(\lambda - \mu)} F(\mu) C(\lambda) \right] \otimes \xi, \end{aligned} \quad (22)$$

$$B(\lambda) B(\mu) = \frac{a(\mu - \lambda)}{b(\mu - \lambda)} B(\mu) B(\lambda) - \frac{1}{b(\mu - \lambda)} B(\lambda) B(\mu), \quad (23)$$

$$\begin{aligned} D(\lambda) B(\mu) = & \frac{b(\lambda - \mu)}{e_n(\lambda - \mu)} B(\mu) D(\lambda) + \frac{1}{e_n(\lambda - \mu)} F(\mu) C^*(\lambda) \\ & - \frac{c_n(\lambda - \mu)}{e_n(\lambda - \mu)} F(\lambda) C^*(\mu) - \frac{d_n(\lambda - \mu)}{e_n(\lambda - \mu)} \xi \cdot \{B^*(\lambda) \otimes \hat{A}(\mu)\}, \end{aligned} \quad (24)$$

where  $\tilde{X}^{(1)}(\lambda)$  is a factorizable auxiliary  $R$ -matrix responsible for the *first* nested Bethe ansatz structure. In what follows we shall see that it is also useful to introduce a second factorizable  $R$ -matrix  $X^{(1)}(\lambda)$ . The matrix elements of  $X^{(1)}$  and  $\tilde{X}^{(1)}(\lambda)$ , however, are related to each other under permutation of their horizontal and vertical spaces, namely  $X^{(1)}(\lambda)_{cd}^{ab} = \tilde{X}^{(1)}(\lambda)_{ab}^{cd}$ . This distinction is necessary in order to include the  $Osp(1|2n)$  model in our discussion. This model is an exception because its auxiliary  $R$ -matrix  $\tilde{X}^{(1)}(\lambda)$  is no longer  $T$  invariant. All the other vertex models, however, have the property  $X^{(1)}(\lambda)_{cd}^{ab} = X^{(1)}(\lambda)_{ab}^{cd}$ , and therefore  $X^{(1)}(\lambda)$  and  $\tilde{X}^{(1)}(\lambda)$  are indeed identical. In Table 3 we have collected the Boltzmann weights of the  $R$ -matrix  $X^{(k)}(\lambda)$  on a certain  $k$ th step of the nested Bethe ansatz. We also show the value of crossing parameter  $\Delta^{(k)}$  corresponding to  $X^{(k)}(\lambda)$ . From this table we note that the pairs of models:  $\{B_n, Osp(2n-1|2)\}$ ,  $\{C_n, Osp(2|2n-2)\}$  and  $\{D_n, Osp(2n-2|2)\}$  share the same auxiliary  $X^{(k)}(\lambda)$  matrix. In fact, this is not that surprising if one takes into account

the similarities between their root structure (see Fig. 1). Furthermore, for these models, the matrix  $X^{(k)}$  is precisely the  $R$ -matrix associated to the  $B_{n-k}$ ,  $C_{n-k}$  and  $D_{n-k}$  vertex models, respectively. However, since the vertex models on a given pair have distinct Boltzmann weights, we shall see that their eigenvectors and eigenvalues will be in fact different too. As before, the  $Osp(1|2n)$  model is an exception. Here, the nesting still keeps the same structure as the original model and we have the embedding  $Osp(1|2n) \subset Osp(1|2n-2) \subset \dots \subset Osp(1|2)$ . Finally, the vector  $\xi$  and its “conjugated”  $\xi^*$  are given in terms of the matrix  $\alpha$ , defining the monoid operator present in the  $R$ -matrix  $X^{(1)}$ . They are defined by

$$\xi = \sum_{a,b=1}^{q-2} \alpha_{ab}^{-1} (\hat{e}_a \otimes \hat{e}_b), \quad \xi^* = \sum_{a,b=1}^{q-2} \alpha_{ab} (\hat{e}_a \otimes \hat{e}_b), \quad (25)$$

where  $\hat{e}_i$  denotes the elementary projection on the  $i$ th position.

We now will give other important commutation relations. The commutation rules between the scalar creation operator  $F(\lambda)$  and the diagonal fields are given by

$$\begin{aligned} \hat{A}_{ab}(\lambda) F(\mu) = & \left[ 1 - \frac{1}{b^2(\lambda - \mu)} \right] F(\mu) \hat{A}_{ab}(\lambda) + \frac{1}{b^2(\lambda - \mu)} F(\lambda) \hat{A}_{ab}(\mu) \\ & - \frac{1}{b(\lambda - \mu)} \left[ [B(\lambda) \otimes B^*(\mu)]_{ba} - [B^*(\lambda) \otimes B(\mu)]_{ab} \right], \end{aligned} \quad (26)$$

$$\begin{aligned} B(\lambda) F(\mu) = & \frac{a(\mu - \lambda)}{e_n(\mu - \lambda)} F(\mu) B(\lambda) - \frac{c_n(\mu - \lambda)}{e_n(\mu - \lambda)} F(\lambda) B(\mu) \\ & - \frac{d_n(\mu - \lambda)}{e_n(\mu - \lambda)} \xi^* \cdot \{B(\lambda) \otimes B(\mu)\}, \end{aligned} \quad (27)$$

$$\begin{aligned} D(\lambda) F(\mu) = & \frac{a(\lambda - \mu)}{e_n(\lambda - \mu)} F(\mu) D(\lambda) - \frac{c_n(\lambda - \mu)}{e_n(\lambda - \mu)} F(\lambda) D(\mu) \\ & - \frac{d_n(\lambda - \mu)}{e_n(\lambda - \mu)} \xi \cdot \{B^*(\lambda) \otimes B^*(\mu)\} \end{aligned} \quad (28)$$

and the commutation relations between the creation operators are

$$\begin{aligned} B(\lambda) \otimes B(\mu) = & \frac{1}{a(\lambda - \mu)} [B(\mu) \otimes B(\lambda)] \cdot X^{(1)}(\lambda - \mu) + \frac{d_n(\lambda - \mu)}{e_n(\lambda - \mu)} \\ & \times \left[ F(\lambda) B(\mu) - \frac{\{1 - \hat{b}(\lambda - \mu)\}}{a(\lambda - \mu)} F(\mu) B(\lambda) \right] \xi, \end{aligned} \quad (29)$$

$$[F(\lambda), F(\mu)] = 0, \quad (30)$$

$$F(\lambda) B(\mu) = \frac{1}{a(\lambda - \mu)} F(\mu) B(\lambda) + \frac{b(\lambda - \mu)}{a(\lambda - \mu)} B(\mu) F(\lambda), \quad (31)$$

$$B(\lambda) F(\mu) = \frac{1}{a(\lambda - \mu)} B(\mu) F(\lambda) + \frac{b(\lambda - \mu)}{a(\lambda - \mu)} F(\mu) B(\lambda). \quad (32)$$

We close this section with the following remark. In this section, we have kept our presentation concerning the basic properties of the quantum inverse scattering approach

as general as possible. All the vertex models have been treated by the same and standard way of formulating the quantum inverse scattering method. We note, however, that the solution of the supersymmetric  $Osp(n|2m)$  models could also be presented in terms of a graded framework from the very beginning. This is possible because the  $R$ -matrix (4) has a null Grassmann parity, and consequently can produce a vertex operator  $\mathcal{L}_{Ai}$  satisfying either the Yang–Baxter equation (standard formulation) or its graded version [11,5]. The last choice, however, is the most natural way of formulating the vertex operator  $\mathcal{L}_{Ai}$  for the  $Osp(n|2m)$  models, if one wants to make a real distinction between the bosonic and fermionic degrees of freedom. In other words, the graded quantum inverse method makes sure that the fermionic degrees of freedom anticommutes even if they act on different lattice sites. In order to accomplish this “non-local” property, one has to use the supersymmetric (graded) formalism developed in Refs. [5,11] (see also Ref. [31]), which basically changes the trace and the tensor product properties by their analogs on the graded space. In Appendix C we have summarized this approach for the  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$  vertex models. We note, however, that the supersymmetric formulation does not simplify the original problem of the diagonalization of the corresponding transfer matrices. We shall come back to this point again in Section 5, where the final results for the eigenvalues and Bethe ansatz equations are presented.

#### 4. The eigenvectors and the eigenvalue construction

In the quantum inverse scattering scheme the eigenvectors are constructed by acting the creation operators on the ferromagnetic pseudovacuum  $|0\rangle$ . The excitations over the pseudovacuum  $|0\rangle$  present a multiparticle feature, characterized by a set of variables  $\{\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)}\}$  which are determined after a posteriori analysis. The way we are going to start our discussion is much inspired from our previous results for the supersymmetric  $sp(2|1)$  vertex model [17]. Indeed, the construction we shall present here is a non-trivial generalization of ideas discussed by us in Ref. [17]. Due to the presence of many kinds of creation fields, it is convenient to represent the  $m_1$ -particle state  $|\Phi_{m_1}(\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)})\rangle$  by the following linear combination:

$$|\Phi_{m_1}(\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)})\rangle = \Phi_{m_1}(\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)}) \cdot \mathcal{F} |0\rangle, \quad (33)$$

where  $\Phi_{m_1}(\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)})$  and  $\mathcal{F}$  are multicomponent vectors with  $(q-2)^{m_1}$  components. The dependence on the creation fields is encoded in the vector  $\Phi_{m_1}(\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)})$  and  $\mathcal{F}$  is a constant in this space of fields. We shall denote the components of  $\mathcal{F}$  by  $\mathcal{F}^{a_{m_1} \dots a_1}$ .

The simplest excitation, i.e. the one-particle state, can be built only in terms of the creation fields  $B(\lambda)$ , namely

$$\Phi_1(\lambda_1^{(1)}) = B(\lambda_1^{(1)}) \quad (34)$$

and as a consequence of Eqs. (33) and (34), the one-particle state is given by

$$\left| \Phi_1(\lambda_1^{(1)}) \right\rangle = B_a(\lambda_1^{(1)}) \mathcal{F}^a |0\rangle, \quad (35)$$

where here and in the following repeated indices denote the sum over  $a = 1, \dots, (q-2)$ . The commutation relations (22)–(24) and properties (19) and (20) can be used to solve the eigenvalue problem for  $|\Phi_1(\lambda_1^{(1)})\rangle$ . The solution is based on the following relations:

$$\begin{aligned} B(\lambda) \left| \Phi_1(\lambda_1^{(1)}) \right\rangle &= \frac{a(\lambda_1^{(1)} - \lambda)}{b(\lambda_1^{(1)} - \lambda)} [a(\lambda)]^L \left| \Phi_1(\lambda_1^{(1)}) \right\rangle \\ &\quad - \frac{1}{b(\lambda_1^{(1)} - \lambda)} [a(\lambda_1^{(1)})]^L B_a(\lambda) F^a |0\rangle, \end{aligned} \quad (36)$$

$$\begin{aligned} D(\lambda) \left| \Phi_1(\lambda_1^{(1)}) \right\rangle &= \frac{b(\lambda - \lambda_1^{(1)})}{e_n(\lambda - \lambda_1^{(1)})} [e_n(\lambda)]^L \left| \Phi_1(\lambda_1^{(1)}) \right\rangle \\ &\quad - \frac{d_n(\lambda - \lambda_1^{(1)})}{e_n(\lambda - \lambda_1^{(1)})} [b(\lambda_1^{(1)})]^L \xi_{ab} B_a^*(\lambda) F^b |0\rangle, \end{aligned} \quad (37)$$

$$\begin{aligned} \sum_{a=1}^{q-2} A_{aa}(\lambda) \left| \Phi_1(\lambda_1^{(1)}) \right\rangle &= \frac{1}{b(\lambda - \lambda_1^{(1)})} \left[ 1 + (q-2)(\lambda - \lambda_1^{(1)}) \right. \\ &\quad \left. - \frac{(\lambda - \lambda_1^{(1)})}{(\lambda - \lambda_1^{(1)} - \Delta^{(1)})} [b(\lambda)]^L \left| \Phi_1(\lambda_1^{(1)}) \right\rangle \right. \\ &\quad \left. - \frac{1}{b(\lambda - \lambda_1^{(1)})} [b(\lambda_1^{(1)})]^L B_a(\lambda) F^a |0\rangle \right. \\ &\quad \left. + \frac{d_n(\lambda - \lambda_1^{(1)})}{e_n(\lambda - \lambda_1^{(1)})} [a(\lambda_1^{(1)})]^L \xi_{ab} B_a^*(\lambda) F^b |0\rangle \right]. \end{aligned} \quad (38)$$

The terms proportional to the eigenstate  $|\Phi_1(\lambda_1^{(1)})\rangle$  are denominated “wanted” terms and contribute to the eigenvalue  $\Lambda(\lambda, \lambda_1^{(1)})$ . The remaining ones are the so-called “unwanted” terms and they have to be cancelled by imposing a further restriction on the variable  $\lambda_1^{(1)}$ . Such constraint goes by the name of Bethe ansatz equation. From relations (36)–(38), for the one-particle state, we find a single equation given by<sup>6</sup>

$$\left[ \frac{a(\lambda_1^{(1)})}{b(\lambda_1^{(1)})} \right]^L = 1. \quad (39)$$

The feature of the two-particle state is a bit more complicated. Now, the scalar operator  $F(\lambda)$  starts to play an important role on the eigenvector construction. This

<sup>6</sup> This means that the degeneracy of the one-particle state is  $(q-2)$ , because no further constraint is necessary for the vector  $\mathcal{F}$ .

becomes clear if one takes into account the commutation rule between two creation fields of type  $B(\lambda)$  (see Eq. (29)). For the two-particle state we have to seek for a combination between two fields of type  $B(\lambda)$  with the scalar field  $F(\lambda)$ . The structure of commutation rule (29) suggests us to take the following combination for the vector  $\Phi_2(\lambda_1^{(1)}, \lambda_2^{(1)})$ :

$$\Phi_2(\lambda_1^{(1)}, \lambda_2^{(1)}) = B(\lambda_1^{(1)}) \otimes B(\lambda_2^{(1)}) + \hat{h}(\lambda_1^{(1)}, \lambda_2^{(1)}) F(\lambda_1^{(1)}) B(\lambda_2^{(1)}) \xi. \quad (40)$$

One way of fixing the function  $\hat{h}(\lambda_1^{(1)}, \lambda_2^{(1)})$  is to note that the eigenvalue problem for the two-particle state generates a certain kind of “unwanted” terms which are proportional to  $\xi \cdot \mathcal{F}$ . They are given by

$$\begin{aligned} & F(\lambda) D(\lambda_1^{(1)}) B(\lambda_2^{(1)}), \\ & B(\lambda) \cdot B^*(\lambda_1^{(1)}) B(\lambda_2^{(1)}), \\ & \xi \cdot [B^*(\lambda) \otimes B^*(\lambda_1^{(1)})] B(\lambda_2^{(1)}). \end{aligned} \quad (41)$$

We call these structures “easy unwanted” terms, because they can be automatically cancelled out by an appropriate choice of the form of the function  $\hat{h}(\lambda_1^{(1)}, \lambda_2^{(1)})$ , namely

$$\hat{h}(\lambda_1^{(1)}, \lambda_2^{(1)}) = \hat{h}(\lambda_1^{(1)} - \lambda_2^{(1)}) = -\frac{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})}. \quad (42)$$

There is a more elegant way, however, to determine the function  $\hat{h}(\lambda_1^{(1)}, \lambda_2^{(1)})$ . In general, we expect that the extra constraints on the variables  $\lambda_i^{(1)}$ , i.e. the Bethe ansatz equations, are invariant under index permutation. This means, for the two-particle state, that the vectors  $\Phi_2(\lambda_1^{(1)}, \lambda_2^{(1)})$  and  $\Phi_2(\lambda_2^{(1)}, \lambda_1^{(1)})$  have to be related in some sense. The commutation rule (29) itself gives us a hint how these two vectors can be related to each other. In fact, it is not difficult to conclude that the following exchange property:

$$\Phi_2(\lambda_1^{(1)}, \lambda_2^{(1)}) = \Phi_2(\lambda_2^{(1)}, \lambda_1^{(1)}) \cdot \frac{X^{(1)}(\lambda_1^{(1)} - \lambda_2^{(1)})}{a(\lambda_1^{(1)} - \lambda_2^{(1)})}, \quad (43)$$

is valid, provided the function  $\hat{h}(\lambda_1^{(1)}, \lambda_2^{(1)})$  is fixed as in Eq. (42). To prove (43) we have used the remarkable identity

$$\xi \cdot X^{(1)}(\lambda_1^{(1)} - \lambda_2^{(1)}) = \frac{\hat{h}(\lambda_1^{(1)}, \lambda_2^{(1)})}{\hat{h}(\lambda_2^{(1)}, \lambda_1^{(1)})} \{1 - \hat{b}(\lambda_1^{(1)} - \lambda_2^{(1)})\} \xi, \quad (44)$$

which relates the vector  $\xi$ , the auxiliary matrix  $X^{(1)}$  and the Boltzmann weights. This is another way to see the role of the vector  $\xi$ , it is an eigenvector of the auxiliary matrix  $X^{(1)}$  with a defined eigenvalue where the function  $\hat{h}(\lambda_1^{(1)}, \lambda_2^{(1)})$  enters in a symmetrical way.

