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# PERTURBATION THEORY CHECK OF A PROPOSED EXACT THIRRING MODEL $S$-MATRIX

P.H. WEISZ
*Institut für Theoretische Physik der Freien Universität Berlin, W. Germany*

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Zamolodchikov's proposed exact solution of the Thirring model $S$-matrix satisfying unitarity and crossing and having certain required properties in the semiclassical limit, is shown to be in agreement with perturbation theory in the Thirring model coupling constant to third order. A useful integral representation of the soliton-soliton phase shift corresponding to Zamolodchikov's solution is exhibited.

## 1. Introduction

Only now, long after the satisfactory solution of the massless Thirring model [1] is the structure of its massive counterpart being uncovered and found to be equally fascinating. The recent progress was initiated by Coleman's [2] remarkable discovery of the equivalence of the massive Thirring model to the quantum sine-Gordon theory. Since then a concentrated effort has been invested in studying various aspects of the models, and many important results have been established [3].

In particular the existence of an infinite set of conserved currents has been established in both theories [4,5] *. The corresponding conservation laws imply that in any scattering process the set of outgoing momenta are identical to the set of incoming momenta. This has drastic effects on the $S$-matrix elements. Firstly absence of particle production implies that elastic unitarity holds for the two-particle $S$-matrix elements to arbitrarily high energies. Secondly the $n$-particle $S$-matrix elements are determined as products of the two-particle $S$-matrix elements (factorisation) [6]. These results are at first surprising from the viewpoint of four dimensions where absence of particle production in conjunction with non-trivial elastic scattering is in contradiction with unitarity and crossing [7]. In two dimensions this is however not the case, as was stressed by Schlitt [8].

The general equations expressing elastic unitarity and crossing in two dimensions are reviewed in sect. 2. It has recently been suggested [9] that the classification of solutions of these equations coupled with consistency conditions on the spectrum

* So far, only the first non-trivial current has been rigorously treated for the Thirring model.

arising from factorisation might lead to a (practically) unique $S$-matrix. In the case of the sine-Gordon theory the semiclassical spectrum is derived. Given the $S$-matrix the problem of determining matrix elements of Heisenberg fields could be initiated [10].

Meanwhile in a very interesting recent article, Zamolodchikov [11] has ingeniously constructed a solution of the coupled unitarity and crossing equations by imposing certain semiclassical results. This solution is elaborated upon in sect. 3. A useful representation (3.11) for the soliton-soliton phase shift is derived.

The main purpose of this work is to report that Zamolodchikov's solution is in agreement with perturbation theory in the Thirring model up to third order; this is presented in sect. 4. This confrontation with perturbation theory is all the more impressive when one compares the failure of the solution of Berezin and Sushko [12] already at first order [13]. It is remarked that the results of perturbation theory at second order can be uniquely obtained from general principles and known structure of the diagrammatic contributions, without evaluating diagrams explicitly. At third order only one constant associated with singular order by order threshold behaviour of one of the amplitudes remains undetermined by the above mentioned principles and requires thorough analysis of a certain class of diagrams.

## 2. Elastic scattering in two dimensions

Consider the scattering of a particle ($s$) with its antiparticle ($\bar{s}$). Assuming the theory to be parity conserving * the elastic $s\bar{s}$ $S$-matrix element may be written as **

$$
\begin{aligned}
&{}^{\text{out}}\langle s(\tilde{p}_1)\bar{s}(\tilde{p}_2)|s(p_1)\bar{s}(p_2)\rangle^{\text{in}} \\
&\quad = (2\pi)^2 2p_1^0 2p_2^0[\delta(\tilde{p}_1^1 - p_1^1)\delta(\tilde{p}_2^1 - p_2^1)F(\nu + i0) + \delta(\tilde{p}_1^1 - p_2^1)\delta(\tilde{p}_2^1 - p_1^1) \\
&\quad \times B(\nu + i0)] \ , \qquad (2.1)
\end{aligned}
$$

where $\nu = p_1 p_2/m^2$ ($\geqslant 1$ in the physical scattering region.) The amplitudes $F$ and $B$ describe forward and backward scattering respectively, which are the only allowed elastic configurations in two dimensions. The physical amplitudes are boundary

