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REDUCTIONS OF THE SINE-GORDON MODEL AS A PERTURBATION OF MINIMAL MODELS OF CONFORMAL FIELD THEORY

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It is shown that for special values of the coupling constant a reduction is possible in the sine-Gordon model which preserves the locality of certain operators. The reduced model corresponds to perturbed $M_{2/(2n+1)}$ models of conformal field theory. It is explained that for any rational coupling constant, the reduction is possible which leads to the perturbed $M_{p/q}$ model.

1. Introduction

Recently interest has arisen in massive two-dimensional field theories which give nontrivial conformal field theories (CFT) in the ultraviolet limit. Zamolodchikov has shown [1–3] that in certain cases massive perturbations of CFT [4] appear to be completely integrable. Massive completely integrable field theories (MCIFT) are much more complicated than CFT. However, great progress has been made in the investigation of MCIFT. First, the theory of factorizable S-matrices allows us to present exact S-matrices for these models [5]. Second, there is a construction which allows us for a given S-matrix to describe exhaustively local operators in MCIFT presenting all their matrix elements in the physical space of states [6–8]. A good example of a combination of these methods with those of CFT is provided by the 3-state Potts model. In ref. [3] it was shown that the corresponding scaling model is completely integrable, being the perturbation of the M_5 CFT by the operator with scaling dimensions $(\frac{2}{5}, \frac{2}{5})$. The special character of the conservation laws allowed us to confirm the validity of the hypothetic S-matrix of scaling theory [9]. In ref. [10] the matrix elements of all the important operators in the theory were obtained. With these it was possible to write down a convergent series for correlation functions in the scaling model.

The present paper considers special reductions of the sine-Gordon model (SG) which lead to nontrivial ultraviolet limits. Let us describe briefly the main results of the paper; the details are given in sects. 2–4.

The SG model is described by the lagrangian

$$\mathscr{L}=\int\left(\tfrac{1}{2}(\partial_\mu\varphi)^2+\frac{m^2}{\gamma}\left(\cos\sqrt{\gamma}\,\varphi-1\right)\right)\mathrm{d}x\,.$$

We use the renormalized coupling constant $\xi=\pi\gamma/(8\pi-\gamma)$. In the SG model for $\xi<\pi$ there are the following excitations: solitons which transfer the topological charge and their bound states (breathers), the number of which being $[\pi/\xi]$.

The S-matrix of the SG model was obtained in ref. [5]. The matrix elements of local operators were calculated in refs. [6,8]. The operator $\varphi(x)$ is normalized by requiring that the topological charge of a soliton is exactly $2\pi/\sqrt{\gamma}$. The energy–momentum tensor $T_{\mu\nu}$ is normalized in order that energy–momentum obtained from $T_{\mu\nu}$ has proper eigenvalues (i.e. coincides with $(1/i)\partial_\mu$). There is also a natural method to normalize the operators $\cos\sqrt{\gamma}\,\varphi$, $\sin\sqrt{\gamma}\,\varphi$ which will be discussed in sect. 2. Provided the operators are correctly normalized, the following equations are valid for the quantum model:

$$T_{\mu\mu}=\frac{M_1^2}{4\sin\xi}\cos\sqrt{\gamma}\,\varphi\,,\qquad \Box\varphi=\frac{\sqrt{2}\,\xi M_1^2}{\sqrt{\gamma}\sin\xi}\sin\sqrt{\gamma}\,\varphi\,, \tag{1}$$

where M_1 is the mass of the lowest breather ($M_1=2M\sin(\xi/2)$, M is the soliton mass). The commutators of local fields have the following leading singularities at the origin of coordinates:

$$[\varphi(x),\varphi_0(0)]=ic_\varphi\delta(x)\,,\qquad [T_{00}(x),T_{01}(0)]=\frac{c}{24\pi}\delta'''(x)+\dots\,,$$

where x is a space variable. The central charge c of the SG model should be equal to 1. We believe also that in our normalization $c_\varphi=1$. These assumptions are discussed in sect. 2.

The principal result of the present paper is the following. For $\xi=2\pi/(2n+1)$ a reduction is possible in the spectrum of the SG model which preserves the locality of some operators. This reduction means contraction of the model to the soliton-free sector. Denote the projection operator on the soliton-free sector by P. The reduction of an arbitrary operator is POP. The operators $PT_{\mu\nu}P$ and $P\varphi P$ are not local but the operator

$$\mathscr{T}_{\mu\nu}=P\left(T_{\mu\nu}+i2^{-5/2}\frac{\sqrt{\gamma}}{\xi}\varepsilon_{\mu\mu'}\varepsilon_{\nu\nu'}\partial_{\mu'}\partial_{\nu'}\varphi\right)P$$

is local:

$$\left[\mathscr{T}_{\mu\nu}(x_0,x_1),\mathscr{T}_{\mu'\nu'}(0,0)\right]=0\,,\qquad x_\mu^2<0\,.$$

The operator $\mathscr{T}_{\mu\nu}^*$ is also local but $\mathscr{T}_{\mu\nu}$ and $\mathscr{T}_{\mu\nu}^*$ are not mutually local:

$$\left[\mathscr{T}_{\mu\nu}(x_0, x_1), \mathscr{T}_{\mu'\nu'}^*(0,0)\right] \neq 0, \qquad x_\mu^2 < 0.$$

More generally there are two algebras of local operators $\mathscr{M}$ and $\mathscr{M}^*$ which are not mutually local. So we are dealing with two equivalent field theories. We have an amusing situation when the energy–momentum tensor is not self-adjoint and can not be made self-adjoint without loss of locality. At the same time energy and momentum are self-adjoint and energy is positive. The situation undermines the usual views, in particular it demonstrates that the definition of positivity accepted in CFT [11], which is nothing other than the assumption of self-adjointness of the energy–momentum tensor, does not necessarily correspond to the principal ideas of positivity of energy and unitarity of the S-matrix.

The above reasonings show that there is nothing strange in the fact that the central charge of the reduced model is equal to

$$c = 1 - 6\frac{(2n+1)^2}{2(2n+3)}$$

for $\xi = 2\pi/(2n+1)$, and that it corresponds to the "nonpositive" [11] minimal models of CFT, $\mathrm{M}_{2/(2n+3)}$. The model $\mathrm{M}_{2/(2n+3)}$ contains the primary fields with scaling dimensions

$$\Delta_l = -\frac{(l-1)(2n+2-l)}{2(2n+3)}, \qquad l = 1, 2, \ldots, 2n+2.$$

Evidently, $\Delta_l = \Delta_{2n+3-l}$. All these dimensions are negative, which is unpleasant for CFT itself (because of the increase of correlations) but quite satisfactory for the ultraviolet limit of the massive theory. This fact only means that the correlator of the primary field goes to zero according to a power law near the origin of coordinates, but decreases exponentially at large distances due to the massive spectrum of the model.

Using eq. (1) one can easily show that

$$\mathscr{T}_{\mu\mu} = \frac{M_1^2}{4\sin\xi} P \exp\left(i\sqrt{\gamma}\,\varphi\right) P.$$

The scaling dimensions of $\mathscr{T}_{\mu\mu}$ are $(-(2n-1)/(2n+3), -(2n-1)/(2n+3))$. So from the point of view of ref. [1] we are dealing with the perturbation of $\mathrm{M}_{2/(2n+3)}$ by the operator of the kind $(1,3)$. The calculations done in ref. [1] do not depend on the "positivity" of the model; they are quite appropriate for any minimal model. According to ref. [1] the perturbed model possesses higher conservation laws with arbitrary odd spins. Part of these conservation laws is equal to zero in concrete cases. As will be clear later, in the reduced SG model for $\xi = 2\pi/(2n+1)$ only those conservation laws occur whose spins are odd and are not multiples of $2n+1$.

2. The sine-Gordon model

As has already been noted, the spectrum of the SG model involves solitons transferring topological charge with mass M, and their bound states (breathers). There are $[\pi/\xi]$ breathers, their masses being $M_j = 2M\sin(\xi j/2)$. The soliton is a two-component particle (soliton–antisoliton); the soliton–soliton S-matrix is

$$S^{\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}(\beta) = S_0(\beta), \qquad S^{\frac{1}{2}-\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}}(\beta) = -S_0(\beta)\frac{\sinh \pi\beta\beta/\xi}{\sinh \pi(\beta-\pi i)/\xi},$$

$$S^{-\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}}(\beta) = S_0(\beta)\frac{\sinh \pi^2 i/\xi}{\sinh \pi(\beta-\pi i)/\xi},$$

$$S^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2}(\beta) = S^{-\varepsilon_1'-\varepsilon_2'}_{-\varepsilon_1-\varepsilon_2}(\beta), \qquad S^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2}(\beta) = 0, \qquad \varepsilon_1+\varepsilon_2 \neq \varepsilon_1'+\varepsilon_2',$$

$$S_0(\beta) = \exp\left(-i\int_0^\infty \frac{\sin k\beta \sinh\frac{1}{2}(\pi-\xi)k}{k\sinh\frac{1}{2}\xi k\cosh\frac{1}{2}\pi k}\,\mathrm{d}k\right),$$

where $\beta = \beta_1 - \beta_2$ (β_1, β_2 are the rapidities of the solitons), $\varepsilon_1, \varepsilon_2$, $\varepsilon_1', \varepsilon_2'$ are equal to $\pm\frac{1}{2}$ and characterize the isotopic states of "in" and "out" particles.