In order to cancel out other remaining “unwanted terms”, it is necessary to impose further restrictions on the variables  $\{\lambda_1^{(1)}, \lambda_2^{(1)}\}$ . In Appendix A we show how many

different kinds of “unwanted terms” can be cancelled out by the following Bethe ansatz equations:

$$\left[ \frac{a(\lambda_i^{(1)})}{b(\lambda_i^{(1)})} \right]^L \prod_{j=1, j \neq i}^2 b(\lambda_i^{(1)} - \lambda_j^{(1)}) \frac{a(\lambda_j^{(1)} - \lambda_i^{(1)})}{b(\lambda_j^{(1)} - \lambda_i^{(1)})} \mathcal{F}^{a_2 a_1} \\ = \tilde{X}^{(1)}(\lambda_i^{(1)} - \lambda_j^{(1)})_{a_2 a_1}^{b_1 b_2} \mathcal{F}^{b_2 b_1}, \quad i = 1, 2. \quad (45)$$

The same kind of reasoning can be used in order to construct the three-particle state. Here, however, it is already interesting to write it in terms of a certain recurrence structure. Inspired by our previous construction for the supersymmetric  $spl(2|1)$  model, we begin with the following ansatz:

$$\Phi_3(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}) = B(\lambda_1^{(1)}) \otimes \Phi_2(\lambda_2^{(1)}, \lambda_3^{(1)}) \\ + \left[ \xi \otimes F(\lambda_1^{(1)}) B(\lambda_3^{(1)}) B(\lambda_2^{(1)}) \right] \hat{h}_1(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}) \\ + \left[ \xi \otimes F(\lambda_1^{(1)}) B(\lambda_2^{(1)}) B(\lambda_3^{(1)}) \right] \hat{h}_2(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}). \quad (46)$$

As before, the functions  $\hat{h}_1(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)})$  and  $\hat{h}_2(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)})$  can be fixed either by collecting together the “easy unwanted” terms or by using the permutation symmetries  $\lambda_2^{(1)} \rightarrow \lambda_3^{(1)}$  and  $\lambda_1^{(1)} \rightarrow \lambda_2^{(1)}$ . We have found that they are given by

$$\hat{h}_1(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}) = - \frac{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \frac{a(\lambda_3^{(1)} - \lambda_2^{(1)})}{b(\lambda_3^{(1)} - \lambda_2^{(1)})} I, \quad (47)$$

$$\hat{h}_2(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}) = - \frac{d_n(\lambda_1^{(1)} - \lambda_3^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_3^{(1)})} \frac{1}{b(\lambda_2^{(1)} - \lambda_3^{(1)})} X_{23}^{(1)}(\lambda_2^{(1)} - \lambda_3^{(1)}). \quad (48)$$

While the permutation  $\lambda_2^{(1)} \rightarrow \lambda_3^{(1)}$  is easily verified, the one concerning the symmetry  $\lambda_1^{(1)} \rightarrow \lambda_2^{(1)}$  is quite cumbersome. One of the main difficulties, for example, is that we have also to commute  $B(\lambda_1^{(1)})$  and  $F(\lambda_2^{(1)})$ . Therefore, the commutation relations (31) and (32) between the creation fields play an important role to disentangle the permutation  $\lambda_1^{(1)} \rightarrow \lambda_2^{(1)}$ . Besides that, certain additional properties between the auxiliary matrix  $X^{(1)}$ , the vector  $\xi$  and the field  $B(\lambda)$  are also necessary. For sake of completeness, we have collected the details of the analysis of the three-particle state in Appendix B. Considering the structures of the two- and three-particle state (Eqs. (40) and (46)), it becomes clear that the  $m_1$ -particle state can be generated by means of a recurrence relation. The general  $m_1$ -particle state can be obtained by using an induction procedure, and our final result is given by<sup>7</sup>

<sup>7</sup> The indices under  $X_{k,k+1}^{(1)}(\lambda_k^{(1)} - \lambda_j^{(1)})$  indicate the positions on the Hilbert space where this matrix acts in a non-trivial way.

$$\begin{aligned}
\Phi_{m_1}(\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_{m_1}^{(1)}) &= B(\lambda_1^{(1)}) \otimes \Phi_{n-1}(\lambda_2^{(1)}, \dots, \lambda_{m_1}^{(1)}) \\
&- \sum_{j=2}^{m_1} \frac{d_n(\lambda_1^{(1)} - \lambda_j^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_j^{(1)})} \prod_{k=2, k \neq j}^{m_1} \frac{a(\lambda_k^{(1)} - \lambda_j^{(1)})}{b(\lambda_k^{(1)} - \lambda_j^{(1)})} \\
&\times \left[ \xi \otimes F(\lambda_1^{(1)}) \Phi_{n-2}(\lambda_2^{(1)}, \dots, \lambda_{j-1}^{(1)}, \lambda_{j+1}^{(1)}, \dots, \lambda_{m_1}^{(1)}) B(\lambda_j^{(1)}) \right] \\
&\times \prod_{k=2}^{j-1} \frac{X_{k,k+1}^{(1)}(\lambda_k^{(1)} - \lambda_j^{(1)})}{a(\lambda_k^{(1)} - \lambda_j^{(1)})}
\end{aligned} \tag{49}$$

and under a consecutive permutation  $\lambda_j^{(1)} \rightarrow \lambda_{j+1}^{(1)}$  the  $m_1$ -particle state satisfies the following relation:

$$\begin{aligned}
\Phi_{m_1}(\lambda_1^{(1)}, \dots, \lambda_j^{(1)}, \lambda_{j+1}^{(1)}, \dots, \lambda_{m_1}^{(1)}) &= \Phi_{m_1}(\lambda_1^{(1)}, \dots, \lambda_{j+1}^{(1)}, \lambda_j^{(1)}, \dots, \lambda_{m_1}^{(1)}) \\
&\times \frac{X_{j,j+1}^{(1)}(\lambda_j^{(1)} - \lambda_{j+1}^{(1)})}{a(\lambda_j^{(1)} - \lambda_{j+1}^{(1)})}.
\end{aligned} \tag{50}$$

The following remark is now in order. So far, we have not commented about the role of the creation field  $B^*(\lambda)$ . From the integrability condition (14), we can verify that the commutation rules between the creation field  $B^*(\lambda)$  with the diagonal operators and with the scalar creation field  $F(\lambda)$  are similar to those in Eqs. (22)–(24) and (26)–(32). Basically, we have to change  $B(\lambda)$  by  $B^*(\lambda)$ , to replace the definitions of  $\hat{A}(\lambda)$  and  $\xi$  by their transpose ones, and to interchange the diagonal operators  $B(\lambda) \leftrightarrow D(\lambda)$ . The three creation fields do not mix all together, but only in pairs as  $\{B^*(\lambda), F(\lambda)\}$  and  $\{B(\lambda), F(\lambda)\}$ . We shall then call  $B^*(\lambda)$  the “dual” of  $B(\lambda)$ . This dual creation field together with  $F(\lambda)$  can also be used to built eigenvectors, and by using the replacements mentioned above we obtain an expression analogous to that of Eq. (49). We think that these two possible ways of constructing the eigenvectors go back to the fact that the vertex operator  $\mathcal{L}_{Ai}(\lambda)$  becomes naturally crossing invariant after multiplying it by the factor  $(\Delta - \lambda)$ . Indeed, it is not difficult to see that their corresponding eigenvalues are related to each other by performing the crossing shift  $x \rightarrow \Delta - x$  both in the transfer matrix parameter and in all the Bethe ansatz rapidities. It seems interesting to analyse to what extent we can benefit from this property of the eigenvectors in order to provide new physical insights for these vertex models.

Let us now turn to the Bethe ansatz conditions on the variables  $\{\lambda_i^{(1)}\}$ . A typical unwanted term, coming from the eigenvalue problem (18), arises when the variables  $\lambda_i^{(1)}$  of the  $m_1$ -particle state is exchanged with the transfer matrix rapidity  $\lambda$ . For instance, this is the case of an unwanted term having the structure  $B_{a_1}(\lambda) B_{a_2}(\lambda_2^{(1)}) \dots B_{a_{m_1}}(\lambda_{m_1}^{(1)})$ . This unwanted term is produced by the action of the diagonal operators  $B(\lambda)$  and  $\sum_a \hat{A}_{aa}(\lambda)$  on the  $m_1$ -particle state. From the commutation rules (22) and (23) we are able to show that they can be collected in the following forms:



$$-\frac{[a(\lambda_1^{(1)})]^L}{b(\lambda_1^{(1)} - \lambda)} \prod_{j=2}^{m_1} \frac{a(\lambda_j^{(1)} - \lambda_1^{(1)})}{b(\lambda_j^{(1)} - \lambda_1^{(1)})} \mathcal{F}^{a_{m_1} \dots a_1} B_{a_1}(\lambda) B_{a_2}(\lambda_2^{(1)}) \dots B_{a_{m_1}}(\lambda_{m_1}^{(1)}) |0\rangle, \quad (51)$$

$$\frac{[b(\lambda_1^{(1)})]^L}{b(\lambda_1^{(1)} - \lambda)} \prod_{j=2}^{m_1} \frac{1}{b(\lambda_1^{(1)} - \lambda_j^{(1)})} T^{(1)}(\lambda = \lambda_1^{(1)}, \{\lambda_i^{(1)}\})_{b_1 \dots b_{m_1}}^{a_1 \dots a_{m_1}} \times \mathcal{F}^{a_{m_1} \dots a_1} B_{b_1}(\lambda) B_{b_2}(\lambda_2^{(1)}) \dots B_{b_{m_1}}(\lambda_{m_1}^{(1)}) |0\rangle, \quad (52)$$

where  $T^{(1)}(\lambda, \{\lambda_i^{(1)}\})$  is the transfer matrix of an inhomogeneous vertex model defined in terms of the auxiliary  $R$ -matrix  $\tilde{X}^{(1)}(\lambda)$  by

$$T^{(1)}(\lambda, \{\lambda_i^{(1)}\})_{b_1 \dots b_{m_1}}^{a_1 \dots a_{m_1}} = [\tilde{X}^{(1)}(\lambda - \lambda_1^{(1)})]_{b_1 d_1}^{c_1 a_1} [\tilde{X}^{(1)}(\lambda - \lambda_2^{(1)})]_{b_2 c_2}^{d_1 a_2} \dots [\tilde{X}^{(1)}(\lambda - \lambda_{m_1}^{(1)})]_{b_{m_1} c_1}^{d_{m_1-1} a_{m_1}}. \quad (53)$$

Similarly, the same kind of reasoning can be pursued for any  $\lambda_i^{(1)}$ , thanks to the property (50) which relates the vector  $\Phi_{m_1}(\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)})$  to  $\Phi_{m_1}(\lambda_i^{(1)}, \dots, \lambda_{m_1}^{(1)})$  by cyclic permutations. Thus, from Eqs. (51) and (52) the unwanted term  $B_{a_1}(\lambda_1^{(1)}) \dots B_{a_i}(\lambda) \dots B_{a_{m_1}}(\lambda_{m_1}^{(1)})$  can be cancelled out, provided that

$$\left[ \frac{a(\lambda_i^{(1)})}{b(\lambda_i^{(1)})} \right]^L \prod_{j=1, j \neq i}^{m_1} b(\lambda_i^{(1)} - \lambda_j^{(1)}) \frac{a(\lambda_j^{(1)} - \lambda_i^{(1)})}{b(\lambda_j^{(1)} - \lambda_i^{(1)})} \mathcal{F}^{a_{m_1} \dots a_1} = T^{(1)}(\lambda = \lambda_i^{(1)}, \{\lambda_j^{(1)}\})_{b_1 \dots b_{m_1}}^{a_1 \dots a_{m_1}} \mathcal{F}^{b_{m_1} \dots b_1}, \quad i = 1, \dots, m_1. \quad (54)$$

This generalizes Eq. (45) for an arbitrary value of the number of “particles”  $m_1$ . In general, the action of the diagonal operators on the  $m_1$ -particle state generates many other species of unwanted terms. A systematic way to collect all of them in families of unwanted terms with a defined structure for a general value of  $m_1$  has eluded us so far. However, we remark that for those we have been able to catalog, such as  $B_{a_1}^*(\lambda_1^{(1)}) \dots B_{a_i}(\lambda) \dots B_{a_{m_1}}(\lambda_{m_1}^{(1)})$ , condition (54) has been explicitly verified. Furthermore, the explicit checks we have performed in two- and three-particle states leaves no doubt that restriction (54) is the unique condition<sup>8</sup> to be imposed on the variables  $\{\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)}\}$ . Anyhow, a rigorous proof should be welcome, because probably it will shed extra light to the mathematical structure we have found for the eigenvectors.

By the same token, the eigenvalue  $\Lambda(\lambda, \{\lambda_i^{(1)}\})$  of the  $m_1$ -particle state can be calculated by keeping only terms proportional to the eigenstate  $|\Phi_{m_1}(\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)})\rangle$ . For instance, that proportional to the term  $B_{a_1}(\lambda_1^{(1)}) \dots B_{a_i}(\lambda_i^{(1)}) \dots B_{a_{m_1}}(\lambda_{m_1}^{(1)})$  can be determined by the commutation relations (22)–(24). By keeping the first terms each time we turn the diagonal operators over one of the  $B(\lambda)$  components, we find that

<sup>8</sup> In a factorizable theory, which is our case here, the two-particle structure already contains the main flavour about the Bethe ansatz equations (see Eq. (45)).

$$\begin{aligned} \Lambda(\lambda, \{\lambda_i^{(1)}\}) = & [a(\lambda)]^L \prod_{i=1}^{m_1} \frac{a(\lambda_i^{(1)} - \lambda)}{b(\lambda_i^{(1)} - \lambda)} + [e_n(\lambda)]^L \prod_{i=1}^{m_1} \frac{b(\lambda - \lambda_i^{(1)})}{e_n(\lambda - \lambda_i^{(1)})} \\ & + [b(\lambda)]^L \prod_{i=1}^{m_1} \frac{1}{b(\lambda - \lambda_i^{(1)})} \Lambda^{(1)}(\lambda, \{\lambda_i^{(1)}\}), \end{aligned} \quad (55)$$

where  $\Lambda^{(1)}(\lambda, \{\lambda_i^{(1)}\})$  is the eigenvalue of the inhomogeneous transfer matrix  $T^{(1)}(\lambda, \{\lambda_i^{(1)}\})$ ,

$$T^{(1)}(\lambda, \{\lambda_i^{(1)}\})_{a_1 \dots a_{m_1}}^{b_1 \dots b_{m_1}} \mathcal{F}^{b_{m_1} \dots b_1} = \Lambda^{(1)}(\lambda, \{\lambda_i^{(1)}\}) \mathcal{F}^{a_{m_1} \dots a_1} \quad (56)$$

and therefore the Bethe ansatz equations (54) are then disentangled in terms of the eigenvalue  $\Lambda^{(1)}(\lambda, \{\lambda_i^{(1)}\})$  as

$$\begin{aligned} \left[ \frac{a(\lambda_i^{(1)})}{b(\lambda_i^{(1)})} \right]^L \prod_{j=1, j \neq i}^{m_1} b(\lambda_i^{(1)} - \lambda_j^{(1)}) \frac{a(\lambda_j^{(1)} - \lambda_i^{(1)})}{b(\lambda_j^{(1)} - \lambda_i^{(1)})} = \Lambda^{(1)}(\lambda = \lambda_i^{(1)}, \{\lambda_j^{(1)}\}), \\ i = 1, \dots, m_1. \end{aligned} \quad (57)$$

This completes the first step of our analysis, because we still need to find the eigenvalue  $\Lambda^{(1)}(\lambda, \{\lambda_i^{(1)}\})$  of the transfer matrix  $T^{(1)}(\lambda, \{\lambda_i^{(1)}\})$ . This is the so-called nested Bethe ansatz problem and we shall deal with it in the next section.