* If parity is not conserved, then each amplitude can be split into a parity even and parity odd part,

$$F(p_1, p_2) = F_{\text{C}}(\nu) + \text{sign}(p_1^\mu \epsilon_{\mu\nu} p_2^\nu) F_{\text{N}}(\nu)$$

where we have used the fact that $(p_1 p_2)^2 - (p_1 \epsilon p_2)^2 = p_1^2 p_2^2$. $C$ parity conservation gives extra constraints in the amplitudes, as is the case for the Federbush model [14]. The Federbush model also possesses an infinite number of conservation laws, however, the $S$-matrix elements are rather uninteresting energy independent phases.

** State normalization is chosen to be

$${}^{\text{in}}\langle s(\tilde{p}_1)\bar{s}(\tilde{p}_2)|s(p_1)\bar{s}(p_2)\rangle^{\text{in}} = (2\pi)^2 2p_1^0 2p_2^0 \delta(\tilde{p}_1^1 - p_1^1)\delta(\tilde{p}_2^1 - p_2^1) \ .$$

values of real analytic functions $F(z)$, $B(z)$ cut along the real axes from 1 to $\infty$ and from $-1$ to $-\infty$.

Under crossing, forward $s\bar{s}$ crosses into $ss$ (or equivalently $\bar{s}\bar{s}$) scattering, for which there is no distinction between forward and backward, whereas backward $s\bar{s}$ crosses into itself (in sharp contrast to the situation in four dimensions). We thus have *

$$^{\text{out}}\langle s(\tilde{p}_1)s(\tilde{p}_2)|s(p_1)s(p_2)\rangle^{\text{in}} = F(-\nu - i0)^{\text{in}}\langle s(\tilde{p}_1)s(\tilde{p}_2)|s(p_1)s(p_2)\rangle^{\text{in}} , \tag{2.2}$$

$$B(z) = B(-z) . \tag{2.3}$$

A physical bound state appears as a pole on the real $z$-axis of the physical sheet.

Elastic unitarity implies for $s\bar{s}$ scattering, (using real analyticity)

$$F(z)F(\bar{z}) + B(z)B(\bar{z}) = 1 , \tag{2.4}$$

$$F(z)B(\bar{z}) + B(z)F(\bar{z}) = 0 , \tag{2.5}$$

and for $ss$ scattering

$$F(-z)F(-\bar{z}) = 1 , \tag{2.6}$$

for $z = \nu + i0$, $1 \leqslant \nu < \nu_{\text{I}}$ where $\nu_{\text{I}}$ is the position of the next threshold.

The amplitudes $F_\pm$ describing scattering of states $|s(p_1)\bar{s}(p_2)\rangle \pm |s(p_2)\bar{s}(p_1)\rangle$ of definite $C$ parity are given by

$$F_\pm(z) = F(z) \pm B(z) . \tag{2.7}$$

These obey separate unitarity equations

$$F_\pm(z)F_\pm(\bar{z}) = 1 . \tag{2.8}$$

In the absence of particle production, which is the case for the Thirring model the elastic unitarity equations hold for the entire range $1 \leqslant \nu < \infty$. Moreover if it is assumed that the amplitudes are meromorphic functions in the rapidity difference variable $\theta$ defined by $z = \cosh\theta$ **, (as is the case in low orders of perturbation theory) then the crossing symmetry relation (2.3) becomes,

$$B(\theta) = B(i\pi - \theta) , \tag{2.9}$$

and the unitarity equations (2.4)–(2.6) can be written

$$F(\theta)F(-\theta) + B(\theta)B(-\theta) = 1 , \tag{2.10}$$

$$F(\theta)B(-\theta) + B(\theta)F(-\theta) = 0 , \tag{2.11}$$

$$F(i\pi + \theta)F(i\pi - \theta) = 1 , \tag{2.12}$$

* Note as yet no reference need be made to the statistics obeyed by the particles $s$.

