LITERATURE CITED

1. A. Galperin, E. Ivanov, S. Kalitzkin, V. Ogievetsky, and E. Sokatchev, "Unconstrained N = 2 matter and supergravity theories in harmonic superspace," Preprint IC/84/43, ICTP, Trieste (1984); Class. Quantum Grav., 1, 469 (1984).
2. A. Galperin, E. Ivanov, S. Kalitzkin, V. Ogievetsky, and E. Sokatchev, "Unconstrained off-shell N = 3 supersymmetric Yang–Mills theory," Preprint E2-84-441 [in English], JINR, Dubna (1984).
3. S. Ivanov, S. Kalitzkin, N. Ai Veit, and V. Ogievetsky, J. Phys. A, 18, 3433 (1985).
4. A. Galperin, E. Ivanov, V. Ogievetsky, and E. Sokatchev, "Conformal invariance in harmonic superspace," Preprint E2-85-363 [in English], JINR, Dubna (1985).
5. A. Galperin, I. Ivanov, V. Ogievetsky, and E. Sokatchev, "Hyper-Kähler metric and harmonic superspace," Preprint E2-85-514 [in English], JINR, Dubna (1985).
6. A. Galperin, E. Ivanov, V. Ogievetsky, and P. K. Townsend, "Eguchi–Hanson type metric from harmonic superspace," Preprint E2-85-732 [in English], JINR, Dubna (1985).
7. E. Sokatchev, Phys. Lett. B, 169, 209 (1986).
8. B. M. Zupnik, Yad. Fiz., 44, 793 (1986).
9. R. Kallosh, "Superstrings and harmonic superspaces," Preprint TH. 4188/85, CERN, Geneva (1985).
10. A. Galperin, E. Ivanov, V. Ogievetsky, and E. Sokatchev, "Harmonic supergraphs, Green's functions," Preprint E2-85-127 [in English], JINR, Dubna (1985).
11. A. Galperin, E. Ivanov, V. Ogievetsky, and E. Sokatchev, "Harmonic supergraphs. Feynman rules and examples," Preprint E2-85-128 [in English], JINR, Dubna (1985).
12. M. Sohnius, Nucl. Phys. B, 136, 461 (1978).
13. J. P. Serre, Lie Algebras and Lie Groups, Benjamin (1965).
14. A. O. Barut and R. Raczka, Theory of Group Representations and Applications, Warsaw (1977).
15. E. Witten, Phys. Lett. B, 77, 394 (1987).
16. A. Rosly, in: Proc. International Seminar on Group Theoretical Methods in Physics, Vol. 1, Nauka, Moscow (1982), p. 263.

ALGEBRAIC BETHE ANSATZ FOR THE IZERGIN–KOREPIN R MATRIX

V. O. Tarasov

A generalization of the algebraic Bethe ansatz to the case of the Izergin–Korepin R matrix is proposed. For this R matrix the simplest L operators are calculated.

Introduction

In the quantum inverse scattering method the algebraic Bethe ansatz and its vector and matrix generalizations [1-6] play an important part. Hitherto it has been the main, although not the only, way of constructing simultaneous eigenvectors for a complete set of commuting integrals of the motion. A necessary condition for applicability of the algebraic Bethe ansatz is that the monodromy matrix of the corresponding quantum model should have a highest vector (vacuum or generating state). It is evident that this condition must also be sufficient, but at the present time the proof of the corresponding assertion is far from complete. In fact, there exist solutions of the Yang–Baxter equation for which monodromy matrices with highest vector have been constructed, but a suitable generalization of the algebraic Bethe ansatz is unknown. This applies particularly to solutions of odd dimension.

In the present paper we propose a generalization of the algebraic Bethe ansatz for the Izergin–Korepin R matrix — the simplest unstudied odd-dimensional solution of the Yang–Baxter equation — and we discuss some related questions. The first section of the paper is an introduction. In the second we indicate a way of generalizing the algebraic Bethe ansatz to the case of the Izergin–Korepin R matrix. The simplest monodromy matrices

(L operators) for this R matrix are described in the third section. The fourth section is devoted to the proof of the proposed generalization of the algebraic Bethe ansatz.

1. R. Matrices and Monodromy Matrices

We recall some basic relations of the quantum inverse scattering method. For us the basic object will be the R matrix $R(\lambda)$, the solution of the Yang–Baxter equation

$$(I \otimes R(\lambda-\mu))(R(\lambda) \otimes I)(I \otimes R(\mu)) = (R(\mu) \otimes I)(I \otimes R(\lambda))(R(\lambda-\mu) \otimes I), \qquad (1.1)$$

where I is the unit matrix in the linear space V, and $R(\lambda)$ is a matrix in the space $V \otimes V$, and it satisfies

$$R(\lambda)|_{\lambda=0} = I \otimes I. \qquad (1.2)$$

From Eqs. (1.1) and (1.2) there follow the unitarity relation

$$R(\lambda)R(-\lambda) = I \otimes I. \qquad (1.3)$$

A quantum integrable system is characterized by monodromy matrix $T(\lambda)$ satisfying the equation

$$R(\lambda-\mu)T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda)R(\lambda-\mu). \qquad (1.4)$$

At the same time, the relation (1.1) guarantees consistency of Eq. (1.4); $T(\lambda)$ is a matrix in the space V with matrix elements that are operators on the state space of the quantum system (the quantum space). The space V is called the auxiliary space of $T(\lambda)$. An example of a monodromy matrix is the matrix $PR(\lambda)$ (P is the matrix of transposition in $V \otimes V$), this following directly from (1.1); here, the quantum space will also be the space V.

The trace of the monodromy matrix in the space V, tr $T(\lambda)$, is the generating function of the family of commuting operators in terms of which the Hamiltonian of the quantum system is expressed.

We note the possibility of multiplying monodromy matrices as matrices in the space V. If $T_1(\lambda)$ and $T_2(\lambda)$ are solutions of the relation (1.4) with the same R matrix and quantum spaces $\mathfrak{H}_1, \mathfrak{H}_2$, respectively, then $T_1(\lambda)T_2(\lambda)$ is also a solution of (1.4) with the same R matrix and quantum space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. The simplest monodromies have become known as L operators. We shall call the monodromy matrix $PR(\lambda)$ the fundamental L operator. The vector $|0\rangle$ in the quantum space of the monodromy matrix $T(\lambda)$ that is annihilated by the operators $T_{ij}(\lambda)$ for i > j and is an eigenvector for the operators $T_{ii}(\lambda)$ is called a highest vector of the monodromy matrix $T(\lambda)$.

A detailed exposition of the quantum inverse scattering method can be found in the reviews [1,2,7,8].

2. The Izergin–Korepin R Matrix

This solution of the Yang–Baxter equation was found in [9]. It has the form

$$R(\lambda) = \frac{1}{c(\lambda)} \begin{pmatrix} c(\lambda) & & & & & & & & \\ & \bar{e}(\lambda) & & b(\lambda) & & & & & \\ & & \bar{f}(\lambda) & & \bar{g}(\lambda) & & d(\lambda) & & \\ \hline & b(\lambda) & & e(\lambda) & & & & & \\ & & \bar{g}(\lambda) & & a(\lambda) & & g(\lambda) & & \\ & & & & & \bar{e}(\lambda) & & b(\lambda) & \\ \hline & & d(\lambda) & & g(\lambda) & & f(\lambda) & & \\ & & & & & b(\lambda) & & e(\lambda) & \\ & & & & & & & & c(\lambda) \end{pmatrix}, \qquad (2.1)$$

where

$$a(\lambda) = \operatorname{sh}(\lambda-3\eta) + \operatorname{sh} 3\eta - \operatorname{sh} 5\eta + \operatorname{sh} \eta, \quad b(\lambda) = \operatorname{sh}(\lambda-3\eta) + \operatorname{sh} 3\eta,$$

$$c(\lambda) = \operatorname{sh}(\lambda-5\eta) + \operatorname{sh} \eta, \quad d(\lambda) = \operatorname{sh}(\lambda-\eta) + \operatorname{sh} \eta,$$

$$e(\lambda) = -\operatorname{sh} 2\eta(e^{-\lambda+3\eta} + e^{-3\eta}), \quad \bar{e}(\lambda) = -\operatorname{sh} 2\eta(e^{\lambda-3\eta} + e^{3\eta}), \qquad (2.2)$$

$$f(\lambda) = -2e^{-\lambda+2\eta} \operatorname{sh} \eta \operatorname{sh} 2\eta - e^{-\eta} \operatorname{sh} 4\eta, \quad \bar{f}(\lambda) = 2e^{\lambda-2\eta} \operatorname{sh} \eta \operatorname{sh} 2\eta - e^{\eta} \operatorname{sh} 4\eta,$$

$$g(\lambda) = e^{2\eta} \operatorname{sh} 2\eta (1 - e^{-\lambda}), \quad \bar{g}(\lambda) = e^{-2\eta} \operatorname{sh} 2\eta (1 - e^{\lambda}),$$

and the omitted matrix elements are zero.

