A NEW SET OF EXACT FORM FACTORS.

FEODOR A. SMIRNOV

Research Institute for Mathematical Sciences, Kyoto University,

Sakyo-ku 606, Kyoto, JAPAN

and

St. Petersburg Branch of Steklov Mathematical Institute,

Fontanka 27, St. Petersburg 191011, RUSSIA

ABSTRACT. We present form factors for a wide list of integrable models which includes marginal perturbations of $SU(2)$ WZNZ model for arbitrary central charge and the principal chiral field model. The interesting structure of these form factors is discussed.

1. INTRODUCTION

The form factor bootstrap is powerful method of study of integrable field theories. There are many models for which the form factors are known, the most important examples are Sine-Gordon model, its asymptotically free limit: $SU(2)$- invariant Thirring model, $O(3)$ - nonlinear sigma model, $SU(N)$-invariant Thirring model [1]. This list seems to be representative enough. However, for better understanding of integrable field theory we still need some more examples. Asymptotically free theories are of primary importance for further investigations in the field. They allow important type of quantum symmetry: Yangian symmetry [2], which is in less modern language the same as Lüsher nonlocal charges with interesting classical limit [3]. The knowledge of exact out of shell solutions of asymptotically free integrable theories should allow to understand how do the usual tools of modern theoretical physics (functional integration) apply to them. We consider this problem as the most important one: we should learn from exact solutions how to perform functional integration. There are also other reasons why such models as perturbations of WZNW model and principal chiral field are interesting for the application of form factor bootstrap which will be clear later.

Suppose we deal with an massive integrable model which contains only one particle in the spectrum (with isotopic degrees of freedom) with the two-particle S-matrix $S(\beta)$ which satisfies Yang-Baxter, crossing, unitarity [4]. Then for the operator to be local it is necessary and sufficient that its form factors (matrix element between vacuum and n-particle state) satisfy the system of equations [1] which naturally splits into two parts: Riemann-Hilbert problem

$$f(\beta_1, \cdots, \beta_i, \beta_{i+1}, \cdots, \beta_n)S(\beta_i - \beta_{i+1}) = f(\beta_1, \cdots, \beta_{i+1}, \beta_i, \cdots, \beta_n),$$
$$f(\beta_1, \cdots, \beta_{n-1}, \beta_n + 2\pi i) = f(\beta_n, \beta_1, \cdots, \beta_{n-1}), \tag{1}$$

Typeset by $\mathcal{AMS}$-TEX

and residue condition

$$2\pi i \ \mathrm{res}_{\beta_n = \beta_{n-1} + \pi i} \ f(\beta_1, \cdots, \beta_{n-2}, \beta_{n-1}, \beta_n) = \tag{2}$$
$$= f(\beta_1, \cdots, \beta_{n-2}) \otimes s_{n-1,n} \big(I - S(\beta_{n-1} - \beta_1) \cdots S(\beta_{n-1} - \beta_{n-2})\big)$$

where the usual [1] conventions are made: f belongs to the tensor product of spaces h^* related to particles (h is one particle isotopic space), $S(\beta_i - \beta_j)$ acts nontrivially only in the tensor product of spaces related to particles with rapidities β_i, β_j , permutation of spaces is applied if corresponding rapidities are permuted, $s_{n-1,n}$ is a vector in the tensor product of $(n-1)$-th and n-th spaces constructed from charge conjugation matrix, f is meromorphic function of all its arguments in finite part of complex plane, as function of β_n it does not have other singularities for $0 \le \mathrm{Im}\beta_n \le 2\pi i$ but simple poles at the points $\beta_n = \beta_j + \pi i, \ j \le n-1$.

Several solutions to this infinite system of equations are known for the SG model which correspond to the most important local operators. We shall consider SG only in repulsive case when two component soliton is the only particle in the spectrum, so the space h is two-dimensional, and 4×4 S-matrix $S_\xi^{SG}(\beta)$ depends on the coupling constant $\xi : \ \pi \le \xi \le \infty$. The explicit formula for this S-matrix will be given later. When $\frac{\xi}{\pi}$ is rational the restriction in the space of states is possible [5] which provides different model ($\phi_{1,3}$-perturbation of minimal model [6]). These restrictions do not respect the Hermitian structure of the space of states of SG and allow their own definite Hermitian structure for $\frac{\xi}{\pi}$ integer. The S-matrix for the restricted models is obtained from SG one by RSOS [7] procedure. Generally, the idea of using RSOS restrictions for constructing physical S-matrices is due to [8]. In this paper we shall not go into much details of the restrictions because the restrictions of form factors occur quite naturally and do not present much difficulty as far as SG form factors are known. It should be mentioned also that for $\xi = \infty$ the SG S-matrix produces the S-matrix of $SU(2)$-invariant Thirring model:

$$S_\infty^{SG}(\beta) = S^{ITM}(\beta)$$

There are several models for which the one-particle isotopic spaces are tensor product of two SG isotopic spaces (or their restrictions) the S-matrices are different particular cases (and restrictions) of the following one

$$-S_{\xi_1}^{SG}(\beta) \otimes S_{\xi_2}^{SG}(\beta) \tag{3}$$

for two different coupling constants [9,10,11,12]. The most interesting examples are the following. For $\xi_1 = \infty, \xi_2 = k + 2$ after restriction of the second S-matrix we deal with the perturbation of WZNW-model on level k with action

$$S = S_{WZNW_k} + \lambda \int d^2 x J^a \bar{J}^a$$

which is the same as k-flavour $SU(2)$-Thirring model due to Polyakov-Wiegmann bosonisation [9]. In extreme case $\xi_1 = \infty, \xi_2 = \infty$ we get principal chiral field model (PCF) [9,12]. These are the models we are mostly interested in. To get the form factors for them we shall consider the most general S-matrix of the type given by (3).

2

Trying to solve the equations (1,2) for the S-matrix (3) we find the following amusing circumstance. Consider the Riemann-Hilbert problem (1) for the S-matrix (3). Evidently, it is satisfied by a slight modification of the tensor product of SG form factors:

$$\exp\Big(\frac{1}{2}\sum\beta_j\Big)\prod_{i<j}\operatorname{cth}\frac{1}{2}(\beta_i-\beta_j)f_{\xi_1}(\beta_1,\cdots,\beta_n)\otimes f_{\xi_2}(\beta_1,\cdots,\beta_n) \qquad (4)$$

Also this function is meromorphic and does not have other singularities as function of β_n but usual simple poles at $\beta_n=\beta_j+\pi i$ (SG form factors vanish when $\beta_i=\beta_j$). However, this anzatz breaks the equation (2). So, we have to look for something more intelligent. The lesson we learn from the naive anzatz (4) is that if we consider any solutions to the Riemann-Hilbert problem (1) for SG S-matrices then their tensor product satisfies the same equations for the tensor product S-matrix, but we should take care of the third equation.

On the other hand the Riemann-Hilbert problem (1) for SG model can be considered as deformation of Knizhnik-Zamolodchikov [13] (KZ) equations on level zero for the algebra $U_q(\widehat{sl}(2))$ with $q=\exp(\frac{2\pi^2 i}{\xi})$. This is rather informal way of thinking because strictly speaking the deformations of KZ equations (thorough vertex operators, highest weight representations etc) are properly defined for $q<1$ [14], for Yangian case $q\to 1$ they can be treated in terms of asymptotic series [15], for $|q|=1$ we still do not know how to treat them, but fortunately on level zero we do know how to solve them!

The same equations for the S-matrix (3) can be called deformed KZ for non semi-simple algebra $U_{q_1}(\widehat{sl}(2))\otimes U_{q_2}(\widehat{sl}(2))$ with $q_n=\exp(\frac{2\pi^2 i}{\xi_n})$. Certainly the solutions for this case should be given by linear combinations of tensor products of different solutions of equations for $U_{q_1}(\widehat{sl}(2))$ and $U_{q_2}(\widehat{sl}(2))$. So, our goal will be achieved if we know enough solutions in these two cases in order to construct linear combination of their tensor products which satisfy the residue condition (2). As it had been mentioned in [1] and explained in details for Yangian case in [15,16] we do know many solutions to the Riemann-Hilbert problem (1) in SG case (actually the same number as for usual KZ equations), but only very special ones were used for SG form factors because we had to satisfy the residue equation. In this paper we shall show that all the solutions are needed in order to construct the form factors for the S-matrix (3) through the procedure explained above.