** Under this transformation the first sheet in the $z$ plane is mapped into the strip $0 < \text{Im}\,\theta < \pi$. Real analyticity now implies $\overline{F(\theta)} = F(-\bar{\theta})$ and similarly for $B$.

for all $\theta$. Define

$$A(\theta) = B(\theta)/F(\theta) . \tag{2.13}$$

Then eq. (2.11) is satisfied if and only if,

$$A(\theta) = -A(-\theta) . \tag{2.14}$$

It then remains to solve the eqs. (2.10), (2.12), which can now be rewritten as

$$B(\theta)B(-\theta) = A(i\pi + \theta)A(i\pi - \theta)$$

$$= \frac{A(\theta)^2}{A(\theta)^2 - 1} , \tag{2.15}$$

subject to the conditions (2.9) and (2.14). Note that to satisfy the crossing symmetry equation (2.9) it is obviously sufficient to make the ansatz

$$B(\theta) = X(\theta)X(i\pi - \theta) . \tag{2.16}$$

## 3. Solutions of the coupled unitarity and crossing relations

There are obviously solutions with $B(z) = 0$, which mean absence of backward $s\bar{s}$ scattering. These can be completely classified by

$$F(z) = \frac{g(z)\sqrt{1 - z^2} + 1}{g(z)\sqrt{1 - z^2} - 1} , \tag{3.1}$$

with $g(z)$ an arbitrary meromorphic function. Examples of this occur in the case of scattering of self conjugate particles, in which case $g(z)$ is even [8,9]. Another famous example is that of Berezin and Sushko [12]

$$F(\theta) = \frac{\sinh \frac{1}{2}(\theta + ig)}{\sinh \frac{1}{2}(\theta - ig)} . \tag{3.2}$$

In a recent work, Zamolodchikov [11] ingeniously constructed a solution with non-vanishing backward scattering. His solution was based on requiring for the full theory the following semiclassical sine-Gordon theory properties: (i) The semiclassical soliton-antisoliton bound state (breathers) spectrum [15] *

$$m_n = 2m \sin \tfrac{1}{2} n\lambda\pi , \qquad n = 1, 2, ..., < \lambda^{-1} , \tag{3.3}$$

where $\lambda$ is related to the conventional sine-Gordon coupling constant [2] by

$$\lambda = \frac{\beta^2}{8\pi}\left[1 - \frac{\beta^2}{8\pi}\right]^{-1} > 0 . \tag{3.4}$$

* Physical bound states appear as poles in $F(\theta)$ (and or $B(\theta)$) on the imaginary $\theta$-axis $0 < (\mathrm{Im}\ \theta = \pi(1 - n\lambda)) < \pi$.

The rationale for this assumption is given in ref. [9]. (ii) Vanishing of backward scattering for the special values $\lambda = 1/n$, with $n$ a positive integer [16]. (iii) The absence of resonances [15].

Zamolodchikov's solution is given by

$$A(\theta) = \frac{-i \sin(\pi/\lambda)}{\sinh(\theta/\lambda)} , \tag{3.5}$$

$$F(\theta) = -\frac{i}{\pi} \sinh\frac{\theta}{\lambda} R(\theta) R(i\pi - \theta) , \tag{3.6}$$

with

$$R(\theta) = \Gamma\left(1 + \frac{i\theta}{\lambda\pi}\right) \prod_{l=1}^{\infty} \frac{\Gamma((1/\lambda)[2l + i\theta/\pi])\Gamma(1 + (1/\lambda)[2l + i\theta/\pi])}{\Gamma((1/\lambda)[2l + 1 + i\theta/\pi])\Gamma(1 + (1/\lambda)[2l - 1 + i\theta/\pi])} . \tag{3.7}$$

Note that at the $n$th bound state position * $\theta = \theta_n = i\pi(1 - n\lambda)$ we have

$$A(\theta_n) = (-1)^{n-1} , \tag{3.8}$$

i.e. the bound states have alternating $C$ parity. This is consistent with viewing the $n$th bound state as a bound state of $n$ lowest bound states [9] and assigning the lowest bound state negative $C$ parity.