## 5. The nested Bethe ansatz problem

In the last section we were left with the problem of the diagonalization of the inhomogeneous transfer matrix  $T^{(1)}(\lambda, \{\lambda_i^{(1)}\})$ . This problem can still be solved by the quantum inverse scattering approach, once the Yang–Baxter algebra (14) is extended to accommodate the presence of inhomogeneities. Formally, the corresponding monodromy matrix can be written as

$$T^{(1)}(\lambda, \{\lambda_i^{(1)}\}) = \mathcal{L}_{\mathcal{A}m_1}^{(1)}(\lambda - \lambda_{m_1}^{(1)}) \mathcal{L}_{\mathcal{A}m_1-1}^{(1)}(\lambda - \lambda_{m_1-1}^{(1)}) \dots \mathcal{L}_{\mathcal{A}1}^{(1)}(\lambda - \lambda_1^{(1)}), \quad (58)$$

where the components of the vertex operator  $[\mathcal{L}^{(1)}(\lambda)]_{ab}^{cd}$  are related to that of the R-matrix  $[\tilde{X}^{(1)}(\lambda)]_{ab}^{cd}$  by the standard permutation

$$[\mathcal{L}^{(1)}(\lambda)]_{ab}^{cd} = [\tilde{X}^{(1)}(\lambda)]_{ba}^{cd}. \quad (59)$$

In order to go on, we need first to analyse the properties of the auxiliary matrix  $\tilde{X}^{(1)}(\lambda)$ . For instance, we have to verify if the triangular property of  $\mathcal{L}^{(1)}(\lambda)$  still has the same structure as that present on the original vertex operator  $\mathcal{L}(\lambda)$  we started with (see Eq. (16)). In other words, we have to check whether or not the basic structure of the ansatz (17) for the monodromy matrix is still appropriate. As we shall see, this depends much on the type of the original vertex model we are diagonalizing and also on their number of states per link  $q$ . We recall that in the first step we already lost

two degrees of freedom, remaining  $(q - 2)$  states per link to represent the Boltzmann weights of the matrix  $\tilde{X}^{(1)}(\lambda)$ . Hence, if  $(q - 2) < 3$  the ansatz (17) certainly is not more the convenient one.

Let us, for the moment, suppose that the  $R$ -matrix  $\tilde{X}^{(1)}(\lambda)$  keeps the basic properties of the original model. In this case, we have basically to adapt our main results of Sections 3 and 4 to include the inhomogeneities  $\{\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)}\}$ . For instance, the pseudovacuum  $|0^{(1)}\rangle$  is the usual ferromagnetic state, but now with length  $(q - 2)$ ,

$$|0^{(1)}\rangle = \prod_{i=1}^{m_1} \otimes |0^{(1)}\rangle_i, \quad |0^{(1)}\rangle_i = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{q-2} \quad (60)$$

and the vertex operator  $\mathcal{L}_{\mathcal{A}i}^{(1)}(\lambda - \lambda_i^{(1)})$  has the following triangular property:

$$\mathcal{L}_{\mathcal{A}i}^{(1)}(\lambda - \lambda_i^{(1)}) |0^{(1)}\rangle_i = \begin{pmatrix} a(\lambda - \lambda_i^{(1)})|0^{(1)}\rangle_i & * & * & \dots & * & * \\ 0 & b(\lambda - \lambda_i^{(1)})|0^{(1)}\rangle_i & 0 & \dots & 0 & * \\ \vdots & 0 & \ddots & & 0 & \vdots \\ \vdots & \vdots & & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \dots & b(\lambda - \lambda_i^{(1)})|0^{(1)}\rangle_i & * \\ 0 & 0 & 0 & \dots & 0 & e_{n-1}(\lambda - \lambda_i^{(1)})|0^{(1)}\rangle_i \end{pmatrix}_{(q-2) \times (q-2)} \quad (61)$$

As before, if we assume the following structure for the monodromy matrix  $\mathcal{T}^{(1)}(\lambda, \{\lambda_i^{(1)}\})$ :

$$\mathcal{T}^{(1)}(\lambda, \{\lambda_i^{(1)}\}) = \begin{pmatrix} B^{(1)}(\lambda, \{\lambda_i^{(1)}\}) & \hat{B}^{(1)}(\lambda, \{\lambda_i^{(1)}\}) & F^{(1)}(\lambda, \{\lambda_i^{(1)}\}) \\ C^{(1)}(\lambda, \{\lambda_i^{(1)}\}) & \hat{A}^{(1)}(\lambda, \{\lambda_i^{(1)}\}) & B^{*(1)}(\lambda, \{\lambda_i^{(1)}\}) \\ C^{(1)}(\lambda, \{\lambda_i^{(1)}\}) & C^{*(1)}(\lambda, \{\lambda_i^{(1)}\}) & D^{(1)}(\lambda, \{\lambda_i^{(1)}\}) \end{pmatrix}_{(q-2) \times (q-2)}, \quad (62)$$

we find that the operators of the monodromy matrix  $\mathcal{T}^{(1)}(\lambda, \{\lambda_i^{(1)}\})$  satisfy the following “diagonal” properties:

$$\begin{aligned} B^{(1)}(\lambda, \{\lambda_i^{(1)}\}) |0^{(1)}\rangle &= \prod_{i=1}^{m_1} a(\lambda - \lambda_i^{(1)}) |0^{(1)}\rangle, \\ D^{(1)}(\lambda, \{\lambda_i^{(1)}\}) |0^{(1)}\rangle &= \prod_{i=1}^{m_1} e_{n-1}(\lambda - \lambda_i^{(1)}) |0^{(1)}\rangle, \\ A_{aa}^{(1)}(\lambda, \{\lambda_i^{(1)}\}) |0^{(1)}\rangle &= \prod_{i=1}^{m_1} b(\lambda - \lambda_i^{(1)}) |0^{(1)}\rangle, \quad a = 1, \dots, q-4, \end{aligned} \quad (63)$$

as well as the annihilation properties,

$$\begin{aligned} C_a^{(1)}(\lambda, \{\lambda_i^{(1)}\}) |0^{(1)}\rangle &= 0, \\ C_a^{*(1)}(\lambda, \{\lambda_i^{(1)}\}) |0^{(1)}\rangle &= 0, \\ C^{(1)}(\lambda, \{\lambda_i^{(1)}\}) |0^{(1)}\rangle &= 0, \\ A_{ab}^{(1)}(\lambda, \{\lambda_i^{(1)}\}) |0^{(1)}\rangle &= 0, \quad a, b = 1, \dots, q-4, \quad a \neq b. \end{aligned} \quad (64)$$

In order to diagonalize the transfer matrix  $T^{(1)}(\lambda, \{\lambda_i^{(1)}\})$  we have to introduce a new set of variables  $\{\lambda_1^{(2)}, \dots, \lambda_{m_2}^{(2)}\}$ , which are going to parameterize the multiparticle states of the nesting problem at step 1. Evidently, the structure of the commutation rules (22)–(24) and (26)–(32) and the eigenvector construction of Section 4 remain precisely the same. We basically have to change  $q$  to  $(q-2)$  in the Boltzmann weights, a given operator  $\hat{O}(\lambda)$  by its corresponding  $\hat{O}^{(1)}(\lambda, \{\lambda_i^{(1)}\})$  and to replace the parameters  $\{\lambda_1^{(1)}, \dots, \lambda_{m_1}^{(1)}\}$  by  $\{\lambda_1^{(2)}, \dots, \lambda_{m_2}^{(2)}\}$  in the eigenvector expression (49). It turns out that the eigenvalue expression  $A^{(1)}(\lambda, \{\lambda_i^{(1)}\})$  and the corresponding Bethe ansatz equations will again depend on another auxiliary inhomogeneous vertex model, having now  $(q-4)$  states per link. Of course, we can repeat this procedure until we reach a certain step  $l$  where the underlying auxiliary  $R$ -matrix  $\tilde{X}^{(l)}$  has lost the basic features presented by the original  $R$ -matrix we began with. In general, by using this “nested” procedure we can find a relation between the eigenvalues  $A^{(l)}(\lambda, \{\lambda_j^{(l)}\}, \dots, \{\lambda_j^{(1)}\})$  and  $A^{(l+1)}(\lambda, \{\lambda_j^{(l+1)}\}, \dots, \{\lambda_j^{(1)}\})$ , on the steps  $l$  and  $l+1$ , respectively. We basically have to “dress” our previous result (55) with inhomogeneities, and to consider the appropriate Boltzmann weights of the step  $l$  we are diagonalizing. This relation can be written as

$$\begin{aligned} A^{(l)}\left(\lambda; \{\lambda_j^{(l)}\}, \dots, \{\lambda_j^{(1)}\}\right) &= \prod_{j=1}^{m_l} a(\lambda - \lambda_j^{(l)}) \prod_{j=1}^{m_{l+1}} \frac{a(\lambda_j^{(l+1)} - \lambda)}{b(\lambda_j^{(l+1)} - \lambda)} \\ &\quad + \prod_{j=1}^{m_l} e_{(n-l)}(\lambda - \lambda_j^{(l)}) \prod_{j=1}^{m_{l+1}} \frac{b(\lambda - \lambda_j^{(l+1)})}{e_{(n-l)}(\lambda - \lambda_j^{(l+1)})} \\ &\quad + \prod_{j=1}^{m_l} b(\lambda - \lambda_j^{(l)}) \prod_{j=1}^{m_{l+1}} \frac{1}{b(\lambda - \lambda_j^{(l+1)})} \\ &\quad \times A^{(l+1)}\left(\lambda; \{\lambda_j^{(l+1)}\}, \{\lambda_j^l\}, \dots, \{\lambda_j^{(1)}\}\right). \end{aligned} \quad (65)$$

This last equation has to be understood as a recurrence relation, beginning on step zero. In order to be consistent, we identify the zero step  $l=0$  with the eigenvalue of the original transfer matrix  $T(\lambda)$  we wish to diagonalize. Therefore, we are assuming the following identifications  $A^{(0)} = A$ ,  $\{\lambda_i^{(0)}\} \equiv 0$  and  $m_0 \equiv L$ . Analogously, the Bethe ansatz restriction on the variables  $\{\lambda_j^{(l+1)}\}$  introduced to parameterize the Fock space of the inhomogeneous transfer matrix  $T^{(l)}(\lambda, \{\lambda_j^{(l)}\}, \dots, \{\lambda_j^{(1)}\})$  on step  $l$  is given by

$$\prod_{i=1}^{m_l} \frac{a(\lambda_j^{(l+1)} - \lambda_i^{(l)})}{b(\lambda_j^{(l+1)} - \lambda_i^{(l)})} \prod_{i=1, i \neq j}^{m_{l+1}} b(\lambda_j^{(l+1)} - \lambda_i^{(l+1)}) \frac{a(\lambda_i^{(l+1)} - \lambda_j^{(l+1)})}{b(\lambda_i^{(l+1)} - \lambda_j^{(l+1)})} \\ = A^{(l+1)}(\lambda = \lambda_j^{(l+1)}, \{\lambda_j^{(l+1)}\}, \dots, \{\lambda_j^{(1)}\}), \quad j = 1, \dots, m_{l+1}. \quad (66)$$

We now will particularize our discussion concerning the nesting structure for each pair of models  $\{B_n, Osp(2n-1|2)\}$ ,  $\{C_n, Osp(2|2n-2)\}$ ,  $\{D_n, Osp(2n-2|2)\}$  and for the  $Osp(1|2n)$  vertex models. For the pair  $\{B_n, Osp(2n-1|2)\}$ , it is not difficult to check that the nesting structure developed above (see Eqs. (65) and (66)) works in any step. In this case, the last step consists of an inhomogeneous  $B_1$  vertex model and the respective  $R$ -matrix acts on the  $C^3 \times C^3$  tensor space. The vertex operator  $\mathcal{L}^{B_1}(\lambda)$  has the following structure:

$$\mathcal{L}^{B_1}(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_1(\lambda) & 0 & d_1(\lambda) & 0 & c_1(\lambda) & 0 & 0 \\ 0 & 1 & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_1(\lambda) & 0 & f_1(\lambda) & 0 & d_1(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & 1 & 0 \\ 0 & 0 & c_1(\lambda) & 0 & d_1(\lambda) & 0 & e_1(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & b(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \quad (67)$$

where the Boltzmann weights  $a(\lambda)$ ,  $b(\lambda)$ ,  $c_1(\lambda)$ ,  $d_1(\lambda)$  and  $e_1(\lambda)$  are listed on Table 2 for  $B_1$ , and  $f_1(\lambda) = b(\lambda) + c_1(\lambda)$ . Due to the isomorphism  $B_1 \equiv O(3) \sim SU(2)_{k=2}$ , this vertex model is equivalent to the isotropic spin-1 XXX model [30]. Therefore, the  $B_1$  model can be solved either by adapting the known results for the spin-1 XXX model [30] to include inhomogeneities or by applying the general approach we have developed in Sections 3 and 4. In the latter case, we remark that our construction for the eigenvalues, eigenvectors and commutation rules reduce to that proposed previously by Tarasov in the context of the Izergin–Korepin vertex model [29]. The final result we have found for the eigenvalue  $A^{B_1}(\lambda, \{\lambda_i\}, \{\mu_j\})$  of the inhomogeneous  $B_1$  vertex model is

$$A^{B_1}(\lambda, \{\lambda_i\}, \{\mu_j\}) = \prod_{i=1}^n (\lambda - \lambda_i + 1) \prod_{j=1}^m \frac{\mu_j - \lambda + 1}{\mu_j - \lambda} \\ + \prod_{i=1}^n (\lambda - \lambda_i) \prod_{j=1}^m \frac{(\lambda - \mu_j + 1/2)}{(\lambda - \mu_j)} \frac{(\lambda - \mu_j - 1)}{(\lambda - \mu_j - 1/2)} \\ + \prod_{i=1}^n \frac{(\lambda - \lambda_i)(\lambda - \lambda_i - 1/2)}{(\lambda - \lambda_i + 1/2)} \prod_{j=1}^m \frac{\lambda - \mu_j + 1/2}{\lambda - \mu_j - 1/2}, \quad (68)$$

where  $\{\lambda_1, \dots, \lambda_n\}$  are the inhomogeneities and the variables  $\{\mu_j\}$  satisfy the following Bethe ansatz equation:

$$\prod_{i=1}^n \frac{\mu_j - \lambda_i + 1}{\mu_j - \lambda_i} = \prod_{k=1, k \neq j}^m \frac{\mu_j - \mu_k + 1/2}{\mu_j - \mu_k - 1/2}, \quad j = 1, \dots, m. \quad (69)$$

