

RELATIVISTIC FACTORIZED S -MATRIX IN TWO DIMENSIONS HAVING $O(N)$ ISOTOPIC SYMMETRY

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The factorized total S -matrix in two space-time dimensions with isotopic $O(N)$ symmetry is constructed. Arguments are presented that this S -matrix is the exact one of the $O(N)$ chiral field.

1. Introduction

Recent progress in the study of two-dimensional quantum field theory has led to the extensive development of some models which have a remarkable property: an infinite set of conservation laws, leading to the absence of multiple production and the conservation of the set of individual momenta of particles in scattering [1,2]. The factorization of the total S -matrix also seems to be an effect of these conservation laws [3]. The classical analog of all these models is connected with non-linear equations, completely integrable by the inverse scattering method.

An example of this type is the massive Thirring model (MTM), or, equivalently, the quantum sine-Gordon model. It turns out that due to the simplified scattering properties of this model, all the elements of the total S -matrix [4–6], and some off-shell matrix elements [7], can be found explicitly.

In a recent paper, Karowski, Thun, Truong and Weisz [8] have shown that the analyticity, unitarity and factorization equations [5,6] of this model can be solved uniquely giving a one-parameter set of solutions, where the parameter can be connected with the MTM coupling constant.

Being the model of charged fermions, the MTM has the phase symmetry $U(1) = O(2)$. In the present paper the factorized S -matrix with isotopic $O(N)$ symmetry is constructed for any $N \geq 3$. We assume the existence of an isovector N -plet of particles of mass m and require $O(N)$ isosymmetry of the S -matrix elements. It turns

out that under these requirements the S -matrix can be determined uniquely *, without parameters except for the overall mass scale. The latter is shown in sects. 2 and 3, where we derive the explicit form of the S -matrix.

At the present time we cannot definitely say which two-dimensional field theory (if any) leads to this S -matrix. We have some arguments, however, that such a theory is an $O(N)$ chiral field model described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2g_0} \sum_{i=1}^N (\partial_\mu n_i)^2, \quad (1.1)$$

with the constraint

$$\sum_{i=1}^N n_i^2 = 1. \quad (1.2)$$

This model is $O(N)$ symmetric, renormalizable and asymptotically free [10,11]. The infrared charge singularity in this model seems to lead to the disintegration of the Goldstone vacuum and to mass transmutation of particles [12], which should form the $O(N)$ multiplets in this case. The $O(N)$ symmetry of the spectrum and in particular the existence of an isovector N -plet of massive particles are clear in the framework of the $1/N$ expansion of the model [13] (see sect. 4).

In asymptotically free theories with spontaneous mass transmutation, the observable characteristics do not depend on the coupling constant (due to the renormalizability) [14]. We should like to mention in this connection that the S -matrix obtained in sect. 3 does not depend on free parameters.

It is worth mentioning that the classical equation of motion of model (1.1) possesses an infinite set of conservation laws [15]. However, in quantum theory, such laws, if they remain in this case at all, are surely modified by quantum corrections. Thus, the existence of an infinite set of conservation laws in quantum theory, or direct factorization of the quantum S -matrix, requires special verification. A suitable device to do this is the $1/N$ perturbation theory of model (1.1).

In sect. 4 we show that in $1/N$ perturbation of model (1.1), there is no particle production and the S -matrix really factorizes to the order of $1/N^2$. Furthermore, the two-particle matrix elements calculated to the order of $1/N$ coincide with the corresponding terms of the $1/N$ expansion of the S -matrix obtained in sect. 3.

Comparison of the ultraviolet asymptotics of the S -matrix of sect. 3 with the results of the ordinary g -perturbations of model (1.1) is another argument in favour of the supposed connection. Although in such perturbation theory one deals with the $(N-1)$ -component multiplet of Goldstone particles instead of the N -component multiplet of massive particles and, hence, faces infrared divergences, one may suppose

* In this case, as well as in the MTM, the unitarity, analyticity and factorization conditions admit, of course, an arbitrariness of the CDD-type, so here we mean the uniqueness of the "minimum" solution, i.e., the solution with the minimum set of singularities (see sect. 3).

that the contribution of ultraviolet logarithms of perturbation theory to the scattering amplitudes gives the correct asymptotic behaviour of these amplitudes (at least up to g^2). Comparison with perturbation theory is performed to order g^2 in sect. 4. The result also supports our hypothesis.