The soliton-m-breather, and m_1-breather–m_2-breather S-matrices are scalar ones:

$$S_m(\beta) = \prod_{j=1}^{m}\frac{i\cos\frac{1}{2}\xi + \sinh\left(\beta - \frac{1}{2}i\xi(m+1-2j)\right)}{i\cos\frac{1}{2}\xi - \sinh\left(\beta - \frac{1}{2}i\xi(m+1-2j)\right)},$$

$$S_{m_1m_2}(\beta) = \coth\tfrac{1}{2}\left(\beta - \tfrac{1}{2}i\xi(m_1+m_2)\right)\tanh\tfrac{1}{2}\left(\beta + \tfrac{1}{2}i\xi(m_1+m_2)\right)\coth\tfrac{1}{2}\left(\beta - \tfrac{1}{2}i\xi|m_1-m_2|\right)$$
$$\times\tanh\tfrac{1}{2}\left(\beta + \tfrac{1}{2}i\xi|m_1-m_2|\right)\prod_{j=1}^{\min(m_1,m_2)-1}\tanh^2\tfrac{1}{2}\left(\beta + \tfrac{1}{2}i\xi(|m_1-m_2|+2j)\right)$$
$$\times\coth^2\tfrac{1}{2}\left(\beta - \tfrac{1}{2}i\xi(|m_1-m_2|+2j)\right).$$

There is an involved hierarchy of bound states. First, the soliton and antisoliton create an m-breather when the rapidity shift is $(\pi - \xi m)i$; evidently, the point $(\pi-\xi m)i$ lies on physical sheet $0 < \operatorname{Im}\beta < \pi$ only for $m < \pi/\xi$. Second, the soliton and the m-breather create a soliton when the rapidity shift is $\frac{1}{2}(\pi+\xi m)i$. Third, the m_1- and m_2-breather create an m_1+m_2 breather when the rapidity shift is $\frac{1}{2}i(m_1+m_2)\xi$.

It is clear that beside the poles which correspond to these physically clear possibilities the soliton–breather and breather–breather S-matrices have many additional poles. It is well known that singularities of the S-matrix are connected with the possibility to construct reduced graphics [12]. The additional poles of the S-matrices correspond to the processes which involve "physical particles" as intermediate ones. Consider for example a 1-breather–1-breather S-matrix

$$S_{11}(\beta)=\frac{\sinh\beta+\sinh i\xi}{\sinh\beta-\sinh i\xi}.$$

For $\xi>\pi/2$ the 2-breather disappears but the corresponding pole $\beta=i\xi$ does not leave the physical sheet. The same can be said about the crossing pole $\beta=(\pi-\xi)i$. This phenomenon is explained as follows: for $\pi>\xi>\pi/2$ the inequality $M_1^2>2M^2$ holds, which provides the possibility to construct the reduced graphics for 1-breather–1-breather scattering which involves "physical solitons" as intermediate particles. In a certain sense it can be said that the 2-breather exists for $\pi>\xi>\pi/2$ as a virtual state; the precise meaning of this interpretation will be clarified later.

Let us pass to the local operators. In refs. [6, 7] all the matrix elements (form factors) of the operators $T_{\mu\nu}$, φ, $\exp(\pm i\sqrt{\gamma}\,\varphi/2)$ were obtained. These form factors satisfy the system of axioms [7, 8] which guarantee the locality of operators. The form factor of an operator is the analytic continuation of the matrix element

$$\langle 0|O(0,0)|\beta_1\ldots\beta_k\rangle^{in}_{\varepsilon_1\ldots\varepsilon_k}$$

from the range $\beta_1>\beta_2\ldots>\beta_k$ to all values of the arguments. We denote the form factor by $f(\beta_1\ldots\beta_k)_{\varepsilon_1\ldots\varepsilon_k}$. For the SG model the indices can be equal to $\frac{1}{2}$ (soliton), $-\frac{1}{2}$ (antisoliton), and m (m-breather). A relation can be derived which expresses the matrix element taken between two arbitrary states in terms of form factors $f(\beta_1\ldots\beta_k)$ [7, 8].

The form factor as a function of β_k has simple poles corresponding to bound states in the strip $0<\operatorname{Im}\beta_k<\pi$, crossing poles in the strip $\pi<\operatorname{Im}\beta_k<2\pi$, and annihilation poles at the line $\operatorname{Im}\beta_k=\pi$. Three main axioms are

$$\text{(I)}\qquad f(\beta_1\ldots\beta_i,\beta_{i+1}\ldots\beta_k)_{\varepsilon_1\ldots\varepsilon_i\varepsilon_{i+1}\ldots\varepsilon_k}S^{\varepsilon_i\varepsilon_{i+1}}_{\varepsilon_i'\varepsilon_{i+1}'}(\beta_i-\beta_{i+1})$$
$$=f(\beta_1\ldots\beta_{i+1},\beta_i\ldots\beta_k)_{\varepsilon_1\ldots\varepsilon_{i+1}'\varepsilon_i'\ldots\varepsilon_k}. \qquad (2)$$

$S^{\varepsilon_i\varepsilon_{i+1}}_{\varepsilon_i'\varepsilon_{i+1}'}$ is diagonal when the ith or $(i+1)$th particle is a breather.

$$\text{(II)}\qquad f(\beta_1\ldots\beta_{k-1},\beta_k+2\pi i)_{\varepsilon_1\ldots\varepsilon_{k-1}\varepsilon_k}=f(\beta_k,\beta_1\ldots\beta_{k-1})_{\varepsilon_k\varepsilon_1\ldots\varepsilon_{k-1}}, \qquad (3)$$

$$\text{(III)}\qquad \operatorname*{res}_{\beta_k=\beta_{k-1}+\pi i} f(\beta_1\ldots\beta_k)_{\varepsilon_1\ldots\varepsilon_k}$$
$$=\frac{1}{2\pi i}c_{\varepsilon_k\varepsilon_{k-1}'}f(\beta_1\ldots\beta_{k-2})_{\varepsilon_1'\ldots\varepsilon_{k-2}'}\Big[\delta^{\varepsilon_1'}_{\varepsilon_1}\ldots\delta^{\varepsilon_{k-1}'}_{\varepsilon_{k-1}}-S^{\varepsilon_{k-1}'\varepsilon_1'}_{\tau_1\varepsilon_1}(\beta_{k-1}-\beta_1)$$
$$\times S^{\tau_1\varepsilon_2'}_{\tau_2\varepsilon_2}(\beta_{k-1}-\beta_2)\ldots S^{\tau_{k-3}\varepsilon_{k-2}'}_{\varepsilon_{k-1}\varepsilon_{k-2}}(\beta_{k-1}-\beta_{k-2})\Big]. \qquad (4)$$

In the presence of bound states the set of axioms should be supplemented with expressions for the residues at the poles which correspond to bound states. For the SG model these expressions are given by

(I)
$$\operatorname{res} f(\beta_1 \ldots \beta_{k-1}\beta_k)_{\varepsilon_1 \ldots \varepsilon_{k-2} - \frac{1}{2}\frac{1}{2}}$$
$$= a_m f\big(\beta_1 \ldots \beta_{k-2}, \beta_{k-1} + \tfrac{1}{2}(\pi i - i\xi m)\big)_{\varepsilon_1 \ldots \varepsilon_{k-2} m}, \tag{5}$$

with

$$\beta_k = \beta_{k-1} + \pi i - i\xi m, \qquad m \geqslant 1,$$

where

$$a_m = \left(\frac{1}{2\pi} S_0(\pi i - i\xi m)\frac{\xi}{\pi}\sin\frac{\pi^2}{\xi}\right)^{1/2};$$

(II)
$$\operatorname{res} f(\beta_1 \ldots \beta_{k-1}, \beta_k)_{\varepsilon_1 \ldots \varepsilon_{k-2}\frac{1}{2}m}$$
$$= a_m^p f\big(\beta_1 \ldots \beta_{k-1} + i\xi(m-p), \beta_{k-1} + \tfrac{1}{2}(\pi i - \xi i p)\big)_{\varepsilon_1 \ldots \frac{1}{2}p}, \tag{6}$$

with

$$\beta_k = \beta_{k-1} + \tfrac{1}{2}\pi i + \tfrac{1}{2}i\xi(m-2p), \qquad p = 0, \ldots, m-1,$$

where

$$f(\beta_1 \ldots \beta_{k-1}\beta_k)_{\varepsilon_1 \ldots \varepsilon_{k-1}0} \equiv f(\beta_1 \ldots \beta_{k-1})_{\varepsilon_1 \ldots \varepsilon_{k-1}},$$

$$a_m^p = 2\pi a_{m-p}^0 a_{m-p,p}, \qquad p > 0,$$

$$a_p^0 = \left(\frac{1}{2\pi i}\operatorname*{res}_{\beta = \frac{1}{2}(\pi + \xi p)i} S_p(\beta)\right)^{1/2};$$

$a_{m_1 m_2}$ will be described later.