On the other hand, the situation for the pairs of models  $\{C_n, Osp(2|2n-2)\}$  and  $\{D_n, Osp(2n-2|2)\}$  is a little bit different. In these cases, we can proceed by using the recurrence relation (65) and the Bethe ansatz condition (66) until we reach the steps  $l = n-1$  and  $l = n-2$ , respectively. For the  $C_n$  and  $Osp(2|2n-2)$  vertex models the nesting level  $l = n-1$  corresponds to the diagonalization of an inhomogeneous transfer matrix whose Boltzmann weights have the 6-vertex symmetry. Indeed, at this level we have to diagonalize the inhomogeneous  $C_1$  system whose vertex operator  $\mathcal{L}^{C_1}(\lambda)$  is given by

$$\mathcal{L}^{C_1}(\lambda) = (\lambda + 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\lambda}{\lambda+2} & \frac{2}{\lambda+2} & 0 \\ 0 & \frac{2}{\lambda+2} & \frac{\lambda}{\lambda+2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (70)$$

The diagonalization of the 6-vertex models on a irregular lattice has appeared in many different contexts in the literature (see for instance Refs. [12,13,15,17,31]). By adapting these results, in order to consider the particular structure of  $\mathcal{L}^{C_1}(\lambda)$ , we find that the eigenvalue  $A^{C_1}(\lambda, \{\lambda_i\}, \{\mu_j\})$  is

$$\begin{aligned} A^{C_1}(\lambda, \{\lambda_i\}, \{\mu_j\}) &= \prod_{i=1}^n (\lambda - \lambda_i + 1) \prod_{j=1}^m \frac{\mu_j - \lambda + 2}{\mu_j - \lambda} \\ &\quad + \prod_{i=1}^n \frac{(\lambda - \lambda_i)(\lambda - \lambda_i + 1)}{(\lambda - \lambda_i + 2)} \prod_{j=1}^m \frac{\lambda - \mu_j + 2}{\lambda - \mu_j}, \end{aligned} \quad (71)$$

where the variables  $\{\mu_j\}$  satisfy the Bethe ansatz condition

$$\prod_{i=1}^n \frac{\mu_j - \lambda_i}{\mu_j - \lambda_i + 2} = \prod_{k=1, k \neq j}^m \frac{\mu_j - \mu_k - 2}{\mu_j - \mu_k + 2}, \quad j = 1, \dots, m. \quad (72)$$

We now turn to the analysis of the last nesting level  $l = n-2$  for the models  $D_n$  and  $Osp(2n-2|2)$ . The final stage for these systems consists of the diagonalization of the  $D_2$  vertex model.<sup>9</sup> It turns out, however, that the  $D_2$  weights can be decomposed in terms of the tensor product of two 6-vertex models. More precisely, the vertex operator  $\mathcal{L}^{D_2}(\lambda)$  can be written as

$$\mathcal{L}^{D_2}(\lambda) = \mathcal{L}_\sigma^{6\text{-ver}} \otimes \mathcal{L}_\tau^{6\text{-ver}}, \quad (73)$$

where the two 6-vertex structures are given by

<sup>9</sup> Naively, one would think that this model can still be reduced to the  $D_1$  vertex model. This is not the case, because the  $R$ -matrix of the  $D_1$  vertex model ( $\Delta = 0$  in Eq. (4)) is no longer regular at  $\lambda = 0$ .

$$\mathcal{L}_\sigma^{6\text{-ver}}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\lambda}{\lambda+1} & \frac{1}{\lambda+1} & 0 \\ 0 & \frac{1}{\lambda+1} & \frac{\lambda}{\lambda+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{L}_\tau^{6\text{-ver}}(\lambda) = \begin{pmatrix} \lambda+1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & \lambda+1 \end{pmatrix}. \quad (74)$$

Consequently, the eigenvalues of the model  $D_2$  are given in terms of the product of the eigenvalues of the two 6-vertex models defined in Eq. (74). In the presence of inhomogeneities  $\{\lambda_1, \dots, \lambda_n\}$  we find that these eigenvalues are given by

$$\Lambda^{D_2}(\lambda, \{\lambda_i\}, \{\mu_j^\pm\}) = \left[ \prod_{j=1}^{m_+} \frac{\lambda - \mu_j^+ - 1}{\lambda - \mu_j^+} + \prod_{i=1}^n \frac{\lambda - \lambda_i}{\lambda - \lambda_i + 1} \prod_{j=1}^{m_+} \frac{\lambda - \mu_j^+ + 1}{\lambda - \mu_j^+} \right] \\ \times \left[ \prod_{j=1}^{m_-} \frac{\lambda - \mu_j^- - 1}{\lambda - \mu_j^-} \prod_{i=1}^n (\lambda - \lambda_i + 1) + \prod_{j=1}^{m_-} \frac{\lambda - \mu_j^- + 1}{\lambda - \mu_j^-} \prod_{i=1}^n (\lambda - \lambda_i) \right], \quad (75)$$

where the variables  $\{\mu_j^+\}$  and  $\{\mu_j^-\}$  parameterize the multiparticle states of the inhomogeneous models related to  $\mathcal{L}_\sigma^{6\text{-ver}}(\lambda)$  and  $\mathcal{L}_\tau^{6\text{-ver}}(\lambda)$ , respectively. They satisfy the following Bethe ansatz equations:

$$\prod_{i=1}^n \frac{\mu_j^\pm - \lambda_i}{\mu_j^\pm - \lambda_i + 1} = \prod_{k=1, k \neq j}^{m_\pm} \frac{\mu_j^\pm - \mu_k^\pm - 1}{\mu_j^\pm - \mu_k^\pm + 1}, \quad j = 1, \dots, m_\pm. \quad (76)$$

Finally, it remains to investigate the  $Osp(1|2n)$  vertex model. In this case, the nesting recurrence relations (65) and (66) are valid for any level. On the last step we have to deal with the inhomogeneous  $Osp(1|2)$  model, possessing the following vertex operator:

$$\mathcal{L}^{Osp(1|2)}(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_1(\lambda) & 0 & -d_1(\lambda) & 0 & c_1(\lambda) & 0 & 0 \\ 0 & 1 & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_1(\lambda) & 0 & f_1(\lambda) & 0 & d_1(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & 1 & 0 \\ 0 & 0 & c_1(\lambda) & 0 & d_1(\lambda) & 0 & e_1(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & b(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \quad (77)$$

where the Boltzmann weights  $a(\lambda)$ ,  $b(\lambda)$ ,  $c_1(\lambda)$ ,  $d_1(\lambda)$  and  $e_1(\lambda)$  are listed in Table 2, and  $f_1(\lambda) = 1 - b(\lambda) + d_1(\lambda)$ . In Ref. [6] such a vertex model was mapped on a certain isotropic branch of the Izergin-Korepin model, and consequently its algebraic Bethe ansatz solution is similar to that developed by Tarasov [29] (see also Ref. [6]).

By including the inhomogeneities  $\{\lambda_1, \dots, \lambda_n\}$ , we conclude that the corresponding eigenvalues are

$$\begin{aligned} \Lambda^{Osp(1|2)}(\lambda, \{\lambda_i\}, \{\mu_j\}) = & \prod_{i=1}^n (\lambda - \lambda_i + 1) \prod_{j=1}^m \frac{\mu_j - \lambda + 1}{\mu_j - \lambda} \\ & + \prod_{i=1}^n (\lambda - \lambda_i) \prod_{j=1}^m -\frac{(\lambda - \mu_j - 1/2)}{(\lambda - \mu_j)} \frac{(\lambda - \mu_j + 1)}{(\lambda - \mu_j + 1/2)} \\ & + \prod_{i=1}^n \frac{(\lambda - \lambda_i)(\lambda - \lambda_i + 1/2)}{(\lambda - \lambda_i + 3/2)} \prod_{j=1}^m \frac{\lambda - \mu_j + 3/2}{\lambda - \mu_j + 1/2}, \end{aligned} \quad (78)$$

where the variables  $\{\mu_j\}$  satisfy the equation

$$\prod_{i=1}^n \frac{\mu_j - \lambda_i + 1}{\mu_j - \lambda_i} = - \prod_{k=1}^m -\frac{(\mu_j - \mu_k - 1/2)(\mu_j - \mu_k + 1)}{(\mu_j - \mu_k + 1/2)(\mu_j - \mu_k - 1)}, \quad j = 1, \dots, m. \quad (79)$$

Now we come to the point where all the results can be gathered together in the following way. Supposing we are interested in the eigenvalues of our original vertex model, we start with the eigenvalue formula (55) and use recurrence relation (65) until we reach the problem of diagonalizing the  $B_1$ ,  $C_1$ ,  $D_2$  and  $Osp(1|2)$  models. Then, we have to take into account our results for the eigenvalues of these systems, which are collected in Eqs. (68), (71), (75) and (78). By using this recipe, it is straightforward to find the eigenvalues expressions for the vertex models we have so far discussed in this paper. We now will list our results for the eigenvalues. For the  $B_n$  ( $n \geq 1$ ) model we have

$$\begin{aligned} \Lambda^{B_n}(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}) = & [a(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} - \frac{1}{2}}{\lambda - \lambda_j^{(1)} + \frac{1}{2}} \\ & + [e_n(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + n}{\lambda - \lambda_j^{(1)} + n - 1} + [b(\lambda)]^L \sum_{l=1}^{2n-1} G_l(\lambda, \{\lambda_j^{(\beta)}\}), \end{aligned} \quad (80)$$

where the functions  $G_l(\lambda, \{\lambda_j^{(\beta)}\})$  are given by

$$\begin{aligned} G_l(\lambda, \{\lambda_j^{(\beta)}\}) = & \begin{cases} \prod_{j=1}^{m_l} \frac{\lambda - \lambda_j^{(l)} + \frac{l+2}{2}}{\lambda - \lambda_j^{(l)} + \frac{l}{2}} \prod_{k=1}^{m_{l+1}} \frac{\lambda - \lambda_k^{(l+1)} + \frac{l-1}{2}}{\lambda - \lambda_k^{(l+1)} + \frac{l+1}{2}}, & l = 1, \dots, n-1, \\ \prod_{k=1}^{m_n} \frac{(\lambda - \lambda_k^{(n)} + \frac{n-2}{2})}{(\lambda - \lambda_k^{(n)} + \frac{n}{2})} \frac{(\lambda - \lambda_k^{(n)} + \frac{n+1}{2})}{(\lambda - \lambda_k^{(n)} + \frac{n-1}{2})}, & l = n, \\ G_{2n-l}(1/2 - n - \lambda, \{-\lambda_j^{(\beta)}\}), & l = n+1, \dots, 2n-1. \end{cases} \end{aligned} \quad (81)$$



For  $C_n$  ( $n \geq 2$ ) we have

$$A^{C_n}(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}) = [a(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} - \frac{1}{2}}{\lambda - \lambda_j^{(1)} + \frac{1}{2}} \\ + [e_n(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + \frac{2n+3}{2}}{\lambda - \lambda_j^{(1)} + \frac{2n+1}{2}} + [b(\lambda)]^L \sum_{l=1}^{2n-2} G_l(\lambda, \{\lambda_j^{(\beta)}\}), \quad (82)$$

where the functions  $G_l(\lambda, \{\lambda_j^{(\beta)}\})$  are

$$G_l(\lambda, \{\lambda_j^{(\beta)}\}) = \begin{cases} \prod_{j=1}^{m_l} \frac{\lambda - \lambda_j^{(l)} + \frac{l+2}{2}}{\lambda - \lambda_j^{(l)} + \frac{l}{2}} \prod_{k=1}^{m_{l+1}} \frac{\lambda - \lambda_k^{(l+1)} + \frac{l-1}{2}}{\lambda - \lambda_k^{(l+1)} + \frac{(l+1)}{2}}, & l = 1, \dots, n-2, \\ \prod_{k=1}^{m_{n-1}} \frac{\lambda - \lambda_k^{(n-1)} + \frac{n+1}{2}}{\lambda - \lambda_k^{(n-1)} + \frac{n-1}{2}} \prod_{k=1}^{m_n} \frac{\lambda - \lambda_k^{(n)} + \frac{n-3}{2}}{\lambda - \lambda_k^{(n)} + \frac{n+1}{2}}, & l = n-1, \\ G_{2n-1-l}(-1-n-\lambda, \{-\lambda_j^{(\beta)}\}), & l = n, \dots, 2n-2. \end{cases} \quad (83)$$

For  $D_n$  ( $n \geq 3$ ) we have

$$A^{D_n}(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}) = [a(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} - \frac{1}{2}}{\lambda - \lambda_j^{(1)} + \frac{1}{2}} \\ + [e_n(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + \frac{2n-1}{2}}{\lambda - \lambda_j^{(1)} + \frac{2n-3}{2}} + [b(\lambda)]^L \sum_{l=1}^{2n-2} G_l(\lambda, \{\lambda_j^{(\beta)}\}), \quad (84)$$

where the functions  $G_l(\lambda, \{\lambda_j^{(\beta)}\})$  are

$$G_l(\lambda, \{\lambda_j^{(\beta)}\}) = \begin{cases} \prod_{j=1}^{m_l} \frac{\lambda - \lambda_j^{(l)} + \frac{l+2}{2}}{\lambda - \lambda_j^{(l)} + \frac{l}{2}} \prod_{k=1}^{m_{l+1}} \frac{\lambda - \lambda_k^{(l+1)} + \frac{l-1}{2}}{\lambda - \lambda_k^{(l+1)} + \frac{l+1}{2}}, & l = 1, \dots, n-3, \\ \prod_{j=1}^{m_{n-2}} \frac{\lambda - \lambda_j^{(n-2)} + \frac{n}{2}}{\lambda - \lambda_j^{(n-2)} + \frac{n-2}{2}} \prod_{j=1}^{m_+} \frac{\lambda - \lambda_j^{(+)} + \frac{n-3}{2}}{\lambda - \lambda_j^{(+)} + \frac{n-1}{2}} \prod_{j=1}^{m_-} \frac{\lambda - \lambda_j^{(-)} + \frac{n-3}{2}}{\lambda - \lambda_j^{(-)} + \frac{n-1}{2}}, & l = n-2, \\ \prod_{j=1}^{m_+} \frac{\lambda - \lambda_j^{(+)} + \frac{n-3}{2}}{\lambda - \lambda_j^{(+)} + \frac{n-1}{2}} \prod_{j=1}^{m_-} \frac{\lambda - \lambda_j^{(-)} + \frac{n+1}{2}}{\lambda - \lambda_j^{(-)} + \frac{n-1}{2}}, & l = n-1, \\ G_{2n-1-l}(1-n-\lambda, \{-\lambda_j^{(\beta)}\}), & l = n, \dots, 2n-2 \end{cases} \quad (85)$$