It is convenient to derive a representation for the soliton-soliton phase shift which is more amenable to perturbation expansions than the infinite product (3.7). This can be achieved by using Malmstén's formula [17]

$$\log \Gamma(z) = \int_0^{\infty} \left[ z - 1 - \frac{1 - \exp\{-(z-1)t\}}{1 - e^{-t}} \right] \frac{e^{-t}}{t} dt , \tag{3.9}$$

valid for $\mathrm{Re}\, z > 0$. One obtains for the soliton-soliton phase shift defined by

$$F(i\pi - \theta) = e^{2i\delta_{ss}(\theta)} , \tag{3.10}$$

the integral representation

$$\delta_{ss}(\theta) = \tfrac{1}{2} \int_0^{\infty} \frac{dt}{t} \frac{\sin(\theta t/\pi) \sinh \frac{1}{2}(\lambda - 1)t}{\sinh \frac{1}{2}\lambda t \cosh \frac{1}{2} t} , \tag{3.11}$$

for $|\mathrm{Im}\, \theta| < \min[\pi, \lambda\pi]$. This exhibits the absence of physical bound states for $\lambda > 1$. It is now easily checked that

(a) this phase shift has the desired semiclassical limit [18] and reproduces the

* The limit $\theta \to \infty$ can also be realized by the formal massless limit $m \to 0$, $s$ fixed. For the $S$-matrix in the zero mass model see ref. [19].

known first quantum correction [19]

$$\delta_{ss}(\theta) \underset{\substack{\beta\to 0\\ \theta\in\mathbb{R}^+}}{\to} \left(\frac{8\pi}{\beta^2}-1\right)\frac{2}{\pi}\int_0^{\tanh\frac{1}{2}\theta}\frac{\log x}{1-x^2}\,\mathrm{d}x+\tfrac{1}{4}\pi\,, \tag{3.12}$$

(b) $\delta_{ss}$ has the normalized threshold behaviour

$$\delta_{ss}(0)=0\,, \tag{3.13}$$

(c) $\delta_{ss}$ tends to a constant as $\theta\to\infty$ [20]

$$\delta_{ss}(\infty)=\tfrac{1}{4}\pi\left[1-\frac{1}{\lambda}\right]. \tag{3.14}$$

## 4. The Thirring model elastic $S$-matrix amplitudes in perturbation theory

In this section we derive the $S$-matrix amplitudes $F, B$ of the massive Thirring model in perturbation theory in the coupling constant $g$ appearing in the (formal) interaction Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}gJ_\mu J^\mu=g\bar{\psi}\psi\bar{\psi}\psi\,. \tag{4.1}$$

This has already been performed to second order by Hählen [13] who contrasted the result with the Berezin-Sushko [12] proposed solution (3.2). Zamolodchikov [11] has checked the agreement with his solution to second order. We extend this calculation to third order but for completeness we repeat the essential steps of the lower order calculations. We define the $T$-matrix elements $f, b$ by

$$F(z)=1+\frac{1}{\sqrt{1-z^2}}\,f(z)\,, \tag{4.2}$$

$$B(z)=\frac{1}{\sqrt{1-z^2}}\,b(z)\,, \tag{4.3}$$

and write $f, b$ as power series in $g$

$$f(z)=\sum_{n=1}^{\infty}f_n(z)g^n\,, \tag{4.4}$$

$$b(z)=\sum_{n=1}^{\infty}b_n(z)g^n\,. \tag{4.5}$$

We now impose the unitarity equations order by order. To first order, unitarity im-

Fig. 1.