(III)
$$\operatorname{res} f(\beta_1 \ldots \beta_{k-2}, \beta_{k-1}\beta_k)_{\varepsilon_1 \ldots \varepsilon_{k-2} m_1 m_2}$$
$$= a_{m_1 m_2}^p f\big(\beta_1 \ldots \beta_{k-1} + \tfrac{1}{2}i\xi(m_2 - p), \beta_{k-1} + \tfrac{1}{2}i\xi(m_1 - p)\big)_{\varepsilon_1 \ldots m_1 + m_2 - p, p} \tag{7}$$

with

$$\beta_k = \beta_{k-1} + \tfrac{1}{2}i\xi(m_1 + m_2 - p), \qquad p = 0, \ldots, \min(m_1 m_2) - 1,$$

where

$$a^p_{m_1m_2} = 2\pi a_{m_1m_2-p} a_{m_2-p,p}, \qquad p>0,$$

$$a_{m_1m_2} = \left(\frac{1}{2\pi i} \operatorname*{res}_{\beta=\frac{1}{2}i\xi(m_1+m_2)} S_{m_1m_2}(\beta) \right)^{1/2}$$

(IV) $$\operatorname{res} f(\beta_1 \dots \beta_{k-2}\beta_{k-1}, \beta_k)_{\varepsilon_1\dots\varepsilon_{k-2}m_1m_2}$$
$$= \tilde{a}^p_{m_1m_2} f\big(\beta_1 \dots \beta_{k-1} + \tfrac{1}{2}i\xi p, \beta_{k-1} + \pi i - \tfrac{1}{2}i\xi(m_1+m_2-2p)\big)_{\varepsilon_1\dots\varepsilon_{k-2},m_1-p,m_2-p}, \tag{8}$$

with

$$\beta_k = \beta_{k-1} + \pi i - \tfrac{1}{2}i\xi(m_1-m_2-p), \qquad p=1,\dots,\min(m_1m_2) - \delta_{m_1m_2},$$

where

$$\tilde{a}^p_{m_1m_2} = 2\pi a_{m_1-p,p} a_{m_2-p,p}.$$

The physical meaning of some of the poles is quite clear, for example the soliton–soliton pole at $\beta_k = \beta_{k-1} + \pi i - i\xi m$ corresponds to the creation of an m-breather, the m_1-breather–m_2-breather pole at $\beta_k = \beta_{k-1} + i\xi(m_1-m_2)/2$ corresponds to the creation of an (m_1+m_2)-breather. Other poles correspond to more complicated processes, for example, the m_1-breather–m_2-breather pole at $\beta_k = \beta_{k-1} + i\xi(\frac{1}{2}(m_1+m_2)-p)$ corresponds to the decay of an m_2-breather into a (m_2-p)-breather and a p-breather, followed by the creation of an (m_1+m_2-p)-breather by an (m_2-p)- and an m_1-breather. The kinematical possibility for these poles to appear is provided by the inequality $(M^{(k-1)})^2 + (M^{(k)})^2 > (\tilde{M}^{(k-1)})^2 + (\tilde{M}^{(k)})^2$, where $M^{(k)}$, $M^{(k-1)}$ and $\tilde{M}^{(k)}$, $\tilde{M}^{(k-1)}$ are the masses of kth and $(k-1)$th particles in the left- and right-hand sides respectively.

It is worth noting that for particular values of the coupling constant, the r.h.s. of eq. (7) loses its direct physical meaning. Consider for example the case $\xi > \pi/2$ which has been treated in connection with singularities of the breather S-matrix. Consider the form factor for which both the kth and $(k-1)$th particles are 1-breathers. The pole at the point $\beta_k = \beta_{k-1} + i\xi$ corresponds as well as in the S-matrix to more complicated processes than just the creation of a 2-breather, which is absent for $\xi > \pi/2$. However, we preserve eq. (7) in this case, interpreting a 2-breather as a virtual particle which does not appear in asymptotic states. Eq. (7) can be considered as a definition of a virtual 2-breather for $\xi > \pi/2$. An essential requirement is that all possible definitions of virtual particles must coincide. For example, the same 2-breather can be obtained as bound state of solitons at the point $\beta_k = \beta_{k-1} + \pi i - 2i\xi$ (5), which for $\xi > \pi/2$ lies out of the physical sheet. Such an understanding of virtual particles ensures locality. Let us discuss the point in more detail.

The local commutativity theorem in the absence of bound states has been proven in refs. [7, 8]. This theorem states that the operators defined by the form factors are local, provided the form factors satisfy the axioms eqs. (2)–(4). The proof is rather straightforward. Consider the commutator $[O(x_0, x_1), O(0,0)]$ (x_μ is space-like). For the products $O(x)O(0)$ and $O(0)O(x)$ one can write down decompositions into the sum over all the intermediate states. The contours of integration in the integrals over the rapidities of intermediate particles for $O(x)O(0)$ can be transformed in such a manner that the integrals turn into those for $O(0)O(x)$. The SG model possesses bound states. That is why one meets the poles of the integrand by deforming the contours. So one has to show that all the pole contributions cancel each other. That is not the case for terms which correspond to the same intermediate state. It can be shown however that eqs. (5)–(8) ensure the cancellation of poles after the sum over all intermediate states is taken. The virtual particles do not occur in physical states but they appear in pole contributions. The requirement that all the possible definitions of these particles coincide provides the cancellation of pole contributions. For example in the case $\pi > \xi > \pi/2$ the pole contributions involving virtual 2-breathers obtained from two 2-breathers and from two solitons cancel each other. For $\xi = 2\pi/3$ however, a virtual 2-breather can be identified with a 1-breather, and for certain operators locality holds in the soliton-free sector. This situation will be considered in the next sections.

Let us consider physically important operators. The energy–momentum tensor can be presented in the form

$$T_{\mu\nu} = \varepsilon_{\mu\mu'}\varepsilon_{\nu\nu'}\partial_{\mu'}\partial_{\nu'}A\,,$$

where A is nonlocal operator. The form factors of A will be denoted by f^+. The form factors of $T_{\mu\nu}$ can be easily expressed in terms of f^+. For example, the form factors of T_{00} are equal to

$$\left(\sum M_{\varepsilon_j}\sinh\beta_j\right)^2 f^+(\beta_1\ldots\beta_k)_{\varepsilon_1\ldots\varepsilon_k}\,,$$

where $M_{\pm 1/2} \equiv M$. The form factors f^+ are normalized in order that $\int T_{00}(x)\,\mathrm{d}x_1$ coincides with the hamiltonian, which requirement is equivalent to the condition

$$\operatorname*{res}_{\beta_2=\beta_1+\pi i}\left(\left(M_{\varepsilon_1}\sinh\beta_1 + M_{\varepsilon_2}\sinh\beta_2\right)f^+_{\varepsilon_1\varepsilon_2}(\beta_1\beta_2)\right) = \frac{c_{\varepsilon_1\varepsilon_2}}{2\pi i}M_{\varepsilon_1}\sinh\beta_1\,. \qquad (9)$$

Operator A generates all the densities of higher conservation laws in the theory. In the SG model there is an infinite series of conservation laws with arbitrary odd spins $s\,(I_s,\,\bar I_s)$. The eigenvalues of these integrals are

$$I_s|\beta_1\dots\beta_k\rangle_{\varepsilon_1\dots\varepsilon_k}=\left(\sum M^{(s)}_{\varepsilon_j}\,\mathrm{e}^{s\beta_j}\right)|\beta_1\dots\beta_k\rangle_{\varepsilon_1\dots\varepsilon_k},$$

$$\bar I_s|\beta_1\dots\beta_k\rangle_{\varepsilon_1\dots\varepsilon_k}=\left(\sum M^{(s)}_{\varepsilon_j}\,\mathrm{e}^{-s\beta_j}\right)|\beta_1\dots\beta_k\rangle_{\varepsilon_1\dots\varepsilon_k},\tag{10}$$

where $M^{(s)}_{\pm\frac{1}{2}}=M$, $M^{(s)}_m=2M\sin(sm\xi/2)$, s is odd. The local densities of these integrals are

$$\partial_{t_s}\partial_{x_1}A,\qquad \partial_{\bar t_s}\partial_{x_1}A,$$

where $t_s,\bar t_s$ means "times" associated with $I_s,\bar I_s$. The form factors of these densities are equal to

$$\left(\sum M^{(s)}_{\varepsilon_j}\,\mathrm{e}^{\pm s\beta_j}\right)\left(\sum M_{\varepsilon_j}\sinh\beta_j\right)f^+(\beta_1\dots\beta_k)_{\varepsilon_1\dots\varepsilon_k}.$$

The operators I_1, $\bar I_1$ coincide with P_0+P_1, P_0-P_1.

The operator φ is not local due to the existence of solitons transferring topological charge. True local operators are the topological currents $j_\mu=\varepsilon_{\mu\mu'}\partial_{\mu'}\varphi$ whose form factors can be easily expressed in terms of φ form factors. We will denote form factors of φ by f^-. Form factors f^- are normalized by the condition

$$\operatorname*{res}_{\beta_2=\beta_1+\pi i}f^-(\beta_1\beta_2)_{\frac{1}{2},-\frac{1}{2}}=2\pi/\sqrt{\gamma}\,,$$

which means that the soliton has proper topological charge.