Let

$$T(\lambda) = \begin{pmatrix} A_1(\lambda) & B_1(\lambda) & B_2(\lambda) \\ C_1(\lambda) & A_2(\lambda) & B_3(\lambda) \\ C_2(\lambda) & C_3(\lambda) & A_3(\lambda) \end{pmatrix} \tag{2.3}$$

be the monodromy matrix corresponding to it with highest vector $|0\rangle$:

$$C_i(\lambda)|0\rangle = 0, \quad A_i(\lambda)|0\rangle = \alpha_i(\lambda)|0\rangle, \quad i = 1, 2, 3. \tag{2.4}$$

It follows from (1.1) and the structure of the R matrix that the complex-valued functions $\alpha_i(\lambda)$ must satisfy the relation

$$\alpha_1(\lambda)\alpha_3(\lambda + 2\eta - i\pi) = d_2(\lambda)\alpha_2(\lambda + 2\eta - i\pi). \tag{2.5}$$

We denote

$$\omega(\lambda) = \frac{c(\lambda)d(\lambda)}{a(\lambda)d(\lambda) - g(\lambda)\bar{g}(\lambda)}, \quad y(\lambda) = \frac{d(\lambda)}{\bar{g}(\lambda)}, \quad z(\lambda) = \frac{c(\lambda)}{b(\lambda)},$$

where

$$\omega(\lambda)\omega(-\lambda) = 1. \tag{2.6}$$

We consider the vector-valued functions $|\Phi_n(\lambda_1, \ldots, \lambda_n)\rangle$ defined by the recursion relation

$$|\Phi_n(\lambda_1, \ldots, \lambda_n)\rangle = B_1(\lambda_1)|\Phi_{n-1}(\lambda_2, \ldots, \lambda_n)\rangle -$$

$$B_2(\lambda_1) \sum_{j=2}^{n} \frac{\alpha_1(\lambda_j)}{y(\lambda_1 - \lambda_j)} \prod_{k=2}^{j-1} \omega(\lambda_j - \lambda_k) \prod_{\substack{k=2 \\ k \neq j}}^{n} z(\lambda_k - \lambda_j) |\Phi_{n-2}(\lambda_2, \ldots, \hat{\lambda}_j, \ldots, \lambda_n)\rangle \tag{2.7}$$

and the initial conditions

$$|\Phi_0\rangle = |0\rangle, \quad |\Phi_1(\lambda)\rangle = B_1(\lambda)|0\rangle. \tag{2.8}$$

If the numbers $\mu_1, \ldots, \mu_n$ satisfy the system of equations

$$\frac{\alpha_1(\mu_j)}{\alpha_2(\mu_j)} = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{z(\mu_j - \mu_k)}{z(\mu_k - \mu_j)} \omega(\mu_k - \mu_j), \tag{2.9}$$

then $|\Phi_n(\mu_1, \ldots, \mu_n)\rangle$ is an eigenvector of the operator $\operatorname{tr} T(\lambda) = A_1(\lambda) + A_2(\lambda) + A_3(\lambda)$ with eigenvalue

$$\Lambda(\lambda; \mu_1, \ldots, \mu_n) = \alpha_1(\lambda) \prod_{k=1}^{n} z(\mu_k - \lambda) + \alpha_2(\lambda) \prod_{k=1}^{n} z(\lambda - \mu_k)\omega(\mu_k - \lambda) + \alpha_3(\lambda) \prod_{k=1}^{n} \frac{b(1 - \mu_k)}{d(\lambda - \mu_k)}. \tag{2.10}$$

As was to be expected, the order of the numbers $\mu_1, \ldots, \mu_n$ is unimportant, since

$$|\Phi_n(\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n)\rangle = \omega(\mu_i - \mu_{i+1})|\Phi_n(\mu_1, \ldots, \mu_n)\rangle. \tag{2.11}$$

If the monodromy matrix $T(\lambda)$ is the product of N fundamental L operators, then $\alpha_1(\lambda) = c^N(\lambda)$, $\alpha_2(\lambda) = b^N(\lambda)$, $\alpha_3(\lambda) = d^N(\lambda)$, and formulas (2.9) and (2.10) reproduce the results of [10] for the eigenvalues of the trace of a monodromy matrix as obtained by the method of the analytic Bethe ansatz.

All the proofs are given in Sec. 4.

3. L Operators for the Izergin-Korepin R Matrix

The elementary L operator for given R matrix can be expressed in the form

$$L(\lambda) = \begin{pmatrix} A_{11}e^{\lambda} + A_{12} + A_{13}e^{-\lambda}, & B_{12} + B_{13}e^{-\lambda}, & B_{22} + B_{23}e^{-\lambda} \\ C_{11}e^{\lambda} + C_{12}, & A_{21}e^{\lambda} + A_{22} + A_{23}e^{-\lambda}, & B_{32} + B_{33}e^{-\lambda} \\ C_{21}e^{\lambda} + C_{22}, & C_{31}e^{\lambda} + C_{32}, & A_{31}e^{\lambda} + A_{32} + A_{33}e^{-\lambda} \end{pmatrix},$$

where A_{ij}, B_{ij}, C_{ij} are constant vectors. To obtain for them explicit formulas, we consider two algebras $\mathscr{A}_1$, $\mathscr{A}_2$, generated by the generators $t_i^{\pm}$, p_i, $i = 1, 2$, and the relations

$$p_1 t_1^{\pm} = t_1^{\pm}(p_1 \pm 1), \qquad t_1^{+} t_1^{-} = 2\,\text{sh}\,4\eta\,(p_1 - 1), \qquad t_1^{-} t_1^{+} = 2\,\text{sh}\,4\eta p_1, \tag{3.1}$$

$$p_2 t_2^{\pm} = t_2^{\pm}(p_2 \pm 1), \qquad [t_2^{+}, t_2^{-}] = \text{sh}\,\eta\,\text{sh}\,2\eta p_2, \tag{3.2}$$

respectively. The elements of the different algebras commute with one another. Such relations have already arisen often in the framework of the quantum inverse scattering method, beginning with [11,12].

Let $\mathbf{C}_2$ be a Casimir operator of the algebra $\mathscr{A}_2$:

$$\mathbf{C}_2 = {}^1\!/_2 (t_2^{+} t_2^{-} + t_2^{-} t_2^{+} + \text{ch}\,\eta\,\text{ch}\,2\eta p_2) = t_2^{-} t_2^{+} + {}^1\!/_2\,\text{ch}(2\eta p_2 + \eta).$$

We introduce the notation

$$\xi_1^{+} = t_1^{+}, \quad \xi_2^{+} = \frac{2\,\text{sh}\,4\eta\,(p_1 + p_2 - 1)}{\text{sh}\,2\eta\,(p_2 - 1)}\,t_2^{+}, \quad \xi_1^{-} = t_1^{-}\,\frac{1}{2\,\text{sh}\,4\eta\,(p_1 + p_2 - 1)}, \quad \xi_2^{-} = t_2^{-}\,\frac{1}{\text{sh}\,2\eta p_2}.$$

