# THE $S$-MATRIX OF THE KINKS OF THE $(\bar{\psi} \psi)^{2}$ MODEL 

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We consider the particle-kink and kink-kink $S$-matrix elements of the two-dimensional $(\bar{\psi} \dot{\psi})^{2}$ model, where the Majorana spinor $\psi$ is an $\mathrm{O}(N)$ isovector. Our results confirm many qualitative ideas about the model, including the mass spectrum, the decoupling at $N=4$, and the isospinor nature of the kinks.

## 1. Introduction

One of the most interesting and best understood two-dimensional field theory models is the multifermion $(\bar{\psi} \psi)^{2}$ model. This model has asymptotic freedom, dynamical symmetry breaking and non-perturbative mass generation [1]. It also has a rich bound state spectrum [2].

Recently A. Zamolodchikov and Al. Zamolodchikov have determined the exact $S$-matrix of the elementary fermions and their bound states in this model [3]. However, the model is believed to have additional states, the kink states, whose existence is related to the dynamical symmetry breaking [4]. The purpose of this paper is to complete the determination of the $S$-matrix of the $(\bar{\psi} \psi)^{2}$ model, by calculating the kink-elementary particle and kink-kink $S$-matrix elements.

When one attempts to calculate a two-dimensional $S$-matrix by the methods of A. Zamolodchikov [5] and of Karowski, Thun, Troung and Weisz. [6], one encounters equations that are strongly overdetermined. If a solution exists, this is a good confirmation of the qualitative assumptions on which the calculation is based. This is probably the greatest interest of our calculation **.

The following are some of the qualitative ideas that our calculation tests.

[^0]For our calculation, we need to know the kink quantum numbers. In a previous paper [8] by one of us it has been argued that the kinks are isospinors, and that there is a $\gamma_{5}$ symmetry in isospin space.
A. Zamolodchikov has suggested [9] that the elementary fermions of this model disappear from the spectrum at $N=4$, and that for $N<4$ the spectrum consists only of kink states. In addition, it is known that at $N=4$, this model consists of two decoupled sine-Gordon equations [10]. Finally we expect the semiclassical spectrum [2] to appear as poles in the kink-kink $S$-matrix. We derive here the kink-elementary particle and kink-kink $S$-matrices. Our results, though not fully explicit, contain the essential features (pole structure, factorization etc.) that corroborate the conjectures we set out to verify. Explicit solutions may always be obtained by a straightforward application of the techniques reviewed in ref. [2]. In the kink-kink case, we restrict ourselves to even $N$, for the odd $N$ case has additional supersymmetry properties [8]. Also, since the computation involves Fierz transformations which are very involved for arbitrary $N$, we give the full details only for $N=4$ and 6 . We emphasize that it is the lack of motivation (given the goals listed above), and not conceptual obstacles that induce us to restrict ourselves in this manner.

## 2. The kink quantum numbers and the disappearance of the elementary particles

Since this is essential in what follows, we would first like to explain why we think that the kink is an isospinor. This was argued in ref. [8] on the basis of some semiclassical arguments of Jackiw and Rebbi [11]. Here we will give a simple argument using a technique, the bosonization of fermions [12], that is special to two-dimensions.

First we will illustrate the Jackiw-Rebbi phenomenon in a simple example. The simplest model that possesses solitons is the scalar field theory

$$
\begin{equation*}
\mathcal{L}=\int \mathrm{d} x\left[\frac{1}{2}\left(\partial_{\mu} \phi\right)^{2}-\lambda\left(\phi^{2}-a^{2}\right)^{2}\right] \tag{1}
\end{equation*}
$$

It possesses, in the classical approximation, two ground states, $\phi= \pm a$, as well as solitons which, for instance, interpolate between $\phi=-a$ at $x=-\infty$ and $\phi=a$ at $x=+\infty$.

Now we couple to fermions

$$
\begin{equation*}
\mathcal{L}=\int \mathrm{d} x\left[\frac{1}{2}\left(\partial_{\mu} \phi\right)^{2}+\bar{\psi} i \not \partial \psi-\lambda\left(\phi^{2}-a^{2}\right)^{2}-g \phi \bar{\psi} \psi\right] \tag{2}
\end{equation*}
$$

in such a way that there is a discrete chiral symmetry, $\psi \rightarrow \gamma_{5} \psi, \phi \rightarrow-\phi(\psi$ is a Dirac fermion).

Jackiw and Rebbi claim that the single soliton state of (1) becomes a pair of states, of fermion number $\pm \frac{1}{2}$, in model (2). To see this in a simple way, we replace $\psi$ by a boson field $\sigma$ in the standard way

$$
\begin{align*}
& \bar{\psi} i \partial \psi=\frac{1}{2}\left(\partial_{\mu} \sigma\right)^{2} \\
& \bar{\psi} \psi=\cos (\sqrt{4 \pi} \sigma) \tag{3}
\end{align*}
$$

The Lagrangian becomes

$$
\begin{equation*}
\mathcal{L}=\int \mathrm{d} x\left[\frac{1}{2}\left(\partial_{\mu} \phi\right)^{2}+\frac{1}{2}\left(\partial_{\mu} \sigma\right)^{2}-\lambda\left(\phi^{2}-a^{2}\right)^{2}-g \phi \cos (\sqrt{4 \pi \sigma})\right] . \tag{4}
\end{equation*}
$$

Now instead of two minima at $\phi= \pm a$, we have two families of minima, one family at $\phi=-a^{\prime}, \sigma=n \sqrt{ } \pi$, and the other at $\phi=+a^{\prime}, \sigma=\left(n+\frac{1}{2}\right) \sqrt{ } \pi$ where $n$ ranges over the integers ( $a^{\prime}$ is the minimum of the function $\left.\left(\phi^{2}-a^{2}\right)^{2}-g \phi\right)$. Previously we considered the soliton state that interpolates between $\phi=-a$ and $\phi=+a$, now we must also specify what is happening to $\sigma$.