The operators $\square A$ and $\square\varphi$ coincide up to a coefficient with $\cos\sqrt{\gamma}\,\varphi$ and $\sin\sqrt{\gamma}\,\varphi$. How could one normalize the form factors of $\cos\sqrt{\gamma}\,\varphi$ and $\sin\sqrt{\gamma}\,\varphi$? The recipe is the following. Let us denote the form factors of $\cos\sqrt{\gamma}\,\varphi$, $\sin\sqrt{\gamma}\,\varphi$ by g^+, g^- respectively. They coincide up to normalization with

$$\left(\sum M_{\varepsilon_j}\mathrm{e}^{\beta_j}\right)\left(\sum M_{\varepsilon_j}\mathrm{e}^{-\beta_j}\right)f^+(\beta_1\dots\beta_k)_{\varepsilon_1\dots\varepsilon_k},$$

$$\left(\sum M_{\varepsilon_j}\mathrm{e}^{\beta_j}\right)\left(\sum M_{\varepsilon_j}\mathrm{e}^{-\beta_j}\right)f^-(\beta_1\dots\beta_k)_{\varepsilon_1\dots\varepsilon_k}.$$

For a certain normalization the following asymptotic expressions hold:

$$g^\#(\beta_1\dots\beta_k\beta_{k+1}+\Lambda\dots\beta_l+\Lambda)_{\varepsilon\dots\varepsilon_l}$$

$$\xrightarrow[\Lambda\to\infty]{}g^\#(\beta_1\dots\beta_k)_{\varepsilon_1\dots\varepsilon_k}g^\#(\beta_{k+1}\dots\beta_l)_{\varepsilon_{k+1}\dots\varepsilon_l},\tag{11a}$$

if $Q|\beta_1 \dots \beta_k\rangle_{\varepsilon_1 \dots \varepsilon_k} = 0$, with Q the topological charge and

$$g^{\#}(\beta_1 \dots \beta_k, \beta_{k+1} + \Lambda, \dots, \beta_l + \Lambda)_{\varepsilon_1 \dots \varepsilon_k \varepsilon_{k+1} \dots \varepsilon_l} = \mathrm{O}(\mathrm{e}^{-\pi\Lambda/2\xi}). \qquad \text{(11b)}$$

if $Q|\beta_1 \dots \beta_k\rangle_{\varepsilon_1 \dots \varepsilon_k} \neq 0$.

Eqs. (11a, b) are very important because they allow us to write down a virial expansion for the scaling dimension of $\exp(i\sqrt{\gamma}\,\varphi)$. An example of this expansion is given in sect. 4. The explicit formulas for breather form factors will be given later; the formulas for soliton form factors can be found in refs. [6, 8]. Using these explicit formulas one makes sure that the asymptotic expressions hold if

$$g^+(\beta_1 \dots \beta_k) = \frac{4 \sin \xi}{M_1^2}\left(\sum M_{\varepsilon_j} \mathrm{e}^{\beta_j}\right)\left(\sum M_{\varepsilon_j} \mathrm{e}^{-\beta_j}\right) f^+(\beta_1 \dots \beta_k),$$

$$g^-(\beta_1 \dots \beta_k) = \frac{\sqrt{\gamma}\, \sin \xi}{\sqrt{2}\, M_1^2 \xi}\left(\sum M_{\varepsilon_j} \mathrm{e}^{\beta_j}\right)\left(\sum M_{\varepsilon_j} \mathrm{e}^{-\beta_j}\right) f^-(\beta_1 \dots \beta_k).$$

These formulas clarify the origin of the quantum equation of motion (1).

The last pair of operators we will deal with is $\cos(\sqrt{\gamma}\,\varphi/2)$ and $\sin(\sqrt{\gamma}\,\varphi/2)$. Their form factors will be denoted by h^+, h^-. For a proper normalization they satisfy asymptotic expressions similar to eqs. (11a, b).

Let us present explicit formulas for breather form factors which are important for what follows. To this end we first define some auxiliary objects.

The function $\mathscr{S}_{m_1 m_2}(\beta)$ is given by

$$\mathscr{S}_{m_1 m_2}(\beta) = c_{m_1 m_2} \frac{2 \prod_{j=0}^{m_1 - m_2} \sinh \frac{1}{2}\left(\beta + \frac{1}{2} i \xi (m_1 - m_2 - 2j)\right)}{\prod_{j=1}^{m_1 + m_2 - 1} \cosh \frac{1}{2}\left(\beta + \frac{1}{2} i \xi (m_1 + m_2 - 2j)\right)}$$

$$\times \exp\left(4 \int_0^\infty \frac{\sin^2 \frac{1}{2} k(\beta + \pi i) \cosh\left(\frac{1}{2}(\pi - \xi m_1) k\right) \cosh \frac{1}{2} \xi k \sinh \frac{1}{2} \xi m_2 k}{k \sinh \frac{1}{2} \xi k \cosh \frac{1}{2} \pi k \sinh \pi k}\, \mathrm{d}k\right),$$

$$c_{m_1 m_2} = \exp\left(2 \int_0^\infty \frac{m_2 \cosh \frac{1}{2} \pi k \sinh \frac{1}{2} \xi k - \cosh\left(\frac{1}{2}\pi - \xi m_1\right) k \cosh \frac{1}{2} \xi k \sinh \frac{1}{2} \xi m_2 k}{k \sinh \frac{1}{2} \xi k \cosh \frac{1}{2} \pi k \sinh \pi k}\, \mathrm{d}k\right).$$

For $m_1 > m_2$,

$$\mathscr{S}_{m_1 m_2}(\beta) = \mathscr{S}_{m_2 m_1}(\beta).$$

This function satisfies the equations

$$\mathscr{S}_{m_1 m_2}(\beta) S_{m_1 m_2}(\beta) = \mathscr{S}_{m_1 m_2}(-\beta) = \mathscr{S}_{m_1 m_2}(\beta - 2\pi i),$$

$$\mathscr{S}_{m_1 m_2}(\beta) \mathscr{S}_{m_1 m_2}(\beta - \pi i) = \varphi_{m_1}^{-1}\left(\beta - \tfrac{1}{2} i \xi m_2\right) \varphi_{m_1}^{-1}\left(\beta - \pi i + \tfrac{1}{2} \xi m_2\right),$$

where

$$\varphi_m(\beta)=\frac{\prod_{j=1}^{m-1}\cosh\frac{1}{2}\left(\beta+\frac{1}{2}i\xi(m-2j)\right)}{2\prod_{j=0}^{m}\sinh\frac{1}{2}\left(\beta+\frac{1}{2}i\xi(m-2j)\right)}.$$

The functions $F_\lambda(\beta_1\dots\beta_k)_{m_1\dots m_k}$ are defined by the integral

$$F_\lambda(\beta_1\dots\beta_k)_{m_1\dots m_k}=\left(\frac{1}{2\pi i}\right)^{k-1}\int_{\Gamma_{m_1}}\mathrm{d}\alpha_1\dots\int_{\Gamma_{m_{k-1}}}\mathrm{d}\alpha_{k-1}\prod_{i=1}^{k-1}\prod_{j=1}^{k}\varphi_{m_j}(\alpha_i-\beta_j)$$
$$\times\prod_{i<j}\sinh(\alpha_1-\alpha_j)\exp\left(\lambda\left(\sum\beta_j-\sum\alpha_i\right)\right),\qquad \lambda=0,\pm1, \tag{12}$$

where Γ_{m_j} is the contour enveloping the points $\beta_j+\frac{1}{2}i\xi(m_j-2l)$, $l=0,1,\dots,m_j$. Evidently, this function is antisymmetric with respect to β_i β_j $(i,j<k)$ if $m_i=m_j$. The argument β_k seems to play a special role because the integration over Γ_{m_k} is absent. In fact this is not the case. Consider the integral over α_j. The integrand decreases when $\alpha_j\to\infty$, thus this integral can be transformed into

$$\int_{\Gamma_{m_j}}\mathrm{d}\alpha_j\to-\sum_{p\neq j}\int_{\Gamma_{m_p}}\mathrm{d}\alpha_j.$$

The integrals over $\Gamma_{m_1}\dots\Gamma_{m_{k-1}}$ can be omitted due to the antisymmetry with respect to $\alpha_1\dots\alpha_{j-1}\alpha_{j+1}\dots\alpha_{k-1}$. Thus the integral over α_j in eq. (12) can be replaced by $-\int_{\Gamma_{m_k}}\mathrm{d}\alpha_j$.

So the integral (12) can be rewritten as follows:

$$F_\lambda(\beta_1\dots\beta_k)_{m_1\dots m_k}=\left(\frac{1}{2\pi i}\right)^{k-1}\int_{\Gamma_{m_1}}\mathrm{d}\alpha_1\dots\widehat{\int\mathrm{d}\alpha_j}\dots\int_{\Gamma_{m_k}}\mathrm{d}\alpha_k\prod_{i\neq j}\varphi(\alpha_i-\beta_l)$$
$$\times\prod_{i<l}\sinh(\alpha_i-\alpha_l)\exp\left(\lambda\left(\sum\beta_l-\sum_{i\neq j}\alpha_i\right)\right). \tag{13}$$

These reasonings show that $\beta_1\dots\beta_k$ participate in $F_\lambda(\beta_1\dots\beta_k)_{m_1\dots m_k}$ on equal footing.