For  $Osp(2n-1|2)$  ( $n \geq 2$ ) we have

$$\begin{aligned}
\Lambda^{Osp(2n-1|2)}\left(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}\right) &= [a(\lambda)]^L \prod_{j=1}^{m_1} -\frac{\lambda - \lambda_j^{(1)} + \frac{1}{2}}{\lambda - \lambda_j^{(1)} - \frac{1}{2}} \\
&+ [e_n(\lambda)]^L \prod_{j=1}^{m_1} -\frac{\lambda - \lambda_j^{(1)} + n - 3}{\lambda - \lambda_j^{(1)} + n - 2} + [b(\lambda)]^L \sum_{l=1}^{2n-1} G_l(\lambda, \{\lambda_j^{(\beta)}\}), \quad (86)
\end{aligned}$$

where the functions  $G_l(\lambda, \{\lambda_j^{(\beta)}\})$  are

$$\begin{aligned}
G_l(\lambda, \{\lambda_j^{(\beta)}\}) &= \begin{cases} \prod_{j=1}^{m_l} \frac{\lambda - \lambda_j^{(l)} + \frac{l}{2}}{\lambda - \lambda_j^{(l)} + \frac{l-2}{2}} \prod_{k=1}^{m_{l+1}} \frac{\lambda - \lambda_k^{(l+1)} + \frac{l-3}{2}}{\lambda - \lambda_k^{(l+1)} + \frac{l-1}{2}}, & l = 1, \dots, n-1, \\ \prod_{k=1}^{m_n} \frac{(\lambda - \lambda_k^{(n)} + \frac{n-1}{2})(\lambda - \lambda_k^{(n)} + \frac{n-4}{2})}{(\lambda - \lambda_k^{(n)} + \frac{n-3}{2})(\lambda - \lambda_k^{(n)} + \frac{n-2}{2})}, & l = n, \\ G_{2n-l}(5/2 - n - \lambda, \{-\lambda_j^{(\beta)}\}), & l = n+1, \dots, 2n-1. \end{cases} \quad (87)
\end{aligned}$$

For  $Osp(2|2n-2)$ , ( $n \geq 2$ ) we have

$$\begin{aligned}
\Lambda^{Osp(2|2n-2)}\left(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}\right) &= [a(\lambda)]^L \prod_{j=1}^{m_1} -\frac{\lambda - \lambda_j^{(1)} + \frac{1}{2}}{\lambda - \lambda_j^{(1)} - \frac{1}{2}} \\
&+ [e_n(\lambda)]^L \prod_{j=1}^{m_1} -\frac{\lambda - \lambda_j^{(1)} + \frac{2n-3}{2}}{\lambda - \lambda_j^{(1)} + \frac{2n-1}{2}} + [b(\lambda)]^L \sum_{l=1}^{2n-2} G_l(\lambda, \{\lambda_j^{(\beta)}\}), \quad (88)
\end{aligned}$$

where the functions  $G_l(\lambda, \{\lambda_j^{(\beta)}\})$  are

$$\begin{aligned}
G_l(\lambda, \{\lambda_j^{(\beta)}\}) &= \begin{cases} \prod_{j=1}^{m_l} \frac{\lambda - \lambda_j^{(l)} + \frac{l}{2}}{\lambda - \lambda_j^{(l)} + \frac{l-2}{2}} \prod_{k=1}^{m_{l+1}} \frac{\lambda - \lambda_k^{(l+1)} + \frac{l-3}{2}}{\lambda - \lambda_k^{(l+1)} + \frac{l-1}{2}}, & l = 1, \dots, n-2, \\ \prod_{j=1}^{m_{n-1}} \frac{\lambda - \lambda_j^{(n-1)} + \frac{n-1}{2}}{\lambda - \lambda_j^{(n-1)} + \frac{n-3}{2}} \prod_{k=1}^{m_n} \frac{\lambda - \lambda_k^{(n)} + \frac{n-5}{2}}{\lambda - \lambda_k^{(n)} + \frac{n-1}{2}}, & l = n-1, \\ G_{2n-1-l}(1 - n - \lambda, \{-\lambda_j^{(\beta)}\}), & l = n, \dots, 2n-2. \end{cases} \quad (89)
\end{aligned}$$

For  $Osp(2n-2|2)$  ( $n \geq 3$ ) we have

$$\begin{aligned}
\Lambda^{Osp(2n-2|2)}\left(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}\right) &= [a(\lambda)]^L \prod_{j=1}^{m_1} -\frac{\lambda - \lambda_j^{(1)} + \frac{1}{2}}{\lambda - \lambda_j^{(1)} - \frac{1}{2}} \\
&+ [e_n(\lambda)]^L \prod_{j=1}^{m_1} -\frac{\lambda - \lambda_j^{(1)} + \frac{2n-7}{2}}{\lambda - \lambda_j^{(1)} + \frac{2n-5}{2}} + [b(\lambda)]^L \sum_{l=1}^{2n-2} G_l(\lambda, \{\lambda_j^{(\beta)}\}), \quad (90)
\end{aligned}$$

where the functions  $G_l(\lambda, \{\lambda_j^{(\beta)}\})$  are

$$G_l(\lambda, \{\lambda_j^{(\beta)}\}) = \begin{cases} \prod_{j=1}^{m_l} \frac{\lambda - \lambda_j^{(l)} + \frac{l}{2}}{\lambda - \lambda_j^{(l)} + \frac{l-2}{2}} \prod_{k=1}^{m_{l+1}} \frac{\lambda - \lambda_k^{(l+1)} + \frac{l-3}{2}}{\lambda - \lambda_k^{(l+1)} + \frac{l-1}{2}}, & l = 1, \dots, n-3, \\ \prod_{j=1}^{m_{n-2}} \frac{\lambda - \lambda_j^{(n-2)} + \frac{n-2}{2}}{\lambda - \lambda_j^{(n-2)} + \frac{n-4}{2}} \prod_{j=1}^{m_+} \frac{\lambda - \lambda_j^{(+)} + \frac{n-5}{2}}{\lambda - \lambda_j^{(+)} + \frac{n-3}{2}} \prod_{j=1}^{m_-} \frac{\lambda - \lambda_j^{(-)} + \frac{n-5}{2}}{\lambda - \lambda_j^{(-)} + \frac{n-3}{2}}, & l = n-2, \\ \prod_{j=1}^{m_+} \frac{\lambda - \lambda_j^{(+)} + \frac{n-5}{2}}{\lambda - \lambda_j^{(+)} + \frac{n-3}{2}} \prod_{j=1}^{m_-} \frac{\lambda - \lambda_j^{(-)} + \frac{n-1}{2}}{\lambda - \lambda_j^{(-)} + \frac{n-3}{2}}, & l = n-1, \\ G_{2n-1-l}(3-n-\lambda, \{-\lambda_j^{(\beta)}\}), & l = n, \dots, 2n-2. \end{cases} \quad (91)$$

Finally, for  $Osp(1|2n)$  ( $n \geq 1$ ) we have

$$A^{Osp(1|2n)}(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}) = [a(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} - \frac{1}{2}}{\lambda - \lambda_j^{(1)} + \frac{1}{2}} + [e_n(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + n + 1}{\lambda - \lambda_j^{(1)} + n} + [b(\lambda)]^L \sum_{l=1}^{2n-1} G_l(\lambda, \{\lambda_j^{(\beta)}\}), \quad (92)$$

where the functions  $G_l(\lambda, \{\lambda_j^{(\beta)}\})$  are

$$G_l(\lambda, \{\lambda_j^{(\beta)}\}) = \begin{cases} \prod_{j=1}^{m_l} \frac{\lambda - \lambda_j^{(l)} + \frac{l+2}{2}}{\lambda - \lambda_j^{(l)} + \frac{l}{2}} \prod_{k=1}^{m_{l+1}} \frac{\lambda - \lambda_k^{(l+1)} + \frac{l-1}{2}}{\lambda - \lambda_k^{(l+1)} + \frac{l+1}{2}}, & l = 1, \dots, n-1, \\ \prod_{k=1}^{m_n} - \frac{(\lambda - \lambda_k^{(n)} + \frac{n-1}{2})}{(\lambda - \lambda_k^{(n)} + \frac{n+1}{2})} \frac{(\lambda - \lambda_k^{(n)} + \frac{n+2}{2})}{(\lambda - \lambda_k^{(n)} + \frac{n}{2})}, & l = n, \\ G_{2n-l}(-1/2 - n - \lambda, \{-\lambda_j^{(\beta)}\}), & l = n+1, \dots, 2n-1. \end{cases} \quad (93)$$

We see that the eigenvalues depend on the parameters  $\{\{\lambda_j^{(1)}\}, \{\lambda_j^{(2)}\}, \dots, \{\lambda_j^{(n)}\}\}$ , which represent the multiparticle Hilbert space of the many steps necessary for the diagonalization of the nesting problem. In the expressions (80)–(93) we have already performed convenient shifts in these parameters,  $\{\lambda_j^{(\beta)}\} \rightarrow \{\lambda_j^{(\beta)}\} - \delta^{(\beta)}$ , in order to present the final results in a more symmetrical way. In Table 4 we show the values for the shifts  $\delta^{(\beta)}$ . These set of variables are constrained by the Bethe ansatz equation, again at each level of the nesting. The same recipe described above for the eigenvalues also works for determining the corresponding Bethe ansatz equations. We start with Eq. (57) and each step of the nesting is disentangled by using the recurrence relation (66). When we reach the last step, we take into account the Bethe ansatz results (69), (72), (76) and (79) for the inhomogeneous  $B_1$ ,  $C_1$ ,  $D_2$  and  $Osp(1|2)$  vertex models. It turns

Table 4

The shifts  $\delta^{(\beta)}$  performed on the Bethe ansatz variables  $\lambda_j^{(\beta)}$ 

	$\delta^{(\beta)}$
$B_n$ and $Osp(1 2n)$	$\beta/2, \beta = 1, \dots, n$
$C_n$	$\begin{cases} \beta/2, & \beta = 1, \dots, n-1 \\ (n+1)/2, & \beta = n \end{cases}$
$D_n$	$\begin{cases} \beta/2, & \beta = 1, \dots, n-2 \\ (n-1)/2, & \beta = \pm \end{cases}$
$Osp(2n-1 2)$	$(\beta-2)/2, \beta = 1, \dots, n$
$Osp(2 2n-2)$	$\begin{cases} (\beta-2)/2, & \beta = 1, \dots, n-1 \\ (n-1)/2, & \beta = n \end{cases}$
$Osp(2n-2 2)$	$\begin{cases} (\beta-2)/2, & \beta = 1, \dots, n-2 \\ (n-3)/2, & \beta = \pm \end{cases}$

out that the nested Bethe ansatz equations have the following structure. All the vertex models share a common part, which can be written as

$$\prod_{k=1}^{m_l-1} \frac{\lambda_j^{(l)} - \lambda_k^{(l-1)} + 1/2}{\lambda_j^{(l)} - \lambda_k^{(l-1)} - 1/2} \prod_{k=1, k \neq j}^{m_l} \frac{\lambda_j^{(l)} - \lambda_k^{(l)} - 1}{\lambda_j^{(l)} - \lambda_k^{(l)} + 1} \prod_{k=1}^{m_{l+1}} \frac{\lambda_j^{(l)} - \lambda_k^{(l+1)} + 1/2}{\lambda_j^{(l)} - \lambda_k^{(l+1)} - 1/2} = 1, \quad (94)$$

$$l = 2, \dots, s(n),$$

where  $s(n) = n-1$  for the  $B_n$ ,  $Osp(2n-1|2)$  and  $Osp(1|2n)$  models;  $s(n) = n-2$  for the  $C_n$  and  $Osp(2|2n-2)$  models;  $s(n) = n-3$  for the  $D_n$  and  $Osp(2n-2|2)$  models. The remaining equations are somewhat model dependent and below we list their particular forms. For the models  $B_n$  ( $n \geq 2$ ),  $C_n$  ( $n \geq 3$ ),  $D_n$  ( $n \geq 4$ ) and  $Osp(1|2n)$  ( $n \geq 2$ ) the equation for the first root  $\{\lambda_j^{(1)}\}$  is given by

$$\left[ \frac{\lambda_j^{(1)} - 1/2}{\lambda_j^{(1)} + 1/2} \right]^L = \prod_{k=1, k \neq j}^{m_1} \frac{\lambda_j^{(1)} - \lambda_k^{(1)} - 1}{\lambda_j^{(1)} - \lambda_k^{(1)} + 1} \prod_{k=1}^{m_2} \frac{\lambda_j^{(1)} - \lambda_k^{(2)} + 1/2}{\lambda_j^{(1)} - \lambda_k^{(2)} - 1/2}, \quad (95)$$

while for the  $Osp(2n-1|2)$  ( $n \geq 2$ ),  $Osp(2|2n-2)$  ( $n \geq 3$ ) and  $Osp(2n-2|2)$  ( $n \geq 4$ ) we have

$$\left[ \frac{\lambda_l^{(1)} + 1/2}{\lambda_l^{(1)} - 1/2} \right]^L = (-1)^{L-m_1-1} \prod_{k=1}^{m_2} \frac{\lambda_l^{(1)} - \lambda_k^{(2)} + 1/2}{\lambda_l^{(1)} - \lambda_k^{(2)} - 1/2}. \quad (96)$$

Due to the peculiar root structure of the models  $C_2$ ,  $D_3$ ,  $Osp(2|2)$  and  $Osp(4|2)$ , their first Bethe ansatz equations are a bit different than that presented in Eqs. (95) and (96), respectively. In order to avoid further confusion we have to quote them separately. For the  $C_2$  model the Bethe ansatz equation for  $\{\lambda_j^{(1)}\}$  is

$$\left[ \frac{\lambda_j^{(1)} - 1/2}{\lambda_j^{(1)} + 1/2} \right]^L = \prod_{k=1, k \neq j}^{m_1} \frac{\lambda_j^{(1)} - \lambda_k^{(1)} - 1}{\lambda_j^{(1)} - \lambda_k^{(1)} + 1} \prod_{k=1}^{m_2} \frac{\lambda_j^{(1)} - \lambda_k^{(2)} + 1}{\lambda_j^{(1)} - \lambda_k^{(2)} - 1}, \quad (97)$$

while for the  $Osp(2|2)$  model we have

$$\left[ \frac{\lambda_l^{(1)} + 1/2}{\lambda_l^{(1)} - 1/2} \right]^L = (-1)^{L-m_1-1} \prod_{k=1}^{m_2} \frac{\lambda_l^{(1)} - \lambda_k^{(2)} + 1}{\lambda_l^{(1)} - \lambda_k^{(2)} - 1}. \quad (98)$$

Furthermore, the first equation for the  $D_3$  model is

$$\begin{aligned} \left[ \frac{\lambda_j^{(1)} - 1/2}{\lambda_j^{(1)} + 1/2} \right]^L &= \prod_{k=1, k \neq j}^{m_1} \frac{\lambda_j^{(1)} - \lambda_k^{(1)} - 1}{\lambda_j^{(1)} - \lambda_k^{(1)} + 1} \prod_{k=1}^{m_+} \frac{\lambda_j^{(1)} - \lambda_k^{(+)} + 1/2}{\lambda_j^{(1)} - \lambda_k^{(+)} - 1/2} \\ &\quad \times \prod_{k=1}^{m_-} \frac{\lambda_j^{(1)} - \lambda_k^{(-)} + 1/2}{\lambda_j^{(1)} - \lambda_k^{(-)} - 1/2} \end{aligned} \quad (99)$$

and for the model  $Osp(4|2)$  we have

$$\left[ \frac{\lambda_l^{(1)} + 1/2}{\lambda_l^{(1)} - 1/2} \right]^L = (-1)^{L-m_1-1} \prod_{k=1}^{m_+} \frac{\lambda_l^{(1)} - \lambda_k^{(+)} + 1/2}{\lambda_l^{(1)} - \lambda_k^{(+)} - 1/2} \prod_{k=1}^{m_-} \frac{\lambda_l^{(1)} - \lambda_k^{(-)} + 1/2}{\lambda_l^{(1)} - \lambda_k^{(-)} - 1/2}. \quad (100)$$

The Bethe ansatz equation for the last variables are common for the pairs  $\{B_n, Osp(2n-1|2)\}$ ,  $\{C_n, Osp(2|2n-2)\}$  and  $\{D_n, Osp(2n-2|2)\}$  and are given as follows. For the  $B_n$  and  $Osp(2n-1|2)$  models the variables  $\{\lambda_j^{(n)}\}$  satisfy the equation

$$\prod_{k=1}^{m_{n-1}} \frac{\lambda_j^{(n)} - \lambda_k^{(n-1)} + 1/2}{\lambda_j^{(n)} - \lambda_k^{(n-1)} - 1/2} \prod_{k=1, k \neq j}^{m_n} \frac{\lambda_j^{(n)} - \lambda_k^{(n)} - 1/2}{\lambda_j^{(n)} - \lambda_k^{(n)} + 1/2} = 1. \quad (101)$$

For  $C_n$  and  $Osp(2|2n-2)$ , the parameters  $\{\lambda_j^{(n-1)}\}$  and  $\{\lambda_j^{(n)}\}$  satisfy the equations<sup>10</sup>

$$\begin{aligned} \prod_{k=1}^{m_{n-2}} \frac{\lambda_j^{(n-1)} - \lambda_k^{(n-2)} + 1/2}{\lambda_j^{(n-1)} - \lambda_k^{(n-2)} - 1/2} \prod_{k=1, k \neq j}^{m_{n-1}} \frac{\lambda_j^{(n-1)} - \lambda_k^{(n-1)} - 1}{\lambda_j^{(n-1)} - \lambda_k^{(n-1)} + 1} \prod_{k=1}^{m_n} \frac{\lambda_j^{(n-1)} - \lambda_k^{(n)} + 1}{\lambda_j^{(n-1)} - \lambda_k^{(n)} - 1} &= 1, \\ \prod_{j=1}^{m_{n-1}} \frac{\lambda_l^{(n)} - \lambda_j^{(n-1)} - 1}{\lambda_l^{(n)} - \lambda_j^{(n-1)} + 1} \prod_{k=1, k \neq j}^{m_n} \frac{\lambda_j^{(n)} - \lambda_k^{(n)} + 2}{\lambda_j^{(n)} - \lambda_k^{(n)} - 2} &= 1. \end{aligned} \quad (102)$$

For  $D_n$  and  $Osp(2n-2|2)$ , the parameters  $\{\lambda_j^{(n-2)}\}$ ,  $\{\lambda_j^{(+)}\}$  and  $\{\lambda_j^{(-)}\}$  satisfy the equations<sup>11</sup>

<sup>10</sup> For  $n=2$  ( $C_2$  and  $Osp(2|2)$  models) we have to consider only the last equation given in (102).