When $\phi$ changes from $-a^{\prime}$ to $+a^{\prime}, \sigma$ must change by a half-integral multiple of $\sqrt{ } \pi$. Presumably the lowest energy states will be those in which $\sigma$ changes by $\pm \frac{1}{2} \sqrt{ } \pi$. Thus, there will be two kinds of soliton states; if the field at $x \rightarrow-\infty$, is, for instance, ( $-a^{\prime}$, 0 ), then the field at $x \rightarrow \infty$ may be $\left(a^{\prime}, \frac{1}{2} \sqrt{ } \pi\right)$ or ( $a^{\prime},-\frac{1}{2} \sqrt{ } \pi$ ). These two states, with two possible values of the field at $x \rightarrow \infty$ are the two states described, from another viewpoint, by Jackiw and Rebbi.

We can also see that these two states have fermion number $\frac{1}{2}$. (By "fermion number" we mean the conserved quantum number $Q=\int_{-\infty}^{\infty} \mathrm{d} x \bar{\psi} \gamma^{0} \psi$.)

To do this, we must first realize that this model also has soliton states in which $\langle\phi\rangle$ does not change, for instance, transitions from $\left(-a^{\prime}, 0\right)$ to $\left(-a^{\prime}, \sqrt{ } \pi\right)$. As shown by Coleman [12], the fermion number is $1 / \sqrt{ } \pi$ times the change in $\sigma$, so these states have fermion number one and can be identified with the original fermion $\psi$ of Lagrangian (2).

On the other hand, the pair of states discussed earlier, with transitions from ( $-a^{\prime}$, 0 ) to ( $a^{\prime}, \pm \frac{1}{2} \sqrt{ } \pi$ ), have $\sigma$ changing by only $\pm \frac{1}{2} \sqrt{ } \pi$, so, by the same reasoning, they have fermion number $\pm_{2}^{1}$. This confirms the claim of Jackiw and Rebbi.

Let us now apply this reasoning to the model of main interest in this paper, the $(\bar{\psi} \psi)^{2}$ model, which has the Lagrangian

$$
\begin{equation*}
\mathcal{L}=\int \mathrm{d} x\left[\sum_{i=1}^{N} \frac{1}{2} \bar{\psi}_{i} i \not \partial \psi_{i}+g\left(\sum_{i=1}^{N} \bar{\psi}_{i} \psi_{i}\right)^{2}\right], \tag{5}
\end{equation*}
$$

where the $\psi_{i}, i=1,2, \ldots, N$, are $N$ Majorana fermions. This Lagrangian has a discrete chiral symmetry $\psi_{i} \rightarrow \gamma_{s} \psi_{i}$. The symmetry is spontaneously broken, resulting in vacuum states with positive or negative $\langle\bar{\psi} \psi\rangle$. Accordingly, they are soliton or kink states, for which, for instance, $\langle\bar{\psi} \psi\rangle\langle 0$ as $x \rightarrow-\infty$, and $\langle\bar{\psi} \psi\rangle>0$ for $x \rightarrow \infty$.

If we assume that $N$ is even, we may replace (5) by an equivalent boson Lagrangian. We group the Majorana Fermi fields in pairs, replacing each pair by a Bose field. Altogether there will be $\frac{1}{2} N$ Bose fields:

$$
\begin{align*}
& \frac{1}{2} \bar{\psi}_{1} i \not \partial \psi_{1}+\frac{1}{2} \bar{\psi}_{2}^{2} i \not \partial \psi_{2}=\frac{1}{2}\left(\partial_{\mu} \phi_{1}\right)^{2}, \\
& \bar{\psi}_{1} \psi_{1}+\bar{\psi}_{2} \psi_{2}=\cos \left(\sqrt{4 \pi} \phi_{1}\right), \\
& \frac{1}{2} \bar{\psi}_{N-1} i \not \partial \psi_{N-1}+\frac{1}{2} \bar{\psi}_{N} i \not \partial \psi_{N}=\frac{1}{2}\left(\partial_{\mu} \phi_{N / 2}\right)^{2}, \\
& \bar{\psi}_{N-1} \psi_{N-1}+\bar{\psi}_{N} \psi_{N}=\cos \left(\sqrt{4 \pi} \phi_{N / 2}\right) . \tag{6}
\end{align*}
$$

The Lagrangian becomes

$$
\begin{equation*}
\mathcal{L}=\int \mathrm{d} x\left[\sum_{i=1}^{N / 2} \frac{1}{2}\left(\partial_{\mu} \phi_{i}\right)^{2}+g\left(\sum_{i=1}^{N / 2} \cos \left(\sqrt{4 \pi} \phi_{i}\right)\right)^{2}\right] \tag{7}
\end{equation*}
$$

The potential energy is now $-\left(\Sigma \cos \left(\sqrt{4 \pi} \phi_{i}\right)\right)^{2}$. There are two families of minima: $\phi_{i}=\left(n_{i}+\frac{1}{2}\right) \sqrt{ } \pi$, the $n_{i}$ being arbitrary integers. We will refer to these as positive and negative vacua respectively, because they correspond to positive and negative values of $\bar{\psi} \psi=\Sigma_{i} \cos \left(\sqrt{4 \pi} \phi_{i}\right)$.

Let us first consider soliton states that interpolate between two positive vacua or between two negative vacua. For such a soliton, each $\phi_{i}$ changes by an integral multiple of $\sqrt{ } \pi$ between $x \rightarrow-\infty$ and $x \rightarrow+\infty$. The lowest energy states presumably are those in which $\frac{1}{2} N \cdot 1$ of the $\phi_{i}$ do not change and the remaining one field changes by $\pm \sqrt{ } \pi$. There are $N$ such states, since we must choose one of the $\frac{1}{2} N \phi_{i}$ fields and then must decide if it is to increase or decrease by $\sqrt{ } \pi$. These $N$ states can be identified with the $N$ elementary fermions of (5).

We also have transitions from a positive vacuum to a negative vacuum. In this case, each $\phi_{i}$ must change by a half-integral multiple of $\sqrt{ } \pi$. Presumably in this case the lowest energy states have each field changing by $\pm \frac{1}{2} \sqrt{ } \pi$. If, for instance, we have a configuration $(0,0,0, \ldots, 0)$ at $x \rightarrow-\infty$, the field values at $x \rightarrow \infty$ may be $\left( \pm \frac{1}{2} \sqrt{ } \pi\right.$, $\pm \frac{1}{2} \sqrt{ } \pi, \ldots, \pm \frac{1}{2} \sqrt{ } \pi$ ). There are $2^{N / 2}$ states of this type, because we must make $\frac{1}{2} N$ independent choices of the sign.