Equation (1.4) for $L(\lambda)$ leads to the following expressions for the operators $A_k(\lambda), B_k(\lambda), C_k(\lambda)$:

$$A_1(\lambda) = 2\,\text{sh}\,(\lambda + 2\eta - 4\eta p_1 - 2\eta p_2) - 2\,\frac{\text{sh}\,(4\eta p_1 + 2\eta p_2 - 2\eta)}{\text{sh}\,2\eta\,(p_2 - 1)\,\text{sh}\,2\eta\,(p_2 + 1)}\,(\text{ch}\,\eta\,\text{ch}\,2\eta p_2 - \mathbf{C}_2\,\text{ch}\,2\eta) +$$

$$\xi_1^{-} e^{2\eta(p_2 - 1)} \xi_2^{+2} + \xi_2^{-2} e^{-2\eta(p_2 - 1)} \xi_1^{+}, \quad A_2(\lambda) = -2i(\text{ch}\,\lambda + 2\,\text{ch}\,\eta\,\text{ch}\,2\eta p_2 - \mathbf{C}_2),$$

$$A_3(\lambda) = -2\,\text{sh}\,(\lambda + 2\eta - 4\eta p_1 - 2\eta p_2) - 2\,\frac{\text{sh}\,(4\eta p_1 + 2\eta p_2 - 2\eta)}{\text{sh}\,2\eta\,(p_2 - 1)\,\text{sh}\,2\eta\,(p_2 + 1)}\,(\text{ch}\,\eta\,\text{ch}\,2\eta p_2 - \mathbf{C}_2\,\text{ch}\,2\eta) +$$

$$\xi_1^{-} e^{-2\eta(p_2 - 1)} \xi_2^{+2} + \xi_2^{-2} e^{2\eta(p_2 - 1)} \xi_1^{+}, \quad B_1(\lambda) = \xi_1^{-}\,(e^{2\eta p_2 - \eta} + e^{-\lambda})\,\xi_2^{+} + \xi_2^{-}\,(e^{-2\eta p_2 + \eta} + e^{-\lambda}),$$

$$B_2(\lambda) = \xi_1^{-2}\,\text{sh}\,4\eta\,(p_1 + p_2 - 1)\,e^{-4\eta p_1 - 2\eta p_2 + 4\eta} - (\xi_1^{-}\xi_2^{+} + \xi_2^{-})^2\,e^{-4\eta p_1 - 2\eta p_2 + 2\eta}\,\frac{e^{-\lambda}}{2\,\text{ch}\,\eta},$$

$$B_3(\lambda) = i(\xi_1^{-}\,(e^{-2\eta p_2 + \eta} + e^{-\lambda})\,\xi_2^{+} + \xi_2^{-}\,(e^{2\eta p_2 - \eta} + e^{-\lambda}))\,e^{-4\eta p_1 - 2\eta p_2 + 3\eta},$$

$$C_1(\lambda) = -i(\xi_2^{-}\,(e^{\lambda} + e^{-2\eta p_2 + \eta})\,\xi_1^{+} + (e^{\lambda} + e^{2\eta p_2 - \eta})\,\xi_2^{+}),$$

$$C_2(\lambda) = e^{4\eta p_1 + 2\eta p_2 - 4\eta}\,2\,\text{sh}\,4\eta\,(p_1 + p_2 - 1)\,\xi_1^{+} - \frac{e^{\lambda}}{2\,\text{ch}\,\eta}\,e^{4\eta p_1 + 2\eta p_2 - 2\eta}(\xi_2^{-}\xi_1^{+} + \xi_2^{+})^2,$$

$$C_3(\lambda) = -e^{4\eta p_1 + 2\eta p_2 - 3\eta}(\xi_2^{-}\,(e^{\lambda} + e^{2\eta p_2 - \eta})\,\xi_1^{+} + (e^{\lambda} + e^{-2\eta p_2 + \eta})\,\xi_2^{+}). \tag{3.3}$$

To obtain an L operator with highest vector, we consider the Verma modulus $\mathfrak{H}$ of the algebra $\mathscr{A}_1 \otimes \mathscr{A}_2$ with highest vector $|0\rangle$:

$$t_1^{+}|0\rangle = t_2^{+}|0\rangle = p_1|0\rangle = 0, \qquad 2\eta p_2|0\rangle = \Lambda|0\rangle. \tag{3.4}$$

In what follows, we shall assume that η is not a rational multiple of $i\pi$. In the generic situation the representation of the operators (3.6) in the space $\mathfrak{H}$ is irreducible. There are also three series of special values of the parameter Λ:

$$\Lambda = 2\eta m + \frac{i\pi}{2} \pmod{i\pi}, \quad m = 1, 2, \ldots, \tag{3.5a}$$

$$\Lambda = 2\eta m - \eta \pmod{i\pi}, \quad m = 1, 2, \ldots, \tag{3.5b}$$

$$\Lambda = 2\eta m \pmod{i\pi}, \quad m = 1, 2, \ldots. \tag{3.5c}$$

In the cases (3.5a) and (3.5b) the representation of the operators (3.3) becomes reducible in the space $\mathfrak{H}$. An irreducible representation is obtained if $\mathfrak{H}$ is factorized with respect to their maximal invariant eigenspace, which is the linear hull of vectors of the type $(t_1^{-})^k (t_2^{-})^{\ell}|0\rangle$, the relation $k + \ell \geq m$ holding in the case (3.5a) and $\ell \geq m + 1$ in the case (3.5b). In the case (3.5a) the quantum space of the obtained L operator has dimension $\frac{1}{2}m(m + 1)$; for $m = 2$, $\Lambda = 4\eta + i\pi/2$ we reproduce the fundamental L operator: $PR(\lambda) = {}^1\!/_2 L(\lambda - 3\eta + i\pi/2)$. In the case (3.5b) the quantum space is the tensor product of the Verma modulus of the algebra $\mathscr{A}_1$ and the $2m$-dimensional space of an irreducible representation of the algebra $\mathscr{A}_2$. For $m = 1$, $\Lambda = \eta$ one can propose simpler formulas for the irreducible L operator with highest weight $|0\rangle$:

$$L(\lambda) = (1 + e^{-\lambda})\tilde{L}(\lambda), \quad \tilde{A}_1(\lambda) = e^{\lambda - 2\eta p + \eta} - e^{2\eta p - \eta}, \quad \tilde{A}_2(\lambda) = -i(e^{\lambda} + 1), \quad \tilde{A}_3(\lambda) = -e^{\lambda + 2\eta p - \eta} + e^{-2\eta p + \eta},$$

$$\tilde{B}_1(\lambda)=t^-, \qquad \tilde{B}_2(\lambda)=(t^-)^2 e^{-2\eta p+3\eta i}/2\operatorname{ch}\eta, \qquad \tilde{B}_3(\lambda)=it^-e^{-2\eta p+2\eta}, \qquad \tilde{C}_1(\lambda)=e^\lambda it^+,$$

$$\tilde{C}_2(\lambda)=e^\lambda \frac{i}{2\operatorname{ch}\eta} e^{2\eta p-3\eta}(t^+)^2, \qquad \tilde{C}_3(\lambda)=e^\lambda e^{2\eta p-2\eta}t^+,$$

and $t^\pm$ and p satisfy the commutation relations (3.1) and (3.4) for the algebra $\mathscr{A}_1$.

In the case (3.5c) singularities appear in formulas (3.3). However, there exists a similarity transformation in the quantum space that is scalar in the auxiliary space, $\hat{L}(\lambda;\Lambda)=U(\Lambda)L(\lambda;\Lambda)U^{-1}(\Lambda)$, such that $\hat{L}(\lambda;\Lambda)$ no longer has singularities and is irreducible for $\Lambda = 2\eta m$, $m \geqq 1$.

4. Algebraic Bethe Ansatz for the Izergin—Korepin

R Matrix

We turn to the proof of the assertions formulated in the second section. Let $T(\lambda)$ be a monodromy matrix of the form (2.3) satisfying the relation (1.4) with the R matrix (2.1). We consider the algebra $\mathscr{P}$ of polynomials of the matrix elements of $T(\lambda)$. Let

$$\mathscr{P}=\overset{\infty}{\underset{n=0}{\oplus}}\mathscr{P}_n \tag{4.1}$$

be its decomposition into a sum of spaces of polynomials of fixed degree. An element of the term $\mathscr{P}_n$ has the form

$$\sum_\pi \sum_{k=1}^n \sum_{i_k=1}^3 \sum_{j_k=1}^3 f^{(\pi)}_{i_1,\ldots,i_n;j_1,\ldots,j_n}(e^{\lambda_1},\ldots,e^{\lambda_n})T_{i_1 j_1}(\lambda_{\pi_1})\cdot\ldots\cdot T_{i_n j_n}(\lambda_{\pi_n}), \tag{4.2}$$

where π is a permutation of the numbers $1, \ldots, n$, and $f^{(\pi)}_{ij}$ is a rational function of its arguments.