2. Solution of Riemann-Hilbert problem for $U_q(\widehat{sl}(2))$, $|q|=1$.

In this section we shall be interested in the solutions to the Riemann-Hilbert problem (1). It is convenient to change the sign in the RHS of the second equation, this is harmless because the solutions to modified in this way equations can be transformed into the solution of original ones via multiplication by $\exp\big(\frac{1}{2}\sum\beta_j\big)$. So, we want to solve the equations:

$$\tilde{f}(\beta_1,\cdots,\beta_i,\beta_{i+1},\cdots,\beta_{2n})S_\xi(\beta_i-\beta_{i+1})=\tilde{f}(\beta_1,\cdots,\beta_{i+1},\beta_i,\cdots,\beta_{2n}),$$
$$\tilde{f}(\beta_1,\cdots,\beta_{2n-1},\beta_{2n}+2\pi i)=-\tilde{f}(\beta_{2n},\beta_1,\cdots,\beta_{2n-1}) \qquad (5)$$

3

where the SG S-matrix is given by [4]

$$S_\xi(\beta) = \frac{S_{\xi,0}(\beta)}{\mathrm{sh}\frac{\pi}{\xi}(\beta - \pi i)}\widehat{S}_\xi(\beta), \qquad S_{\xi,0}(\beta) = -\exp\Big(-i\int\limits_0^\infty \frac{\sin(k\beta)\mathrm{sh}(\frac{\pi-\xi}{2}k)}{k\,\mathrm{sh}(\frac{\pi k}{2})\mathrm{ch}(\frac{\xi k}{2})}\Big),$$

$$\widehat{S}_\xi(\beta) = \frac{S_{\xi,0}(\beta)}{\mathrm{sh}\frac{\pi}{\xi}(\beta - \pi i)}\begin{pmatrix} \mathrm{sh}\frac{\pi}{\xi}(\beta - \pi i), & 0 & 0 & 0 \\ 0 & -\mathrm{sh}\frac{\pi}{\xi}\beta & \mathrm{sh}\frac{\pi^2 i}{\xi} & 0 \\ 0 & \mathrm{sh}\frac{\pi^2 i}{\xi} & -\mathrm{sh}\frac{\pi}{\xi}\beta & 0 \\ 0 & 0 & 0 & \mathrm{sh}\frac{\pi}{\xi}(\beta - \pi i) \end{pmatrix},$$

Later we shall omit index ξ when only one $U_q(\widehat{sl}(2))$ is involved. We shall consider the solution to these equations of special isotopic character. The algebra $U_q(\widehat{sl}(2))$ contains two finite dimensional subalgebras isomorphic to $U_q(sl(2))$. The isotopic spaces of particles can be considered as the spaces of two-dimensional representations for these two subalgebras (the explicit formulae can be found in [5]). We shall restrict ourselves with the consideration of those solutions which belong to invariant with respect to one of these subgroups spaces in the tensor product (the solutions will carry an index $\gamma = \pm$ to indicate that they are invariant with respect to one or another subalgebra). For SG case these solutions describe the most fundamental local operators: energy-momentum tensor and current. This consideration can be generalized for $|q| < 1$ case [17].

Let us mention that the solutions to the equations (5) are defined up to multiplication by arbitrary quasiconstant: $2\pi i$-periodic, symmetric function of β_j. The solutions which will be presented later are supposed to constitute the full set of meromorphic solutions of given isotopic structure up to quasiconstants.

For what follows we shall need two special functions: $\varphi(\beta)$ and $\zeta(\beta)$. We shall not write down explicit formulae for these functions which can be found in [1], but just present their most important properties:

$$\varphi(\beta - 2\pi i) = \varphi(\beta)\frac{\mathrm{sh}\frac{\pi}{\xi}(\beta - \frac{\pi i}{2})}{\mathrm{sh}\frac{\pi}{\xi}(\beta - \frac{3\pi i}{2})}, \qquad \varphi(\beta - \pi i)\varphi(\beta) = \frac{1}{2\mathrm{sh}\frac{\pi}{\xi}(\beta - \frac{\pi i}{2})\mathrm{sh}(\beta - \frac{\pi i}{2})}$$

$$\frac{\varphi(\beta + \frac{\pi i}{2})}{\varphi(\beta - \frac{\pi i}{2})} = S_0(\beta), \qquad \zeta(\beta)S_0(\beta) = \zeta(-\beta),$$

$$\zeta(\beta - 2\pi i) = \zeta(-\beta), \qquad \zeta(\beta)\zeta(\beta - \pi i) = \big(\varphi(\beta + \frac{\pi i}{2})\big)^{-1} \tag{6}$$

We shall also need the vector-functions $\mathfrak{F}_n^\gamma(\alpha_1, \cdots, \alpha_{n-1}|\beta_1, \cdots, \beta_{2n})$ defined by the following list of requirements:
1) they take values in the tensor product of $2n$ spaces h^* ($h \sim \mathbb{C}^2$),
2) they are antisymmetric, entire, periodic with period $2\xi i$ functions of $\alpha_1, \cdots, \alpha_{n-1}$
3) they are antiperiodic with period $2\xi i$ functions of $\beta_1, \cdots, \beta_{2n}$ satisfying the following symmetry property:

$$\mathfrak{F}_n^\gamma(\alpha_1, \cdots, a_{n-1}|\beta_1, \cdots, \beta_i, \beta_{i+1}, \cdots, \beta_{2n})\widehat{S}_\xi(\beta_i - \beta_{i+1}) =$$
$$= \mathfrak{F}_n^\gamma(\alpha_1, \cdots, \alpha_{n-1}|\beta_1, \cdots, \beta_{i+1}, \beta_i, \cdots, \beta_{2n}),$$

4) for $\gamma = +$ or $-$ they are singlets with respect to one or another finite-dimensional quantum group as explained above,

4

5) they do not have other singularities but simple poles at the points $\beta_j = \beta_i + \pi i + k\xi i$ for $j > i$, $k \in \mathbb{Z}$

6) they satisfy the recurrent relations:

$$\mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} \qquad \mathfrak{F}_n^\gamma(\alpha_1, \cdots, \alpha_{n-1}|\beta_1, \cdots, \beta_{2n-2}, \beta_{2n-1}, \beta_{2n}) =$$

$$= \frac{\xi}{\pi}(e_{2n-1,-} \otimes e_{2n,+} + e_{2n-1,+} \otimes e_{2n,-}) \otimes$$

$$(\sum_{l=1}^{n-1}(-1)^l \mathfrak{F}_{n-1}^\gamma(\alpha_1, \cdots, \widehat{\alpha_l}, \cdots, \alpha_{n-1}|\beta_1, \cdots, \beta_{2n-2}) \exp(\frac{\pi}{\xi}(n-1)(\alpha_l - \beta_{2n-1}))$$

$$\times \Big[\prod_{p=1}^{2n-2} \mathrm{sh}\frac{\pi}{\xi}(\alpha_l - \beta_p - \frac{\pi i}{2}) - q^{(n-1)} \prod_{p=1}^{2n-2} \mathrm{sh}\frac{\pi}{\xi}(\alpha_l - \beta_p - \frac{\pi i}{2})\Big])$$

$$\times \prod_{q=1}^{n-1} \mathrm{sh}\frac{\pi}{\xi}(\alpha_q - \beta_{2n-1} - \frac{\pi i}{2})\Big(\prod_{i=1}^{2n-2} \mathrm{sh}\frac{\pi}{\xi}(\beta_{2n-1} - \beta_i)\Big)^{-1} \qquad (7)$$

where $e_{i,\pm}$ is basis in i-th space, the vector

$$e_{2n-1,-} \otimes e_{2n,+} + e_{2n-1,+} \otimes e_{2n,-} \qquad (8)$$

is nothing but $s_{2n-1,2n}$ from (2).

7) for n=1 we have

$$\mathfrak{F}_1^\gamma(\beta_1, \beta_2) =$$
$$\frac{\exp(\gamma\frac{\pi}{2\xi}(\beta_2 - \beta_1 - \pi i))\big(e_{1,+} \otimes e_{2,-}\big) + \exp(-\gamma\frac{\pi}{2\xi}(\beta_2 - \beta_1 - \pi i))\big(e_{1,-} \otimes e_{2,+}\big)}{\mathrm{sh}\frac{\pi}{\xi}(\beta_2 - \beta_1 - \pi i)}$$

It is possible to give explicit formulae for these functions in terms of certain determinants [1], we shall need it only in the last two sections of this paper.

Now we are ready to write down the solutions. Different solution will be counted by $\gamma = \pm$ which specifies the isotopic structure (singlet with respect to one or another finite-dimensional quantum subalgebra) and a set of integers $\{k_1, \cdots, k_{n-1}\}$ such that $|k_i| < n - 1$, $\forall i$. The sets $\{k_1, \cdots, k_{n-1}\}$ play the same role as different contours in integral formulae for solutions of KZ equations [13,18,19]. The solutions are given by

$$\tilde{f}_{k_1,\cdots,k_{n-1}}^\gamma(\beta_1, \cdots, \beta_{2n}) = d^n \prod_{i<j} \zeta(\beta_i - \beta_j) \qquad (9)$$

$$\int_{-\infty}^{+\infty} d\alpha_1 \cdots \int_{-\infty}^{+\infty} d\alpha_{n-1} \prod_{i=1}^{n-1}\prod_{j=1}^{2n} \varphi(\alpha_i - \beta_j)\exp\big(\sum_{i=1}^{n-1}\alpha_i k_i\big)\mathfrak{F}_n^\gamma(\alpha_1, \cdots, \alpha_{n-1}|\beta_1, \cdots, \beta_{2n})$$

where

$$d = \frac{1}{4\pi\xi i\zeta(\pi i)}$$

The integrals in (9) are regularized in a special way explained in [1], this regularization makes sense only for k_i integer, the limitation $|k_i| < n - 1$ comes from

requirement of convergency of regularized integrals. As it is shown in [1] the functions $\tilde{f}^\gamma_{k_1,\cdots,k_{n-1}}(\beta_1,\cdots,\beta_{2n})$ do satisfy (5).