plies that $f_1, b_1$ are meromorphic functions in the $z$ plane. In the tree graph approximation, fig. 1, one obtains

$$f_1(z) = 1 + z \,, \tag{4.6}$$

$$b_1(z) = 2 \,. \tag{4.7}$$

An extra information that we have is that in a renormalisable field theory higher order Feynman diagrams give contributions to the amplitudes with asymptotic behaviour only logarithmically worse than those of the tree graphs. Given this additional information one immediately asks why it is not possible to derive the Thirring model results to all orders by iterating the tree graph results (4.6) (4.7) in the unitarity and crossing symmetry and using dispersion relations for the amplitudes. The answer lies in the fact that without recourse to the explicit diagrams the threshold behaviours of the amplitudes are not completely specified order by order *. We only have partial knowledge of the threshold behaviour, explicitly order by order in $g$

$$f_n(-1) = 0 \,, \tag{4.8}$$

and

$$(f - b)_n(1) = 0 \,, \tag{4.9}$$

however $(f + b)_n$ diverges as $z \to 1$.

We now proceed to the second order. Very little additional information is needed other than observing that in perturbation theory we must have amplitudes of the form

$$f_2(z) = P_1(z)r(z) + P_2(z)r(-z) + (az + b)r(-1) \,, \tag{4.10}$$

$$b_2(z) = P_3(z)r(z) + P_3(-z)r(-z) + cr(-1) \,, \tag{4.11}$$

where $r(z)$ is the function appearing in the one-loop diagram, fig. 2. This function has just a right hand cut **

$$r(z) = \frac{1}{\pi} \int_1^\infty \frac{dz'}{z' - z} \frac{1}{\sqrt{z'^2 - 1}} \,. \tag{4.12}$$

$r(z)$ behaves as $(1/z) \log z$ as $z \to \infty$ which implies that $P_1, P_2$ are *a priori* second or-

* This point has also been remarked by Flume and Glaser.
** Note the relation $r(z) + r(-z) = 1/\sqrt{1 - z^2}$.

Fig. 2.

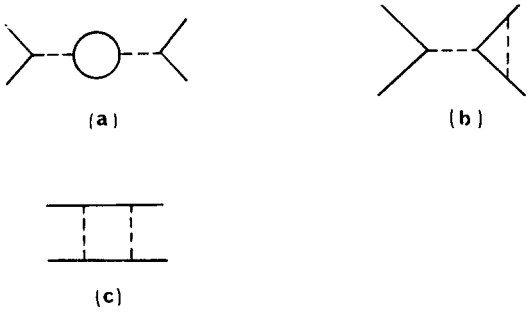


Fig. 3.

der polynomials in $z$ and $P_3$ first order. Substituting (4.10) and (4.11) into the second order unitarity and crossing relations yields unambiguously the desired result [13]

$$P_1(z) = \tfrac{1}{2}[(1+z)^2 + 4] \ , \tag{4.13}$$

$$P_2(z) = \tfrac{1}{2}(1+z)^2 \ , \tag{4.14}$$

$$P_3(z) = 2(1+z) \ . \tag{4.15}$$

The threshold relations (4.8), (4.9) yield

$$\tfrac{1}{2}c = a = 2 + b \ . \tag{4.16}$$

Thus to second order the amplitudes can be obtained without specific diagram calculations which involve tedious $\gamma$-matrix algebra. For completeness however we exhibit in fig. 3 the diagrams which contribute to this order. Here equivalent infinite mass scalar boson exchange (dashed lines) is used, which clearly indicates the various ways (up to direction) that the fermion charges can flow. The contributions to the 4-point vertex function from diagrams 3a and 3b exactly cancel and one is left only with contributions from 3c.

Now we finally turn to the third order calculation. As far as external fermion contractions are concerned there are just two classes of diagrams I, II which are displayed in fig. 4. The structure of the contributions from class I are clear. Using Cutkosky rules for the diagrams with external particles on shell *, one can convince oneself

* See e.g. ref. [8].