We denote by $\mathscr{B}$ the subspace of polynomials of the elements $A_1(\lambda)$, $B_1(\lambda)$, $B_2(\lambda)$; for an R matrix of the form (2.1) it is a subalgebra.

We say that the monomial $T_{i_1 j_1}(\lambda_1)\cdot\ldots\cdot T_{i_n j_n}(\lambda_n)$ has the standard ordering. It follows from the relation (1.4), expressed in the form

$$R(\lambda-\mu)T(\lambda)\otimes T(\mu)R^{-1}(\lambda-\mu)=T(\mu)\otimes T(\lambda), \tag{4.3}$$

and the Yang—Baxter equation for $R(\lambda)$ that any element of the algebra $\mathscr{P}$ can be reduced in a unique manner to a form with standard ordering. Thus, the monomials of degree n with standard ordering form a basis of the space $\mathscr{P}_n$.

We shall say that a monomial is normally ordered if in it all elements of the type $B_i(\lambda)$ are on the left, and all elements of the type $C_i(\lambda)$ on the right of all elements of the type $A_i(\lambda)$, the elements of one given type having standard ordering. It is readily verified that in the space $\mathscr{P}_n$ there is the same number of monomials with standard ordering and normal ordering. Reducing the normally ordered monomials to ones with standard ordering and using Eq. (1.2), we can prove that they are linearly independent. Thus, we establish that the normally ordered monomials of degree n also form a basis of the space $\mathscr{P}_n$, and any element $\Phi\in\mathscr{P}$ can be uniquely reduced to a normally ordered form, which will be denoted by $:\Phi:$.

We also introduce one further characteristic of a monomial — the order (ord). The order of a monomial is equal to the sum of the orders of the factors, which we define as follows:

$$\operatorname{ord}A_i(\lambda)=0, \qquad \operatorname{ord}B_1(\lambda)=\operatorname{ord}B_3(\lambda)=1,$$

$$\operatorname{ord}B_2(\lambda)=2, \qquad \operatorname{ord}C_1(\lambda)=\operatorname{ord}C_3(\lambda)=-1, \qquad \operatorname{ord}C_2(\lambda)=-2. \tag{4.4}$$

It follows from the form of the R matrix (2.1) that polynomials of fixed order form a subspace $\mathcal{O}_n$ in the algebra $\mathscr{P}$ and

$$\mathscr{P}=\overset{\infty}{\underset{n=0}{\oplus}}\mathcal{O}_n. \tag{4.5}$$

LEMMA 1. There exists a unique element $\Phi_n\in\mathscr{B}$ with the following properties:

Property 1: $\Phi_n\in\mathscr{P}_n\cap\mathcal{O}_n$.

Property 2:

$$\Phi_n(\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\lambda_i,\lambda_{i+2},\ldots,\lambda_n)=\omega(\lambda_i-\lambda_{i+1})\Phi_n(\lambda_i,\ldots,\lambda_n). \tag{4.6}$$

Property 3: $\vdots\Phi_n(\lambda_1,\ldots,\lambda_n)\vdots$ does not contain $B_2(\lambda_n)$.

Property 4: $\vdots\Phi_n(\lambda_1,\ldots,\lambda_n)\vdots$ contains the monomial $B_1(\lambda_1)\cdot\ldots\cdot B_1(\lambda_n)$ with coefficient unity.

LEMMA 2. The element Φ_n satisfies the recursion relations

$$\Phi_n(\lambda_1,\ldots,\lambda_n)=B_1(\lambda_1)\Phi_{n-1}(\lambda_2,\ldots,\lambda_n)-B_2(\lambda_1)\sum_{j=2}^{n}\frac{1}{y(\lambda_1-\lambda_j)}\prod_{k=2}^{j-1}\omega(\lambda_j-\lambda_k)\prod_{\substack{k=2\\k\neq j}}^{n}z(\lambda_k-$$

$$\lambda_j)\Phi_{n-2}(\lambda_2,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)A_1(\lambda_j), \tag{4.7}$$

with the initial conditions $\Phi_0=1$, $\Phi_1(\lambda)=B_1(\lambda)$.

LEMMA 3. We consider an element $\Phi_{mn}\in\mathscr{B}$ with the following properties:

Property 1: $\Phi_{mn}\in\mathscr{P}_m\cap\mathscr{O}_n$.

Property 2:

$$\Phi_{mn}(\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\lambda_i,\lambda_{i+2},\ldots,\lambda_m)=\omega(\lambda_i-\lambda_{i+1})\Phi_{mn}(\lambda_1,\ldots,\lambda_m). \tag{4.8}$$

Property 3: $\vdots\Phi_{mn}(\lambda_1,\ldots,\lambda_m)\vdots$ does not contain $B_2(\lambda_m)$. Then: a) if $m < n$, then $\Phi_{mn} = 0$; b) if $m \geq n$, then Φ_{mn} can be represented in the form

$$\Phi_{mn}(\lambda_1,\ldots,\lambda_m)=\sum_{\sigma}g(\lambda_{\sigma_1},\ldots,\lambda_{\sigma_{m-n}}|\lambda_1,\ldots,\hat{\lambda}_{\sigma_1},\ldots,\hat{\lambda}_{\sigma_{m-n}},$$

$$\ldots,\lambda_m)\prod_{a=1}^{m-n}\prod_{\substack{k=1\\k\neq\sigma_1,\ldots,\sigma_{a-1}}}^{\sigma_a-1}\omega(\lambda_{\sigma_a}-\lambda_k)\Phi_n(\lambda_1,\ldots,\hat{\lambda}_{\sigma_1},\ldots,\hat{\lambda}_{\sigma_{m-n}},\ldots,\lambda_m)A_1(A_{\sigma_1})\cdot\ldots\cdot A_1(\lambda_{\sigma_{m-n}}), \tag{4.9}$$

where $\sigma=\{\sigma_i\in\{1,\ldots,m\}\}_{i=1}^m$, $\sigma_1<\sigma_2<\ldots<\sigma_{m-n}$, and $g(\lambda_1,\ldots,\lambda_{m-n}|\lambda_{m-n+1},\ldots,\lambda_m)$ is a symmetric function of its arguments of the second group.

LEMMA 4.

$$A_1(\mu)\Phi_n(\lambda_1,\ldots,\lambda_n)=\prod_{k=1}^{n}z(\lambda_k-\mu)\Phi_n(\lambda_1,\ldots,\lambda_n)A_1(\mu)-$$

$$B_1(\mu)\sum_{j=1}^{n}\frac{e(\lambda_j-\mu)}{b(\lambda_j-\mu)}\prod_{\substack{k=1\\k\neq j}}^{n}z(\lambda_k-\lambda_j)\prod_{k=1}^{j-1}\omega(\lambda_j-\lambda_k)\Phi_{n-1}(\lambda_1,\ldots$$

$$\hat{\lambda}_j,\ldots,\lambda_n)A_1(\lambda_j)+B_2(\mu)\sum_{j=2}^{n}\sum_{l=1}^{j-1}\left(\frac{f(\lambda_l-\mu)}{d(\lambda_l-\mu)y(\lambda_l-\lambda_j)}+\right.$$

$$\left.\frac{z(\lambda_l-\mu)e(\lambda_j-\mu)\omega(\mu-\lambda_l)}{y(\mu-\lambda_l)b(\lambda_j-\mu)}\right)\prod_{k=1}^{l-1}\omega(\lambda_l-\lambda_k)\prod_{\substack{k=1\\k\neq l}}^{j-1}\omega(\lambda_j-\lambda_k)\times$$

$$\prod_{\substack{k=1\\k\neq j,l}}^{n}z(\lambda_k-\lambda_j)z(\lambda_k-\lambda_l)\Phi_{n-2}(\lambda_1,\ldots,\hat{\lambda}_l,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)A_1(\lambda_l)A_1(\lambda_j). \tag{4.10}$$