These states are the usual kink states of (5) because they represent transitions between positive and negative expectation values of $\bar{\psi} \psi$. They form an irreducible representation of the $\mathrm{O}(N)$ symmetry of (5), because they are rotated into one another by the discrete symmetries of (7), which is a subgroup of the original $O(N)$ symmetry. The only irreducible representation of $\mathrm{O}(N)$ of dimension $2^{N / 2}$ is the isospinor representation; therefore, the kinks of (5) are isospinors.

We can also see that the kinks are isospinors because they have half-integer quantum numbers. They involve half-integral changes in the fields $\phi_{i}$ and this corresponds to half-integer values of the $\mathrm{O}(N)$ quantum numbers. Only the isospinor representation (and larger representations built by combining it with tensors) has half-integer quantum numbers.

Finally, we may give a heuristic argument in support of A. Zamolodchikov's idea that for small enough $N$, the elementary particle becomes unstable and disappears from the spectrum.

A kink, which involves a change in $\langle\bar{\psi} \psi\rangle$, can certainly not be built from a finite number of elementary particles, each of which involves no such change. But can an elementary particle be built from kinks?

As far as the quantum numbers are concerned, an elementary particle, in which, for instance, the field jumps from $(0,0, \ldots, 0)$ to $(\sqrt{ } \pi, 0,0, \ldots, 0)$, can certainly be regarded as a succession of two jumps, first from $(0,0, \ldots, 0)$ to $\left(\frac{1}{2} \sqrt{ } \pi, \frac{1}{2} \sqrt{ } \pi, \ldots, \frac{1}{2} \sqrt{ } \pi\right)$, then from $\left(\frac{1}{2} \sqrt{ } \pi, \frac{1}{2} \sqrt{ } \pi, \ldots, \frac{1}{2} \sqrt{ } \pi\right)$ to $(\sqrt{ } \pi, 0,0, \ldots, 0)$. Each of these two jumps corres-
ponds to a kink state, since in each case each field changes by $\pm \sqrt{ } \pi$. Thus, the elementary particle has the same quantum numbers as a pair of kinks.

We thus have a dynamical question. a question of energetics: is the elementary particle unstable against decay into a pair of kinks?

We do not have enough dynamical information to answer this, so let us make a crude guess. We will guess that the energy of a soliton state is proportional to the distance in field space by which the field jumps.

This distance is $\sqrt{ } \pi$ for elementary particles but $\sqrt{\frac{1}{8} N \pi}$ for kink states. Thus, for large $N$, the kinks are much heavier (in confirmation of what is known from ref. [4]). But for small $N$, the masses become comparable, and it is perfectly conceivable that the elementary particle becomes unstable for small enough $N$ against decay into a kink pair. It could then disappear from the spectrum. A. Zamolodchikov has argued that this happens at $N=4$.

## 3. The elementary particle-kink scattering amplitudes

We now turn to a determination of the $S$-matrix elements that involves kinks. In this section we will determine the amplitude for the scattering of elementary fermions by kinks.

The $S$-matrix for the scattering of two elementary fermions has already been obtained [3]. Denoting as $|a(\theta)\rangle$ an elementary fermion of isospin $a$ and rapidity $\theta$, the $S$-matrix for scattering of two particles $a\left(\frac{1}{2} \theta\right)$ and $b\left(-\frac{1}{2} \theta\right)$ to $c\left(\frac{1}{2} \theta\right)$ and $d\left(-\frac{1}{2} \theta\right)$ is

$$
\begin{align*}
& \left\langle d\left(-\frac{1}{2} \theta\right), \left.c\left(\frac{1}{2} \theta\right) \right\rvert\, a\left(\frac{1}{2} \theta\right), b\left(-\frac{1}{2} \theta\right)\right\rangle \\
& \quad=S(\theta)\left(\delta^{a c} \delta^{b d}-\frac{2 \pi i}{N-2} \frac{\delta^{a b} \delta^{c d}}{i \pi-\theta}-\frac{2 \pi i}{N-2} \frac{\delta^{a d} \delta^{b c}}{\theta}\right), \tag{8}
\end{align*}
$$

where $S(\theta)$ is a function described by A. and Al. Zamolodchikov in ref. [3].
Our strategy is now to use the identity indicated in fig. 1 to determine the elementary particle-kink amplitudes. (The ideas behind this were developed in refs. [3,5, 6], and are reviewed in ref. [7].) The identity in fig. 1 is a cubic identity that involves the particle-particle and particle-kink amplitudes. Since the particle-particle amplitudes are already known (eq. (8)), the only unknown in fig. 1 is the particle-kink $S$-matrix, and fig. 1 can be used to determine it.


Fig. 1. The identity that determines the kink-elementary particle $S$-matrix. Solid lines are elementary particles; wavy lines the kinks.

To do this in detail, we first need some notation. Let $\gamma^{i}, i=1,2, \ldots, N$, be the usual $\gamma$ matrices of $\mathrm{O}(N)$, satisfying

$$
\begin{equation*}
\gamma^{i} \gamma^{j}+\gamma^{j} \gamma^{i}=2 \delta^{i j} \tag{9}
\end{equation*}
$$

We also define

$$
\begin{equation*}
\sigma^{i j}=\frac{1}{2}\left(\gamma^{i} \gamma^{j}-\gamma^{j} \gamma^{i}\right) \tag{10}
\end{equation*}
$$

If we denote a kink state of isospin $\alpha$ and rapidity $\theta$ as $\mid \alpha(\theta)$, then the most general form allowed by isospin conservation for the particle-kink $S$-matrix is

$$
\begin{equation*}
\left\langle b\left(-\frac{1}{2} \theta\right), \beta\left(\frac{1}{2} \theta\right) \left\lvert\, \alpha\left(\frac{1}{2} \theta\right)\right., a\left(-\frac{1}{2} \theta\right)\right\rangle=U_{1}(\theta) \delta^{b a} \delta_{\beta \alpha}+U_{2}(\theta) \sigma_{\beta \alpha}^{b a} \tag{11}
\end{equation*}
$$