2. Analyticity, unitarity and factorization equations for the $O(N)$ symmetric S -matrix

Consider an $O(N)$ isovector N -plet of particles of mass m . The S -matrix element of the $2 \rightarrow 2$ scattering can be taken in the form

$$i_{ik} S_{jl} = \frac{p_1}{p_2} \frac{l}{k} \frac{j}{l} \frac{p'_1}{p'_2} = \delta(p_1 - p'_1) \delta(p_2 - p'_2) [\delta_{ik} \delta_{jl} \sigma_1(s) + \delta_{ij} \delta_{kl} \sigma_2(s) + \delta_{il} \delta_{jk} \sigma_3(s)], \quad (2.1)$$

where $s = (p_1 + p_2)^2$. Further it will be convenient to use the rapidities θ_i instead of the momenta p_i

$$p_i^0 = m \operatorname{ch} \theta_i, \quad p_i^1 = m \operatorname{sh} \theta_i. \quad (2.2)$$

Then σ_1 , σ_2 and σ_3 will be functions of the rapidity difference of the initial particles $\theta = |\theta_1 - \theta_2|$, which is simply connected with s :

$$s = 2m^2(1 + \operatorname{ch} \theta). \quad (2.2a)$$

Note that under the transformation (2.2a) the threshold points $s = 0$ and $s = 4m^2$ of the functions $\sigma(s)$ (which are square-root branch points due to two-particle unitarity) become non-branch points of $\sigma(\theta)$. Thus σ_1 , σ_2 and σ_3 are meromorphic functions of θ .

The two-particle unitarity conditions and the crossing-symmetry relations of the two-particle S -matrix (2.1) can be represented as the functional equations

$$\sigma_2(\theta) \sigma_2(-\theta) + \sigma_3(\theta) \sigma_3(-\theta) = 1, \quad (2.3a)$$

$$\sigma_2(\theta) \sigma_3(-\theta) + \sigma_3(\theta) \sigma_2(-\theta) = 0, \quad (2.3b)$$

$$[N\sigma_1(\theta) + \sigma_2(\theta) + \sigma_3(\theta)] [N\sigma_1(-\theta) + \sigma_2(-\theta) + \sigma_3(-\theta)] = 1, \quad (2.3c)$$

$$\sigma_2(\theta) = \sigma_2(i\pi - \theta), \quad (2.4a)$$

$$\sigma_3(\theta) = \sigma_3(i\pi - \theta). \quad (2.4b)$$

Eqs. (2.3) and (2.4) do not determine the functions $\sigma(\theta)$. In addition to unitarity and analyticity let us require factorization of the multiparticle S -matrix.

Factorization means the following special structure of the multiparticle S -matrix: the multiparticle S -matrix elements are sums of terms, each being the product of

two-particle S -matrix elements, as if multiparticle scattering were the consequence of two-particle collisions [5,6,16,17].

The factorized S -matrix can be represented by a simple algebraic construction [5], which in our case consists of N types of special non-commutative symbols $A_i(\theta)$; $i = 1, 2, \dots, N$, each symbol corresponding to a certain component of the isovector multiplet. The asymptotic states of the scattering theory should be identified with products of these symbols, each symbol $A_i(\theta_a)$ corresponding to the particle with rapidity θ_a in the state. We identify the in (out) states with the products in which all symbols are arranged in order of decreasing (increasing) θ . Any in state can be reordered in terms of out states by means of the commutation rules

$$A_i(\theta_1) A_j(\theta_2) = \delta_{ij} \sigma_1(\theta_{12}) \sum_{k=1}^N A_k(\theta_2) A_k(\theta_1) + \sigma_2(\theta_{12}) A_j(\theta_2) A_i(\theta_1) + \sigma_3(\theta_{12}) A_i(\theta_2) A_j(\theta_1), \quad (2.5)$$

which correspond to the two-particle S -matrix (2.1). This construction represents the factorized total S -matrix.