Using the functions $\mathscr{S}_{m_1m_2}$ and F_λ we can define the form factors of the operators $\partial_\pm A$ $(f^+_\pm)$, $\partial_\pm\varphi$ $(f^-_\pm)$ ($\partial_\pm$ are light-cone derivatives), $\sin\frac{1}{2}\sqrt{\gamma}\,\varphi$ (h^-),

$\cos\frac{1}{2}\sqrt{\gamma}\,\varphi\ (h^+)$:

$$\prod_{k=1}^{k} d_{m_j}\prod_{i<j}\mathscr{S}_{m_im_j}(\beta_i-\beta_j)F_{\pm}(\beta_1\ldots\beta_k)_{m_1\ldots m_k}$$

$$=\begin{cases}\dfrac{4\sin\xi}{M_1}f_{\pm}^{+}(\beta_1\ldots\beta_k)_{m_1\ldots m_k} & \text{if}\quad m_1+\cdots+m_k\equiv 0\ (\mathrm{mod}\,2)\,,\\ \dfrac{\sin\xi\sqrt{\gamma}}{\sqrt{2M_1\xi}}f_{\pm}^{-}(\beta_1\ldots\beta_k)_{m_1\ldots m_k} & \text{if}\quad m_1+\cdots+m_k\equiv 1\ (\mathrm{mod}\,2)\,,\end{cases}\tag{14}$$

$$\prod_{j=1}^{k} d_{m_j}\prod_{i<j}\mathscr{S}_{m_im_j}(\beta_i-\beta_j)F_0(\beta_1\ldots\beta_k)_{m_1\ldots m_k}$$

$$=\begin{cases}h^{+}(\beta_1\ldots\beta_k)_{m_1\ldots m_k} & \text{if}\quad m_1+\cdots+m_k\equiv 0\ (\mathrm{mod}\,2)\,,\\ h^{-}(\beta_1\ldots\beta_k)_{m_1\ldots m_k} & \text{if}\quad m_1+\cdots+m_k\equiv 1\ (\mathrm{mod}\,2)\,,\end{cases}\tag{15}$$

where

$$d_m=\prod_{l=1}^{m-1}\sin\xi l\,c_{mm}^{-1/2}\left(\frac{2\sin\xi m}{\pi}\right)^{1/2}.\tag{16}$$

Eqs. (14) need some comment. It is evident that $f_{\pm}^{\pm}$ should be connected with $f^{\pm}$ by the relations:

$$f_{\pm}^{\pm}(\beta_1\ldots\beta_k)_{m_1\ldots m_k}=\left(\sum M_{m_j}\mathrm{e}^{\pm\beta_j}\right)f^{\pm}(\beta_1\ldots\beta_k)_{m_1\ldots m_k}.\tag{17}$$

So definitions of $f_{-}^{\pm}$ and $f_{+}^{\pm}$ should satisfy the condition of self-consistency:

$$\left(\sum M_{m_j}\mathrm{e}^{-\beta_j}\right)f_{+}^{\pm}(\beta_1\ldots\beta_k)_{m_1\ldots m_k}=\left(\sum M_{m_j}\mathrm{e}^{\beta_j}\right)f_{-}^{\pm}(\beta_1\ldots\beta_k)_{m_1\ldots m_k},\tag{18}$$

which means that $\partial_+\partial_-=\partial_-\partial_+$. Eq. (18) can be proven using the method similar to the one used in ref. [7]. Thus eqs. (14) can be regarded as two equivalent definitions of $f^{\pm}$. The structure of $f_{\pm}^{\pm}$ and eq. (18) ensure that by dividing $f_{\pm}^{\pm}$ by $(\Sigma M_{m_j}\exp(\pm\beta_j))$, one gets no singularities in addition to those of $f_{\pm}^{\pm}$. The structure of the singularities can be easily investigated and eqs. (4), (7), and (8) follow.

The normalization of $h^{\pm}$ is chosen in order to satisfy eq. (11). The normalization of f^{+} is chosen in such a way that

$$2\pi i \operatorname*{res}_{\beta_2=\beta_1+\pi i} f^{+}_{\pm}(\beta_1,\beta_2)_{m_1m_2}=\delta_{m_1m_2}M_{m_1}\,e^{\pm\beta_1},$$

in agreement with eq. (9). To check the normalization of f^{-} one has to know the 2-soliton form factor which is [6, 13]

$$f^{-}(\beta_1,\beta_2)_{\frac12-\frac12}=\frac{1}{2\sqrt{\gamma}}\,\frac{\tanh\frac12\beta_{12}}{\cosh\pi/2\xi(\beta_{12}+\pi i)}$$

$$\times\exp\left(\int_0^{\infty}\frac{\sin^2\frac12(\beta_{12}+\pi_i)k\,\sinh\frac12(\pi-\xi)k}{k\sinh\frac12\xi k\,\sinh\pi k\,\cosh\frac12\pi k}\,\mathrm{d}k\right).$$

This form factor satisfies the requirement

$$2\pi i \operatorname*{res}_{\beta_2=\beta_1-\pi i} f^{-}(\beta_1,\beta_2)=\frac{2\pi}{\sqrt{\gamma}},$$

which means that the topological charge of the soliton is equal to $2\pi/\sqrt{\gamma}$. On the other hand, due to eq. (5)

$$\operatorname*{res}_{\beta_2=\beta_1+\pi i-i\xi} f^{-}(\beta_1,\beta_2)_{-\frac12\frac12}=a_1 f^{-}(\beta_1)_1.$$

This equation fixes the normalization of f^{-} in eq. (14).

Eqs. (14) and (15) are suitable for virtual breathers discussed above as well as for physical ones, the square root $(\sin m\xi)^{1/2}$ in eq. (16) for $m>\pi/\xi$ is understood as $i(\sin(\pi-\xi m))^{1/2}$. This choice is connected with the interpretation of the square root $(\operatorname{res} S_{m-l,l})^{1/2}$ in eq. (7); for $m>\pi/\xi$ the residue is negative and we interpret $(\operatorname{res} S_{m-l,l})^{1/2}$ as $i(|\operatorname{res} S_{m-l,l}|)^{1/2}$.

Let us turn to the constants c and c_{φ}. Evidently, they can be expressed as follows:

$$c_{\varphi}=2\pi\sum_{\varepsilon_1\ldots\varepsilon_k}\int_{\beta_k>\beta_{k-1}>\ldots>\beta_2>0}\left|f^{-}(0,\beta_2\ldots\beta_k)\right|^2\mathrm{d}\beta_2\ldots\mathrm{d}\beta_k,$$

$$c=48\pi^2\sum_{\varepsilon_1\ldots\varepsilon_k}\int_{\beta_k>\beta_{k-1}>\ldots>\beta_2>0}\left|f^{+}(0,\beta_2\ldots\beta_k)\right|^2\mathrm{d}\beta_2\ldots\mathrm{d}\beta_k.$$

It was claimed in sect. 1 that

$$c=1,\qquad c_{\varphi}=1. \tag{19}$$

Unfortunately, we do not know a method for the exact summation of the series. However, one can present c, c_φ for small ξ as a power series in ξ. Soliton form factors give no contribution to the expansion, being of order $O(e^{-\pi/\xi})$ for $\xi \to 0$. Breather form factors $f^{\pm}(\beta_1 \dots \beta_k)_{m_1 \dots m_k}$ give contributions of order $O(\xi^{m_1+\cdots+m_k})$, so one only needs to use a few breather form factors to calculate c, c_φ up to arbitrary power of ξ. Such calculations were performed up to ξ^4 yielding

$$c = 1 + O(\xi^4), \qquad c_\varphi = 1 + O(\xi^4).$$

These results provide us with good evidence in favour of eq. (19). Note that the calculation equivalent to that of c_φ up to ξ^3 has also been performed in ref. [13].

3. Reductions of the SG model for $\xi = 2\pi/(2n+1)$

Consider the SG model for $\xi = 2\pi/(2n+1)$. The spectrum of the model contains a soliton and n breathers. The poles of the breather form factors are connected with n physical breathers and n virtual breathers with numbers $n+1, \dots, 2n$. One can formally consider the S-matrices describing the scattering of both physical and virtual breathers. These S-matrices possess the important property

$$S_{lm}(\beta) = S_{2n+1-l,m}(\beta),$$

which means that the virtual l-breather ($l > n$) can be "identified" with the physical $(2n+1-l)$-breather.

We want to show that such an identification can be made for breather form factors. Consider eqs. (14) and (15). A direct calculation shows that for $\xi = 2\pi/(2n+1)$ the following equation holds:

$$\frac{\mathscr{S}_{m_1 m_2}(\beta)}{\sinh \frac{2n+1}{2}\left(\beta + \frac{\pi i(m_1+m_2)}{2n+1}\right)} = \frac{\mathscr{S}_{2n+1-m_1, m_2}(\beta)}{\sinh \frac{2n+1}{2}\left(\beta + \frac{\pi i(2n+1-m_1+m_2)}{2n+1}\right)}. \tag{20}$$

Let us evaluate $F_\lambda(\beta_1 \dots \beta_k)_{m_1 \dots m_k}$. First, rewrite the expression for $F_\lambda(\beta_1 \dots \beta_k)_{m_1 \dots m_k}$ in terms of the variables $x_j = e^{\beta_j}$, $t_i = e^{\alpha_i}$:

$$F_\lambda(\beta_1 \dots \beta_k)_{m_1 \dots m_k} = \exp\left(\sum \beta_j(\lambda - k)\right)\left(\frac{1}{2\pi i}\right)^{k-1} \int_{\gamma_{m_1}} dt_1 \dots \int_{\gamma_{m_{k-1}}} dt_{k-1} \varphi_{m_j}(t_i, x_j)$$

$$\times \prod_{i<j}\left(t_i^2 - t_j^2\right) \prod t_i^{1-\lambda},$$

where

$$\varphi_m(e^{\alpha}, e^{\beta}) = e^{-\alpha-\beta}\varphi_m(\alpha-\beta), \qquad w = \exp\left(\frac{2\pi i}{2n+1}\right),$$

and the contour γ_{m_j} envelops the points $w^{m/2-p}b_j$, $p=0,\dots,m_j$. The structure of the poles allows us to use the following trick: divide the integral over t_j by $\prod_{p\neq j}(x_j^{2n+1}(-1)^{m_j} - x_p^{2n+1}(-1)^{m_p})$ and multiply the integrand by $\prod_{p\neq j}(t_i^{2n+1} - (-1)^{m_p}x_p^{2n+1})$. The value of the integral does not change after that. Thus the integral in the r.h.s. can be replaced by

$$\frac{1}{\prod_{j=1}^{k}\prod_{l\neq j}\left(x_j^{2n+1}(-1)^{m_j} - x_l^{2n+1}(-1)^{m_l}\right)}$$