Let us make some comments on the sets $\{k_1,\cdots,k_{n-1}\}$ which count the solutions. First, due to antisymmetry of $\mathfrak{F}^\gamma_n$ with respect to α's we must consider only those sets with all k_i different. Second, for "good" entire, $2\xi i$-periodic function $F(\alpha)$ we have the following relation:

$$\sum_{i=1}^{n} \sigma_{2i-1}(e^{\beta_1},\cdots,e^{\beta_{2n}}) \int\limits_{-\infty}^{+\infty} d\alpha \prod_{j=1}^{2n} \varphi(\alpha-\beta_j)\exp((n+1-2i)\alpha)F(\alpha) = 0 \quad (10)$$

where σ_k is elementary symmetrical polynomial of degree k. It is explained in [1,16] what kind of functions is "good", we would not go into details here, at least the function $\mathfrak{F}^\gamma_n$ considered as a function of one α is "good". The formula (10) gives one (and only one) relation of linear dependence with quasiconstant coefficients between the integrals with different k. It provides, certainly, linear dependence between some solutions to (5), to get really different solutions we could, for example, require $k_i \neq n-1\ \forall i$.

There is a beautiful way of understanding the solutions (9) explained in details (for Yangian case) in [15,16]. The point is that the solutions of KZ equations for the case of $sl(2)$, on level zero are given by determinants composed of periods of certain second kind differentials on hyperelliptic surface. The size of these determinants is equal to genus of the surface while the number of different cycles in twice bigger, that gives rise to different solutions counted by different subsets of cycles. The deformation of this picture should be understood as follows. The points $\beta_1,\cdots\beta_{2n}$ are branching points of "quantum hyperelliptic surface", the integral

$$\exp(-\frac{1}{2}\sum\beta_p)\sigma_{2i-1}(e^{\beta_1},\cdots,e^{\beta_{2n}}) \int\limits_{-\infty}^{+\infty} d\alpha \prod_{j=1}^{2n} \varphi(\alpha-\beta_j)\exp(k\alpha)F(\alpha)$$

is the period of differential defined by $2\xi i$-periodic function $F(\alpha)$ taken over the cycle around two branching points: β_{n-k}, β_{n-k+1}. The "good" functions mentioned above are analogues of those differentials which do not have simple poles on surface. If we consider first the Yangian limit ($\xi \to \infty$) and then classical limit of Yangian then the correspondence can be explicitly understood in terms of asymptotics [15,16,20].

3. Form factors for SG model and its restrictions.

Let us return to SG model. The form factors should satisfy not only the Riemann-Hilbert problem (1), but also the residue condition

$$2\pi i\ \mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i}\ f(\beta_1,\cdots,\beta_{2n-2},\beta_{2n-1},\beta_{2n}) =$$
$$= f(\beta_1,\cdots,\beta_{2n-2}) \otimes s_{2n-1,2n}\big(I - S(\beta_{2n-1}-\beta_1)\cdots S(\beta_{2n-1}-\beta_{2n-2})\big)$$

So, we have to consider the problem of calculation of residues of the kind.

It is easy to show that generally the expressions (9) have simple poles at $\beta_{2n} = \beta_{2n-1} + \pi i$. The recurrent relations (7) are useful for the calculation of residues

6

because together with the properties of φ (6) they provide that the integral with respect to α_l in l-th term of (7) substituted to (9) can be replaced by contour integral over the contour: $(-\infty, \infty)$ $(\infty, \infty + 2\pi i)$ $(\infty + 2\pi i, -\infty + 2\pi i)$ $(-\infty + 2\pi i, -\infty)$. So, this integral is calculated via the poles of the integrand which are situated at the points $\alpha_l = \beta_{2n-1} + \frac{\pi i}{2}$, $\alpha_l = \beta_{2n-1} + \frac{3\pi i}{2}$. However, generally the expressions for the residues can not be expressed in terms of functions of the type (9) with $2n - 2$ points β_j. There are only three possibilities to combine the functions (9) with different $\{k_i\}$ in order to have nice expressions:

$$
2\pi i \ \mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} \ d^n \prod_{i<j} \zeta(\beta_i - \beta_j) \int_{-\infty}^{+\infty} d\alpha_1 \cdots \int_{-\infty}^{+\infty} d\alpha_{n-1} \prod_{i=1}^{n-1}\prod_{j=1}^{2n} \varphi(\alpha_i - \beta_j)
$$

$$
\times \prod_{i=1}^{n-1} 2\mathrm{ch}(\alpha_i - \beta_{2n-1})\exp\Big(\sum_{i=1}^{n-1} \alpha_i l_i\Big)\mathfrak{F}_n^\gamma(\alpha_1, \cdots, \alpha_{n-1}|\beta_1, \cdots, \beta_{2n}) = 0,
$$

$$
2\pi i \ \mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} \ d^n \prod_{i<j} \zeta(\beta_i - \beta_j) \int_{-\infty}^{+\infty} d\alpha_1 \cdots \int_{-\infty}^{+\infty} d\alpha_{n-1} \prod_{i=1}^{n-1}\prod_{j=1}^{2n} \varphi(\alpha_i - \beta_j)
$$

$$
\times \big(1 + ie^{\alpha_1 - \beta_{2n-1}}\big) \prod_{i=2}^{n-1} 2\mathrm{ch}(\alpha_i - \beta_{2n-1})\exp\Big(\sum_{i=1}^{n-1} \alpha_i l_i\Big) \tag{11}
$$

$$
\times \mathfrak{F}_n^\gamma(\alpha_1, \cdots, \alpha_{n-1}|\beta_1, \cdots, \beta_{2n}) = 2s_{2n-1,2n} \otimes \tilde{f}_{l_1,\cdots,l_{n-1}}^\gamma(\beta_1, \cdots, \beta_{2n}),
$$

$$
2\pi i \ \mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} \ d^n \prod_{i<j} \zeta(\beta_i - \beta_j) \int_{-\infty}^{+\infty} d\alpha_1 \cdots \int_{-\infty}^{+\infty} d\alpha_{n-1} \prod_{i=1}^{n-1}\prod_{j=1}^{2n} \varphi(\alpha_i - \beta_j)
$$

$$
\times \big(1 - ie^{\alpha_1 - \beta_{2n-1}}\big) \prod_{i=2}^{n-1} 2\mathrm{ch}(\alpha_i - \beta_{2n-1})\exp\Big(\sum_{i=1}^{n-1} \alpha_i l_i\Big)
$$

$$
\times \mathfrak{F}_n^\gamma(\alpha_1, \cdots, \alpha_{n-1}|\beta_1, \cdots, \beta_{2n}) =
$$

$$
= -2s_{2n-1,2n} \otimes \tilde{f}_{l_1,\cdots,l_{n-1}}^\gamma(\beta_1, \cdots, \beta_{2n})S(\beta_{2n-1} - \beta_1)\cdots S(\beta_{2n-1} - \beta_{2n-2}),
$$

The limitations for the values of l_i are clear in every case.

The solutions for the complete system of relations (1,2) for $n > 1$ are given by

$$
f_+^\gamma(\beta_1, \cdots, \beta_{2n}) = \exp\Big(\frac{1}{2}\sum \beta_j\Big)\tilde{f}_{-(n-1),-(n-3),\cdots,(n-5),(n-3)}^\gamma(\beta_1, \cdots, \beta_{2n}),
$$

$$
f_+^\gamma(\beta_1, \cdots, \beta_{2n}) = \exp\Big(\frac{1}{2}\sum \beta_j\Big)\tilde{f}_{-(n-3),-(n-5),\cdots,(n-3),(n-1)}^\gamma(\beta_1, \cdots, \beta_{2n})
$$
$$
\tag{12}
$$

To prove that these functions do satisfy the residue condition we notice that due to antisymmetry of $\mathfrak{F}_n^\gamma$ under the integral in (9) the expression

$$
\exp\Big(\sum_{i=1}^{n-1}(n + 1 - 2i)\alpha_i\Big)
$$

7

can be replaced by

$$\frac{1}{2}\exp((n-2)\beta_{2n-1}-(n-3)\alpha_1)\left[\left(1+ie^{\alpha_1-\beta_{2n-1}}\right)+\left(1-ie^{\alpha_1-\beta_{2n-1}}\right)\right]$$
$$\times\prod_{i=2}^{n-1}2\mathrm{ch}(\alpha_i-\beta_{2n-1})\exp\left(\sum_{i=2}^{n-1}(n-2i)\alpha_i\right)$$

Now one just uses (11) to prove that the residue condition is satisfied.