(Since the kink and elementary particle do not have the same mass, the outgoing kink must have the same rapidity as the incoming one, in order to satisfy the infinite sequence of conservation laws of this model.) The cubic identity of fig. 1 determines $U_{2}$ in terms of $U_{1}$,

$$
\begin{equation*}
U_{2}(\theta)=\frac{-2 U_{1}(\theta)}{(N-2)(1+2 i \theta / \pi)} \tag{12}
\end{equation*}
$$

(one derives this by following the standard procedure of the previous references.) Crossing symmetry implies that

$$
\begin{align*}
& U_{1}(\theta)=U_{1}(i \pi-\theta) \\
& U_{2}(\theta)=-U_{2}(i \pi-\theta) \tag{13}
\end{align*}
$$

while unitary implies

$$
\begin{equation*}
U_{1}(\theta) U_{1}(-\theta)=1+\frac{4(N-1)}{(N-2)^{2}\left(1+4 \theta^{2} / \pi^{2}\right)} \tag{14}
\end{equation*}
$$

In addition, $U_{1}$ and $U_{2}$ are expected to be metomorphic functions of $\theta$.
Eq. (14) and the first equation of (13), plus the meromorphic nature of $U_{1}$, suffice to determine $U_{1}$. The details will not be given here; for analogous calculations, see refs. [3,5-7]. (Strictly speaking, $U_{1}$ is determined only up to CDD ambiguities; the ambiguities can, as usual, be fixed by looking for a minimal solution.) Once $U_{1}$ is known, $U_{2}$ is known from (12).

In the calculations leading to eqs. (12)-(14), one encounters many cancellations. Without these cancellations, there would be extra equations, and no consistent solution would exist. The fact that a solution exists is a good indication that the isospinor kink spectrum we have assumed is correct.

## 4. Kink-kink scattering

In this section we consider the much more difficult problem of the kink-kink $S$-matrix. We will be considering only the case of even $N$. Our strategy will be to use



Fig. 2. The identity that determines the kink-kink $S$-matrix. Solid lines are elementary particles; wavy lines the kinks.
the identity indicated in fig. 2 . In this identity the particle-kink and kink-kink $S$-matrix elements appear. Since the particle-kink $S$-matrix is already known from the previous section, the only unknown quantity in this identity is the kink-kink $S$-matrix, and this identity enables us to determine it.

Once again, we must introduce some notation. Let

$$
\begin{align*}
& \sigma^{0}=I, \quad \sigma_{i}^{1}=\gamma_{i}, \\
& \sigma_{i j}^{2}=\frac{1}{2}\left(\gamma_{i} \gamma_{j}-\gamma_{j} \gamma_{i}\right), \\
& \vdots \\
& \sigma_{i_{1} \ldots i_{n}}^{n}=\frac{1}{n!}\left(\gamma_{i_{1}}: \ldots \gamma_{i_{n}}\right)_{\mathrm{A}}, \tag{15}
\end{align*}
$$

where A represents complete antisymmetrization, Also, let

$$
\begin{equation*}
\sigma_{\delta \alpha}^{n} \otimes \sigma_{\gamma \beta}^{n}=\sum_{i_{1}, i_{2} \ldots i_{n}}\left(\sigma_{i_{1} \ldots i_{n}}^{n}\right)_{\delta \alpha}\left(\sigma_{i_{1} \ldots i_{n}}^{n}\right)_{\gamma \beta} \tag{16}
\end{equation*}
$$

Now, we wish to write down the most general form for the kink-kink $S$-matrix allowed by the $\mathrm{O}(N)$ symmetry. What makes the kink-kink problem difficult is that the number of invariant amplitudes is large, of order $N$. Moreover, there does not seem to be any one choice of these amplitudes in which all of the necessary conditions (the cubic identity, crossing, and unitarity) can be stated easily.

We will consider two ways of expanding the $S$-matrix in invariant amplitudes

$$
\begin{align*}
& \left\langle\left.\delta\left(-\frac{1}{2} \theta\right) \gamma\left(\frac{1}{2} \theta\right) \right\rvert\, \alpha\left(\frac{1}{2} \theta\right) \beta\left(-\frac{1}{2} \theta\right)\right\rangle=\sum_{n=0}^{N} \frac{V_{n}(\theta)}{n!} \sigma_{\delta \alpha}^{n} \otimes \sigma_{\gamma \beta}^{n} \\
& \quad=\sum_{n=0}^{N} \frac{w_{n}(\theta)}{n!} \sigma_{\gamma \alpha}^{n} \otimes \sigma_{\delta \beta}^{n} . \tag{17}
\end{align*}
$$

The $V_{n}$ are amplitudes with particles $\delta$ and $\alpha$ coupled to definite isospin $n$; the $W_{n}$ have $\gamma$ and $\alpha$ coupled to definite isuspin $n$. By a Fierz transformation, the $W_{n}$ can be
written as linear combinations of the $V_{n}$, (or viceversa), but unfortunately Fierz transformations for general $N$ are rather complicated.

Before proceeding, let us recall that from ref. [8], we expect a $\gamma_{5}$ invariance in isospin space. This means that $W_{n}=0$ for odd $n$, or equivalently, that

$$
\begin{equation*}
V_{N-n}=(-1)^{(N / 2)+n} V_{n} \tag{18}
\end{equation*}
$$

However, we will not impose (18) as input in the calculation; it will emerge as part of the output.

With the kinematics aside, we now consider the identity of fig. 2. After some fairly long calculations, with many cancellations, one finds that this identity is equivalent to the recursion relation

$$
\begin{equation*}
V_{n+2}(\theta)=V_{n}(\theta) \frac{i \theta / \pi+2 n /(N-2)}{i \theta / \pi+2(N-n-2) /(N-2)} . \tag{19}
\end{equation*}
$$

This formula determines all $V_{n}$ in terms of $V_{0}$ and $V_{1}$. From this recursion relation, one can see that (18) is, in fact, satisfied.