I II

Fig. 4.

that the only new integral appearing in a general contribution is

$$\mathcal{g}(z) = \frac{1}{\pi} \int_1^\infty \frac{\mathrm{d}z'}{(z'-z)} \frac{r(-z')}{\sqrt{z'^2-1}} \ . \tag{4.17}$$

This can be evaluated to yield *

$$\mathcal{g}(z) = -\tfrac{1}{2} r(z)^2 - \frac{1}{4} \frac{1}{z-1} \ . \tag{4.18}$$

The upshot of this analysis is that the diagrams of class II are also expressible in terms of the one loop function $r(z)$. Note, moreover, that the threshold behaviour of $\mathcal{g}(z)$ is only like that of $r(z)$. Summing up what we have learnt so far, the perturbation result must have the structure

$$\begin{aligned} f_3(z) &= Q_1(z) r(z)^2 + Q_2(z) r(-z)^2 + Q_3(z) r(z) r(-1) \\ &\quad + Q_4(z) r(-z) r(-1) + \frac{d}{z-1} + \frac{e}{z+1} + (fz+h) r(-1)^2 \ , \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} b_3(z) &= Q_5(z) r(z)^2 + Q_5(-z) r(-z)^2 + Q_6(z) r(z) r(-1) \\ &\quad + Q_6(-z) r(-z) r(-1) + k\left(\frac{1}{z-1} - \frac{1}{z+1}\right) + l r(-1)^2 \ , \end{aligned} \tag{4.20}$$

where $Q_1, Q_2$ are *a priori* third order polynomials; $Q_3, Q_4, Q_5$ are second order and $Q_6$ is first order. Plugging the ansatz into the third order unitarity equations yields the polynomials $Q_i$

$$Q_1(z) = 5z + 1 \ , \tag{4.21}$$

$$Q_2(z) = -(1+z) \ , \tag{4.22}$$

$$Q_3(z) = (1+z)(az+b) + 2c \ , \tag{4.23}$$

$$Q_4(z) = (1+z)(az+b) \ , \tag{4.24}$$

$$Q_5(z) = 2(z^2+z+1) \ , \tag{4.25}$$

$$Q_6(z) = 2(az+b) + (1+z)c \ . \tag{4.26}$$

* Note the property $\mathcal{g}(z) + \mathcal{g}(-z) = r(z) r(-z)$; a property which comes in so effectively in evaluating $S$-matrix elements in $\Phi^4$ theory to third order [8].

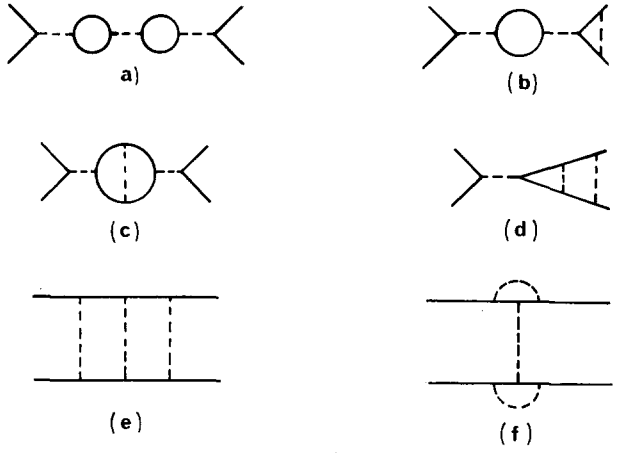


Fig. 5.

Threshold conditions give the further constraints

$$e = 0 \,, \tag{4.27}$$

$$\tfrac{1}{2}(d+1) = [4(a-1) + h - f]\, r(-1)^2 \,, \tag{4.28}$$

$$k = d \,, \tag{4.29}$$

$$\tfrac{1}{2}(d+1) = [-4(a-1) - h - f + l]\, r(-1)^2 \,. \tag{4.30}$$

As mentioned above, we have unfortunately thus far been unable to discover a general property that would specify the precise threshold behaviour of $(f+b)$ order by order; in particular at third order the constant $d$ remains to be determined. We thus resorted to the extremely tedious perturbation theory calculation *. We evaluated both class I and class II contributions, despite the fact that by previous considerations it would have sufficied to extract $d$ which arises directly only from class II contributions, or alternatively determine the leading threshold behaviour which comes from the class I $s\bar{s}$ $s$-channel chains. The full calculation, at the very least, served as a good consistency check.