The functions (12) are not independent, due to (10) they satisfy the relation

$$\left(\sum e^{-\beta_j}\right)f_+^\gamma(\beta_1,\cdots,\beta_{2n})=\left(\sum e^{+\beta_j}\right)f_-^\gamma(\beta_1,\cdots,\beta_{2n})$$

Analyzing the two particle form factor one realizes that the form factors of energy-momentum tensor $T_{\sigma_1\sigma_2}$ and $U(1)$ current J_σ (we use light-cone components, $\sigma=\pm$) are given by [1]:

$$f_{++}(\beta_1,\cdots,\beta_{2n})=\left(\sum_{j=1}^{2n}e^{\beta_j}\right)\sum_\gamma f_+^\gamma(\beta_1,\cdots,\beta_{2n}),$$

$$f_{--}(\beta_1,\cdots,\beta_{2n})=\left(\sum_{j=1}^{2n}e^{-\beta_j}\right)\sum_\gamma f_-^\gamma(\beta_1,\cdots,\beta_{2n}),$$

$$f_{+-}(\beta_1,\cdots,\beta_{2n})=$$
$$=\left(\sum_{j=1}^{2n}e^{-\beta_j}\right)\sum_\gamma f_+^\gamma(\beta_1,\cdots,\beta_{2n})=\left(\sum_{j=1}^{2n}e^{\beta_j}\right)\sum_\gamma f_-^\gamma(\beta_1,\cdots,\beta_{2n}),$$

$$f_\pm(\beta_1,\cdots,\beta_{2n})=\sum_\gamma(-)^\gamma f_\pm^\gamma(\beta_1,\cdots,\beta_{2n}) \tag{13}$$

There is one more set of solutions which have good residues:

$$\tilde{f}^\gamma_{-(n-2),-(n-4),\cdots,(n-4),(n-2)}(\beta_1,\cdots,\beta_{2n}) \tag{14}$$

but in that case we keep minus in the RHS of (1), also we have plus instead of minus between two terms in the RHS of (2), hence these solutions corresponds to disorder operators [1].

Let us explain briefly the restrictions of SG. Consider the modified energy-momentum tensor whose form factors are given by

$$f_{++}^+(\beta_1,\cdots,\beta_{2n})=\left(\sum_{j=1}^{2n}e^{\beta_j}\right)f_+^+(\beta_1,\cdots,\beta_{2n}),$$

$$f_{--}^+(\beta_1,\cdots,\beta_{2n})=\left(\sum_{j=1}^{2n}e^{-\beta_j}\right)f_-^+(\beta_1,\cdots,\beta_{2n}),$$

$$f_{+-}^+(\beta_1,\cdots,\beta_{2n})=$$
$$=\left(\sum_{j=1}^{2n}e^{-\beta_j}\right)f_+^+(\beta_1,\cdots,\beta_{2n})=\left(\sum_{j=1}^{2n}e^{\beta_j}\right)f_-^+(\beta_1,\cdots,\beta_{2n})$$

8

These operators is invariant with respect to one of $U_q(sl(2))$ subalgebras. That is why for rational ξ the intermediate states in the correlations of these operators among themselves happen to be truncated [5]. So, we can construct a new restricted theory (RSG) with smaller operator content and truncated space of states. This theory is well known to coincide with $\phi_{1,3}$-perturbation of minimal model [6]. The truncation does not respect the Hermitian structure of SG space of states, so, to have RSG to be defined intrinsically we have to introduce new Hermitian structure. It can be made positively defined in the region on coupling constant we deal with only for $\frac{\xi}{\pi}$ integer and $\frac{\xi}{\pi} = \frac{3}{2}$ (in what follows we shall not be interested in the latter case). One of disorder operators allows restriction, it coincides with the operator $\phi_{1,2}$ in ultraviolet limit.

Thus, only very small part of solutions we know is useful for SG form factors. However, in the next section we shall show that all of them are needed for the models with the tensor product S-matrices.

4. The form factors for tensor product S-matrix.

The Riemann-Hilbert problem (1) looks in that case as

$$f(\beta_1, \cdots, \beta_i, \beta_{i+1}, \cdots, \beta_{2n})S_{\xi_1\xi_2}(\beta_i - \beta_{i+1}) = f(\beta_1, \cdots, \beta_{i+1}, \beta_i, \cdots, \beta_{2n}),$$
$$f(\beta_1, \cdots, \beta_{2n-1}, \beta_{2n} + 2\pi i) = f(\beta_{2n}, \beta_1, \cdots, \beta_{2n-1})$$

with

$$S_{\xi_1\xi_2}(\beta) = -S_{\xi_1}^{SG}(\beta) \otimes S_{\xi_2}^{SG}(\beta) \tag{15}$$

Evidently, these two equations are satisfied by any expression of the kind:

$$\exp\left(\pm\frac{1}{2}\sum \beta_j\right) \prod_{i<j} \mathrm{cth}\frac{1}{2}(\beta_i - \beta_j)\tilde{f}_{\xi_1,\ K}^{\gamma_1}(\beta_1, \cdots, \beta_{2n}) \otimes \tilde{f}_{\xi_2,\ L}^{\gamma_2}(\beta_1, \cdots, \beta_{2n})$$

where we denoted the ordered subsets of integers $\{k_1, \cdots, k_{n-1}\} : |k_i| \leq n-1$ and $\{l_1, \cdots, l_{n-1}\} : |l_i| \leq n-1$ by K and L. The problem is to satisfy the residue condition:

$$2\pi i \ \mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} \ f(\beta_1, \cdots, \beta_{2n-2}, \beta_{2n-1}, \beta_{2n}) =$$
$$= f(\beta_1, \cdots, \beta_{2n-2}) \otimes \tilde{s}_{2n-1,2n}\big(I - S_{\xi_1\xi_2}(\beta_{2n-1} - \beta_1) \cdots S_{\xi_1\xi_2}(\beta_{2n-1} - \beta_{2n-2})\big) \tag{16}$$

where $\tilde{s}_{2n-1,2n}$ is the tensor product of two vectors like (8). We shall show that these equations are satisfied by

$$f_{\pm\ \xi_1\xi_2}^{\gamma_1\gamma_2}(\beta_1, \cdots, \beta_{2n}) = (2\pi)^n \exp\left(\pm\frac{1}{2}\sum \beta_j\right) \prod_{i<j} \mathrm{cth}\frac{1}{2}(\beta_i - \beta_j)$$

$$\times \sum_{N_{\pm}=K\cup L} \tilde{f}_{\xi_1,\ K}^{\tilde{\gamma}_1}(\beta_1, \cdots, \beta_{2n}) \otimes \tilde{f}_{\xi_2,\ L}^{\gamma_2}(\beta_1, \cdots, \beta_{2n}) \tag{17}$$

where $N_+ = \{-(n-1), -(n-2), \cdots, (n-3), (n-2)\}$, $N_- = \{-(n-2), -(n-3), \cdots, (n-2), (n-1)\}$. To prove that we have to calculate the residue. It is not

complicated, first let us write more explicit formulae:

$$f_{\pm}^{\gamma_1\gamma_2}{}_{\xi_1\xi_2}(\beta_1,\cdots,\beta_{2n}) = \frac{1}{((n-1)!)^2}d_{\xi_1\xi_2}^m\exp\left(\pm\frac{1}{2}\sum\beta_j\right)\prod_{i<j}\zeta_{\xi_1\xi_2}(\beta_i-\beta_j)\times$$

$$\int\limits_{-\infty}^{+\infty}d\alpha_1\cdots\int\limits_{-\infty}^{+\infty}d\alpha_{n-1}\int\limits_{-\infty}^{+\infty}d\tilde{\alpha}_1\cdots\int\limits_{-\infty}^{+\infty}d\tilde{\alpha}_{n-1}\prod_{i=1}^{n-1}\prod_{j=1}^{2n}\varphi_{\xi_1}(\alpha_i-\beta_j)\prod_{i=1}^{n-1}\prod_{j=1}^{2n}\varphi_{\xi_2}(\tilde{\alpha}_i-\beta_j)$$

$$\times\exp\left(\mp\frac{1}{2}\sum(\alpha_i+\tilde{\alpha}_i)\right)\prod_{i<j}2\mathrm{sh}\frac{1}{2}(\alpha_i-\alpha_j)\prod_{i<j}2\mathrm{sh}\frac{1}{2}(\tilde{\alpha}_i-\tilde{\alpha}_j)\prod_{i,j}2\mathrm{ch}\frac{1}{2}(\alpha_i-\tilde{\alpha}_j)$$

$$\times\mathfrak{F}_n^{\gamma_1}{}_{\xi_1}(\alpha_1,\cdots,\alpha_{n-1}|\beta_1,\cdots,\beta_{2n})\otimes\mathfrak{F}_n^{\gamma_2}{}_{\xi_2}(\tilde{\alpha}_1,\cdots,\tilde{\alpha}_{n-1}|\beta_1,\cdots,\beta_{2n}) \qquad (18)$$

where

$$d_{\xi_1\xi_2}=\pi d_{\xi_1}d_{\xi_2}, \qquad \zeta_{\xi_1\xi_2}(\beta)=\mathrm{cth}\frac{1}{2}(\beta)\zeta_{\xi_1}(\beta)\zeta_{\xi_2}(\beta)$$