<sup>11</sup> Here we have assumed the identifications  $\lambda_j^{(n-1)} \equiv \lambda_j^{(+)}$  and  $\lambda_j^{(n)} \equiv \lambda_j^{(-)}$ . Moreover, for  $n=3$  ( $D_3$  and  $Osp(4|2)$  models) only the last equation in (103) matters.

$$\begin{aligned}
& \prod_{\gamma=\pm} \prod_{k=1}^{m_\gamma} \frac{\lambda_j^{(n-2)} - \lambda_k^{(\gamma)} + 1/2}{\lambda_j^{(n-2)} - \lambda_k^{(\gamma)} - 1/2} \prod_{k=1, k \neq j}^{m_{n-2}} \frac{\lambda_j^{(n-2)} - \lambda_k^{(n-2)} - 1}{\lambda_j^{(n-2)} - \lambda_k^{(n-2)} + 1} \\
& \times \prod_{k=1}^{m_{n-3}} \frac{\lambda_j^{(n-2)} - \lambda_k^{(n-3)} + 1/2}{\lambda_j^{(n-2)} - \lambda_k^{(n-3)} - 1/2} = 1, \\
& \prod_{k=1}^{m_{n-2}} \frac{\lambda_j^{(\pm)} - \lambda_k^{(n-2)} + 1/2}{\lambda_j^{(\pm)} - \lambda_k^{(n-2)} - 1/2} \prod_{k=1, k \neq j}^{m_\pm} \frac{\lambda_j^{(\pm)} - \lambda_k^{(\pm)} - 1}{\lambda_j^{(\pm)} - \lambda_k^{(\pm)} + 1} = 1 \quad (103)
\end{aligned}$$

and finally for the  $Osp(1|2n)$  vertex model the equation for the variables  $\{\lambda_j^{(n)}\}$  is<sup>12</sup>

$$\prod_{k=1}^{m_{n-1}} \frac{\lambda_j^{(n)} - \lambda_k^{(n-1)} + 1/2}{\lambda_j^{(n)} - \lambda_k^{(n-1)} - 1/2} \prod_{k=1}^{m_n} - \frac{(\lambda_j^{(n)} - \lambda_k^{(n)} + 1/2)(\lambda_j^{(n)} - \lambda_k^{(n)} - 1)}{(\lambda_j^{(n)} - \lambda_k^{(n)} - 1/2)(\lambda_j^{(n)} - \lambda_k^{(n)} + 1)} = 1. \quad (104)$$

We would like to close this section with the following remarks. We begin by discussing and comparing our results to previous work in the literature. It is not difficult to check that our results for the Bethe ansatz equations are equivalent to the analyticity of the eigenvalues as a function of the variables  $\{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}$ , i.e. all the residues on the direct and crossed poles vanish. This is precisely the main important ingredient entering in the framework of the analytical Bethe ansatz. Indeed, the results of this paper for the eigenvalues and the Bethe equations of the  $B_n$ ,  $C_n$ ,  $D_n$  and  $Osp(1|2n)$ <sup>13</sup> models are in accordance to those obtained from the analytical Bethe ansatz approach in Refs. [2–4,6]. As we have already commented in the introduction, the  $D_n$  vertex model has also been solved by the algebraic Bethe ansatz in Ref. [15]. The authors argued that the nesting problem, in a convenient basis, could be transformed to two commuting eigenvalue problems. One of them has the permutation operator as the main intertwiner, and the algebraic Bethe solution goes along to that known to work for the multistate 6-vertex models [11–13]. The other one was related to the Temperley–Lieb operator (in our notation of Section 2), no explicit algebraic solution was attempted, and the eigenvalue results were obtained via the crossing property. In our approach, however, we deal with these two operators together in the diagonalization problem and the explicit use of crossing it is not needed. We also noted that while our formulation works for vertex models having both even or odd numbers of states  $q$  per link, the basis used in Ref. [15] appears to be suitable only for  $q$  even. Since our approach and the one of Ref. [15] are quite different from the very beginning, we were not able to find a simple way of comparing the results for the eigenvectors.

For the models based on the superalgebras  $Osp(n|2m)$ , we note that the results for the eigenvalues and Bethe ansatz equations present some additional phase factors when

<sup>12</sup> We remark that the Bethe ansatz equations and eigenvalues for the models  $B_1$ ,  $C_1$ ,  $D_2$  and  $Osp(1|2)$  can be obtained from Eqs. (68), (69), (71), (72), (75), (76), (78) and (79) by setting the inhomogeneities to zero.

<sup>13</sup> We remark that in Ref. [6] the grading  $f \dots f b f \dots f$  is used and here we have taken the sequence  $b \dots b f b \dots b$ . This is the reason why the phase factors of Ref. [6] and that of Eq. (104) are different. We also have noticed misprints in the first paper in Ref. [3] concerning some eigenvalue expressions.

compared to that obtained for the pure “bosonic”  $B_n$ ,  $C_n$  and  $D_n$  models. These phases are the only liquid difference between the standard and the graded formulations of the quantum inverse scattering method. They are indeed twisted boundary conditions (periodic and anti-periodic) having a fermionic sector dependence (see e.g. Ref. [6]). In Appendix C we show how such phase factors can be absorbed in the supersymmetric formulation of the  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$  vertex models. Without these extra factors, the nested Bethe ansatz equations become more symmetrical, similar to what happens for the “bosonic” models. It turns out that such symmetrical Bethe ansatz equations can even be formulated in a more compact form, reflecting the underlying group symmetry of these vertex models. In order to see this, we first need to make the following definitions. Let  $C_{ab}$  be a Cartan matrix associated to the Dynkin diagrams of Fig. 1, and  $\eta_a$  the normalized length of the root of type  $\beta_a$ . Here we assume that the length of the long root is 2. This means that  $\eta_a = 2/(\alpha_a, \alpha_a)$  and  $\eta_a = 1$  for a long root.<sup>14</sup> Taking into account these definitions, it is possible to rewrite the Bethe ansatz equations (94)–(103) as<sup>15</sup>

$$\left[ \frac{\lambda_j^{(a)} - \frac{\delta_{a,1}}{2\eta_a}}{\lambda_j^{(a)} + \frac{\delta_{a,1}}{2\eta_a}} \right]^L = \prod_{b=1}^r \prod_{k=1, k \neq j}^{m_b} \frac{\lambda_j^{(a)} - \lambda_k^{(b)} - \frac{C_{a,b}}{2\eta_a}}{\lambda_j^{(a)} - \lambda_k^{(b)} + \frac{C_{a,b}}{2\eta_a}}, \quad j = 1, \dots, m_a, \quad a = 1, \dots, r, \quad (105)$$

where in general  $r$  is the number of roots of the underlying algebra. For the vertex models solved in this papers  $r = n$ .

The idea that the Bethe ansatz equations can be compactly written in terms of their corresponding Lie algebra goes back to the work of Ogievetsky et al. [32] who have conjectured similar formulas for the Bethe ansatz equations of factorizable  $S$ -matrices based on the standard Lie algebras  $A_n$ ,  $B_n$ ,  $C_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$  and  $G_2$ . Thus, our results can be seen as a rigorous proof of this conjecture for the rational  $B_n$ ,  $C_n$  and  $D_n$  vertex models. Moreover, together with the results of Ref. [5] for the  $Sl(n|m)$  algebra (see also Ref. [31]), they also show that this conjecture can be extended to the case of superalgebras. An exception in this construction is again the  $Osp(1|2n)$  vertex model. In order to fit Eq. (105), we need to give a special meaning to the black bullet  $\bullet$  of Fig. 1 (root  $\{\lambda_j^{(n)}\}$  in the Bethe ansatz). One could interpret it as a peculiar two-body self-interaction for root  $\{\lambda_j^{(n)}\}$ , as the one present in the right-hand side of Eq. (104).

Our last remark is concerned with the underlying spin chain associated to the braid-monoid vertex models. The  $R$ -matrix (4) is regular at  $\lambda = 0$ , thus the local conserved charges can be obtained through the logarithmic expansion of the corresponding transfer matrix around  $\lambda = 0$ . The first charge is the momentum itself (we are assuming periodic

<sup>14</sup> More precisely, for  $D_n$  and  $Osp(2n-2|2)$   $\eta_a = 1$ ,  $a = 1, \dots, n$ ; for  $B_n$  and  $Osp(2n-1|2)$   $\eta_a = 1$ ,  $a = 1, \dots, n-1$ , and  $\eta_n = 2$ ; for  $C_n$  and  $Osp(2|2n-2)$   $\eta_a = 2$ ,  $a = 1, \dots, n-1$ , and  $\eta_n = 1$  (see for instance Ref. [27]).

<sup>15</sup> We note that for the  $C_n$  and  $Osp(2|2n-2)$  vertex models one should rescale all  $\{\lambda_j^{(a)}\}$  by a factor 2,  $\{\lambda_j^{(a)}\} \rightarrow \{\lambda_j^{(a)}\}/2$  in Eq. (105) to recover the previous results (94), (95), (97), (98) and (100).

boundary conditions) and the next one, i.e. the logarithmic derivative of  $T(\lambda)$  at  $\lambda = 0$ , is the Hamiltonian. From the expression of the  $R$ -matrix  $R(\lambda)$  we found

$$\mathcal{H} = \sum_{i=1}^L P_i^s + \frac{1}{\Delta} E_i. \quad (106)$$

Analogously, the eigenenergies  $E(L)$  of the corresponding Hamiltonian (106) can be calculated by taking the logarithmic derivative of  $\Lambda(\lambda, \{\lambda_i^{(1)}\}, \dots, \{\lambda_j^{(n)}\})$  at the regular point  $\lambda = 0$ . The eigenenergies  $E(L)$  are parametrized in terms of the variables  $\{\lambda_i^{(1)}\}$  by<sup>16</sup>

$$E(L) = \pm \sum_{i=1}^{m_1} \frac{1}{[\lambda_i^{(1)}]^2 - 1/4} \pm L, \quad (107)$$

where the signs  $\pm$  are related to the two possibilities for the Boltzmann weights of type  $a(\lambda)$ , i.e.  $a(\lambda) = 1 \pm \lambda$ . The variables  $\{\lambda_i^{(1)}\}$  satisfy the general nested Bethe ansatz equations (94)–(104), and therefore the spectrum of all  $Osp(n|2m)$  chains can be determined solving these equations. A preliminary study of the root structures of the Bethe ansatz equations (94)–(104) have shown an intricate behaviour for the general  $Osp(n|2m)$ . We leave a detailed analysis of these root structure, as well the ground state and the low-lying excitations for a forthcoming paper.

## 6. Conclusions

In this paper we have solved exactly a series of rational vertex models based on the braid–monoid algebra from a rather unified perspective. The general construction for the  $R$ -matrices presented in Section 2 played an important role in many steps of our formulation of the quantum inverse scattering method. In particular, it is notable how such a construction becomes useful to determine the universal character of the fundamental commutations rules presented in Section 3. It appears natural to us to blame the properties of the braid–monoid algebra as the main mathematical structure behind such a general picture.

The commutation relations allowed us to determine the eigenvectors and the eigenvalues of the transfer matrix of the vertex models invariant by the  $B_n$ ,  $C_n$ ,  $D_n$ ,  $Osp(2n - 1|2)$ ,  $Osp(2|2n - 2)$ ,  $Osp(2n - 2|2)$  and  $Osp(1|2n)$  symmetries from a systematic point of view. As a consequence, we have been able to establish the solution of the general  $Osp(n|2m)$  invariant spin chain. Their Bethe ansatz equations have been formulated in terms of basic properties of the underlying group symmetry such as the root structure. This means that, in principle, by solving Eqs. (105) together with (107) we can determine the spectrum of the general  $Osp(n|2m)$  spin chain (106).

<sup>16</sup> It has become a tradition in the literature to normalize the Bethe ansatz variables by a pure imaginary factor as  $\lambda_j^{(\beta)} \rightarrow \lambda_j^{(\beta)}/i$ . In this case, Eq. (107) reads  $E(L) = \mp \sum_{i=1}^{m_1} 1/([\lambda_i^{(1)}]^2 + 1/4) \pm L$ .



One possible extension of this work is to consider the trigonometric analogs of the vertex models solved in this paper. In general, we expect that the trigonometric vertex models based on the Birman–Wenzl–Murakami algebra [1,22] can be solved by introducing a few adaptations to our construction of Sections 3 and 4. In addition, we have noted that similar ideas also work for the one-dimensional Hubbard model. In this case, however, some other peculiarities such as the non-additive property of the  $R$ -matrix need to be disentangled too. Due to the special character of the Hubbard  $R$ -matrix and its recognized importance in condensed matter physics, we have dedicated a brief account of our results in a separated publication [28]. It remains to be seen whether these are isolated examples or that our construction could even be more widely applicable than one would think at first sight. The last possibility would suggest that a more deep and essential mathematical structure is still to be grasped from the Yang–Baxter algebra.

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### Note added in proof

After completing this work, we have been able to collect the unwanted terms in a closed form with the help of the permutation symmetry (50). The main steps will be presented elsewhere.