Next we must consider unitarity. We can write unitarity equations for the $V_{n}$ in either the $s$-channel or the $u$-channel. The $u$-channel equations are particularly simple because the $V_{n}$ are amplitudes of definite $u$-channel isospin. Unitarity in the $u$-channel implies

$$
\begin{align*}
& 16 V_{0}(i \pi-\theta) V_{0}(i \pi+\theta)=1 \\
& 16 V_{1}(i \pi-\theta) V_{1}(i \pi+\theta)=1 \tag{20}
\end{align*}
$$

Eq. (20) corresponds to unitarity of the $S$-matrix amplitudes with $u$-channel isospin zero or one. The $u$-channel unitarity conditions for isospin greater than one are identities if eqs. (19), (20) are satisfied.

One can also write $s$-channel unitarity equations. There are $N+1$ equations in all, corresponding to $N+1$ channels of definite isospin. Two of these equations are

$$
\begin{align*}
& \sum_{n=0}^{N} \frac{N!}{(N-n)!n!} V_{n}(\theta) V_{n}(-\theta)=1 \\
& \sum_{n=1}^{N} \frac{N!}{(N-n)!n!} V_{n}(\theta) V_{N-n}(-\theta)=0 \tag{21}
\end{align*}
$$

The remaining $N-1$ equations for $s$-channel unitarity are identities if eqs. (19), (21) are satisfied. (We do not have a general proof, but have checked this explicitly for $N=4,6$, and 8. )

Finally, we must consider crossing. Unfortunately, the amplitudes $V_{n}$ do not have simple crossing properties. For the Fierz transformed amplitudes $W_{n}$ we can, however, write rather simple crossing relations

$$
\begin{equation*}
W_{n}(i \pi-\theta)=(-1)^{n(n-1) / 2} W_{n}(\theta) \tag{22}
\end{equation*}
$$

The deviation of (22) is rather subtle and is explained in the appendix.
Eqs. (19)-(22), together with the meromorphic nature of the $V_{n}$ and $W_{n}$, completely determine the $S$-matrix, apart from CDD ambiguities. (It is not even necessary to use both (20) and (21).) In determining the $S$-matrix, however, one must make a Fierz transformation, either to re-express (22) in terms of the $V_{n}$ or to re-express (19)-(21) in terms of the $W_{n}$. Instead of attempting to carry out a Fierz transformation for general $N$, we will here consider in detail the cases $N=4$ and $N=6$.
(i) $N=4$.

In this case the Fierz relations are

$$
\begin{align*}
& 4 V_{0}=W_{0}+W_{4}-6 W_{2}, \\
& 4 V_{1}=W_{0}-W_{4} . \\
& 4 V_{2}=-\left(W_{0}+W_{4}\right)-2 W_{2} . \tag{23}
\end{align*}
$$

By means of eqs. (19), (23) one can determine $W_{2}$ in terms of $W_{0}$ and $W_{4}$ :

$$
\begin{equation*}
W_{2}=-\frac{\left(W_{0}+W_{4}\right)}{2(1+2 i \theta / \pi)} . \tag{24}
\end{equation*}
$$

The crossing and unitarity equations are simplest in terms of the sum and difference $W_{0}+W_{4}$ and $W_{0}-W_{4}$. We find

$$
\begin{align*}
& W_{0}(\theta)+W_{4}(\theta)=W_{0}(i \pi-\theta)+W_{4}(i \pi-\theta), \\
& \left(W_{0}(\theta)+W_{4}(\theta)\right)\left(W_{0}(-\theta)+W_{4}(-\theta)\right)=\frac{1+4 \theta^{2} / \pi^{2}}{4\left(1+\theta^{2} / \pi^{2}\right)},  \tag{25}\\
& W_{0}(\theta)-W_{4}(\theta)=W_{0}(i \pi-\theta)-W_{4}(i \pi-\theta), \\
& \left(W_{0}(\theta)-W_{4}(-\theta)\right)\left(W_{0}(-\theta)-W_{4}(-\theta)\right)=1 . \tag{26}
\end{align*}
$$

Eqs. (26) has the obvious solution $W_{0}-W_{4}=1$. If as is usual, one looks for the "minimal" solution without CDD zeroes or poles, then this solution is unique. Eq. (25) can be solved by the method of A. Zamolodchikov.

The fact that $W_{0}-W_{4}=1$, written in terms of states of definite (isotopic) chirality (that is, isospinor states with $\gamma_{5}=+1$ or $\gamma_{5}=-1$ ), has a very simple interpretation. $W_{0}-W_{4}$ is the $S$-matrix element for the scattering of a $\gamma_{5}=+1$ kink by a $\gamma_{5}=-1$ kink. The fact that $W_{0}-W_{4}=1$ means that $\gamma_{5}=-1$ kinks are decoupled from $\gamma_{5}=+1$ kinks. This decoupling is expected from field theory arguments $[10,8]$.

Actually, it is known from field theory arguments that the $N=4$ model should be equivalent to two decoupled sine-Gordon equations. Specifically, the two $\gamma_{5}=-1$ kinks of the $N=4$ model can be regarded as the soliton and anti-soliton of a sineGordon system. The sine-Gordon coupling should be taken to be $\beta^{2}=8 \pi^{-}$: at this value of the coupling the sine-Gordon spectrum consists solely of the soliton and anti-soliton. Likewise, the two $\gamma_{5}=+1$ kinks of the $N=4$ model are the soliton and anti-soliton of a second sine-Gordon model, decoupled from the first one. One can
check that the sine-Gordon $S$-matrix, of refs. [5,6] at $\beta^{2}=8 \pi^{-}$; coincides with the $N=4 S$-matrix in the $\gamma_{5}=-1$ (or $\gamma_{5}=+1$ ) sector, as described in our eqs. (24) $-(26$ ). The correspondence between our formulae and the sine-Gordon $S$-matrix is that $W_{0}+$ $W_{4}$ equals $f_{1}+\frac{1}{2} f_{2}$ and $W_{2}$ equals $\frac{1}{2} f_{2}$, where $f_{1}$ and $f_{2}$ are the soliton-antisoliton forward and backward scattering amplitudes of the sine-Gordon model.