Proof of Lemmas 1-4. It will be convenient to prove all the lemmas together by induction on n, and in doing this we shall use the commutation relations

$$B_1(\mu)B_1(\lambda)=\omega(\lambda-\mu)\left(B_1(\lambda)B_1(\mu)-\frac{1}{y(\lambda-\mu)}B_2(\lambda)A_1(\mu)\right)+\frac{1}{y(\mu-\lambda)}B_2(\mu)A_1(\lambda), \tag{4.11}$$

$$A_1(\mu)B_1(\lambda)=z(\lambda-\mu)B_1(\lambda)A_1(\mu)-\frac{e(\lambda-\mu)}{b(\lambda-\mu)}B_1(\mu)A_1(\lambda), \tag{4.12}$$

$$A_1(\mu)B_2(\lambda)=\frac{c(\lambda-\mu)}{d(\lambda-\mu)}B_2(\lambda)A_1(\mu)-\frac{g(\lambda-\mu)}{d(\lambda-\mu)}B_1(\lambda)B_1(\mu)-\frac{f(\lambda-\mu)}{d(\lambda-\mu)}B_2(\mu)A_1(\lambda), \tag{4.13}$$

$$B_1(\lambda)B_2(\mu) = \frac{1}{z(\lambda-\mu)}B_2(\mu)B_1(\lambda) + \frac{\bar{e}(\lambda-\mu)}{c(\lambda-\mu)}B_1(\mu)B_2(\lambda), \qquad (4.14)$$

$$B_2(\lambda)B_1(\mu) = \frac{1}{z(\lambda-\mu)}B_1(\mu)B_2(\lambda) + \frac{e(\lambda-\mu)}{c(\lambda-\mu)}B_2(\mu)B_1(\lambda). \qquad (4.15)$$

We shall not describe in detail how the coefficients of the monomials that we need are to be calculated from these relations, since there is complete analogy with the standard calculations in, for example, the XXZ model [2,7,8].

The induction basis (n = 0, 1) can be verified directly. We establish the induction transition. We begin with assertion a) of Lemma 3. We use induction on m. The assertion is obvious if m < n/2, and this provides the induction basis. We represent Φ_{mn} as follows:

$$\colon \Phi_{mn}(\lambda_1,\ldots,\lambda_m) \colon = B_1(\lambda_1)\Psi_1(\lambda_2,\ldots,\lambda_m) + B_2(\lambda_1)\Psi_2(\lambda_2,\ldots,\lambda_m) + \Psi_3(\lambda_2,\ldots,\lambda_m)A_1(\lambda_1). \qquad (4.16)$$

Here, Ψ_1, Ψ_2, Ψ_3 satisfy the condition of Lemma 3. By the induction hypothesis $\Psi_1 = \Psi_3 = 0$. If m < n − 1, then $\Psi_2 = 0$ and the assertion is proved. If m = n − 1,

$$\Psi_2(\lambda_2,\ldots,\lambda_m) = g(\lambda_2,\ldots,\lambda_m)\Phi_{n-2}(\lambda_2,\ldots,\lambda_m). \qquad (4.17)$$

We substitute (4.17) in (4.16) and consider the relation (4.8) for i = 1. The comparison of the coefficients of the monomial $B_1(\lambda_1)B_2(\lambda_2)B_1(\lambda_3)\cdot\ldots\cdot B_1(\lambda_{n-1})A_1(\lambda_n)A_1(\lambda_{n+1})$ on the two sides of the equation is done by means of (4.15) and Property 4 of Lemma 1, which gives the required assertion: $g(\lambda_2,\ldots,\lambda_m) = 0$,

We turn to Lemma 2. We expand Φ_n in the same way as (4.16):

$$\colon \Phi_n(\lambda_1,\ldots,\lambda_n) \colon = B_1(\lambda_1)\Psi_1(\lambda_2,\ldots,\lambda_n) + B_2(\lambda_1)\Psi_2(\lambda_2,\ldots,\lambda_n) + \Psi_3(\lambda_2,\ldots,\lambda_n)A_1(\lambda_1). \qquad (4.18)$$

By Lemma 3, $\Psi_3 = 0$,

$$\Psi_1(\lambda_2,\ldots,\lambda_n) = g_1(\lambda_2,\ldots,\lambda_n)\Phi_{n-1}(\lambda_2,\ldots,\lambda_n),$$

$$\Psi_2(\lambda_2,\ldots,\lambda_n) = \sum_{j=2}^{n} g_2(\lambda_j|\lambda_2,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\prod_{k=2}^{j=1}\omega(\lambda_j-\lambda_k)\Phi_{n-2}(\lambda_2,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)A_1(\lambda_j). \qquad (4.19)$$

Using Properties 2 and 4 of Lemma 1 and the commutation relations (4.11) and (4.12), we find that

$$g_2(\lambda_2|\lambda_3,\ldots,\lambda_n) = -\frac{1}{y(\lambda_1-\lambda_2)}\prod_{j=3}^{n}z(\lambda_j-\lambda_2)g_1(\lambda_2,\ldots,\lambda_n).$$

The equality is extracted from the condition of canceling of terms of the type $B_2(\lambda_2)B_1(\lambda_3)\cdot\ldots\cdot B_1(\lambda_n)A_1(\lambda_1)$ for normal ordering of the left-hand side of Eq. (4.6); on the right-hand side of (4.6) such terms are absent. Property 4 of Lemma 1 also gives $g_1(\lambda_2,\ldots,\lambda_n) = 1$, and we have thereby verified the relation (4.7).

Lemma 4 can be proved similarly. We write down the decomposition

$$\colon A_1(\mu)\Phi_n(\lambda_1,\ldots,\lambda_n) \colon = \Psi_0(\lambda_1,\ldots,\lambda_n)A_1(\mu) + B_1(\mu)\Psi_1(\lambda_1,\ldots,\lambda_n) + B_2(\mu)\Psi_2(\lambda_1,\ldots,\lambda_n)$$

and represent Ψ_0, Ψ_1, Ψ_2 in the form (4.9). The key polynomials, whose coefficients make it possible, on the one hand, to determine the unknown functions and which, on the other, can be calculated fairly simply from the commutation relations (4.11)-(4.15), are $B_1(\lambda_1)\cdot\ldots\cdot B_1(\lambda_n)A_1(\mu)$, $B_1(\mu)B_1(\lambda_2)\cdot\ldots\cdot B_1(\lambda_n)A_1(\lambda_1)$ and $B_2(\mu)B_1(\lambda_3)\cdot\ldots\cdot B_1(\lambda_n)A_1(\lambda_1)A_1(\lambda_2)$.

We now prove the existence of the element Φ_n. In fact, almost all the properties of Φ_n required in Lemma 1 follow directly from the recursion relation (4.7). An exception is Eq. (4.6) for i = 1. To verify it, we expand (4.7) by one step:

$$\Phi_n(\lambda_1,\ldots,\lambda_n) = B_1(\lambda_1)B_1(\lambda_2)v_{n-2}(\lambda_3,\ldots,\lambda_n) -$$

$$B_2(\lambda_1)\frac{1}{y(\lambda_1-\lambda_2)}\prod_{k=3}^{n}z(\lambda_k-\lambda_2)\Phi_{n-2}(\lambda_3,\ldots,\lambda_n)A_1(\lambda_2) -$$

$$B_1(\lambda_1)B_2(\lambda_2)\sum_{j=3}^{n}\frac{1}{y(\lambda_2-\lambda_j)}\prod_{k=3}^{j-1}\omega(\lambda_j-\lambda_k)\prod_{\substack{k=3\\k\neq j}}^{n}z(\lambda_k-\lambda_j)\Phi_{n-3}(\lambda_3,\ldots$$

$$, \hat{\lambda}_j, \ldots, \lambda_n) A_1(\lambda_j) - B_2(\lambda_1) B_1(\lambda_2) \sum_{j=3}^{n} \frac{1}{y(\lambda_1 - \lambda_j)} \prod_{k=2}^{j-1} \omega(\lambda_j - \lambda_k) \times$$

$$\prod_{\substack{k=2 \\ k \neq j}}^{n} z(\lambda_k - \lambda_j) \Phi_{n-3}(\lambda_3, \ldots, \hat{\lambda}_j, \ldots, \lambda_n) A_1(\lambda_j) + B_2(\lambda_1) B_2(\lambda_2) \sum_{j=3}^{n} \times$$

$$\sum_{l=3}^{n} \frac{1}{y(\lambda_1 - \lambda_j) y(\lambda_2 - \lambda_l)} \prod_{k=2}^{j-1} \omega(\lambda_j - \lambda_k) \prod_{k=3}^{l-1} \omega(\lambda_l - \lambda_k) \prod_{\substack{k=2 \\ k \neq j}}^{n} z(\lambda_k - \lambda_j) \times$$

$$\prod_{\substack{k=3 \\ k \neq j, l}}^{n} z(\lambda_k - \lambda_l) \Phi_{n-4}(\lambda_3, \ldots, \hat{\lambda}_j, \ldots, \hat{\lambda}_l, \ldots, \lambda_n) A_1(\lambda_j) A_1(\lambda_l), \tag{4.20}$$

substitute (4.20) in (4.6), and, using the relations (4.10)-(4.15), reduce the left-hand side of the equation to normally ordered form. As a result we find that for the fulfillment of Property 2 it is sufficient if the elements of the R matrix satisfy the identities (A1.4) and (A1.5) given in the Appendix. For the R matrix (2.1), (2.2) the identities can be verified directly.