The diagrams occurring in class I are exhibited in fig. 5. Contributions to the vertex function from 5a and 5b cancel exactly, as do those from 5c and 5d. We are left only with 5e, 5f which yield contributions $f_3^{\mathrm{I}}, b_3^{\mathrm{I}}$ to $f_3, b_3$ as follows,

$$f_3^{\mathrm{I}}(z) = \tfrac{1}{4}(1+z)\{[(1+z)^2 + 12]\, r(z)^2 + (1+z)^2 r(-z)^2$$

* The author in fact calculated these contributions before the appearance of Zamolodchikov's work, to gain clues as to the nature of the exact solution!

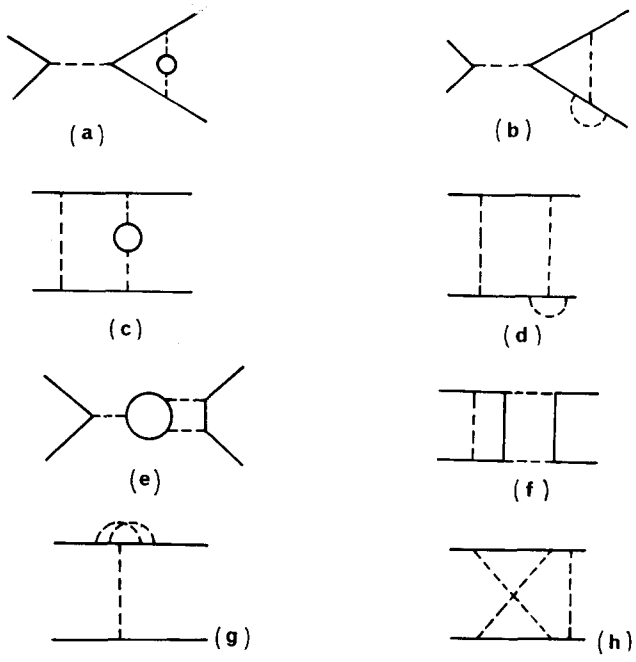


Fig. 6.

$$+ \text{ terms linear in } r(z), r(-z)\} \,, \tag{4.31}$$

$$b_3^{\mathrm{I}}(z) = \tfrac{1}{2}\{[3(1+z)^2+1]r(z)^2 + [3(1-z)^2+1]r(-z)^2$$

$$+ \text{ terms linear in } r(z), r(-z)\} \,. \tag{4.32}$$

Comparing the polynomials occurring here with the correct results (4.21)–(4.26), it is dubious whether the iterated one-loop chain approximation gives a sensible approximation to the correct amplitudes in any kinematic domain except possibly close to the $s\bar{s}$ $s$-channel threshold.

The various diagrams contributing to class II are shown in fig. 6. Here again cancellations occur; contributions 6a cancel with 6b and 6c cancel with 6d. We are left with contributions from 6c, 6d, 6e, 6f among which (at least on shell) certain simplifications occur *. As mentioned above the result for $d$ can be obtained by equating the leading threshold behaviour of $f_3(z)$ (4.19) with that of $f_3^{\mathrm{I}}(z)$ (4.31). One

* Because there is no (infinite) coupling constant renormalization in the Thirring model the vertex functions are finite in every order; however, individual diagrams are divergent. We employed the BPHZ renormalization scheme in our calculations.

obtains

$$d = -1 \,. \tag{4.33}$$

These sorts of considerations of leading threshold behaviours can certainly be extended to higher orders. Now one constant remains undetermined at every order $n > 1$. This is of course only a reflection of the fact that we have not defined the coupling constant $g$ by a normalization condition. We choose to define $g$ by the requirement

$$F(-z) \to \mathrm{e}^{-ig} \qquad \text{as } z \to \infty + i0 \,. \tag{4.34}$$