Let us prove that for $n > 1$ the functions (17) do satisfy the residue condition (the case $n = 1$ is special). To do that it is sufficient to realize that under the integral due to antisymmetry of $\mathfrak{F}$ we can perform a replacement:

$$\prod_{i<j}\mathrm{sh}\frac{1}{2}(\alpha_i-\alpha_j)\prod_{i<j}\mathrm{sh}\frac{1}{2}(\tilde{\alpha}_i-\tilde{\alpha}_j)\prod_{1\le i}\prod_{1\le j}\mathrm{ch}\frac{1}{2}(\alpha_i-\tilde{\alpha}_j)\to 2^{-2(n-2)}e^{(n-2)\beta_{2n-1}}$$

$$\times\left\{(-1)^n(n-1)^2\exp\frac{1}{2}\left(-(n-2)(\alpha_1+\tilde{\alpha}_1)\right)\mathrm{ch}\frac{1}{2}(\alpha_1-\tilde{\alpha}_1)\prod_{i=2}^{n-1}\mathrm{ch}(\alpha_i-\beta_{2n-1})\right.$$

$$\times\prod_{i=2}^{n-1}\mathrm{ch}(\tilde{\alpha}_i-\beta_{2n-1})\prod_{2\le i<j}\mathrm{sh}\frac{1}{2}(\alpha_i-\alpha_j)\prod_{2\le i<j}\mathrm{sh}\frac{1}{2}(\tilde{\alpha}_i-\tilde{\alpha}_j)\prod_{2\le i}\prod_{2\le j}\mathrm{ch}\frac{1}{2}(\alpha_i-\tilde{\alpha}_j)+$$

$$\tag{19}$$

$$+\frac{(n-1)(n-2)}{2}\exp\frac{1}{2}\left(-(n-2)(\alpha_1+\alpha_2)\right)\mathrm{sh}\frac{1}{2}(\alpha_1-\alpha_2)\prod_{i=3}^{n-1}\mathrm{ch}(\alpha_i-\beta_{2n-1})$$

$$\times\prod_{i=1}^{n-1}\mathrm{ch}(\tilde{\alpha}_i-\beta_{2n-1})\prod_{3\le i<j}\mathrm{sh}\frac{1}{2}(\alpha_i-\alpha_j)\prod_{1\le i<j}\mathrm{sh}\frac{1}{2}(\tilde{\alpha}_i-\tilde{\alpha}_j)\prod_{3\le i}\prod_{1\le j}\mathrm{ch}\frac{1}{2}(\alpha_i-\tilde{\alpha}_j)+$$

$$+\frac{(n-1)(n-2)}{2}\exp\frac{1}{2}\left(-(n-2)(\tilde{\alpha}_1+\tilde{\alpha}_2)\right)\mathrm{sh}\frac{1}{2}(\tilde{\alpha}_1-\tilde{\alpha}_2)\prod_{i=3}^{n-1}\mathrm{ch}(\tilde{\alpha}_i-\beta_{2n-1})$$

$$\times\prod_{i=1}^{n-1}\mathrm{ch}(\alpha_i-\beta_{2n-1})\prod_{3\le i<j}\mathrm{sh}\frac{1}{2}(\tilde{\alpha}_i-\tilde{\alpha}_j)\prod_{1\le i<j}\mathrm{sh}\frac{1}{2}(\alpha_i-\alpha_j)\prod_{3\le i}\prod_{1\le j}\mathrm{ch}\frac{1}{2}(\tilde{\alpha}_i-\alpha_j)\right\}$$

This formula looks quite terrible, but it has a simple origin: the LHS is proportional to Vandermonde determinant composed of e^{α_i} and $-e^{\tilde{\alpha}_i}$, so, to get (19) we added in this determinant to k-th (for $k \geq 3$) row the $(k-2)$-th one multiplied by $e^{2\beta_{2n-1}}$, and then decomposed with respect to first two rows. Notice now that we are

interested in the second order pole at $\beta_{2n} = \beta_{2n-1} + \pi i$ of the integral from (18) since $\zeta_{\xi_1,\xi_2}(\beta_{2n} - \beta_{2n-1})$ has zero at this point. But the contributions from the last two terms in (19) do not have such singularity. Consider, for example the last term. The integral with respect to $\tilde{\alpha}$'s will produce first order pole, but the integral with respect to α's is regular because it contains $\prod_{i=1}^{n-1} \mathrm{ch}(\alpha_i - \beta_{2n-1})$ (see (11)). Thus the only interesting term is the first one from (19). In this term we can replace $\mathrm{ch}\frac{1}{2}(\alpha_1 - \tilde{\alpha}_1)$ by

$$(1 - ie^{\alpha_1 - \beta_{2n-1}})(1 + ie^{-\tilde{\alpha}_1 + \beta_{2n-1}})\big]$$

and then use the formulae (11). That proves the formula (16) for $n \geq 2$. Let us mention one more important property of these form factors: they satisfy the relations:

$$\Big(\sum_{j=1}^{2n} e^{-\beta_j}\Big) f_{+\,\xi_1\xi_2}^{\gamma_1\gamma_2}(\beta_1,\cdots,\beta_{2n}) = \Big(\sum_{j=1}^{2n} e^{\beta_j}\Big) f_{-\,\xi_1\xi_2}^{\gamma_1\gamma_2}(\beta_1,\cdots,\beta_{2n}) \qquad (20)$$

these relations are proven using the formula (10). As usual, to understand what kind of operators do these form factors correspond to we have to investigate better the two-particle ones. But before doing that we have to explain what kind of theory we deal with. The point is that the S-matrix in question describes many different models.

5. Different models described by tensor product S-matrix

The simplest way is to understand the S-matrix as it is, and to introduce the scalar product in the space of states such that it respects the Hermitian conjugation of S-matrix:

$$(S_{\xi_1\xi_2}(\beta))^* = S_{\xi_1\xi_2}(-\beta)$$

Then the symmetry of the model is $U(1)\otimes U(1)$, two-particle form factors are given by

$$f_{\pm\,\xi_1\xi_2}^{\gamma_1\gamma_2}(\beta_1,\beta_2) = d_{\xi_1\xi_2}\exp\big(\pm\frac{1}{2}\sum\beta_j\big)\frac{\zeta_{\xi_1\xi_2}(\beta_1 - \beta_2)}{\mathrm{sh}\frac{\pi}{\xi_1}(\beta_2 - \beta_1 - \pi i)\mathrm{sh}\frac{\pi}{\xi_2}(\beta_2 - \beta_1 - \pi i)}$$
$$\times \big[e^{\gamma_1\frac{\pi}{2\xi_1}(\beta_2-\beta_1-\pi i)}\big(e_{1,+}\otimes e_{2,-}\big) + e^{-\gamma_1\frac{\pi}{2\xi_1}(\beta_2-\beta_1-\pi i)}\big(e_{1,-}\otimes e_{2,+}\big)\big]\otimes$$
$$\otimes \big[e^{\gamma_2\frac{\pi}{2\xi_2}(\beta_2-\beta_1-\pi i)}\big(e_{1,+}\otimes e_{2,-}\big) + e^{-\gamma_2\frac{\pi}{2\xi_2}(\beta_2-\beta_1-\pi i)}\big(e_{1,-}\otimes e_{2,+}\big)\big]$$

From these expressions an from the formula (20) one realizes that the form factors of energy-momentum tensor $T_{\sigma_1\sigma_2}$ and two $U(1)$ currents J_σ^L, J_σ^R (we use light-cone

components, $\sigma = \pm$) are given by

$$f_{++}(\beta_1, \cdots, \beta_{2n}) = (\sum_{j=1}^{2n} e^{\beta_j}) \sum_{\gamma_1 \gamma_2} f_{+ \ \xi_1 \xi_2}^{\gamma_1 \gamma_2}(\beta_1, \cdots, \beta_{2n}),$$

$$f_{--}(\beta_1, \cdots, \beta_{2n}) = (\sum_{j=1}^{2n} e^{-\beta_j}) \sum_{\gamma_1 \gamma_2} f_{- \ \xi_1 \xi_2}^{\gamma_1 \gamma_2}(\beta_1, \cdots, \beta_{2n}),$$

$$f_{+-}(\beta_1, \cdots, \beta_{2n}) =$$

$$= (\sum_{j=1}^{2n} e^{-\beta_j}) \sum_{\gamma_1 \gamma_2} f_{+ \ \xi_1 \xi_2}^{\gamma_1 \gamma_2}(\beta_1, \cdots, \beta_{2n}) = (\sum_{j=1}^{2n} e^{\beta_j}) \sum_{\gamma_1 \gamma_2} f_{+ \ \xi_1 \xi_2}^{\gamma_1 \gamma_2}(\beta_1, \cdots, \beta_{2n}),$$

$$f_{\pm}^L(\beta_1, \cdots, \beta_{2n}) = \sum_{\gamma_1 \gamma_2} (-)^{\gamma_1} f_{\pm \ \xi_1 \xi_2}^{\gamma_1 \gamma_2}(\beta_1, \cdots, \beta_{2n}),$$

$$f_{\pm}^R(\beta_1, \cdots, \beta_{2n}) = \sum_{\gamma_1 \gamma_2} (-)^{\gamma_2} f_{\pm \ \xi_1 \xi_2}^{\gamma_1 \gamma_2}(\beta_1, \cdots, \beta_{2n})$$

The important limit of this model is $\xi_1, \xi_2 \to \infty$. In this limit we get the PCF.