It remains to verify assertion b) of Lemma 3. In $\,\vdots\Phi_{mn}\vdots\,$ we separate all monomials that do not contain factors of the type $B_2(\lambda)$ in a separate term Ψ_1: $\,\vdots\Phi_{mn}\vdots = \Psi_1 + \Psi_2$. From the commutation relations (4.11)-(4.15) have the following representation for Ψ_1:

$$\Psi_1(\lambda_1, \ldots, \lambda_m) = \sum_{\sigma} g(\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_{m-n}} | \lambda_1, \ldots$$

$$, \hat{\lambda}_{\sigma_1}, \ldots, \hat{\lambda}_{\sigma_{m-n}}, \ldots, \lambda_m) \prod_{a=1}^{m-n} \prod_{\substack{k=1 \\ k \neq \sigma_1, \ldots, \sigma_{a-1}}}^{\sigma_a - 1} \omega(\lambda_{\sigma_a} - \lambda_k) B_1(\lambda_1) \cdots$$

$$\cdot \overline{B_1(\lambda_{\sigma_1})} \cdots \overline{B_1(\lambda_{\sigma_{m-n}})} \cdots B_1(\lambda_m) A_1(\lambda_{\sigma_1}) \cdots A_1(\lambda_{\sigma_{m-n}}), \tag{4.21}$$

which is analogous to (4.9). We consider the difference

$$\Psi_0(\lambda_1, \ldots, \lambda_m) = \Phi_{mn}(\lambda_1, \ldots, \lambda_m) - \sum_{\sigma} g(\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_{m-n}} | \lambda_1, \ldots$$

$$, \hat{\lambda}_{\sigma_1}, \ldots, \hat{\lambda}_{\sigma_{m-n}}, \ldots, \lambda_m) \prod_{a=1}^{m-n} \prod_{\substack{k=1 \\ k \neq \sigma_1, \ldots, \sigma_{a-1}}}^{\sigma_a - 1} \omega(\lambda_{\sigma_a} - \lambda_k) \times$$

$$\Phi_n(\lambda_1, \ldots, \hat{\lambda}_{\sigma_1}, \ldots, \hat{\lambda}_{\sigma_{m-n}}, \ldots, \lambda_m) A_1(\lambda_{\sigma_1}) \cdots A_1(\lambda_{\sigma_{m-n}}), \tag{4.22}$$

where the function $g(\lambda_1, \ldots, \lambda_{m-n} | \lambda_{m-n+1}, \ldots, \lambda_m)$ can be found from (4.21). Now Ψ_0 satisfies all the conditions of Lemma 3. In addition, $\,\vdots\Psi_0\vdots\,$ does not contain monomials without factors of the type $B_2(\lambda)$. We decompose Ψ_0 in the usual manner:

$$\vdots\Psi_0(\lambda_1, \ldots, \lambda_m)\vdots = B_1(\lambda_1) \Psi_{01}(\lambda_2, \ldots, \lambda_m) + B_2(\lambda_1) \Psi_{02}(\lambda_2, \ldots, \lambda_m) + \Psi_{03}(\lambda_2, \ldots, \lambda_m) A_1(\lambda_1). \tag{4.23}$$

By the induction hypothesis, Ψ_{0i} can be expressed in the form (4.9). Allowance for the additional restriction on $\,\vdots\Psi_0\vdots\,$ now tells us that $\Psi_{01} = \Psi_{03} = 0$. Substituting (4.23) in (4.8) for i = 1, reducing the left-hand side of the equation to normally ordered form, and comparing the coefficients of the monomial $B_1(\lambda_1) B_2(\lambda_2) B_1(\lambda_3) \cdots B_1(\lambda_{n-1}) A_1(\lambda_n) \cdots A_1(\lambda_m)$, we find that $\Psi_{02} = 0$, completing thereby the proof of Lemmas 1-4.

LEMMA 5.

$$A_2(\mu) \Phi_n(\lambda_1, \ldots, \lambda_n) = \prod_{k=1}^{n} \frac{z(\mu - \lambda_k)}{\omega(\mu - \lambda_k)} \Phi_n(\lambda_1, \ldots, \lambda_n) A_2(\mu) -$$

$$B_1(\mu) \sum_{j=1}^{n} \frac{\bar{e}(\mu - \lambda_j)}{b(\mu - \lambda_j)} \prod_{k=1}^{j-1} \omega(\lambda_j - \lambda_k) \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{z(\lambda_j - \lambda_k)}{\omega(\lambda_j - \lambda_k)} \Phi_{n-1}(\lambda_1, \ldots$$

$$, \hat{\lambda}_j, \ldots, \lambda_n) A_2(\lambda_j) + B_3(\mu) \sum_{j=1}^{n} \frac{1}{y(\mu-\lambda_j)} \prod_{k=1}^{j-1} \omega(\lambda_j-\lambda_k) \times$$

$$\prod_{\substack{k=1 \\ k \neq j}}^{n} z(\lambda_k-\lambda_j) \Phi_{n-1}(\lambda_1, \ldots, \hat{\lambda}_j, \ldots, \lambda_n) A_1(\lambda_j) + B_2(\mu) \times$$

$$\sum_{j=2}^{n} \sum_{l=1}^{j-1} \prod_{\substack{k=1 \\ k \neq l}}^{j-1} \omega(\lambda_j-\lambda_k) \prod_{k=1}^{l-1} \omega(\lambda_l-\lambda_k) \Phi_{n-2}(\lambda_1, \ldots, \hat{\lambda}_l, \ldots, \hat{\lambda}_j, \ldots$$

$$, \lambda_n)^{\Gamma} \frac{\bar{e}(\mu-\lambda_l)}{b(\mu-\lambda_l)} \left(\frac{\bar{e}(\lambda_l-\lambda_j)}{y(\mu-\lambda_l) b(\lambda_b-\lambda_j)} + \frac{z(\mu-\lambda_l)}{y(\mu-\lambda_j) \omega(\mu-\lambda_l)} - \right.$$

$$\left. \frac{\bar{e}(\mu-\lambda_l)}{y(\lambda_l-\lambda_j) b(\mu-\lambda_j)} \right) \prod_{\substack{k=1 \\ k \neq j,l}}^{n} \frac{z(\lambda_l-\lambda_k) z(\lambda_k-\lambda_j)}{\omega(\lambda_l-\lambda_k)} A_1(\lambda_j) A_2(\lambda_l) +$$

$$\frac{1}{y(\mu-\lambda_l)} \left(\frac{z(\mu-\lambda_l)\bar{e}(\mu-\lambda_j)}{b(\mu-\lambda_j)} - \frac{\bar{e}(\mu-\lambda_l)\bar{e}(\lambda_l-\lambda_j)}{b(\mu-\lambda_l) b(\lambda_l-\lambda_j)} \prod_{\substack{k=1 \\ k \neq j,l}}^{n} \frac{z(\lambda_k-\lambda_l) z(\lambda_j-\lambda_k)}{\omega(\lambda_j-\lambda_k)} A_1(\lambda_l) A_2(\lambda_j) \right] + \ldots, \qquad (4.24)$$

$$A_3(\mu) \Phi_n(\lambda_1, \ldots, \lambda_n) = \prod_{k=1}^{n} \frac{b(\mu-\lambda_k)}{d(\mu-\lambda_k)} \Phi_n(\lambda_1, \ldots, \lambda_n) A_3(\mu) -$$