In particular, it follows from this that the soliton and anti-soliton of the sineGordon model at $\beta^{2}=8 \pi^{-}$are related by an $\mathrm{O}(3)$ symmetry; they transform as an isodoublet.
(ii) $N=6$.

We will also consider in some detail the next simplest case, $N=6$.
The Fierz relations between the amplitudes $V_{i}$ and $W_{i}$ of eq. (17) are now

$$
\begin{align*}
& 6 V_{0}=W_{0}-W_{6}-15\left(W_{2}-w_{4}\right) \\
& 6 V_{1}=w_{0}+W_{6}-5\left(W_{0}+w_{4}\right) \\
& -6 V_{2}=W_{0}-W_{6}+\left(W_{2}-W_{4}\right) \\
& -6 V_{3}=W_{0}+w_{6}+3\left(W_{2}+w_{4}\right) \tag{27}
\end{align*}
$$

( $V_{4}, V_{5}$, and $V_{6}$ are known in terms of $V_{0}, V_{1}$, and $V_{2}$ from eq. (19).) With the $V$ 's expressed by (27) in terms of the $W$ 's, we can now conveniently combine the information contained in eqs. (19)-(22). In terms of $W$, (19) becomes

$$
\begin{align*}
& \frac{W_{0}-W_{6}}{W_{2}-W_{4}}=-\left(1+\frac{8 i \theta}{\pi}\right), \\
& \frac{W_{0}+W_{6}}{W_{2}+W_{4}}=-\left(7+\frac{i \theta}{\pi}\right) . \tag{28}
\end{align*}
$$

If one now writes the unitarity in terms of $W$ and eliminates $W_{2}$ and $W_{4}$ by means of (28), one can obtain two coupled equations for $W_{0}$ and $W_{6}$ :

$$
\begin{align*}
& \left(W_{0}(\theta)+W_{6}(\theta)\right)\left(W_{0}(-\theta)+W_{6}(-\theta)\right)=\frac{\left(\frac{7}{8}\right)^{2}+\theta^{2} / \pi^{2}}{1+\theta^{2} / \pi^{2}} \\
& \left(W_{0}(\theta)-W_{6}(\theta)\right)\left(W_{0}(-\theta)-W_{6}(-\theta)\right)=\frac{\left(\frac{1}{8}\right)^{2}+\theta^{2} / \pi^{2}}{\frac{1}{4}+\theta^{2} / \pi^{2}} \tag{29}
\end{align*}
$$

These equations, combined with crossing,

$$
\begin{align*}
& W_{0}(\theta)=W_{0}(i \pi-\theta) \\
& W_{6}(\theta)=-W_{6}(i \pi-\theta), \tag{30}
\end{align*}
$$

are a closed system of equations for $W_{0}$ and $W_{6}$ which determine $W_{0}$ and $W_{6}$, up to CDD ambiguities. For instance, one could use eqs. (29), (30) to determine, iteratively, the locations of the zeroes and poles in $W_{0}+W_{6}$ and $W_{0}-W_{6}$, following $\mathbf{A}$.
Zamolodchikov's reasoning in the sine-Gordon case.

A similar procedure can, in principle, be used for any $N$, to eliminate all amplitudes in favor of $W_{0}$ and $W_{N}$, obtaining thus a closed system of equations for $W_{0}$ and $W_{N}$, but we will not attempt to pursue this further.

However, there is one qualitative question that we wish to discuss here. This model is known to have an extensive mass spectrum, with particles of mass

$$
\begin{equation*}
M_{n}=2 M \sin \left(\frac{n \pi}{N-2}\right), \quad n=1, \ldots, \frac{N-4}{2}, \tag{31}
\end{equation*}
$$

where $M$ is the kink mass. The elementary particle is $n=1$; higher values of $n$ are bound states. This spectrum was discovered in a semiclassical calculation by Dashen, Hasslacher and Neveu [2], and was shown to be exact by A. and Al. Zamolodchikov [3]. We would like to ask whether these particles (the elementary particle and the bound states) do, indeed, show up as poles in the kink-kink $S$-matrix.

According to (2) and (3), there is a large isospin degeneracy in the spectrum. If we describe the channel of antisymmetric tensors of rank $n$ as the $n$th isospin channel, then particles of mass $2 M \sin (\pi n /(N-2))$ are expected to be present in the isospin channels $n, n-2, n-4, \ldots, 0$ for even $n$, and $n, n-2, n-4, \ldots, 1$ for odd $n$.

A particle of mass $2 M \sin (\pi n /(N-2))$ will correspond to a pole at $\theta=i \pi-$ $(2 \pi i n) /(N-2)$ in the $s$-channel; the crossed pole will show up at $\theta=2 \pi i n /(N-2)$ in the $u$-channel. The amplitudes $V_{n}$ described earlier have the quantum numbers of $u$-channel isospin $n$. Therefore, we expect that $V_{n}$ will have poles at $\theta=2 \pi i n /(N-2)$ $2 \pi i(n+2) /(N-2), 2 \pi i(n+4) /(N-2) \ldots \pi i(N-4) /(N-2)$, for $n$ even; the last pole is at $\pi i(N-5) /(N-2)$ if $n$ is odd.

Now we wish to compare this with what we can learn from the recursion relation (19), which determines $V_{n+2}$ in terms of $V_{n}$. Replacing $n$ by $n+2$, one will have the same series of poles, but starting at $2 \pi i(n+2) /(N-2)$ instead of $2 \pi i n /(N-2)$. In other words, $V_{n}$ should have the same poles as $V_{n+2}$, plus an extra one at $2 \pi i n /(N-$ 2).

