

EXACT QUANTUM SINE-GORDON SOLITON FORM FACTORS

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Soliton form factors are constructed using Zamolodchikov's proposed exact massive Thirring model S -matrix. The asymptotic behaviour of the electromagnetic form factor is $\sim (-t)^{-g/2}$. The quasiclassical limit is not the previously accepted result.

In a recent paper Zamolodchikov [1] proposed an exact Sine-Gordon soliton S -matrix which satisfies unitarity and crossing, has certain required semiclassical properties and is in agreement with perturbation theory in the Thirring model coupling constant to third order [2]. It is the purpose of this letter to construct soliton form factors corresponding to this solution.

The existence of the infinite set of conservation laws in the Sine-Gordon [3] (massive Thirring [4]) model which forbids particle production is crucial for the solubility, since it reduces unitarity to elastic unitarity [1, 2, 5]. We define the soliton (s)-antisoliton (\bar{s}) forward and backward scattering amplitudes by

$$\text{out} \langle s(\tilde{p}_1) \bar{s}(\tilde{p}_2) | s(p_1) \bar{s}(p_2) \rangle^{\text{in}} = (2\pi)^2 2p_1^0 2p_2^0 [\delta(\tilde{p}_1'' - p_1') \delta(\tilde{p}_2' - p_2') F(\theta) + \delta(\tilde{p}_1' - p_2') \delta(\tilde{p}_2'' - p_1') B(\theta)], \quad (1)$$

where $\cosh \theta = p_1 p_2 / m^2 = \nu$, and the physical sheet in the ν -plane being mapped into the strip $0 < \text{Im } \theta < \pi$ in the θ -plane. The soliton-soliton S -matrix element is related to the $s\bar{s}$ forward scattering amplitude by crossing,

$$F_{ss}(\theta) = F(i\pi - \theta). \quad (2)$$

Zamolodchikov's [1] solution is given by,

$$F_{ss}(\theta) = \prod_{l=1}^{\infty} \frac{\Gamma\left[\frac{1}{\lambda}\left(2l + \frac{i\theta}{\pi}\right)\right] \Gamma\left[1 + \frac{1}{\lambda}\left(2l - 2 + \frac{i\theta}{\pi}\right)\right] \Gamma\left[\frac{1}{\lambda}\left(2l - 1 - \frac{i\theta}{\pi}\right)\right] \Gamma\left[1 + \frac{1}{\lambda}\left(2l - 1 - \frac{i\theta}{\pi}\right)\right]}{\Gamma\left[\frac{1}{\lambda}\left(2l - \frac{i\theta}{\pi}\right)\right] \Gamma\left[1 + \frac{1}{\lambda}\left(2l - 2 - \frac{i\theta}{\pi}\right)\right] \Gamma\left[\frac{1}{\lambda}\left(2l - 1 + \frac{i\theta}{\pi}\right)\right] \Gamma\left[1 + \frac{1}{\lambda}\left(2l - 1 + \frac{i\theta}{\pi}\right)\right]}, \quad (3)$$

and

$$B(\theta) = \frac{-i \sin(\pi/\lambda)}{\sinh(\theta/\lambda)}, \quad (4)$$

so that the $s\bar{s}$ reflection coefficient is given by

$$\mathcal{R} = \frac{\sin^2(\pi/\lambda)}{\sinh^2(\theta/\lambda) + \sin^2(\pi/\lambda)}. \quad (5)$$

The connection with the Thirring model coupling constant is given by [1, 2]

$$g \equiv -2\delta_{ss}(\infty) = \frac{\pi}{2} \left(\frac{1}{\lambda} - 1 \right). \quad (6)$$

The solution is based on the following requirements:

(a) The exact semiclassical $s\bar{s}$ bound state spectrum [5, 6]

$$m_n = 2m \sin(n\lambda\pi/2), \quad n = 1, 2, \dots < \lambda^{-1}. \quad (7)$$

(b) Vanishing of $s\bar{s}$ backward scattering at special values $\lambda = 1/n$.

(c) Meromorphy of the amplitudes in the θ -plane with singularities occurring only on the imaginary θ -axis.

Using this solution we now wish to construct soliton form factors $G_{\pm}(t)$ occurring in the soliton matrix elements $\langle s(p_2) | J_{\pm}(x) | s(p_1) \rangle$ of general local operators $J_{\pm}(x)$, where (\pm) indicates definite C -parity ± 1 . We must have two important general properties;

(i) The form factors are analytic in the t -plane ($t = (p_1 - p_2)^2$), cut along the real axis from $4m^2$ to $+\infty$.