$$B_3(\mu) \sum_{j=1}^{n} \frac{1}{y(\mu-\lambda_j)} \prod_{k=1}^{j-1} \omega(\lambda_j-\lambda_k) \prod_{k=1}^{n} \frac{z(\lambda_j-\lambda_k)}{\omega(\lambda_j-\lambda_k)} \Phi_{n-1}(\lambda_1, \ldots, \hat{\lambda}_j, \ldots, \lambda_n) A_2(\lambda_j) + B_2(\mu) \sum_{j=2}^{n} \times$$

$$\sum_{l=1}^{j-1} \left(\frac{\bar{f}(\mu-\lambda_l)}{d(\mu-\lambda_l) y(\lambda_l-\lambda_j)} - \frac{\bar{e}(\mu-\lambda_l)}{d(\mu-\lambda_l) y(\mu-\lambda_j)} \right) \times$$

$$\prod_{\substack{k=1 \\ k \neq j,l}}^{n} \frac{z(\lambda_j-\lambda_k) z(\lambda_l-\lambda_k)}{\omega(\lambda_j-\lambda_k) \omega(\lambda_l-\lambda_k)} \Phi_{n-2}(\lambda_1, \ldots, \hat{\lambda}_l, \ldots, \hat{\lambda}_j, \ldots, \lambda_n) A_2(\lambda_l) A_2(\lambda_j) + \ldots, \qquad (4.25)$$

where the ellipses denote normally ordered terms containing factors of the type $C_i(\lambda)$.

Proof. We consider the necessary commutation relations (A2.1)-(A2.14). It is evident from them that $:A_i\Phi_n:$ contains only monomials containing not more than two factors of the type $B_3(\lambda)$, and the factor $B_3(\lambda_n)$ is certainly absent. Using the symmetry properties of $A_i(\mu)\Phi_n(\lambda_1, \ldots, \lambda_n)$ inherited from $\Phi_n(\lambda_1, \ldots, \lambda_n)$, and the commutation relations of the element $B_3(\lambda)$, we can find that the factors $B_3(\lambda_i)$ are altogether absent and just the one factor $B_3(\mu)$ is possible. We finally arrive at the decomposition

$$A_i(\mu)\Phi_n(\lambda_1, \ldots, \lambda_n) = \Psi_{0i}(\lambda_1, \ldots, \lambda_n) A_i(\mu) +$$

$$B_1(\mu) \Psi_{1i}(\lambda_1, \ldots, \lambda_n) + B_2(\mu) \Psi_{2i}(\lambda_1, \ldots, \lambda_n) + B_3(\mu) \Psi_{3i}(\lambda_1, \ldots, \lambda_n), \qquad (4.26)$$

where Ψ_{ai} satisfy the conditions of Lemma 3. After this the validity of (4.24) and (4.25) can be established in the same way as we proved Lemmas 2 and 4.

Let $T(\lambda)$ be a monodromy matrix with highest vector $|0\rangle$. We prove the formulas of the algebraic Bethe ansatz (2.5)-(2.11). Indeed, the relations (2.7), (2.8), and (4.7) signify that $|\Phi_n(\lambda_1, \ldots, \lambda_n)\rangle = \Phi_n(\lambda_1, \ldots, \lambda_n)|0\rangle$, and formula (2.11) follows from (4.6). To obtain formulas (2.9) and (2.10), we use Lemmas 4 and 5. Adding the relations (4.10), (4.24), and (4.25) and applying the sum to the highest vector, we calculate the vector $\mathrm{tr}\, T(\lambda)|\Phi_n(\mu_1, \ldots, \mu_n)\rangle$. The requirement of cancellation of the undesirable terms containing $B_1(\lambda)$, $B_2(\lambda)$, $B_3(\lambda)$ leads us to the system (2.9) and to the identities (A1.1) and (A1.6) for the elements of the R matrix. For the Izergin-Korepin R matrix these identities can be directly verified.

The connection (2.5) follows from the commutation relation (A2.7) for $\lambda = \mu + 2\eta - i\pi$, applied to the highest vector. We note that this connection does not hinder our constructions, since it is nonlocal.

Conclusions

I believe that the most promising generalization of the proposed scheme of the algebraic Bethe ansatz is its application to Z_2-graded models with three-dimensional auxiliary space, in particular, to the supersymmetric sine–Gordon model [13] and the osp $(\frac{1}{2})$-invariant nonlinear Schrödinger equation [14]. The elements $B_1(\lambda)$ and $B_2(\lambda)$ in these models will be fermion and boson creation operators, respectively. On the transition to models with grading the necessary identities (A1.1)-(A1.6) for the matrix elements of the R matrix are somewhat modified. Their precise form can be obtained by carrying out all calculations analogous to the case considered in the paper.

Appendix 1

In Secs. 2 and 4 we have used the following identities satisfied by the elements of the Izergin–Korepin R matrix (2.1), (2.2):

$$\frac{\bar{e}(\lambda-\mu)}{b(\lambda-\mu)} = -\frac{e(\mu-\lambda)}{b(\mu-\lambda)}, \tag{A1.1}$$

$$\frac{a(\lambda-\mu)\,d(\lambda-\mu)-\bar{g}(\lambda-\mu)\,g(\lambda-\mu)}{f(\lambda-\mu)\,\bar{g}(\lambda-\mu)-d(\lambda-\mu)\,g(\lambda-\mu)} = y(\mu-\lambda), \tag{A1.2}$$

$$\omega(\lambda-\mu)\,\omega(\mu-\lambda)=1, \tag{A1.3}$$

$$\frac{z(\lambda-\mu)\,\omega(\mu-\lambda)}{y(\nu-\mu)} + \frac{e(\mu-\lambda)}{b(\mu-\lambda)\,y(\nu-\lambda)} = \frac{z(\nu-\lambda)\,\omega(\lambda-\nu)}{y(\nu-\mu)} + \frac{e(\lambda-\nu)}{b(\lambda-\nu)\,y(\lambda-\mu)}, \tag{A1.4}$$

$$z_{34}\omega_{43}\left(\frac{z_{14}\omega_{41}}{y_{13}y_{24}} - \frac{z_{24}\omega_{42}\omega_{12}}{y_{14}y_{23}}\right) + z_{43}\left(\frac{z_{13}\omega_{31}}{y_{14}y_{23}} - \frac{z_{23}\omega_{32}\omega_{12}}{y_{13}y_{24}}\right) +$$
$$\frac{1}{y_{21}}\left(\frac{f_{31}}{d_{31}y_{34}} + \frac{z_{31}\omega_{13}e_{41}}{y_{13}b_{41}}\right) - \frac{\omega_{12}}{y_{12}}\left(\frac{f_{32}}{d_{32}y_{34}} + \frac{z_{32}\omega_{23}e_{42}}{y_{23}b_{42}}\right) = 0, \tag{A1.5}$$

$$\frac{f_{14}}{d_{14}y_{12}} + \frac{\bar{f}_{41}}{d_{41}y_{12}} - \frac{\bar{e}_{41}}{d_{41}y_{42}} - \frac{\omega_{12}}{y_{12}}\left(\frac{\bar{e}_{41}}{y_{41}z_{12}}\right)^2 + \frac{z_{14}\omega_{41}e_{24}}{y_{41}b_{24}} + \frac{\omega_{12}}{z_{12}{}^2}\left(\frac{\bar{e}_{41}\bar{e}_{12}}{b_{41}b_{12}y_{41}} + \frac{z_{41}\omega_{14}\bar{e}_{41}}{y_{42}b_{41}}\right) +$$
$$\frac{\omega_{21}}{z_{21}{}^2}\left(\frac{z_{41}\bar{e}_{42}}{y_{41}b_{42}} - \frac{\bar{e}_{41}\bar{e}_{12}}{y_{41}b_{41}b_{12}}\right) = 0 \tag{A1.6}$$

(we have here used the notation $k_{ij} \equiv k(\lambda_i - \lambda_j)$, $\omega = cd/(ad - \bar{g}g)$). Equation (A1.2) was used in the derivation of the commutation relation (4.11).

These identities are obviously direct consequences of the Yang–Baxter equations and the matrix structure (2.1) of the R matrix. This can be completely verified only for the identities (A1.1)-(A1.4). The first three of them follow from the unitarity relation (1.3). The fourth can be verified by reducing the monomial $B_1(\lambda_3)B_1(\lambda_2)B_1(\lambda_1)$ to normally ordered form in two different ways and comparing the coefficients of, for example, the monomial $B_1(\lambda_1)B_2(\lambda_2)A_1(\lambda_3)$. A proof of the identities (A1.5) and (A1.6) must exist in the same manner.