$$\times\left(\frac{1}{2\pi i}\right)^{k-1}\int_{\gamma_{m_1}} dt_1 \dots \int_{\gamma_{m_{k-1}}} dt_{k-1}\prod_{j=1}^{k-1}\varphi_{m_j}(t_j, x_j)$$

$$\times\prod_{j=1}^{k-1}\prod_{l\neq j}\psi_{m_l}(t_j, x_l)\prod_{i<j}\left(t_i^2 - t_j^2\right)\prod t_i^{1-\lambda}, \tag{21}$$

$$\varphi_m(t,x) = y_m(t,x)\left(t^{2n+1} - (-1)^m x^{2n+1}\right)$$

$$= \prod_{j=1}^{m-1}(t + w^{m/2-j}x)\prod_{j=1}^{2n-m}(t - w^{-m/2-j}x).$$

Now the integrand in the integral over t_j is regular at the point $t_j = 0$ and has no singularities except for the poles at the points $b_j w^{m_j/2-p}$. That is why all the contours γ_j can be replaced by the contour γ which envelops the points $b_j w^{m_j/2-p}$ and zero. Rewrite the function $\varphi_{m_j}(t_j, x_j)$ in the form

$$\varphi_{m_j}(t_j, x_j) = \psi_{m_j}(t_j, x_j)\frac{1}{t_j^{2n+1} - (-1)^{m_j}x_j^{2n+1}}. \tag{22}$$

On the contour γ one has the inequality $t_j > x_j$ which allows us to expand the denominator in eq. (22),

$$\frac{1}{t_i^{2n+1} - (-1)^{m_j}x_j^{2n+1}} = \sum_{q=1}^{\infty}(-1)^{qm_j}x_j^{(2n+1)(q-1)}t_j^{-(2n+1)q}. \tag{23}$$

Note that after substitution of eqs. (22) and (23) into the integral (21), the terms with $q > k-1$ can be omitted because the integrand vanishes for $t_j \to \infty$. Then one can antisymmetrize the integrand with respect to $t_1 \dots t_{k-1}$ and obtain

$$\frac{1}{\prod_{i<j}^{k}\left(x_i^{2n+1}(-1)^{m_i} - (-1)^{m_j} x_j^{2n+1}\right)} \left(\frac{1}{2\pi i}\right)^{k-1}$$

$$\times \int_\gamma \mathrm{d}t_1 \dots \int_\gamma \mathrm{d}t_{k-1} \prod \psi(t_i, x_j) \prod_{i<j} \left(t_i^2 - t_j^2\right) t_i^{1-\lambda-(2n+1)i}. \tag{24}$$

Eq. (24) is equivalent to

$$F_\lambda(\beta_1 \dots \beta_k)_{m_1 \dots m_k} = \frac{1}{\prod_{i<j} \sinh \frac{2n+1}{2}\left(\beta_i - \beta_j + \frac{\pi i}{2n+1}(m_i + m_j)\right)}$$

$$\times \left(\frac{1}{2\pi i}\right)^{k-1} \int_\Gamma \mathrm{d}\alpha_1 \dots \int_\Gamma \mathrm{d}\alpha_{k-1} \prod \psi_{m_j}(\alpha_i - \beta_j) \prod_{i<j} \sinh(\alpha_i - \alpha_j)$$

$$\times \exp\left((2n+1) \sum_{i=1}^{k-1} (k-2i)\alpha_i + \lambda\left(\sum \beta_j - \sum \alpha_i\right)\right), \tag{25}$$

where

$$\psi_m(\alpha) = \prod_{j=1}^{m-1} \cosh \tfrac{1}{2}\left(\alpha + \tfrac{1}{2} i\xi(m-2j)\right) \prod_{j=1}^{2n-m} \sinh \tfrac{1}{2}\left(\alpha - \tfrac{1}{2} i\xi(m+2j)\right),$$

and $\Gamma = (\tau, \tau + 2\pi i)$ with τ an arbitrary number.

The functions $\psi_m(\alpha)$ satisfy the important identity

$$\psi_m(\alpha) = \psi_{2n+1-m}(\alpha).$$

This identity together with eq. (25) show that the form factor of a C-even operator (say f^+), containing a virtual $(2n+1-m)$-breather is proportional to the form factor of the corresponding C-odd operator (f^-) containing an m-breather. More precisely, consider the operators

$$B = A + i2^{-5/2} \frac{\sqrt{\gamma}}{\xi} \varphi, \qquad B^* = A - i2^{-5/2} \frac{\sqrt{\gamma}}{\xi} \varphi,$$

$$\exp\left(i \frac{\sqrt{\gamma}\,\varphi}{2}\right), \qquad \exp\left(-\frac{i\sqrt{\gamma}\,\varphi}{2}\right). \tag{26}$$

Their form factors will be denoted by $f, \bar{f}, h, \bar{h}$ respectively. Evidently,

$$f_{m_1\dots m_k} = \begin{cases} f^{+}_{m_1\dots m_k} & \text{if } m_1 + \cdots + m_k \equiv 0 \pmod 2, \\ i2^{-5/2}\dfrac{\sqrt{\gamma}}{\xi} f^{-}_{m_1\dots m_k} & \text{if } m_1 + \cdots + m_k \equiv 1 \pmod 2, \end{cases}$$

etc. Using eq. (25) one obtains

$$f(\beta_1 \dots \beta_k)_{m_1 \dots m_{k-1} 2n+1-m_k} = f(\beta_1 \dots \beta_k)_{m_1 \dots m_k}, \tag{27}$$

$$\bar{f}(\beta_1 \dots \beta_k)_{m_1 \dots m_{k-1} 2n+1-m_k} = -\bar{f}(\beta_1 \dots \beta_k)_{m_1 \dots m_k}. \tag{28}$$

The same equations hold for $h, \bar{h}$.

In sect. 2 it was mentioned that for the proof of local commutativity one has to use eqs. (5)–(8) which guarantee the cancellation of pole contributions. Eqs. (5)–(8) link essentially solitons and breathers. However, for $\xi = 2\pi/(2n+1)$ it can be shown that due to the possibility of identifying physical and virtual breathers, locality of operators whose form factors satisfy eqs. (27) or (28) takes place in the soliton-free sector. This means that if we calculate the matrix element of the commutator $[O(x_0, x_1), O(0,0)]$, $x_\mu^2 < 0$ taken between two soliton-free states and only use soliton-free states as intermediates, it appears to be equal to zero. The pole contributions cancel each other due to eqs. (27) and (28). It is very important to note that the operator whose form factors satisfy eq. (27) appears to be nonlocal with respect to the operator whose form factors satisfy eq. (28). So we have two sets of local but not mutually local operators as was claimed in sect. 1. These two algebras of operators are equivalent. Let us denote by $\mathscr{M}$ the algebra containing operators whose form factors satisfy eq. (27) (the second algebra is equal evidently to $\mathscr{M}^*$). Let us denote by P the projection operator on soliton-free sector. The operator $\mathscr{B} = PBP$ is not local because its two particle form factors have poles. The local operator $\mathscr{T}_{\mu\nu}$ can be obtained from $\mathscr{B}$,

$$\mathscr{T}_{\mu\nu} = \varepsilon_{\mu\mu'}\varepsilon_{\nu\nu'}\partial_{\mu'}\partial_{\nu'}\mathscr{B}.$$

This operator is equal to the energy–momentum tensor of the reduced model (RSG $(2/(2n+1))$). Note that

$$\mathscr{T}_{\mu\mu} = \frac{M_1^2}{4\sin\xi} P \exp\left(i\sqrt{\gamma}\,\varphi\right) P.$$

Obviously, the operator $\mathscr{P}_\mu = \int_{-\infty}^{\infty} \mathscr{T}_{0\mu}(x)\, dx$ is the well-defined energy–momentum:

$$\mathscr{P}_\mu |\beta_1 \dots \beta_k\rangle_{m_1 \dots m_k} = \left(\sum M_{m_j}\left(e^{\beta_i} + (-1)^k e^{-\beta_j}\right)\right)|\beta_1 \dots \beta_k\rangle_{m_1 \dots m_k}.$$

The same can be said about the operator $\mathscr{T}^*_{\mu\nu}$.

The existence of two equivalent but not mutually local RSG $(2/(2n+1))$ models is connected with the violation of C invariance which occurs in the SG model (the 1-breather has negative intrinsic C parity). At the same time SG T invariance is violated. The SG CT invariance is not violated. Does it mean that RSG $(2/(2n+1))$ is not C and T invariant? Strictly speaking not. One can consider 1-breathers in the reduced model as particles with positive intrinsic C parity. Then RSG $(2/(2n+1))$ is C invariant. One can also redefine $a^*_{in,1}(\beta)$ as $ia^*_{in,1}(\beta)$. Then, due to anti-unitarity of T, the theory appears to be T invariant. However, RSG $(2/(2n+1))$ possesses some strange features from the point of view of canonical field theory. The symmetry of the S-matrix is richer than that of off-shell theory: on shell we can consider the 1-breather as a particle with negative C-parity, but then we destroy the locality. These strange features of the theory are connected, in our opinion, with the following circumstance: the operator φ is not local in RSG $(2/(2n+1))$; we can take as interpolating field for the 1-breather the field $\mathscr{T}_{\mu\mu}$. The properties of $\mathscr{T}_{\mu\mu}$ and of the particle it interpolates are different ($\mathscr{T}_{\mu\mu}$ is not self-adjoint and can not be made self-adjoint without loss of locality). So we deal with the interesting phenomenon which is worth considering in frameworks of canonical axiomatic field theory.