This is possible since the asymptotic behaviour of $f$ is in each order no worse than that of the tree diagram, which is in agreement with the existence of a finite massless limit [20]. The requirement (4.34) requires that we set

$$a = 0 \,, \qquad f = -\frac{1}{6r(-1)^2} \,. \tag{4.35}$$

Using the explicit expression for $r$ in the rapidity variable

$$r(\theta) = \frac{1}{\pi}\frac{i\pi - \theta}{\sinh\theta} \,, \tag{4.36}$$

one obtains to third order

$$A(\theta) = \frac{i2g}{\sinh\theta}\left[1 - \frac{g}{\pi}2\theta\coth\theta + \left(\frac{g}{\pi}\right)^2 2\left\{\theta^2\frac{(1+\cosh^2\theta)}{\sinh^2\theta} - \tfrac{1}{3}\pi^2\right\} + \ldots\right] , \tag{4.37}$$

$$\delta_{ss}(\theta) = -\tfrac{1}{2}g\tanh\frac{\theta}{2} + \frac{g^2}{\pi}\frac{1}{\sinh\theta}\left[\frac{\theta}{\sinh\theta} - 1\right]$$
$$+\frac{g^3}{\pi^2}\frac{2}{\sinh\theta}\left[1 - \frac{\theta^2\cosh\theta}{\sinh^2\theta} + \tfrac{1}{12}\pi^2\tanh^2\frac{\theta}{2}\right] + \ldots \,. \tag{4.38}$$

We now wish to compare these perturbation theoretic results with the Zamolodchikov solution. The connection between $\lambda$ and $g$ is supplied by comparing the normalization condition (4.34) with the Zamolodchikov result (3.14). This gives us the (conventional) identification

$$\frac{2g}{\pi} = \frac{1}{\lambda} - 1 \,. \tag{4.39}$$

Expanding Zamolodchikov's solution (3.5) and (3.11) in powers of $g$ using the Fourier transform,

$$\int_{-\infty}^{\infty} \mathrm{d}t\,\mathrm{e}^{it\theta/\pi}\tanh\frac{ta}{2} = \frac{2\pi i}{a\sinh(\theta/a)} \,, \tag{4.40}$$

(and its derivatives), one finds exact agreement with the perturbation theoretic results (4.37) and (4.38).

## 5. Conclusion

The general classification of the solutions to the coupled unitarity and crossing equations, with non-trivial backward $s\bar{s}$ scattering, remains to be completed. Such a classification is desirable for the application of the previously proposed programme [9] of deriving the Thirring model from such solutions, subject also to consistency conditions on the spectrum in the $n$-particle channels. This is in turn constitutes the first step in the larger programme of deriving matrix elements of the Heisenberg fields and ultimately the construction of the full Green functions. An intermediate question is of course, when does a particular solution correspond to a local quantum field theory?

A particular solution with non-trivial backward scattering has recently been obtained by Zamolodchikov [11]. This solution is an excellent candidate for the exact Thirring model $S$-matrix since it has, by very construction, known $S$-matrix properties in the semiclassical limit and furthermore, as we have demonstrated, agrees with the Thirring model up to third order in perturbation theory. The investigation of the matrix elements of Heisenberg fields, in particular the soliton electromagnetic form factor corresponding to this solution, is in progress [10,11].

As we have conjectured above, the massive Thirring model is probably as soluble as is the massless model. The solubility is a direct consequence of the validity of an infinite set of conservation laws which forbid particle production. From the point of view of strong interaction physics, this property is however in many ways unfortunate, since it would seem to render the model even less interesting for high energy phenomenology than its very two-dimensional nature. The generalizations of the model to include more fermions, with various internal symmetries, are in this respect more interesting since they do not in general exclude particle production. It is hoped that the study of the Thirring model, and its generalizations will serve in the future, as an inspiration for new general ideas applicable also in four dimensions, as it has in the past.

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