This agrees with the recursion relation (19), which can be rewritten as

$$
\begin{equation*}
V_{n}(\theta)=-V_{n+2}(\theta) \frac{i \theta / \pi+2(N-n-2) /(N-2)}{i \theta / \pi+2 n /(N-2)}, \tag{32}
\end{equation*}
$$

and strongly suggests that $V_{n}$ will have a pole at $2 \pi i n /(N-2)$ that is absent in $V_{n+2}$.
The conclusion is not rigorous, because $V_{n+2}(\theta)$ might have a zero at $2 \pi i n /(N-2)$ that would cancel the explicit denominator factor on the right-hand side of (32).
Because of CDD ambiguities, we cannot expect a rigorous conclusion; the factorization, unitarity, and analyticity equations alone cannot uniquely determine the structure of poles and zeroes in the $S$-matrix. At most we can hope that a particularly simple solution of these equations has the expected poles, and (32) seems to show this.

This thus provides support for the idea that the exact mass spectrum is $M_{n}=2 M \times$ $\sin n \pi /(N-2)$. As A. Zamolodchikov has pointed out, this formula has the partic-
ularly striking consequence that for $N \leqslant 4$, the elementary particle mass exceeds twice the kink mass, and the elementary particle would then, presumably, be absent from the spectrum.

## 5. Conclusion

We have in this paper analyzed the particle-kink and kink-kink $S$-matrix elements of the $(\bar{\psi} \psi)^{2}$ model. The results are in satisfying agreement with qualitative properties that the model is believed to have, including the mass spectrum, the decoupling at $N=4$, the isospinor nature of the kinks, and the absence of the elementary particle from the physical spectrum for $N \leqslant 4$.

## Appendix

Here we would like to explain some facts about the $O(N) \gamma$-matrices, and also to explain the derivation of eq. (22).

The gamma matrices of $\mathrm{O}(4)$ are well known to physicists. For example, one may choose

$$
\gamma^{1}=\left(\begin{array}{ll}
0 & \sigma_{1}  \tag{33}\\
\sigma_{1} & 0
\end{array}\right), \quad \gamma^{2}=\left(\begin{array}{ll}
0 & \sigma_{2} \\
\sigma_{2} & 0
\end{array}\right), \quad \gamma^{3}=\left(\begin{array}{ll}
0 & \sigma_{3} \\
\sigma_{3} & 0
\end{array}\right), \quad \gamma^{4}=\left(\begin{array}{ll}
0 & i I \\
-i I & 0
\end{array}\right)
$$

We wish to generalize this to build up higher dimensional $\gamma$ matrices. Suppose that the $\gamma$ matrices have been defined for $\mathrm{O}(N)$. Then to define them for $\mathrm{O}(N+2)$ we write

$$
\begin{align*}
& \gamma_{N+2}^{k}=\gamma_{N}^{k} \otimes I, \quad k=1, \ldots N, \\
& \gamma_{N+2}^{N+1}=(i)^{N(N-1) / 2+1} \gamma_{N}^{P} \otimes \sigma_{1}, \\
& \gamma_{N+2}^{N+2}=(i)^{N(N-1) / 2+1} \gamma_{N}^{P} \otimes \sigma_{3}, \tag{34}
\end{align*}
$$

where $\sigma_{1}$ and $\sigma_{3}$ are Pauli matrices and $\gamma_{N}^{P}=\gamma_{N}^{1} \gamma_{N}^{2} \ldots \gamma_{N}^{N}$ is the generalization of $\gamma_{5}$ for $\mathrm{O}(N)$ gamma matrices. The peculiar factor of $(i)^{N(N-1) / 2+1}$ is needed in (34) because $\gamma_{N}^{P}$ by itself is Hermitian only when $N=2$ modulo 4 , but when multiplied by this factor it is always Hermitian. It is easy to check that all $N+2$ matrices in (34) are Hermitian, have square one, and anticommute with each other.

Note that, in going from $\mathrm{O}(N)$ to $\mathrm{O}(N+2)$ gamma matrices in (34), the dimension of our vector space has doubled; we took the tensor product with the $2 \times 2$ space of the sigma matrices. This is how one can see that for any even $N$, the $\gamma$ matrices, and therefore also the isospinor representation, have dimension $2^{N / 2}$.

Now we must consider crossing, to derive (22). What makes crossing non-trivial is the following. Under crossing, a particle with isospinor $U_{\alpha}$ will cross into a par-
ticle with the complex conjugate spinor $U_{\alpha}^{*}$. The isospinor representation is equivalent to its complex conjugate and therefore the isospinor kinks can cross into themselves, which is what actually occurs. But the isospinor representation is not $o b$ viously equivalent to its complex conjugate, in the sense that the group generators cannot be chosen all to be real. As a result, a non-trivial crossing matrix is needed, which relates the group generators to their complex conjugates.

The semiclassical analysis of ref. [8], and the bosonization argument of sect. (2), show that the kinks form a single isospinor representation, which must therefore cross into itself. But we will see that the kinks cross into themselves in a non-trivial way.

To be more specific, the kink states $|\alpha\rangle$ transform under an infinitessimal isospin transformation as

$$
\begin{equation*}
|\alpha\rangle \rightarrow|\alpha\rangle+\omega_{i j} o_{\alpha \beta}^{i j}|\beta\rangle \tag{35}
\end{equation*}
$$

where $\omega_{i j}$ are real, infinitessimal rotation angles and $\sigma_{\alpha \beta}^{i j}=\frac{1}{2}\left(\gamma^{i} \gamma^{j}-\gamma^{j} \gamma^{i}\right)_{\alpha \beta}$. The antiparticles $|\bar{\alpha}\rangle$ will then transform as

$$
|\bar{\alpha}\rangle \rightarrow|\bar{\alpha}\rangle+\omega_{i j} i_{\alpha \beta}^{i j}|\bar{\beta}\rangle
$$

where $\sigma^{i j *}$ is the complex conjugate of $\sigma^{i j}$. If the $\sigma^{i j}$ matrices were all real, then the states $|\bar{\alpha}\rangle$ would be transforming the same way that the states $|\alpha\rangle$ transform, and one could identify the particles and antiparticles.