Let us consider one of the equivalent RSG $(2/(2n+1))$ theories; namely that which has $\mathscr{T}_{\mu\nu}$ as energy–momentum tensor. The operator $\sigma = P\exp(i\sqrt{\gamma}\,\varphi/2)P$ is local in this theory. Can we point out other local fields? First of all we can try to define the densities of higher conservation laws by their form factors

$$\left(\sum M^{(s)}_{m_j} e^{\pm s\beta_j}\right)\left(\sum M^{(s)}_{m_j} \sinh \beta_j\right) f(\beta_1 \dots \beta_k)_{m_1 \dots m_k} \tag{29}$$

(see eq. (10)). In contrast with SG, where every odd spin is allowed, in RSG $(2/(2n+1))$ we also have the restriction $s \neq 0 \pmod{2n+1}$ because the integrals with spins $(2n+1)P$ are identically equal to zero. Every operator (say σ) has an infinite number of local descendants

$$\partial_{t_{k_1}} \dots \partial_{t_{k_p}} \partial_{\bar{t}_{l_1}} \dots \partial_{\bar{t}_{l_q}} \sigma \qquad \left(k_i, l_i \neq 0 \,(\mathrm{mod}\, 2, 2n+1)\right).$$

Let us consider the problem of the ultraviolet behaviour of the RSG $(2/(2n+1))$ model. We have to calculate the central charge:

$$[\mathscr{T}_{00}(0, x_1), \mathscr{T}_{01}(0,0)] = \frac{c}{24\pi}\delta'''(x_1) + \dots .$$

According to eq. (26) one has

$$c=\left(1-c^{(s)}\right)-3\frac{(2n+1)^2}{2n+3}\left(1-c_{\varphi}^{(s)}\right),$$

where $c^{(s)}$ and $c_{\varphi}^{(s)}$ are soliton contributions to c and c_φ which have disappeared in the reduced model. Soliton contributions are of order $O(e^{-n})$ for $n \to \infty$, hence

$$c=1-3\frac{(2n+1)^2}{2n+3}+o\left(\frac{1}{n^{\infty}}\right). \tag{30}$$

It is natural to suppose that eq. (30), being correct to every order of the $1/n$ expansion, is precise ($c^{(s)}$ and $\xi[(2n+1)^2/(2n+3)]c_{\varphi}^{(s)}$ cancel each other). So we suppose that

$$c=1-6\frac{(2n+1)^2}{2(2n+1)}.$$

This value of the central charge corresponds to the minimal CFT which describes the ultraviolet limit of RSG $(2/(2n+1))$.

At this point an intriguing analogy with refs. [14, 15] arises. In these papers the energy–momentum tensor

$$\tfrac{1}{2}(\partial_\mu\varphi)(\partial_\nu\varphi)+i\frac{1}{2\sqrt{\pi}}\frac{(p-q)}{\sqrt{pq}}\varepsilon_{\mu\mu'}\varepsilon_{\nu\nu'}\partial_{\mu'}\partial_{\nu'}\varphi$$

for the description of $M_{p/q}$ models is introduced (φ is the free massless field). One need also to introduce screening operators [14, 15]. RGS $(2/(2n+1))$ is a massive perturbation of $M_{2/(2n+3)}$; the energy – momentum tensor is expressed in the form (26) through the SG field φ which is free in the ultraviolet limit. Thus restriction of SG to the soliton-free sector should be equivalent to the introduction of screening operators in the ultraviolet limit. It would be very interesting to develop this point in detail.

Let us turn to the anomalous dimensions of operators in the ultraviolet limit. There are n primary fields in $M_{2/(2n+3)}$ with dimensions

$$\Delta_k=-\frac{(k-1)(2n+2-k)}{2(2n+3)}.$$

Usually these fields are denoted as $\phi_{1,k}$. The dimensions of the operators $\exp(i\sqrt{\gamma}\,\varphi)$ and $\exp(i\sqrt{\gamma}\,\varphi/2)$ in the SG model are equal to $2/(2n+3)$ and $1/2(2n+3)$ respectively. The above reasonings show that it is natural to suppose that the dimensions of the operators $\mathscr{T}_{\mu\mu} = (M_1^2/4\sin\xi)P\exp(i\sqrt{\gamma}\,\varphi)P$ and $\sigma = P\exp(i\sqrt{\gamma}\,\varphi/2)P$ can be calculated using the procedure of ref. [15]:

$$\Delta_{\mathscr{T}_{\mu\mu}} = \frac{2}{2n+3} - \sqrt{\frac{2(2n+1)^2}{2(2n+3)(2n+3)}} = -\frac{2n-1}{2n+3},$$

$$\Delta_\sigma = \frac{1}{2(2n+3)} - \sqrt{\frac{(2n+1)^2}{2(2n+3)2(2n+3)}} = -\frac{n}{2n+3}.$$

These dimensions correspond to the fields $\phi_{1,3}$, $\phi_{1,2}$ respectively. Thus RSG $(2/(2n+1))$ is a perturbation of $\mathrm{M}_{2/(2n+3)}$ via the operator $\phi_{1,3}$.

4. The RSG ($\frac{2}{3}$) model

Let us consider in more detail the simplest reduced model RSG ($\frac{2}{3}$). The spectrum of RSG ($\frac{2}{3}$) contains only one particle (1-breather). The S-matrix is equal to

$$S(\beta) = \frac{\sinh\beta + \sinh 2\pi i/3}{\sinh\beta - \sinh 2\pi i/3}.$$

My interest in RSG models arose from considering this S-matrix to which my attention was directed by V. Fateev. The pole $\beta = 2\pi i/3$ corresponds to a virtual 2-breather which is identified with a 1-breather. Note that the residue at this point is negative, i.e. the three 1-breather vertex is imaginary. This is one more amusing feature of reduced models.

Let us rewrite for this particular case eqs. (14), (15) and (25):

$$\begin{aligned}\mathscr{S}_{11}(\beta) &= 2\coth\tfrac{1}{2}\beta\exp\left(4\int_0^\infty \frac{\sin^2\frac{1}{2}(\beta+\pi i)\sinh\frac{1}{6}\pi k\sinh\frac{1}{3}\pi k}{k\sinh\pi k\cosh\frac{1}{2}\pi k}\,\mathrm{d}k\right)\\ &\equiv \tanh\tfrac{1}{2}\beta\mathscr{S}(\beta),\end{aligned}$$

$$c = \exp\left(2\int_0^\infty \frac{\sinh\frac{1}{6}\pi k\sinh\frac{1}{3}\pi k}{k\sinh\pi k\cosh\frac{1}{2}\pi k}\,\mathrm{d}k\right).$$

Eqs. (25) appear to be very effective for RSG ($\frac{2}{3}$):

$$F_\lambda(\beta_1\dots\beta_k) = \frac{1}{\prod_{i<j}\sinh 3(\beta_i-\beta_j)}\exp\left(-\tfrac{1}{2}(k-1-2\lambda)\sum\beta_j\right)P_\lambda(\mathrm{e}^{\beta_1}\dots\mathrm{e}^{\beta_k}),\tag{31}$$

where

$$P_\lambda(x_1 \dots x_k) = \left(\frac{1}{2\pi i}\right)^{k-1} \int_\gamma \mathrm{d}t_1 \dots \int_\gamma \mathrm{d}t_{k-1} \prod (t_i + x_j) \prod_{i<j} (t_i^2 - t_j^2) \prod t_i^{1-\lambda-3i}.$$

The polynomials $P_\lambda(x_1 \dots x_k)$ can be rewritten as the determinant of $(k-1)\times(k-1)$-matrices M_λ with the following matrix elements:

$$(M_\lambda)_{ij} = \sigma_{3i-2j-\lambda}(x_1 \dots x_k), \qquad i, j = 1, \dots, k-1,$$

where σ_l is an elementary symmetric polynomial ($\sigma_l \equiv 0$ if $l < 0$, $l > k$). It is easy to show that

$$\det M_0 = \sigma_1 \sigma_{k-1} \tilde{P},$$

$$\det M_1 = \sigma_1 \sigma_0 \tilde{P},$$

$$\det M_{-1} = \sigma_k \sigma_{k-1} \tilde{P}, \tag{32}$$

where $\tilde{P} = \det \tilde{M}$, $\tilde{M}$ is a $(k-3)\times(k-3)$-matrix with the following matrix elements:

$$\tilde{M}_{ij} = \sigma_{3i-2j+1}(x_1 \dots x_k), \quad i, j = 1, \dots, k-3.$$

Eqs. (31) and (32) mean that for $\xi = 2\pi/3$ the following notable identification holds:

$$P \exp\left(\pm i\sqrt{\gamma}\,\varphi\right) P = P \exp\left(\pm i\sqrt{\gamma}\,\varphi/2\right) P. \tag{33}$$

This identification is very important for the ultraviolet limit. This limit coincides with the theory $\mathrm{M}_{2/5}$, in which the operators $\phi_{1,2}$ and $\phi_{1,3}$ should be identified because $\Delta_2 = \Delta_3$. Eq. (33) carries over this fact to the perturbed model. The $\mathrm{M}_{2/5}$ theory is the simplest CFT because it contains only one primary field. The perturbed theory RSG $(2/(2n+1))$ also should possess unique properties and is worth special investigation.