(ii) Watson's theorem [7], which fixes the phase of the form factors above the right hand cut from $4m^2$ to the next threshold to be equal to the $C = \pm 1$ $s\bar{s}$ phase shifts $\delta_{\pm}(\theta) \equiv (1/2i) \log [F(\theta) \mp B(\theta)]$. Absence of further thresholds in the Thirring model makes this a very powerful constraint.

We define θ by

$$t = 4m^2 \cosh^2(\theta/2), \quad (8)$$

and make the Zamolodchikov assumption that the form factors are also meromorphic in the θ -plane. Then the properties (i), (ii) above give, respectively

$$G_{\pm}(i\pi - \theta) = G_{\pm}(i\pi + \theta), \quad \text{and} \quad \frac{G_{\pm}(\theta)}{G_{\pm}(-\theta)} = e^{2i\delta_{\pm}(\theta)}. \quad (9, 10)$$

We wish to solve these equations subject to the further condition that the form factors have no other dynamical singularities in the physical strip other than simple poles corresponding to bound states with masses given by eq. (7), n odd(even) for C odd(even). Such solutions can be obtained by matching the pole and zero structure in eq. (10) and iterating using eq. (9), and are given by

$$G_{-}(i\pi - \theta) = \frac{\cosh(\theta/2)}{\cosh(\theta/2\lambda)} e^{T(\theta)}, \quad G_{+}(i\pi - \theta) = \frac{\sinh(\theta/2)}{\sinh(\theta/2\lambda)} e^{T(\theta)}, \quad (11, 12)$$

with

$$e^{T(\theta)} = \left\{ \prod_{l=0}^{\infty} \prod_{m=1}^{\infty} \frac{\tilde{\Gamma} \left[1 + \frac{1}{\lambda} \left(2(l+m) - \frac{i\theta}{\pi} \right) \right] \tilde{\Gamma} \left[1 + \frac{1}{\lambda} \left(2(l+m) + \frac{i\theta}{\pi} \right) \right]}{\tilde{\Gamma} \left[1 + \frac{1}{\lambda} \left(2(l+m) - 1 - \frac{i\theta}{\pi} \right) \right] \tilde{\Gamma} \left[1 + \frac{1}{\lambda} \left(2(l+m) - 1 + \frac{i\theta}{\pi} \right) \right]} \right\}^2 \\ \times \prod_{n=1}^{\infty} \tilde{\Gamma} \left[1 + \frac{1}{\lambda} \left(2n - 1 - \frac{i\theta}{\pi} \right) \right] \tilde{\Gamma} \left[1 + \frac{1}{\lambda} \left(2n - 1 + \frac{i\theta}{\pi} \right) \right] \prod_{k=1}^{\infty} \frac{\tilde{\Gamma} \left[\frac{1}{2} \left(2k - \frac{i\theta}{\pi} \right) \right] \tilde{\Gamma} \left[\frac{1}{2} \left(2k + \frac{i\theta}{\pi} \right) \right]}{\tilde{\Gamma} \left[\frac{1}{2} \left(2k + 1 - \frac{i\theta}{\pi} \right) \right] \tilde{\Gamma} \left[\frac{1}{2} \left(2k + 1 + \frac{i\theta}{\pi} \right) \right]}, \quad (13)$$

where we have normalized $G_{\mp}(i\pi) = 1$ by defining $\tilde{\Gamma}[f(\theta)] \equiv (\Gamma[f(\theta)]/\Gamma[f(0)])$.

These solutions are obviously ambiguous up to kinematic zeros and singularities of the form $(\cosh(\theta/2))^p (\sinh(\theta/2))^q$ which can be absorbed in the definition of the form factors. The horrendous expression (13) merely expresses the multipole (and multizero) structure on the unphysical sheets. Using Malmstén's representation for $\log \Gamma(z)$ [8], one can obtain a more manageable integral representation

$$T(\theta) = \int_0^{\infty} dt \frac{\sin^2(\theta t/2\pi)}{t \sinh t} \frac{\sinh(t/2)(1-\lambda)}{\sinh \frac{1}{2} \lambda t \cosh \frac{1}{2} t}, \quad (14)$$

valid for $|\text{Im } \theta| < \text{Min}[2\pi, (1+\lambda)\pi]$.

We now claim that the soliton electromagnetic form factor defined by

$$\langle s(p_2) | J_{\mu}(0) | s(p_1) \rangle = G_{-}(\theta) \bar{u}(p_2) \gamma_{\mu} u(p_1), \quad (15)$$

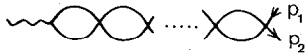


Fig. 1.

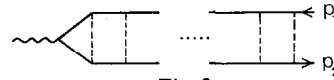


Fig. 2.

is given precisely by eqs. (11) and (13).