Since the $\sigma^{i j}$ for general $N$, cannot be chosen all to be real, we introduce instead a unitary matrix $G$ with

$$
\begin{equation*}
G \sigma^{i j *}=\sigma^{i j} G \tag{37}
\end{equation*}
$$

and now consider the states

$$
\begin{equation*}
|\alpha\rangle^{\prime}=G_{\alpha \beta}|\bar{\beta}\rangle \tag{38}
\end{equation*}
$$

These states transform under an infinitessimal isospin rotation as

$$
\begin{aligned}
|\alpha\rangle^{\prime} & =G_{\alpha \beta}|\bar{\beta}\rangle \rightarrow G_{\alpha \beta}|\bar{\beta}\rangle+G_{\alpha \beta} \omega_{i j} \sigma_{\beta \gamma}^{i j *}|\bar{\gamma}\rangle \\
& =G_{\alpha \beta}|\bar{\beta}\rangle+\omega_{i j} \sigma_{\alpha \beta}^{i j} G_{\beta \gamma}|\bar{\gamma}\rangle \\
& =|\alpha\rangle^{\prime}+\omega_{i j} \sigma_{\alpha \beta}^{i j}|\beta\rangle^{\prime} .
\end{aligned}
$$

Thus, the states $|\alpha\rangle^{\prime}$ transform as the states $|\alpha\rangle$ do, and it is consistent with $O(N)$ symmetry to assume $|\alpha\rangle^{\prime}=|\alpha\rangle$. This leads to a relation between particles and antiparticles: $|\alpha\rangle=G_{\alpha \beta}|\bar{\beta}\rangle$, or equivalently $|\bar{\beta}\rangle=G_{\beta \alpha}^{-1}|\alpha\rangle$.

If one accepts the semiclassical and bosonization arguments that the kink states are a single isospinor that crosses into itself, then we must identify particles with antiparticles in some way, and this is the only identification consistent with $\mathrm{O}(N)$.

To find a $G$ that satisfies (37) is very simple. Of the $\gamma$ matrices in (34), some are real and symmetric, some are imaginary and antisymmetric. We define $G$ as the product of all the imaginary $\gamma$ matrices.

To discuss crossing, we note that by general rules, if

$$
S_{\delta \gamma, \beta \alpha}(\theta)=\langle\delta(0) \gamma(\theta) \mid \alpha(\theta) \beta(0)\rangle
$$

then

$$
S_{\gamma \delta, \beta \alpha}(i \pi-\theta)=\langle\delta(0) \bar{\alpha}(\theta) \mid \bar{\gamma}(\theta) \beta(0)\rangle
$$

With our identification of particles and antiparticles, $|\bar{\gamma}\rangle=G_{\gamma \delta}^{-1}|\delta\rangle$ and therefore $\langle\bar{\alpha}|=\langle\sigma| G_{\sigma \alpha}$ so

$$
\begin{align*}
S_{\delta \gamma, \beta \alpha}(i \pi-\theta) & =G_{a \alpha} G_{\gamma \delta}^{-1}\langle\delta(0) \sigma(\theta) \mid \delta(\theta) \beta(0)\rangle \\
& =G_{\sigma \alpha} G_{\gamma \delta}^{-1} S_{\delta \sigma, \beta \delta}(\theta) \tag{40}
\end{align*}
$$

With the definition of $W_{n}$ from the text

$$
\begin{equation*}
S_{\delta \gamma, \gamma \alpha}(\theta)=\sum_{n=0}^{N} \frac{W_{n}(\theta)}{n!} \sigma_{\delta \beta}^{n} \otimes \sigma_{\gamma \alpha}^{n}, \tag{41}
\end{equation*}
$$

we see that (40) can be written

$$
\begin{equation*}
S_{\delta \gamma, \beta \alpha}(i \pi-\theta)=\sum_{n=0}^{N} \frac{W_{n}(\theta)}{n!} \sigma_{\delta \beta}^{n} \otimes\left(G_{\sigma \alpha} G_{\gamma \delta}^{-1} \sigma_{\sigma \delta}^{n}\right) \tag{42}
\end{equation*}
$$

But (41), with $\theta$ replaced by $i \pi-\theta$ and $\alpha$ replaced by $\gamma$, says

$$
\begin{equation*}
S_{\delta \alpha, \beta \gamma}(i \pi-\theta)=\sum_{n=0}^{N} \frac{W_{n}(i \pi-\theta)}{n!} \sigma_{\delta \beta}^{n} \otimes \sigma_{\gamma \alpha}^{n} \tag{43}
\end{equation*}
$$

If, therefore, we can show

$$
\begin{equation*}
\sigma_{\gamma \alpha}^{n}=(-1)^{n(n-1) / 2} G_{\sigma \alpha} G_{\gamma \delta}^{-1} \sigma_{\sigma \delta}^{n}, \tag{44}
\end{equation*}
$$

it will follow that $W_{n}(i \pi-\theta)=(-1)^{n(n-1) / 2} W_{n}(\theta)$, which is eq. (22) of the text.
Let us first consider in (44) the case $n=1$. We must show

$$
\begin{equation*}
\gamma_{\gamma \alpha}^{i}=G_{\sigma \alpha} G_{\gamma \delta}^{-1} \gamma_{\sigma \delta}^{i} \tag{45}
\end{equation*}
$$

for each $i$.
Now for each $i, \gamma^{i}$ is real and symmetric or imaginary and antisymmetric. Suppose $\gamma^{i}$ is real and symmetric. Then is commutes with $G$, which is a product of an even number of imaginary, antisymmatric $\gamma$ matrices. In this case

$$
\begin{align*}
G_{\sigma \alpha} G_{\gamma \delta}^{-1} \gamma_{\sigma \delta}^{i} & =G_{\sigma \alpha} G_{\gamma \delta}^{-1} \gamma_{\delta \sigma}^{i} \\
& =G_{\delta \sigma} G_{\gamma \delta}^{-1} \gamma_{\sigma \alpha}^{i} \\
& =\gamma_{\gamma \alpha}^{i} \tag{46}
\end{align*}
$$

as was to be proved. If instead $\gamma^{i}$ is imaginary and antisymmetric, it anticommutes
with $G$, and the calculation in (44) is still valid, except that the sign in one of the intermediate steps should be chaged.

An analogous argument, keeping track of the anticommutativity of the gamma matrices, suffices when $n>1$ in (44).

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    ** For a review of refs. $[3,5,6]$ see ref. [7].