### Appendix A. The two-particle state

In this appendix we present some details concerning the complete solution of the eigenvalue problem for the two-particle state. Here we shall need a few extra commutation relations between the fields  $B(\lambda)$ ,  $B^*(\lambda)$ ,  $C(\lambda)$  and  $C^*(\lambda)$ . They are given by

$$C_a(\lambda)B_b(\mu) = B_b(\mu)C_a(\lambda) - \frac{1}{b(\lambda - \mu)} [B(\lambda)A_{ab}(\mu) - B(\mu)A_{ab}(\lambda)], \quad (\text{A.1})$$

$$B_a^*(\lambda)B_b(\mu) = B_b(\mu)B_a^*(\lambda) - \frac{1}{b(\lambda - \mu)} [F(\lambda)A_{ab}(\mu) - F(\mu)A_{ab}(\lambda)], \quad (\text{A.2})$$

$$C_a^*(\lambda) B_b(\mu) |0\rangle = \xi_{ab} \frac{d_n(\lambda - \mu)}{e_n(\lambda - \mu)} [B(\mu) D(\lambda) - A_{aa}(\lambda) A_{bb}(\mu)] |0\rangle, \quad (\text{A.3})$$

where  $a, b = 1, \dots, q-2$ .

We start our discussion by considering the wanted terms for the two-particle state. Since the terms proportional to  $B_i(\lambda_1^{(1)}) B_j(\lambda_2^{(2)}) \mathcal{F}^{ji}$  have been already cataloged in Section 4 (see Eq. (55)) we turn our attention to those proportional to

$$-\frac{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})} F(\lambda_1^{(1)}) B(\lambda_2^{(1)}) \xi \cdot \mathcal{F}.$$

In other words, we have to show that the wanted terms proportional to the second part of the two-particle eigenvector (40) produce the eigenvalue expression (55) for  $m_1 = 2$ . It is not difficult to collect the contributions proportional to  $[a(\lambda)]^L$  and  $[e_n(\lambda)]^L$ . They are given, respectively, by

$$\left[ \frac{a(\lambda_1^{(1)} - \lambda)}{e_n(\lambda_1^{(1)} - \lambda)} + \frac{1}{b(\lambda_2^{(1)} - \lambda)} \frac{a(\lambda_1^{(1)} - \lambda)}{b(\lambda_1^{(1)} - \lambda)} \frac{d_n(\lambda_1^{(1)} - \lambda)}{e_n(\lambda_1^{(1)} - \lambda)} \frac{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \right] [a(\lambda)]^L, \quad (\text{A.4})$$

$$\left[ \frac{a(\lambda - \lambda_1^{(1)})}{e_n(\lambda - \lambda_1^{(1)})} - \frac{1}{e_n(\lambda - \lambda_1^{(1)})} \frac{d_n(\lambda - \lambda_2^{(1)})}{e_n(\lambda - \lambda_2^{(1)})} \frac{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \right] [e_n(\lambda)]^L. \quad (\text{A.5})$$

Now, if we use that the Boltzmann weights satisfy the following identities:

$$\begin{aligned} & \left[ \frac{a(\lambda_1^{(1)} - \lambda)}{e_n(\lambda_1^{(1)} - \lambda)} + \frac{1}{b(\lambda_2^{(1)} - \lambda)} \frac{a(\lambda_1^{(1)} - \lambda)}{b(\lambda_1^{(1)} - \lambda)} \frac{d_n(\lambda_1^{(1)} - \lambda)}{e_n(\lambda_1^{(1)} - \lambda)} \frac{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \right] \\ &= \prod_{i=1}^2 \frac{a(\lambda_i^{(1)} - \lambda)}{b(\lambda_i^{(1)} - \lambda)}, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} & \left[ \frac{a(\lambda - \lambda_1^{(1)})}{e_n(\lambda - \lambda_1^{(1)})} - \frac{1}{e_n(\lambda - \lambda_1^{(1)})} \frac{d_n(\lambda - \lambda_2^{(1)})}{e_n(\lambda - \lambda_2^{(1)})} \frac{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \right] \\ &= \prod_{i=1}^2 \frac{b(\lambda - \lambda_i^{(1)})}{e_n(\lambda - \lambda_i^{(1)})}, \end{aligned} \quad (\text{A.7})$$

it is immediately seen that these two contributions are precisely the first and the second terms of Eq. (55) for  $m_1 = 2$ .

The analysis of the wanted term proportional to  $[b(\lambda)]^L$  is a bit more involved. The basic reason for that is the presence of some “asymmetric” terms of the kind

$$[b(\lambda) a(\lambda_2^{(1)})]^L F(\lambda_1^{(1)}) \sum_{ij} \xi_{ij} \mathcal{F}^{ij}$$

and the peculiar form of the nested eigenvalue  $A^{(1)}(\lambda, \{\lambda_i^{(1)}\})$ . We find that the terms contributing to this last wanted piece are given by

$$\begin{aligned}
& -(q-2) \left[ 1 - \frac{1}{[b(\lambda - \lambda_1^{(1)})]^2} \right] [b(\lambda) a(\lambda_2^{(1)})]^L \frac{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \\
& \times F(\lambda_1^{(1)}) \sum_{ij} \xi_{ij} \mathcal{F}^{ji}, \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
& -[b(\lambda) a(\lambda_2^{(1)})]^L \frac{1}{[b(\lambda - \lambda_1^{(1)})]^2} \frac{d_n(\lambda - \lambda_2^{(1)})}{e_n(\lambda - \lambda_2^{(1)})} \\
& \times \sum_{ijlm} \xi_{mj} [X^{(1)}(\lambda - \lambda_1^{(1)})]_{li}^{lm} F(\lambda_1^{(1)}) \mathcal{F}^{ji}, \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
& -[b(\lambda) a(\lambda_2^{(1)})]^L \frac{d_n(\lambda - \lambda_1^{(1)})}{e_n(\lambda - \lambda_1^{(1)})} \frac{[1 - \hat{b}(\lambda - \lambda_1^{(1)})]}{b(\lambda - \lambda_1^{(1)})} \frac{1}{b(\lambda - \lambda_2^{(1)})} \\
& \times F(\lambda_1^{(1)}) \sum_{ij} \xi_{ji} \mathcal{F}^{ji}, \tag{A.10}
\end{aligned}$$

where the first term (A.8) comes from the eigenvector part proportional to  $F(\lambda_1^{(1)})$  while the remaining ones originate from the term  $B_i(\lambda_1^{(1)}) B_j(\lambda_2^{(1)}) \mathcal{F}^{ji}$ . In order to go on we need to take advantage of the identity

$$A^{(1)}(\lambda, \{\lambda_i^{(1)}\}) \mathcal{F}^{ml} = [T^{(1)}(\lambda, \{\lambda_i^{(1)}\})]_{ij}^{lm} \mathcal{F}^{ji}. \tag{A.11}$$

By expanding expressions (A.8)–(A.10) with the help of identity (A.11) we have checked that these three terms together produce the third piece of the eigenvalue expression (55).

We now turn our attention to the unwanted terms. They are generated when the parameter  $\lambda$  of the diagonal operators  $B(\lambda)$ ,  $A_{aa}(\lambda)$  and  $D(\lambda)$  exchange with the variables  $\{\lambda_1^{(1)}, \lambda_2^{(1)}\}$  parameterizing the eigenvector  $|\Phi_2(\lambda_1^{(1)}, \lambda_2^{(1)})\rangle$ . Basically, we have three kinds of such terms and they are proportional to

$$B_i(\lambda) B_j(\lambda_i^{(1)}), \quad B_i^*(\lambda) B_j(\lambda_i^{(1)}), \quad F(\lambda) \tag{A.12}$$

When  $\lambda_i^{(1)} = \lambda_2^{(1)}$  we just have two contributions for the first two unwanted terms in (A.12), and they are cancelled out by same procedure presented in Section 4 (see Eqs. (51) and (52)). By contrast, when  $\lambda_i^{(1)} = \lambda_1^{(1)}$ , several terms appear and the simplifications are more involved. For example, in the case of the unwanted term  $B_i(\lambda) B_j(\lambda_1^{(1)}) \mathcal{F}^{ji}$  we have four contributions coming from  $B_i(\lambda_1^{(1)}) B_j(\lambda_2^{(1)})$  which are given by

$$\begin{aligned}
& -[b(\lambda_2^{(1)})] \frac{a(\lambda - \lambda_1^{(1)})}{b(\lambda - \lambda_1^{(1)})} \frac{1}{b(\lambda - \lambda_2^{(1)})} \sum_{ij} B_j(\lambda) B_i(\lambda_1^{(1)}) \mathcal{F}^{ji}, \\
& +[b(\lambda_2^{(1)})]^L \frac{1}{b(\lambda - \lambda_1^{(1)})} \frac{1}{b(\lambda_1^{(1)} - \lambda_2^{(1)})} \sum_{ij} B_j(\lambda) B_i(\lambda_1^{(1)}) \mathcal{F}^{ji},
\end{aligned}$$

$$\begin{aligned}
& -[a(\lambda_2^{(1)})]^L \frac{1}{b(\lambda_1^{(1)} - \lambda)} \frac{1}{b(\lambda_2^{(1)} - \lambda)} \sum_{ijlm} [X^{(1)}(\lambda_1^{(1)} - \lambda)]_{lm}^{ij} B_l(\lambda) B_m(\lambda_1^{(1)}) \mathcal{F}^{ji}, \\
& + [a(\lambda_2^{(1)})]^L \frac{1}{b(\lambda_1^{(1)} - \lambda)} \frac{1}{b(\lambda_2^{(1)} - \lambda_1^{(1)})} \sum_{ij} B_i(\lambda) B_j(\lambda_1^{(1)}) \mathcal{F}^{ji}, \quad (\text{A.13})
\end{aligned}$$

and only one coming from the term  $F(\lambda_1^{(1)}) \xi \cdot \mathcal{F}$ , which is given by

$$[a(\lambda_2^{(1)})]^L \frac{d_n(\lambda_1^{(1)} - \lambda)}{e_n(\lambda_1^{(1)} - \lambda)} \frac{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \sum_{ijlm} \xi_{ml}^* B_l(\lambda) B_m(\lambda_1^{(1)}) \xi_{ij} \mathcal{F}^{ji}. \quad (\text{A.14})$$

The simplest way of checking that the terms (A.13) and (A.14) together cancel out is by using explicitly the Bethe ansatz equations (45) for the variable  $\lambda_2^{(1)}$ , namely

$$\begin{aligned}
[b(\lambda_2^{(1)})]^L \mathcal{F}^{lm} &= \frac{b(\lambda_2^{(1)} - \lambda_1^{(1)})}{b(\lambda_1^{(1)} - \lambda_2^{(1)})} \frac{1}{a(\lambda_2^{(1)} - \lambda_1^{(1)})} [a(\lambda_2^{(1)})]^L \\
&\times \sum_{ij} [X^{(1)}(\lambda_1^{(1)} - \lambda_2^{(1)})]_{ij}^{lm} \mathcal{F}^{ji}. \quad (\text{A.15})
\end{aligned}$$

Substituting this expression in the first two terms of equation (A.13), all the unwanted terms become proportional to  $[a(\lambda_2^{(1)})]^L$ . Then we have verified that such a final expression gives a null result with the help of some additional identities between the Boltzmann weights. Analogously, a similar procedure can be used to deal with the unwanted term  $B_l^*(\lambda) B_j(\lambda_1^{(1)})$ . The five contributions for this unwanted term are collected below,

$$\begin{aligned}
& [a(\lambda_2^{(1)})]^L \frac{1}{b(\lambda - \lambda_1^{(1)})} \frac{d_n(\lambda - \lambda_2^{(1)})}{e_n(\lambda - \lambda_2^{(1)})} \sum_{aijlm} [X^{(1)}(\lambda - \lambda_1^{(1)})]_{ai}^{lm} B_a^*(\lambda) B_l(\lambda_1^{(1)}) \xi_{mj} \mathcal{F}^{ji}, \\
& - [a(\lambda_2^{(1)})]^L \frac{d_n(\lambda - \lambda_1^{(1)})}{e_n(\lambda - \lambda_1^{(1)})} \frac{1}{b(\lambda_2^{(1)} - \lambda_1^{(1)})} \sum_{aij} \xi_{ai} B_a^*(\lambda) B_j(\lambda_1^{(1)}) \mathcal{F}^{ji}, \\
& - [b(\lambda_2^{(1)})]^L \left[ \frac{b(\lambda - \lambda_1^{(1)})}{e_n(\lambda - \lambda_1^{(1)})} \frac{d_n(\lambda - \lambda_2^{(1)})}{e_n(\lambda - \lambda_2^{(1)})} - \frac{d_n(\lambda - \lambda_1^{(1)})}{e_n(\lambda - \lambda_1^{(1)})} \frac{1}{b(\lambda_1^{(1)} - \lambda_2^{(1)})} \right] \\
& \times \sum_{ijl} \xi_{lj} B_l^*(\lambda) B_i(\lambda_1^{(1)}) \mathcal{F}^{ji}, \\
& - [a(\lambda_2^{(1)})]^L \frac{1}{b(\lambda - \lambda_1^{(1)})} \frac{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \sum_{ijl} \xi_{ij} B_l^*(\lambda) B_l(\lambda_1^{(1)}) \mathcal{F}^{ji}. \quad (\text{A.16})
\end{aligned}$$

Finally, we have nine contributions to the third term proportional to  $F(\lambda)$ . The first seven come from  $B_i(\lambda_1^{(1)}) B_j(\lambda_2^{(1)})$  and the other two from  $F(\lambda_1^{(1)})$ . An easy way of verifying that all these nine terms cancel out is by using explicitly the Bethe ansatz equations (45) both for  $\lambda_1^{(1)}$  and  $\lambda_2^{(1)}$ . The final expression can be disentangled only in terms of  $[a(\lambda_1^{(1)}) a(\lambda_2^{(1)})]^L F(\lambda)$ , and we have checked that it gives a null result by using Mathematica<sup>TM</sup>.

## Appendix B. The three-particle state symmetrization

This appendix deals with the symmetric properties of the three-particle state. We begin our discussion with the  $\lambda_2^{(1)} \leftrightarrow \lambda_3^{(1)}$  permutation. After this permutation the (vector) three-particle state (46) looks like

$$\begin{aligned} \Phi_3(\lambda_1^{(1)}, \lambda_3^{(1)}, \lambda_2^{(1)}) &= B(\lambda_1^{(1)}) \otimes \Phi_2(\lambda_3^{(1)}, \lambda_2^{(1)}) \\ &+ \left[ \xi \otimes F(\lambda_1^{(1)}) B(\lambda_2^{(1)}) B(\lambda_3^{(1)}) \right] \hat{h}_1(\lambda_1^{(1)}, \lambda_3^{(1)}, \lambda_2^{(1)}) \\ &+ \left[ \xi \otimes F(\lambda_1^{(1)}) B(\lambda_3^{(1)}) B(\lambda_2^{(1)}) \right] \hat{h}_2(\lambda_1^{(1)}, \lambda_3^{(1)}, \lambda_2^{(1)}). \end{aligned} \quad (\text{B.1})$$

This vector can be related to  $\Phi_3(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)})$  by commuting the fields  $B(\lambda_2^{(1)})$  and  $B(\lambda_3^{(1)})$  in expression (46). In order to do that, we have to use the commutation rule (29), which now reads

$$\begin{aligned} B(\lambda_2^{(1)}) \otimes B(\lambda_3^{(1)}) &= \frac{1}{a(\lambda_2^{(1)} - \lambda_3^{(1)})} [B(\lambda_3^{(1)}) \otimes B(\lambda_2^{(1)})] \cdot X^{(1)}(\lambda_2^{(1)} - \lambda_3^{(1)}) \\ &+ \frac{d_n(\lambda_2^{(1)} - \lambda_3^{(1)})}{e_n(\lambda_2^{(1)} - \lambda_3^{(1)})} \left[ F(\lambda_2^{(1)}) B(\lambda_3^{(1)}) \right. \\ &\quad \left. - \frac{\{1 - \hat{b}(\lambda_2^{(1)} - \lambda_3^{(1)})\}}{a(\lambda_2^{(1)} - \lambda_3^{(1)})} F(\lambda_3^{(1)}) B(\lambda_2^{(1)}) \right] \xi. \end{aligned} \quad (\text{B.2})$$