(A) Perturbation theory in g : Knowledge of the soliton scattering amplitudes (3), (4) and the structure of the contributing graphs, together with the requirements (9), (10) fix the form factor to second order to be [2]

$$G_{-}(\theta) = 1 + \frac{g}{2} [(3 + \cosh \theta) r(\theta) - 2r(i\pi)] + \frac{g^2}{8} \left[(1 + \cosh \theta)(13 + \cosh \theta) r(\theta)^2 - 4(5 + \cosh \theta) r(\theta) r(i\pi) - d \coth^2 \frac{\theta}{2} + 16r(i\pi)^2 \right] + \dots, \quad (16)$$

where $r(\theta)$ is the 1-loop integral

$$r(\theta) = \frac{i - (\theta/\pi)}{\sinh \theta}. \quad (17)$$

Only one constant d , remains to be determined. This is specified by knowledge of the leading $\theta = 0$ threshold behaviour, which at n th order is given simply by the $(n-1)$ 1-loop chain. Fig. 1, using equivalent infinite mass scalar boson exchange, the only non-vanishing diagrams which contribute to the chains are given by the ladder diagrams fig. 2 (C-parity conservation). These ladder diagrams yield a geometric series which give the ladder approximation to the form factor

$$G_{-}(\theta)_{\text{ladder}} = \{1 - (g/2) [(3 + \cosh \theta) r(\theta) - 2r(i\pi)]\}^{-1}. \quad (18)$$

From this one sees that the leading threshold behaviour at n th order is given by $(2ig/\theta)^n$. Firstly it can be checked that this is also the leading threshold behaviour order by order in g of the exact expression proposed above. Secondly this leading threshold behaviour at second order requires $d = 1$, and then it can be checked that expression (16) indeed coincides with the perturbation expansion of (11).

(B) The semiclassical limit $\lambda \rightarrow 0$: In this limit, the form factor behaves like

$$G_{-}(i\pi - \theta) \underset{\lambda \rightarrow 0}{\sim} \frac{\cosh(\theta/2)}{\cosh(\theta/2\lambda)} \exp \left\{ \frac{1}{\lambda} \int_0^\infty \frac{dt \sin^2(\theta t/2\pi)}{t^2 \cosh^2(t/2)} \right\}, \quad (19)$$

for $|\text{Im } \theta| < \pi$.

This is to be contrasted with the conventional expression [9]

$$G_{-}(t)_{\text{classical}} = \frac{\sqrt{1 - (t/4m^2)}}{\cosh(1/2\lambda) \sqrt{-t/m^2}}, \quad (20)$$

obtained by associating the form factor in the brick-wall frame with the Fourier transform $\tilde{f}(\sqrt{-t})$ of the classical static solution

$$f(x) = \frac{4}{\beta} \tan^{-1}(\exp(\sqrt{\alpha}x)), \quad (21)$$

of the sine-Gordon equation,

$$\square \Phi = -\frac{\alpha}{\beta} \sin \beta \Phi. \quad (22)$$

Above, we have also used the identification [10]

$$J_\mu = -\frac{\beta}{2\pi} \epsilon_{\mu\nu} \partial^\nu \Phi, \quad (23)$$

and

$$\sqrt{\alpha} \sim m_1 \sim m\lambda\pi \sim m\beta^2/8. \quad (24)$$

Despite certain agreeable analytic properties, it appears that the classical expression (20) is only good in the "non-relativistic region" $t \approx 0$ and demonstrates that suitable care should be taken in interpreting comparable recipes.

Indeed, the classical expression (20) should strictly only be compared with the exact expression in the limit t fixed, $m \rightarrow \infty$, $m\lambda$ fixed, whereupon agreement is obtained[†].

C. Asymptotic behaviour: The exact form factor is power behaved and decays or grows as $t \rightarrow -\infty$ according as the Thirring coupling is attractive or repulsive

$$G_-(t) \sim (-t)^{-g/2\pi} \quad \text{as} \quad t \rightarrow -\infty. \quad (25)$$

We remark that the ladder approximation (18) and the classical expression (20) fail to describe the asymptotic behaviour correctly. Finally, it is noteworthy that the behaviour does not involve the anomalous dimension d_ψ of the spinor field in the often speculated form $t^{-(ad_\psi+b)}$, but is rather more directly associated with the anomalous dimension of the mass term $\bar{\psi}\psi$. It remains to be seen whether this is a property of the particular model or a more general feature of a wide class of models.

Accepting the Zamolodchikov solution as an excellent candidate for the soliton S -matrix, we have determined the soliton form factors. This constitutes part of the programme proposed in ref. [5] of solving the Thirring model using the higher conserved currents and general principles. The ultimate ambition is the exact determination of the Wightman functions.

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