Now we can consider two levels of restriction of the model. First, let us introduce the modified energy-momentum tensor and modified left current as

$$f_{++}^+(\beta_1, \cdots, \beta_{2n}) = (\sum_{j=1}^{2n} e^{\beta_j}) \sum_{\gamma_1} f_{+ \ \xi_1 \xi_2}^{\gamma_1 \ +}(\beta_1, \cdots, \beta_{2n}),$$

$$f_{--}^+(\beta_1, \cdots, \beta_{2n}) = (\sum_{j=1}^{2n} e^{-\beta_j}) \sum_{\gamma_1} f_{- \ \xi_1 \xi_2}^{\gamma_1 \ +}(\beta_1, \cdots, \beta_{2n}),$$

$$f_{+-}^+(\beta_1, \cdots, \beta_{2n}) =$$

$$= (\sum_{j=1}^{2n} e^{-\beta_j}) \sum_{\gamma_1} f_{+ \ \xi_1 \xi_2}^{\gamma_1 \ +}(\beta_1, \cdots, \beta_{2n}) = (\sum_{j=1}^{2n} e^{\beta_j}) \sum_{\gamma_1} f_{+ \ \xi_1 \xi_2}^{\gamma_1 \ +}(\beta_1, \cdots, \beta_{2n}),$$

$$f_{\pm}^{L,+}(\beta_1, \cdots, \beta_{2n}) = \sum_{\gamma_1} (-)^{\gamma_1} f_{\pm \ \xi_1 \xi_2}^{\gamma_1 \ +}(\beta_1, \cdots, \beta_{2n}),$$

These operators are not selfajoint in SG Hermitian structure, but they are invariant under the action of one $U_{q_2}(sl(2))$. That is why for rational $\frac{\xi_2}{\pi}$ the correlations of these operators among themselves contain only RSOS restricted with respect to $U_{q_2}(sl(2))$ states. For $\frac{\xi_2}{\pi} = k+2, k \in \mathbb{Z}$ the model with restricted set of operators and truncated space can be equipped with positively defined scalar product. It is no wonder that J_σ^R is lost in the restricted model: right $U(1)$ is broken. The important limit of the restricted model is $\xi_1 \to \infty$ for given k. In this limit we get perturbations of WZNW on level k.

It is instructive to recover SG model itself. It should coincide with the restricted model for $\xi_1 = \xi$, $\xi_2 = 3\pi$. In that case in the truncated space of states right degrees of freedom are frozen and the restriction of $S_{\xi_2}(\beta)$ is just -1 (Ising S-

12

matrix). Consider, for example, the formula

$$f_{++}^+(\beta_1, \cdots, \beta_{2n}) = (2\pi)^n \Big(\sum_{j=1}^{2n} e^{\beta_j}\Big) \exp\Big(\pm\frac{1}{2}\sum \beta_j\Big) \prod_{i<j} \mathrm{cth}\frac{1}{2}(\beta_i - \beta_j)$$

$$\times \sum_\gamma \sum_{N_+ = K \cup L} \tilde{f}_{\xi,\ K}^\gamma(\beta_1, \cdots, \beta_{2n}) \otimes \tilde{f}_{3\pi,\ L}^+(\beta_1, \cdots, \beta_{2n})$$

It can be shown that the restriction of

$$\tilde{f}_{3\pi,\ L}^+(\beta_1, \cdots, \beta_{2n})$$

differs from zero only for $L = \{n - 2j, \ j = 1, \cdots, n - 1\}$. But in the latter case the restriction coincides (14) with the form factor of Ising disorder operator which is given by

$$\Big(\frac{1}{2\pi}\Big)^n \prod_{i<j} \mathrm{th}\frac{1}{2}(\beta_i - \beta_j)$$

this expression cancels in (17) and we recover SG energy-momentum tensor form factors (13).

At last if we consider another modification of the energy-momentum tensor with form factors

$$f_{++}^{++}(\beta_1, \cdots, \beta_{2n}) = \Big(\sum_{j=1}^{2n} e^{\beta_j}\Big) f_{+\ \xi_1\xi_2}^{++}(\beta_1, \cdots, \beta_{2n}),$$

$$f_{--}^{++}(\beta_1, \cdots, \beta_{2n}) = \Big(\sum_{j=1}^{2n} e^{-\beta_j}\Big) f_{-\ \xi_1\xi_2}^{++}(\beta_1, \cdots, \beta_{2n}),$$

$$f_{+-}^{++}(\beta_1, \cdots, \beta_{2n}) =$$

$$= \Big(\sum_{j=1}^{2n} e^{-\beta_j}\Big) f_{+\ \xi_1\xi_2}^{++}(\beta_1, \cdots, \beta_{2n}) = \Big(\sum_{j=1}^{2n} e^{\beta_j}\Big) f_{+\ \xi_1\xi_2}^{++}(\beta_1, \cdots, \beta_{2n}),$$

then for $\frac{\xi_1}{\pi}$, $\frac{\xi_2}{\pi}$ integers we can perform restriction which will give the perturbations of coset models discussed in [11].

It should be said that we restricted ourselves in this paper with consideration of repulsive SG coupling constants ($\xi \geq \pi$). The consideration of attractive case is also important, after restriction we can find quite unexpected models.

6. THE MATHEMATICAL STRUCTURE OF THE SOLUTION.

The S-matrix $-S_{\xi_1}^{SG}(\beta) \otimes S_{\xi_2}^{SG}(\beta)$ is constructed in amusing way: as if we have two types of particles confined together. The formula for form factors

$$f_{\pm\ \xi_1\xi_2}^{\gamma_1\gamma_2}(\beta_1, \cdots, \beta_{2n}) = \exp\Big(\pm\frac{1}{2}\sum \beta_j\Big) \prod_{i<j} \mathrm{cth}\frac{1}{2}(\beta_i - \beta_j)$$

$$\times \sum_{N_\pm = K \cup L} \tilde{f}_{\xi_1,\ K}^{\gamma_1}(\beta_1, \cdots, \beta_{2n}) \otimes \tilde{f}_{\xi_2,\ L}^{\gamma_2}(\beta_1, \cdots, \beta_{2n}) \qquad (21)$$

13

shows that these two types of particles "interact" in quite interesting fashion. In this section we shall explain the most attractive features of this "interaction".

When considering the solutions to Riemann-Hilbert problem we explained that the sets K which count different solutions play the same role as choice of different contours in integral formulae for the solutions of KZ equations. From the algebraic point of view these data count different passes composed of intermediate Verma modules for the products of vertex operators whose vacuum expectations provide the solutions to KZ and deformed KZ equations. This interaction through passes looks quite interesting, it is similar to combining together holomorphic and anti-holomorphic conformal blocks in CFT.

However, we suppose the following circumstance to be of the main importance. We deal with $k = 0$ equations, in that case the contours in question are quite special, as it has been mentioned in Section 2. Let us explain this point in more details, but before doing that we have to provide some more information.

Let us concentrate on one SG S-matrix with coupling constant ξ. First, we shall give more explicit description of the vector $\mathfrak{F}_n^\gamma(\alpha_1, \cdots, \alpha_{n-1} | \beta_1, \cdots, \beta_{2n})$.

We introduce the notation $B = \{1, 2, \cdots, 2n\}$. To every $T \subset B : \#T = n$ vector $w_T(\beta_1, \cdots, \beta_{2n-1}, \beta_{2n})$ from the tensor product of isotopic spaces is related such that

$$w_T(\beta_1, \cdots, \beta_i, \beta_{i+1}, \cdots, \beta_{2n})\widehat{S}_\xi(\beta_i - \beta_{i+1}) = w_{T\backslash i}(\beta_1, \cdots, \beta_{i+1}, \beta_i, \cdots, \beta_{2n}),$$

where

$$T\backslash i = \begin{cases} T & \text{if } i, i+1 \in T \text{ or } i, i+1 \notin T \\ T \circ i \circ \{i+1\} & \text{if } i \in T, \ i+1 \notin T \text{ or } i+1 \in T, \ i \notin T \end{cases}$$

where $A \circ B = (A \cup B) \backslash (A \cap B)$. These notation are different from those used in [1], but one can easily understand the relation. It would simplify a lot further formulae if we introduce notations

$$v_T^\gamma(\beta_1, \cdots, \beta_{2n}) = \frac{\exp\frac{\gamma\pi}{2\xi}\left(\sum_{j\in T}\beta_j - \sum_{j\notin T}\beta_j\right)}{\prod_{i\in T, j\notin T} \mathrm{sh}\frac{\pi}{\xi}(\beta_i - \beta_j)} w_T(\beta_1, \cdots, \beta_{2n})$$

We have [1]:

$$\mathfrak{F}_n^\gamma(\alpha_1, \cdots, \alpha_{n-1} | \beta_1, \cdots, \beta_{2n}) =$$

$$= \sum_{T\in B, \#T=n} \Delta^T(\alpha_1, \cdots, \alpha_{n-1} | \beta_1, \cdots, \beta_{2n}) v_T^\gamma(\beta_1, \cdots, \beta_{2n})$$

where Δ^T is $(n-1) \times (n-1)$ matrix with matrix elements:

$$A_{i,j}^T = A_i^T(\alpha_j | \beta_1, \cdots, \beta_{2n}),$$

where

$$A_i^T(\alpha | \beta_1, \cdots, \beta_{2n}) = \exp(-\frac{\pi}{\xi}((n-2)\alpha + \sum\beta_p))$$

$$\times \left\{ \prod_{q\in T}(e^{\frac{2\pi}{\xi}\alpha} - q^{\frac{1}{2}}e^{\frac{2\pi}{\xi}\beta_q}) \sum_{k=0}^{i-1}(1-q^{i-k})e^{\frac{2\pi}{\xi}(i-k-1)\alpha}q^{\frac{k}{2}}\sigma_{k, \ B\backslash T} + \right.$$

$$\left. + q^i \prod_{q\in B\backslash T}(e^{\frac{2\pi}{\xi}\alpha} - q^{-\frac{1}{2}}e^{\frac{2\pi}{\xi}\beta_q}) \sum_{k=0}^{i-1}(1-q^{i-k})e^{\frac{2\pi}{\xi}(i-k-1)\alpha}q^{\frac{k}{2}}\sigma_{k,T} \right\}$$

14

with $\sigma_{k,T}$, $\sigma_{k,\,B\setminus T}$ are elementary symmetric polynomials of degree k with arguments $\exp\frac{2\pi}{\xi}(\sum \beta_p)$, $p \in T$ and $\exp\frac{2\pi}{\xi}(\sum \beta_p)$, $p \notin T$ respectively. We remind also that $q = \exp\frac{2\pi^2 i}{\xi}$.