Appendix 2

We write down the commutation relations needed to prove Lemma 5:

$$A_2(\mu)B_1(\lambda) = \frac{z(\mu-\lambda)}{\omega(\mu-\lambda)}B_1(\lambda)A_2(\mu) - \frac{\bar{e}(\mu-\lambda)}{b(\mu-\lambda)}B_1(\mu)A_2(\lambda) +$$
$$\cdot\frac{1}{y(\mu-\lambda)}B_3(\mu)A_1(\lambda) + \frac{\bar{e}(\mu-\lambda)}{y(\mu-\lambda)\,b(\mu-\lambda)}B_2(\lambda)C_1(\mu) - \frac{z(\mu-\lambda)}{\omega(\mu-\lambda)\,y(\lambda-\mu)}B_2(\mu)C_1(\lambda), \tag{A2.1}$$

$$A_2(\mu)B_2(\lambda) = z(\lambda-\mu)\,z(\mu-\lambda)B_2(\lambda)A_2(\mu) + \frac{\bar{e}(\mu-\lambda)}{b(\mu-\lambda)}\left(B_3(\mu)B_1(\lambda) - B_1(\lambda)B_3(\mu) + \frac{\bar{e}(\mu-\lambda)}{b(\mu-\lambda)}B_2(\mu)A_2(\lambda)\right), \tag{A2.2}$$

$$A_3(\mu)B_1(\lambda) = \frac{b(\mu-\lambda)}{b(\mu-\lambda)}B_1(\lambda)A_3(\mu) - \frac{1}{y(\mu-\lambda)}B_3(\mu)A_2(\lambda) + \frac{e(\mu-\lambda)}{d(\mu-\lambda)}B_2(\lambda)C_3(\mu) - \frac{\bar{f}(\mu-\lambda)}{d(\mu-\lambda)}B_2(\mu)C_3(\lambda), \tag{A2.3}$$

$$A_3(\mu)B_2(\lambda) = \frac{c(\mu-\lambda)}{d(\mu-\lambda)}B_2(\lambda)A_3(\mu) - \frac{1}{y(\mu-\lambda)}B_3(\mu)B_3(\lambda) - \frac{\bar{f}(\mu-\lambda)}{d(\mu-\lambda)}B_2(\mu)A_3(\lambda), \tag{A2.4}$$

$$C_1(\mu)B_1(\lambda) = B_1(\lambda)C_1(\mu) - \frac{\bar{e}(\mu-\lambda)}{b(\mu-\lambda)}(A_1(\mu)A_2(\lambda) - A_1(\lambda)A_2(\mu)), \tag{A2.5}$$

$$d(\lambda-\mu)C_1(\mu)B_2(\lambda)=b(\lambda-\mu)B_2(\lambda)C_1(\mu)+e(\lambda-\mu)B_3(\lambda)A_1(\mu)-f(\lambda-\mu)B_3(\mu)A_1(\lambda)-g(\lambda-\mu)A_2(\mu)B_1(\lambda), \quad (A2.6)$$

$$d(\mu-\lambda)C_3(\mu)B_1(\lambda)=a(\mu-\lambda)B_1(\lambda)C_3(\mu)+g(\mu-\lambda)B_2(\lambda)\times$$
$$C_2(\mu)-\bar{f}(\mu-\lambda)B_1(\mu)C_3(\lambda)+\bar{g}(\mu-\lambda)(A_1(\lambda)A_3(\mu)-A_2(\lambda)A_2(\mu)). \quad (A2.7)$$

$$b(\lambda-\mu)C_3(\mu)B_2(\lambda)=d(\lambda-\mu)B_2(\lambda)C_3(\mu)+g(\lambda-\mu)B_3(\lambda)A_2(\mu)+f(\lambda-\mu)A_3(\lambda)B_1(\mu)-e(\lambda-\mu)A_3(\mu)B_1(\lambda), \quad (A2.8)$$

$$b(\lambda-\mu)C_2(\mu)B_1(\lambda)=d(\lambda-\mu)B_1(\lambda)C_2(\mu)+g(\lambda-\mu)A_2(\lambda)C_1(\mu)+f(\lambda-\mu)C_3(\lambda)A_1(\mu)-e(\lambda-\mu)C_3(\mu)A_1(\lambda), \quad (A2.9)$$

$$d(\lambda-\mu)C_3(\mu)A_1(\lambda)=b(\lambda-\mu)A_1(\lambda)C_3(\mu)+e(\lambda-\mu)B_1(\lambda)C_2(\mu)-\bar{g}(\lambda-\mu)A_2(\mu)C_1(\lambda)-\bar{f}(\lambda-\mu)B_1(\mu)C_2(\lambda), \quad (A2.10)$$

$$C_2(\mu)B_2(\lambda)=B_2(\lambda)C_2(\mu)+\frac{g(\lambda-\mu)}{d(\lambda-\mu)}(B_3(\lambda)C_1(\mu)-C_3(\mu)B_1(\lambda))+\frac{f(\lambda-\mu)}{d(\lambda-\mu)}(A_3(\lambda)A_1(\mu)-A_3(\mu)A_1(\lambda)), \quad (A2.11)$$

$$B_1(\lambda)B_3(\mu)=B_3(\mu)B_1(\lambda)+\frac{\bar{e}(\mu-\lambda)}{b(\mu-\lambda)}B_2(\mu)A_2(\lambda)-\frac{e(\mu-\lambda)}{b(\mu-\lambda)}B_2(\lambda)A_2(\mu), \quad (A2.12)$$

$$B_2(\lambda)B_3(\mu)=\frac{1}{z(\mu-\lambda)}B_3(\mu)B_2(\lambda)+\frac{\bar{e}(\mu-\lambda)}{c(\mu-\lambda)}B_2(\mu)B_3(\lambda), \quad (A2.13)$$

$$B_3(\lambda)B_2(\mu)=\frac{1}{z(\mu-\lambda)}B_2(\mu)B_3(\lambda)+\frac{e(\mu-\lambda)}{c(\mu-\lambda)}B_3(\mu)B_2(\lambda). \quad (A2.14)$$

LITERATURE CITED

1. L. D. Faddeev, "Quantum completely integrable field theory models," Preprint R-2-79 [in Russian], Leningrad Branch, V. A. Steklov Mathematics Institute, USSR Academy of Sciences, Leningrad (1979).
2. L. A. Takhtadzhyan and L. D. Faddeev, Usp. Mat. Nauk, $\underline{34}$, 13 (1979).
3. E. K. Sklyanin, L. A. Takhtadzhyan, and L. D. Faddeev, Teor. Mat. Fiz., $\underline{40}$, 194 (1979).
4. P. P. Kulish and N. Yu. Reshetikhin, J. Phys. A, $\underline{16}$, L591 (1983).
5. L. A. Takhtadzhyan, Zap. Nauchn. Semin. LOMI, $\underline{101}$, 158 (1981).
6. N. Yu. Reshetikhin, Teor. Mat. Fiz., $\underline{63}$, 347 (1985).
7. A. G. Izergin and V. E. Korepin, Fiz. Elem. Chastits At. Yadra, $\underline{13}$, 501 (1982).
8. P. P. Kulish and E. K. Sklyanin, Lect. Notes in Phys., $\underline{151}$, 61 (1982).
9. A. G. Izergin and V. E. Korepin, Commun. Math. Phys., $\underline{79}$, 303 (1981).
10. V. I. Vichirko and N. Yu. Reshetikhin, Teor. Mat. Fiz., $\underline{56}$, 260 (1983).
11. P. P. Kulish and N. Yu. Reshetikhin, Zap. Nauchn. Semin. LOMI, $\underline{101}$, 101 (1981).
12. E. K. Sklyanin, Funktsional. Analiz i Ego Prilozhen., $\underline{16}$, 27 (1982).
13. P. P. Kulish and S. A. Tsyplyaev, Zap. Nauchn. Semin. LOMI, $\underline{120}$, 122 (1982).
14. P. P. Kulish, "Quantum osp-invariant nonlinear Schrödinger equation," Preprint IC/85/39, ICTF, Miramare, Trieste (1985).