By substituting this relation in Eq. (46) and by comparing the final result with the vector (B.1), we find that the symmetric property (50) holds for the variables  $\{\lambda_2^{(1)}, \lambda_3^{(1)}\}$  provided the functions  $\hat{h}_1(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)})$  and  $\hat{h}_2(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)})$  satisfy

$$\hat{h}_2(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}) = \hat{h}_1(\lambda_1^{(1)}, \lambda_3^{(1)}, \lambda_2^{(1)}) \frac{X_{23}^{(1)}(\lambda_2^{(1)} - \lambda_3^{(1)})}{a(\lambda_2^{(1)} - \lambda_3^{(1)})}, \quad (\text{B.3})$$

$$\hat{h}_1(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}) = \hat{h}_2(\lambda_1^{(1)}, \lambda_3^{(1)}, \lambda_2^{(1)}) \frac{X_{23}^{(1)}(\lambda_2^{(1)} - \lambda_3^{(1)})}{a(\lambda_2^{(1)} - \lambda_3^{(1)})}. \quad (\text{B.4})$$

In fact, the two expressions above are equivalent due to the unitarity property of the auxiliary matrix  $X^{(1)}(\lambda)$ , namely

$$X^{(1)}(u) X^{(1)}(-u) = a(u) a(-u) I. \quad (\text{B.5})$$

Therefore, in order to determine these functions we still need to find an extra constraint. This can be found if we perform the permutation between variables  $\lambda_1^{(1)}$  and  $\lambda_2^{(1)}$ . The procedure is similar to the one described above, but now some additional commutation rules are needed. For example, besides commuting the creation fields  $B(\lambda_1^{(1)})$  and  $B(\lambda_2^{(1)})$ , we have to commute  $B(\lambda_1^{(1)})$  and  $F(\lambda_2^{(1)})$  and also  $F(\lambda_1^{(1)})$  and  $B(\lambda_2^{(1)})$ . This latter step is worked out with the help of commutations rules (31) and (32), respectively. We then are able to determine the function  $\hat{h}_1(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)})$  by imposing

the necessary restriction that the terms proportional to  $\xi \otimes F(\lambda_1^{(1)}) B(\lambda_3^{(1)}) B(\lambda_2^{(1)})$  need to be cancelled out. We find that this condition is verified provided

$$\hat{h}_1(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}) = -\frac{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \frac{a(\lambda_3^{(1)} - \lambda_2^{(1)})}{b(\lambda_3^{(1)} - \lambda_2^{(1)})} I \quad (\text{B.6})$$

and by substituting (B.6) in (B.3) we are able to fix  $\hat{h}_2(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)})$  (see Eq. (48)).

Besides that, some other non-trivial checks need to be performed in order to verify the consistency of the procedure mentioned above. In the process of using the three commutation rules mentioned above, certain extra identities emerge, and they are given by

$$\begin{aligned} & [\xi \otimes B(\lambda_2^{(1)}) F(\lambda_1^{(1)}) B(\lambda_3^{(1)})] b(\lambda_1^{(1)} - \lambda_2^{(1)}) \\ & \times \left[ -\frac{1}{b(\lambda_3^{(1)} - \lambda_2^{(1)})} \frac{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})} + \hat{h}_2(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}) \right] \\ & = \frac{d_n(\lambda_2^{(1)} - \lambda_3^{(1)})}{e_n(\lambda_2^{(1)} - \lambda_3^{(1)})} [B(\lambda_2^{(1)}) F(\lambda_1^{(1)}) B(\lambda_3^{(1)}) \otimes \xi] \\ & - \frac{d_n(\lambda_1^{(1)} - \lambda_3^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_3^{(1)})} [B(\lambda_2^{(1)}) F(\lambda_1^{(1)}) B(\lambda_3^{(1)}) \otimes \xi] X_{12}^{(1)}(\lambda_1^{(1)} - \lambda_2^{(1)}) \end{aligned} \quad (\text{B.7})$$

and

$$\begin{aligned} & [\xi \otimes F(\lambda_2^{(1)}) B(\lambda_1^{(1)}) B(\lambda_3^{(1)})] \\ & \times \left[ \frac{[1 - \hat{b}(\lambda_1^{(1)} - \lambda_2^{(1)})] d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{b(\lambda_3^{(1)} - \lambda_1^{(1)}) e_n(\lambda_1^{(1)} - \lambda_2^{(1)})} - \frac{1}{b(\lambda_3^{(1)} - \lambda_2^{(1)})} \frac{d_n(\lambda_1^{(1)} - \lambda_2^{(1)})}{e_n(\lambda_1^{(1)} - \lambda_2^{(1)})} \right] \\ & - b(\lambda_1^{(1)} - \lambda_2^{(1)}) \frac{d_n(\lambda_2^{(1)} - \lambda_3^{(1)})}{e_n(\lambda_2^{(1)} - \lambda_3^{(1)})} [F(\lambda_2^{(1)}) B(\lambda_1^{(1)}) B(\lambda_3^{(1)}) \otimes \xi] \\ & = [\xi \otimes F(\lambda_2^{(1)}) B(\lambda_1^{(1)}) B(\lambda_3^{(1)})] \\ & \times \left[ -\hat{h}_2(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}) + X_{12}^{(1)}(\lambda_1^{(1)} - \lambda_2^{(1)}) \hat{h}_2(\lambda_2^{(1)}, \lambda_1^{(1)}, \lambda_3^{(1)}) \right]. \quad (\text{B.8}) \end{aligned}$$

In order to show that these relations are satisfied, we end up proving remarkable properties between  $\xi$  and the auxiliary matrix  $X^{(1)}(u)$ . They are listed below as

$$[\xi_{12} B_3(y)] X_{12}^{(1)}(u) = \frac{d_n(u)}{e_n(u)} \frac{e_n(-u)}{d_n(-u)} [1 - \hat{b}(u)] [\xi_{12} B_3(y)], \quad (\text{B.9})$$

$$[B_1(y) \xi_{23}] X_{23}^{(1)}(u) = \frac{d_n(u)}{e_n(u)} \frac{e_n(-u)}{d_n(-u)} [1 - \hat{b}(u)] [B_1(y) \xi_{23}], \quad (\text{B.10})$$

$$\begin{aligned} [B_1(y) \xi_{23}] X_{12}^{(1)}(u) &= [B_1(y) \xi_{23}] + u [B_1(y) \xi_{23} P_{12}^{(1)g}] \\ &\quad - \frac{u}{u - \Delta^{(1)}} [\xi_{12} B_3(y)], \end{aligned} \quad (\text{B.11})$$

$$[\xi_{12} B_3(y)] X_{23}^{(1)}(u) = [\xi_{12} B_3(y)] + u[\xi_{12} B_3(y) P_{23}^{(1)g}] - \frac{u}{u - \Delta^{(1)}} [B_1(y) \xi_{23}], \quad (B.12)$$

$$[\xi_{12} B_3(y) P_{23}^{(1)g}] X_{12}^{(1)}(u) = [\xi_{12} B_3(y) P_{23}^{(1)g}] + u[B_1(y) \xi_{23}] - \frac{u}{u - \Delta^{(1)}} \hat{t}[\xi_{12} B_3(y)], \quad (B.13)$$

where the lower indices indicate the position where the vector  $\xi$ , the vector  $B(y)$  and the matrix  $X^{(1)}(u)$  act in a non-trivial way.  $P^{(1)g}$  means the permutator entering in the construction of  $X^{(1)}(\lambda)$ . Besides that, we also note the following useful identity:  $\xi_{12} B_3(y) P_{23}^{(1)g} = B_1(y) \xi_{23} P_{12}^{(1)g}$ .

### Appendix C. The graded quantum inverse approach

The purpose of this appendix is to discuss the main modifications occurring in the commutation rules, eigenvalues and Bethe ansatz equations for the models  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$ ,  $Osp(2n-2|2)$  and  $Osp(1|2n)$  when we formulate their solution in terms of the graded inverse scattering framework [11]. In this formalism, the vertex operator  $\mathcal{L}_{Ai}(\lambda)$  entering in the monodromy matrix (12) satisfies the graded Yang–Baxter equation and is given by acting the graded permutation operator  $P^g$  on the  $R$ -matrix (4), namely

$$\mathcal{L}_{ab}^{cd}(\lambda) = (-1)^{p(a)p(b)} R_{ab}^{cd}(\lambda). \quad (C.1)$$

In order to accomplish this change, the integrability condition (14) now reads

$$R(\lambda, \mu) T(\lambda) \overset{s}{\otimes} T(\mu) = T(\mu) \overset{s}{\otimes} T(\lambda) R(\lambda, \mu), \quad (C.2)$$

where the symbol  $\overset{s}{\otimes}$  stands for the supertensor product

$$\left( A \overset{s}{\otimes} B \right)_{ab}^{cd} = (-1)^{p(b)[p(a)+p(c)]} A_{ac} B_{bd}.$$

Moreover, the transfer matrix  $T(\lambda)$  is written in terms of a supertrace of the monodromy matrix  $\mathcal{T}(\lambda)$  as

$$T(\lambda) = \text{Str } \mathcal{T}(\lambda) = \sum_a (-1)^{p(a)} \mathcal{T}_{aa}(\lambda). \quad (C.3)$$

The modifications (C.1)–(C.3) are responsible for the appearance of extra signs in some terms of the commutation rules, in the “diagonal” conditions (19) and in the eigenvalue problem (18). Furthermore, the way that such signs enter in these relations depends much on the grading sequence we have chosen. We recall that we have taken the  $fb \dots bfb$  grading for the first three models and  $b \dots bfb \dots b$  for the  $Osp(1|2n)$  vertex model. Therefore, we have to present the modifications for these two classes of grading separately. We begin by listing the results for the  $Osp(2n-1|2)$ ,  $Osp(2|2n-2)$

and  $Osp(2n-2|2)$  models. For the commutation rules, the extra signs appear only in relations (22)–(24) and (29) and for the following right-hand side pairs of terms:

$$B_i B_j \rightarrow -B_i B_j, \quad B_i B \rightarrow -B_i B, \quad B_i D \rightarrow -B_i D, \quad B_i^* B \rightarrow -B_i^* B. \quad (C.4)$$

All the other fundamental commutation relations remain unchanged. Furthermore, the action of the diagonal operators  $B(\lambda)$ ,  $A_{ab}(\lambda)$  and  $D(\lambda)$  on the reference state (15) becomes

$$B(\lambda) |0\rangle = [-a(\lambda)]^L |0\rangle, \quad D(\lambda) |0\rangle = [-e_n(\lambda)]^L |0\rangle, \\ A_{aa}(\lambda) |0\rangle = [b(\lambda)]^L |0\rangle, \quad a = 1, \dots, q-2, \quad (C.5)$$

while the eigenvalue problem (18) becomes

$$\left[ -B(\lambda) + \sum_{a=1}^{q-2} A_{aa}(\lambda) - D(\lambda) \right] |\Phi\rangle = \Lambda(\lambda) |\Phi\rangle. \quad (C.6)$$

Taking into account these relations, one can easily verify that the phase factor  $(-1)^{L-m_1-1}$  present in the Bethe ansatz equations of these models is cancelled out. For sake of completeness we also present the final results for the eigenvalues in the graded formalism. For  $Osp(2n-1|2)$  ( $n \geq 2$ ) we have

$$\Lambda^{Osp(2n-1|2)}(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}) = -[-a(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + \frac{1}{2}}{\lambda - \lambda_j^{(1)} - \frac{1}{2}} \\ - [-e_n(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + n-3}{\lambda - \lambda_j^{(1)} + n-2} + [b(\lambda)]^L \sum_{l=1}^{2n-1} G_l(\lambda, \{\lambda_j^{(\beta)}\}) \quad (C.7)$$

and for  $Osp(2|2n-2)$ , ( $n \geq 2$ ) we find

$$\Lambda^{Osp(2|2n-2)}(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}) = -[-a(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + \frac{1}{2}}{\lambda - \lambda_j^{(1)} - \frac{1}{2}} \\ - [-e_n(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + \frac{2n-3}{2}}{\lambda - \lambda_j^{(1)} + \frac{2n-1}{2}} + [b(\lambda)]^L \sum_{l=1}^{2n-2} G_l(\lambda, \{\lambda_j^{(\beta)}\}), \quad (C.8)$$

while for  $Osp(2n-2|2)$  ( $n \geq 3$ ) we have

$$\Lambda^{Osp(2n-2|2)}(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}) = -[-a(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + \frac{1}{2}}{\lambda - \lambda_j^{(1)} - \frac{1}{2}} \\ - [-e_n(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + \frac{2n-7}{2}}{\lambda - \lambda_j^{(1)} + \frac{2n-5}{2}} + [b(\lambda)]^L \sum_{l=1}^{2n-2} G_l(\lambda, \{\lambda_j^{(\beta)}\}), \quad (C.9)$$



where the functions  $G_l(\lambda, \{\lambda_j^{(\beta)}\})$  remain unchanged and are given by Eqs. (87), (89) and (91), respectively.

The analysis for the  $Osp(1|2n)$  model is a bit simpler because we just have one “central” fermionic species in the grading sequence. In this case, we have only extra signs for the terms  $B_i A_{jl}$  and  $[B^*]_i A_{jl}$  in relations (22) and (24), respectively. The change in the eigenvalue (92) now appears in the  $n$ th term, and the final result is

$$\begin{aligned} \Lambda^{Osp(1|2n)}(\lambda; \{\lambda_j^{(1)}\}, \dots, \{\lambda_j^{(n)}\}) &= [a(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} - \frac{1}{2}}{\lambda - \lambda_j^{(1)} + \frac{1}{2}} \\ &+ [e_n(\lambda)]^L \prod_{j=1}^{m_1} \frac{\lambda - \lambda_j^{(1)} + n + 1}{\lambda - \lambda_j^{(1)} + n} + [b(\lambda)]^L \sum_{l=1}^{n-1} G_l(\lambda, \{\lambda_j^{(\beta)}\}) \\ &+ [b(\lambda)]^L \sum_{l=n+1}^{2n-1} G_l(\lambda, \{\lambda_j^{(\beta)}\}) - [b(\lambda)]^L \tilde{G}_n(\lambda, \{\lambda_j^{(\beta)}\}), \end{aligned} \quad (C.10)$$

where the functions  $G_l(\lambda, \{\lambda_j^{(\beta)}\})$  for  $l = 1, \dots, n-1, n+1, \dots, 2n-1$  are still given by (93), but  $\tilde{G}_n(\lambda, \{\lambda_j^{(\beta)}\})$  is

$$\tilde{G}_n(\lambda, \{\lambda_j^{(\beta)}\}) = \prod_{k=1}^{m_{(n)}} \frac{(\lambda - \frac{\lambda_k^{(n)}}{i} + \frac{(n-1)}{2}) (\lambda - \frac{\lambda_k^{(n)}}{i} + \frac{(n+2)}{2})}{(\lambda - \frac{\lambda_k^{(n)}}{i} + \frac{(n+1)}{2}) (\lambda - \frac{\lambda_k^{(n)}}{i} + \frac{n}{2})}. \quad (C.11)$$

Analogously, the Bethe ansatz equations are only modified for the last root  $\{\lambda_j^{(n)}\}$ . Instead of Eq. (104) we now have

$$\prod_{k=1}^{m_{n-1}} \frac{\lambda_j^{(n)} - \lambda_k^{(n-1)} + 1/2}{\lambda_j^{(n)} - \lambda_k^{(n-1)} - 1/2} \prod_{k=1}^{m_n} \frac{(\lambda_j^{(n)} - \lambda_k^{(n)} + 1/2)(\lambda_j^{(n)} - \lambda_k^{(n)} - 1)}{(\lambda_j^{(n)} - \lambda_k^{(n)} - 1/2)(\lambda_j^{(n)} - \lambda_k^{(n)} + 1)} = 1. \quad (C.12)$$

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