These formulae provide that the functions $\tilde{f}^{\gamma}_{\xi,\,K}(\beta_1, \cdots, \beta_{2n})$ can be written in the form:

$$\tilde{f}^{\gamma}_{\xi,\,K}(\beta_1, \cdots, \beta_{2n}) =$$

$$= \sum_{T \in B, \#T=n} \det \left| \int \prod_p \varphi(\alpha - \beta_p) A_i^T(\alpha|\beta_1, \cdots, \beta_{2n})\exp(k_j\alpha)d\alpha \right| v_T^{\gamma}(\beta_1, \cdots, \beta_{2n}) \tag{22}$$

where $K = \{k_1, \cdots, k_{n-1}\}$, $k_1 < k_2 < \cdots < k_{n-1}$.

To rewrite this answer in more beautiful way let us introduce two vector spaces: the space H of dimension $2n-2$ with basis A_i, B_i $i = 1, \cdots, n-1$, and the space V of dimension $n-1$ with basis Z_i, $i = 1, \cdots, n-1$. Now for every $T \subset B$, $\#T = n$ we introduce two forms

$$\omega^{\pm}(T) = \omega_{i,j}(T)A_i \wedge Z_j \pm \tilde{\omega}_{i,j}(T)B_i \wedge Z_j \tag{23}$$

where

$$\omega_{i,j}(T) = \exp(-\frac{1}{2}\sum \beta_p)\{\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \sigma_{2n-2j}(e^{\beta_1}, \cdots, e^{\beta_{2n}})$$

$$\times \int \prod_p \varphi(\alpha - \beta_p)A_i^T(\alpha|\beta_1, \cdots, \beta_{2n})\exp((2p-n)\alpha)d\alpha\}$$

$$\tilde{\omega}_{i,j}(T) = \exp(-\frac{1}{2}\sum \beta_p)\{\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{l=1}^{j} \sigma_{2n-2l+1}(e^{\beta_1}, \cdots, e^{\beta_{2n}})$$

$$\times \int \prod_p \varphi(\alpha - \beta_p)A_i^T(\alpha|\beta_1, \cdots, \beta_{2n})\exp((2l-n-1)\alpha)d\alpha\}$$

Then different solutions (22) can be found as coefficients in decomposition with respect to

$$(\pm)^q A_{j_1} \wedge \cdots \wedge A_{j_q} \wedge B_{k_1} \wedge \cdots \wedge B_{k_{n-q-1}} \tag{24}$$

of the expression

$$\sum_{T \in B, \#T=n} \wedge^{(n-1)}(\omega^{\pm}(T))v_T^{\gamma}(\beta_1, \cdots, \beta_{2n}) \tag{25}$$

We can think of this decomposition as of one with respect to Grassmanian of $(n-1)$-dimensional subspaces of H.

The "interaction" in the formula for the form factor $f^{\gamma_1\gamma_2}_{+\,\xi_1\xi_2}$ (21) can be understood as follows. Take two copies of the spase V (with basises Z_i, W_i). Then the

form factors are defined by the inner product of forms $\wedge^{(n-1)}\omega_{\xi_1}^+$ and $\wedge^{(n-1)}\omega_{\xi_2}^-$ as follows

$$
d_{\xi_1\xi_2}^n \exp\Big(\frac{n}{2}\sum\beta_j\Big)\prod_{i<j}\zeta_{\xi_1\xi_2}(\beta_i-\beta_j)\Big(\prod_{p=1}^{2n-1}\sigma_1(e^{\beta_1},\cdots,e^{\beta_{2n}})\Big)^{-2}\times
$$

$$
\sum_{T_1\in B,\#T=n}\sum_{T_2\in B,\#T=n}\wedge^{(n-1)}\left(\omega_{\xi_1,\ i,j}(T_1)A_i\wedge Z_j+\tilde\omega_{\xi_1,\ i,j}(T_1)B_i\wedge Z_j\right)\wedge
$$

$$
\wedge^{(n-1)}\left(\omega_{\xi_2,\ i,j}(T_2)A_i\wedge W_j-\tilde\omega_{\xi_2,\ i,j}(T_2)B_i\wedge W_j\right)
$$

$$
\times\, v_{\xi_1,T_1}^{\gamma_1}(\beta_1,\cdots,\beta_{2n})\otimes v_{\xi_2,T_2}^{\gamma_2}(\beta_1,\cdots,\beta_{2n})=
$$

$$
= f_{\pm\ \xi_1\xi_2}^{\gamma_1\gamma_2}(\beta_1,\cdots,\beta_{2n})
$$

$$
\times A_1\wedge\cdots\wedge A_{n-1}\wedge B_1\wedge\cdots\wedge B_{n-1}\wedge Z_1\wedge\cdots\wedge Z_{n-1}\wedge W_1\wedge\cdots\wedge W_{n-1}
$$

Certainly, the formula (25) presents rather formal way of rewriting (22), but it is not an empty exercise since the form $\omega(T)$ independently has interesting meaning. It has been said in Section 2 that the expression

$$
\exp\Big(-\frac{1}{2}\sum\beta_p\Big)\sigma_{n-k}(e^{\beta_1},\cdots,e^{\beta_{2n}})
$$

$$
\times\int\prod_p\varphi(\alpha-\beta_p)A_i^T(\alpha|\beta_1,\cdots,\beta_{2n})\exp(k\alpha)d\alpha
$$

can be considered as quantum deformation of the period of special second type hyperelliptic differential ζ_i^T over the contour around two c_k around two branching points β_{n+k} and β_{n+k+1}. The canonical choice of homology basis on the surface is the following:

$$
a_i=c_{2i-n},\ \ b_i=c_{-n+1}+c_{-n+3}+\cdots+c_{-n+2i-1}
$$

for $i=1,\cdots,(n-1)$, genus of surface equals $n-1$. From that point of view it is natural to identify A_i,B_i with basic vectors of the lattice of periods and the vectors Z_i with differentials along Jacobian. Then the form $\omega(T)$ has nice meaning [21]. We shall explain this point in some more details in the next section. The combination of ω^+ with ω^- reminds one more time combining holomorphic and antiholomorphic pieces in CFT: the b-cycles are imaginary ones.

7. REMARKS ON THE CLASSICAL LIMIT.

Let us discuss the perturbation of WZNW model on level k. In that case we have to put $\xi_1=\infty$, $\xi_2=k+2$, to fix γ_2 (say $\gamma_2=+$) and to perform restriction. Let us consider, for example, the form factors of one component of energy-momentum tensor:

$$
f_{++}(\beta_1,\cdots,\beta_{2n})=(2\pi)^n\exp\Big(\frac{1}{2}\sum\beta_j\Big)\Big(\sum e^{\beta_j}\Big)\prod_{i<j}\mathrm{cth}\frac{1}{2}(\beta_i-\beta_j)
$$

$$
\times\sum_{N_+=K\cup L}\tilde f_K(\beta_1,\cdots,\beta_{2n})\otimes\tilde f_L^k(\beta_1,\cdots,\beta_{2n}) \tag{26}
$$

16

where the notations are used

$$\tilde{f}_K(\beta_1, \cdots, \beta_{2n}) = \lim_{\xi \to \infty} \tilde{f}^\gamma_{\xi_1, \ K}(\beta_1, \cdots, \beta_{2n}),$$
$$\tilde{f}^k_L(\beta_1, \cdots, \beta_{2n}) = (\tilde{f}^+_{\pi(k+2), \ L}(\beta_1, \cdots, \beta_{2n}))_{\text{restricted}} \tag{27}$$