The form factors $f(\beta_1 \dots \beta_k)$ can be written as follows:

$$\frac{d^k}{4\sin\xi} \prod_{i<j} \mathscr{S}(\beta_i - \beta_j) \frac{\sinh \frac{1}{2}\beta_{ij}}{2\cosh \frac{1}{2}\beta_{ij} \sinh \frac{1}{2}(\beta_{ij} - 2\pi i/3) \sinh \frac{1}{2}(\beta_{ij} - 2\pi i/3)}$$

$$\times \tilde{P}_k(\mathrm{e}^{\beta_1} \dots \mathrm{e}^{\beta_k}) \exp\left(-\frac{n-3}{2} \sum \beta_j\right) = \begin{cases} f(\beta_1 \dots \beta_k), & k \equiv 0 \ (\mathrm{mod}\,2), \\ \frac{1}{i} f(\beta_1 \dots \beta_k), & k \equiv 1 \ (\mathrm{mod}\,2), \end{cases}$$

where

$$d = c^{-1/2} 3^{1/4} 2^{-1} \pi^{-1/2}, \qquad P_1(x) \equiv 1/x, \qquad P_2(x_1 x_2) \equiv 1/(x_1 + x_2).$$

The form factors of the operator $\sigma = (4 \sin \xi / M_1^2) T_{\mu\mu}$ are equal to

$$h(\beta_1 \dots \beta_k) = \left(\sum \mathrm{e}^{\beta_j}\right)\left(\sum \mathrm{e}^{-\beta_j}\right) f(\beta_1 \dots \beta_k).$$

Consider the euclidean Green function

$$\langle 0 | \sigma(z, \bar{z}) \sigma(0,0) | 0 \rangle = \sum_{k=0}^{\infty} \frac{(-1)^k}{\cancel{k!}} \int_{-\infty}^{\infty} H_k(\beta_1 \dots \beta_k) \mathrm{e}^{-\rho \Sigma \cosh \beta_j} \mathrm{d}\beta_1 \dots \mathrm{d}\beta_k, \quad (34)$$

where $\rho = (z\bar{z})^{1/2}$, and

$$H_k = (-1)^k h(\beta_1 \dots \beta_k) h(\beta_k \dots \beta_1), \quad H_k > 0$$

is a symmetric function of its arguments. We believe that the ultraviolet limit of RSG $(\frac{2}{3})$ coincides with $\mathrm{M}_{2/5}$, which implies that

$$\langle 0 | \sigma(z, \bar{z}) \sigma(0,0) | 0 \rangle \underset{\rho \to 0}{\longrightarrow} \rho^{4/5}.$$

It would be very nice to obtain this result directly from eq. (34). A straightforward investigation of the series (34) can not be very fruitful because the kth term behaves as $(\ln \rho)^k$ when $\rho \to 0$, and one has to sum the series of increasing logarithms to decreasing powers. However, the equation

$$H_k(\beta_1 \dots \beta_l, \beta_{k+1} + \Lambda \dots \beta_k + \Lambda) \underset{\Lambda \to \infty}{\longrightarrow} H_l(\beta_1 \dots \beta_l) H_{k-l}(\beta_{l+1} \dots \beta_k) + \mathrm{O}(\mathrm{e}^{-\Lambda}), \quad (35)$$

(see eq. (11)) allows us to apply a variant of the virial expansion as has been claimed in sect. 2. Let us demonstrate the techniques on this particular example.

Let us take the logarithm of the series (34):

$$\ln\langle 0|\sigma(\bar z, z)\sigma(0,0)|0\rangle = -\sum_{k=1}^{\infty}\int \tilde H_k(\beta_1\dots\beta_k)\mathrm{e}^{-\rho\Sigma\cosh\beta_j}\,\mathrm{d}\beta_1\dots\mathrm{d}\beta_k\,,$$

$$\tilde H_k(\beta_1\dots\beta_k) = \sum_{q=1}^{k}\sum_{k=k_1+\cdots+k_q}\frac{(-1)^{k+q}}{q}\frac{1}{k_1!k_2!\dots k_q!}$$

$$\times H_{k_1}(\beta_1\dots\beta_{k_1})H_{k_2}(\beta_{k_1+1}\dots\beta_{k_1+k_2})H_{k_q}(\beta_{k_1+\cdots+k_{q-1}+1}\dots\beta_k)\,.$$

The function $\tilde H_k(\beta_1\dots\beta_k)$ is a symmetric function of its arguments. Due to eq. (35) it has the property

$$\tilde H_k(\beta_1\dots\beta_l,\beta_{l+1}+\Lambda,\dots,\beta_k+\Lambda) \underset{\Lambda\to\infty}{\simeq} \mathrm{O}(\mathrm{e}^{-\Lambda})\,,\qquad l\neq 0, l\neq k\,. \tag{36}$$

This is why the integral

$$\int \tilde H_k(\beta_1\dots\beta_k)\mathrm{e}^{-\rho\Sigma\cosh\beta_j}\,\mathrm{d}\beta_1\dots\mathrm{d}\beta_k$$

for $\rho\to 0$ behaves as

$$2\ln\rho\int\tilde H_k(0,\beta_2\dots\beta_k)\,\mathrm{d}\beta_2\dots\mathrm{d}\beta_k\,.$$

The integral is convergent due to eq. (36). Thus

$$\ln\langle 0|\sigma(z,\bar z)\sigma(0,0)|0\rangle \sim -4\Delta\ln\rho\,,$$

where

$$\Delta = \tfrac{1}{2}\sum_{k=1}^{\infty}\int\mathrm{d}\beta_2\dots\mathrm{d}\beta_k\,\tilde H_k(0,\beta_2\dots\beta_k)\,.$$

It can be shown that $\tilde H_i > 0$, $\forall i$, so we have a series composed of positive terms, and one can check the equation $\Delta = -\frac{1}{5}$ by computer calculations.

5. Conclusions

The RSG $(2/(2n+1))$ models described in this paper present an example of a very interesting phenomenon, which can be regarded as a kind of confinement. Actually, we extract from SG theory, which contains solitons (fermions), the subsector which does not contain solitons but only their bound states (bosons), preserving locality of the theory.

Let us discuss further prospects of the methods developed in this paper. The equations

$$T_{\mu\mu} = \frac{M_1^2}{4\sin\xi}\cos\sqrt{\gamma}\,\varphi, \qquad \Box\,\varphi = \frac{M_1^2\sqrt{2}}{\sqrt{\gamma}\,\sin\xi}\xi\sin\sqrt{\gamma}\,\varphi$$

are valid for arbitrary ξ. Let us consider the tensor

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + i2^{-2/5}\frac{\sqrt{\gamma}}{\xi}\varepsilon_{\mu\mu'}\varepsilon_{\nu\nu'}\partial_{\mu'}\partial_{\nu'}\varphi\,.$$

The central charge of the tensor $\tilde{T}_{\mu\nu}$ is equal to

$$c = 1 - 6\pi\frac{\gamma}{8\xi^2} = 1 - 6\left(\frac{\gamma}{8\pi} + \frac{8\pi}{\gamma} - 2\right),$$

where $\xi = \gamma\pi/(8\pi - \gamma)$. If ξ/π is rational ($\xi/\pi = p/(q-p)$), then

$$c = 1 - 6\frac{(p-q)^2}{pq},$$

i.e. it coincides with the central charge of the model $\mathrm{M}_{p/q}$. At the same time

$$\tilde{T}_{\mu\mu} = \frac{M_1^2}{4\sin\xi}\exp\left(i\sqrt{\gamma}\,\varphi\right).$$

In the ultraviolet model one can introduce screening operators and restrict the free model to the model $\mathrm{M}_{p/q}$. The dimension of $\exp(i\sqrt{\gamma}\,\varphi)$ then becomes $(2p-q)/q$, i.e. $\exp(i\sqrt{\gamma}\,\varphi)$ leads to the primary field $\phi_{1,3}$.

It is natural to suppose that for every rational $\xi = p/(q-p)$ one can construct the reduction of the SG model which describes the perturbation of $\mathrm{M}_{p/q}$ via the operator $\phi_{1,3}$. The reduced models RSG $(2/(2n+3))$ considered in this paper should be the simplest example of the phenomenon. In general, we should be able to construct the models RSG $(p/(q-p))$ for arbitrary p, q.

What kind of limitations should be imposed on the spectrum of SG to get RSG $(p/(q-p))$? The case $p = 2$, $q = 2n+3$ seems to be unique: only in that case solitons can be completely omitted. Usually, only some limitations on soliton degrees of freedom should be imposed. Bazhanov and Reshetikhin claimed that it is possible to use the RSOS [16] restriction of the SG soliton S-matrix as the physical S-matrix of some relativistic model (private communication). It seems that this limitation is just what we need. The RSG models are of special interest because they describe the perturbation of "positive" models M_p considered in ref. [1].

From the point of view of statistical mechanics the SG model is the scaling limit of Baxter's 8-vertices model [17]. The critical 8-vertices model has central charge equal to 1 (evidently, it coincides with the ultraviolet limit of SG). So we have a diagram

$$\text{8-vertices model} \xrightarrow{\text{scaling}} \text{SG} \xrightarrow{\rho \to 0} \text{massless free field}.$$

For a rational coupling constant one can restrict the 8-vertices model to obtain the RSOS model [16]. We can also restrict the $c = 1$, model to get $\mathrm{M}_{p/q}$. Our statement is that we should be able to restrict also the SG model to obtain RSG $(p/(q-p))$. Thus we should be able to obtain the following restricted diagram:

$$\text{RSOS} \xrightarrow{\text{scaling}} \text{RSG}(p/(q-p)) \xrightarrow{\rho \to 0} \mathrm{M}_{p/q}.$$

Of course this diagram needs further development.

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