Two pieces of (26) looking quite similar have very different analytical structure. The function $\tilde{f}_K(\beta_1, \cdots, \beta_{2n})$ is a solution to Yangian Riemann-Hilbert problem i.e. that with rational in β S-matrix. That is why it makes perfect sense to consider the following classical limit for this piece [20]:

$$\beta_j = \frac{2\pi\lambda_j}{\hbar} \qquad \hbar \to +0$$

In that limit the asymptotics of $\tilde{f}_K(\beta_1, \cdots, \beta_{2n})$ is described by a solution to usual KZ equations on level zero. On the other hand the S-matrix for $\tilde{f}^k_L(\beta_1, \cdots, \beta_{2n})$ depends typically on $\exp(\frac{\beta_j}{k+2})$, and the limit (27) does not make much sense for this function. Our dream would be to present the form factor as functional integral (with special boundary conditions) of the original action of the theory. The theory has two main features: it is asymptotically free and it contains WZNW term in action. These two features should lead to different effects: the asymptotic freedom should provide reasonable perturbation theory with respect to Plank constant while the WZNW term should provide nontrivial nonperturbative effects. The exact solution (26) shows that these two effects are combined in rather special way. Notice that the first piece of (26) is independent of k, also it allows the classical limit (27), $\hbar$ should be identified with Plank constant: it rescales the rapidities (logarithms of momenta) which makes perfect sense in asymptotically free situation. The second piece is the one depending upon k, it should be related to nonperturbative effects due to WZNW term. The formula (26) and the Grassmanian of the previous section suggest that by introducing certain fermionic field we should be able to treat these two pieces independently and perform the averaging over this fermionic field (summation over K, L) afterwards. We hope to return to the consideration of functional integral in feature, but now let us concentrate on the quasiclassical theory of $\tilde{f}_K(\beta_1, \cdots, \beta_{2n})$.

What we shall do now is certain extension of the consideration of papers [20,21]. Let us take the form $\omega(T)$ from the previous section for the Yangian case ($\xi = \infty$). Now we calculate the asymptotics (27) the result being [20]:

$$\omega^\pm(T) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (\int_{a_j} \zeta^T_i \ A_j \wedge Z_i \pm \int_{b_j} \zeta^T_i \ B_j \wedge Z_i) \tag{28}$$

where a_i, b_i are basic contours on the hyperelliptic surface $\tau^2 = P(\lambda) \equiv \prod(\lambda - \lambda_p)$ of genus $g = n - 1$, ζ^T_i is the following second kind differential:

$$\zeta^T_i = \frac{1}{\sqrt{P(\lambda)}}$$

$$\times \left\{ \prod_{p \in T}(\lambda - \lambda_p) \left[\frac{d}{d\lambda} \frac{\prod_{p \notin T}(\lambda - \lambda_p)}{\lambda^{n-i}} \right]_+ + \prod_{p \notin T}(\lambda - \lambda_p) \left[\frac{d}{d\lambda} \frac{\prod_{p \in T}(\lambda - \lambda_p)}{\lambda^{n-i}} \right]_+ \right\}$$

where $[]_+$ means that only polynomial part of the expression in brackets is taken. When substituting $\omega^\pm(T)$ into (25) we can perform certain transformations. Little modification of formulae from [20] provides that (25) can be rewritten in the limit as

$$\hbar^{\frac{3n}{4}}C^{-3}\sum_{T\in B,\#T=n}\wedge^{(n-1)}(\hat{\omega}^\pm(T))\;\theta[\eta_T](0)^4E_T$$

where E_T is basic vector in the tensor product of isotopic spaces. We introduced Riemann theta-function taking a_i, b_i for the canonical basis, η_T is even nonsingular (such that $\theta[\eta_T](0)\neq 0$) half-period related to the subset of branching points defined by T [22],

$$C = \prod_{i<j}(\lambda_i - \lambda_j)^{\frac{1}{4}}\Delta, \tag{29}$$

Δ is given by the determinant

$$\Delta = \det\left|\int_{a_i}\frac{\lambda^{j-1}}{\sqrt{P(\lambda)}}\right|_{(n-1)\times(n-1)}$$

The form $\hat{\omega}^\pm(T)$ is rewritten as

$$\tilde{\omega}(T) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\big(\partial_i\partial_j\log\theta[\eta_T](0)A_i\wedge dz_j \pm \sum_{l=1}^{n-1}(\Omega_{i,l}\partial_l\partial_j\log\theta[\eta_T](0)+2\pi i\delta_{i,j})B_l\wedge dz_j\big) \tag{30}$$

where we replaced quite formally Z_i by differential along Jacobian dz_i.

Let us consider the form on the Jacobian which interpolates between (30) for different T:

$$\hat{\omega}^\pm(z) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\big(\partial_i\partial_j\log\theta(z)A_i\wedge dz_j \pm \sum_{l=1}^{n-1}(\Omega_{i,l}\partial_l\partial_j\log\theta(z)+2\pi i\delta_{i,j})B_l\wedge dz_j\big)$$

where z varies over Jacobian: $z\in\mathbb{C}^{n-1}/\mathbb{Z}^{n-1}\times\Omega\mathbb{Z}^{n-1}$ The mathematical meaning of this form can be explained as averaging of the simplectic form induced on Jacobian after embedding into projective space by means of theta-functions of second order [21].

Probaaly the best possible situation takes place for PCF. Here both pieces of the form factor allow classical limit, and the form factors are special values (at even non-singular half-periods) of the form:

$$\hbar^{\frac{3n}{2}}C^{-6}\theta(z)^4\theta(w)^4(\wedge^{(n-1)}\hat{\omega}^+(z))\wedge(\wedge^{(n-1)}\hat{\omega}^-(w)) \tag{31}$$

It is interesting that all the formulae of this paper can be considered as deformations of (31).

8. Conclusion.

To conclude this paper let us formulate several problems which, to our mind, are worth investigation.

1. It would be interesting to generalize the considerations of this paper to other Lie algebras. The necessary preliminary information for $SU(N)$ case can be found in [1].

2. It is interesting to consider the deformation of the construction of this paper to the lattice models [23]. In that case different choice of contours in KZ is replaced not by introducing the exponents under the integrals, as it happened in the situation of this paper, but to introducing different theta-functions. The generalization for the lattice analog of PCF should be not complicated to find, however, the generalization for lattice version of perturbed WZNW on level k which coincides with integrable version of 6-vertex model of spin $2k$,$|q| < 1$ is not easy to describe because it will require the knowledge of solutions to elliptic version of Riemann-Hilbert problem (1) which are not known (the same is needed for form factors of 8-vertex model).

3. We suppose that the most important question is that of understanding the origin of form factor formulae in terms of functional integral. The results of this paper should be important in understanding of this problem. Formula (31) should be crucial for this goal.

REFERENCES

[1] F.A.Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, Adv. Series in Math. Phys. 14, World Scientific, Singapore, 1992.

[2] D. Bernard, Comm. Math. Phys. **137** (1991), 191.

[3] M. Lüscher, Nucl. Phys. **135B** (1978), 1.

[4] A.B.Zamolodchikov, Al.B.Zamolodchikov,, Annals. Phys. **120** (1979), 253.

[5] N.Yu. Reshetikhin, F.A. Smirnov, Comm. Math. Phys. **131** (1990), 157.

[6] A.B. Zamolodchikov, JETP Lett. **46** (1987), 160.

[7] G. Andrews, R. Baxter, P. Forrester, J. Stat. Phys. **35** (1984), 193.

[8] V.V. Bazhanov, N.Yu. Reshetikhin, Progress of Theor. Phys. **120, Suppl.** (1990), 301.

[9] A.M. Polyakov. P.B. Wiegmann, Physics Letters **131B** (1983), 121.

[10] N.Yu. Reshetikhin, J. Phys. **A24** (1991), 3299.

[11] C. Ahn, D. Bernard, A. LeClair, Nucl. Phys. **B346** (1990), 409.

[12] L.D. Faddeev, N.Yu. Reshetikhin, Annals of Physics **167** (1986), 227.

[13] A.G. Knizhnik, A.B. Zamolodchikov,, Nucl. Phys. **B297** (1984), 83.

[14] I.B. Frenkel, N.Yu. Reshetikhin, Comm. Math. Phys. **146** (1992), 1.

[15] F.A.Smirnov, Int.Jour.Math.Phys. **7A,suppl.1B** (1992), 813.

[16] F.A. Smirnov, *Remarks on deformed and undeformed KZ equations*, RIMS preprint-860, 1992.

[17] M. Jimbo, T. Kojima, T. Miwa, Y.-H. Quano, *Smirnov's integrals and quantum Knizhnik-Zamolodchikov equations on level zero.*, RIMS preprint RIMS-945, 1993.

[18] V.A. Fateev, A.B.Zamolodchikov, Yad. Fizika **43** (1986), 1031.

[19] V.V. Schechtman, A.N. Varchenko, *Integral Representations of N-point Conformal Correlations in the WZW Model*, Max-Plank-Institute preprint, 1987.

[20] F.A. Smirnov, Comm. Math. Phys. **155** (1993), 459.

[21] F.A. Smirnov, *What are we quantizing in integrable field theory?*, RIMS preprint RIMS-935, 1993.

[22] D. Mumford, *Tata Lectures on Theta* I,II, Birkhäuser, Boston, 1983.

[23] M. Idzumi,T. Ikihiro,K. Iohara,M. Jimbo,T. Miwa,T. Nakashima, Int.Jour.Mod.Phys. **8A** (1993), 1479.