The self-consistency of the above construction (namely the associativity property of A 's) requires σ_1 , σ_2 and σ_3 to satisfy certain functional identities. One obtains them by rearranging the product of three symbols $A_i(\theta_1) A_j(\theta_2) A_k(\theta_3)$ in two possible successions and requiring the results to be equal. The factorization property necessarily forces these identities, so we shall refer to them as the factorization equations.

The factorization equations have a simple meaning. Consider, for example, the collision of three particles with rapidities $\theta_1 > \theta_2 > \theta_3$. In the infinite past they have spatial coordinates $x_1 < x_2 < x_3$. The particles collide with each other sequentially in the interaction region, the succession of the collisions depending on the initial positions of particles, as is shown in fig. 1a and b.

In quantum mechanics both these possibilities contribute to the same outgoing wave. The conservation of the set of momenta implies the monochromatic nature of this wave; hence, the outgoing waves of the processes in figs. 1a and b should be coherent. The factorization equations ensure this coherence.

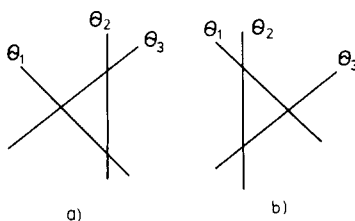


Fig. 1. Two possible successions in the collision of three particles.

The number and the form of factorization identities turn out to be different for the cases $N = 2$ and $N \geq 3$. For $N = 2$ the factorization equations are given in [5,6,8] and their solution is the sine-Gordon S -matrix. For the case $N \geq 3$ they acquire the form

$$\sigma_2 \sigma_3 \sigma_3 + \sigma_3 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3, \quad (2.6a)$$

$$\sigma_2 \sigma_1 \sigma_1 + \sigma_3 \sigma_2 \sigma_1 = \sigma_3 \sigma_2 \sigma_1, \quad (2.6b)$$

$$N \sigma_1 \sigma_3 \sigma_1 + \sigma_1 \sigma_3 \sigma_2 + \sigma_2 \sigma_3 \sigma_1 + \sigma_1 \sigma_3 \sigma_3 + \sigma_3 \sigma_3 \sigma_1 + \sigma_1 \sigma_2 \sigma_1 + \sigma_1 \sigma_1 \sigma_1 = \sigma_3 \sigma_1 \sigma_3, \quad (2.6c)$$

where the first, second and third σ in each term are functions of θ , $\theta + \theta'$ and θ' , respectively.

3. Solution of the unitarity, analyticity and factorization equations

In terms of the ratio $h(\theta) = \sigma_2(\theta)/\sigma_3(\theta)$, eq. (2.6a) takes the form

$$h(\theta) + h(\theta') = h(\theta + \theta'), \quad (3.1)$$

i.e.,

$$\sigma_3(\theta) = -i \frac{\lambda}{\theta} \sigma_2(\theta), \quad (3.2)$$

where λ is a certain parameter. The crossing-symmetry equations (2.4) lead to

$$\sigma_1(\theta) = -i \frac{\lambda}{i\pi - \theta} \sigma_2(\theta). \quad (3.3)$$

Note that (3.2) and (3.3) satisfy equations (2.3b) and 2.6b) identically. It is notable also that, after substitution of eqs. (3.2) and (3.3), eqs. (2.3c) and (2.6c) lead to the same algebraic equation for the parameter λ , which has (except for the trivial case $\lambda = 0$) the unique solution

$$\lambda = \frac{2\pi}{N-2}. \quad (3.4)$$

The remaining eq. (2.3a) acquires the form

$$\sigma_2(\theta) \sigma_2(-\theta) = \frac{\theta^2}{\theta^2 + \lambda^2}. \quad (3.5)$$

Eqs. (3.5) and (2.4a) form the system for $\sigma_2(\theta)$.

It is clear that these equations permit σ_2 to be multiplied by any $2\pi i$ periodic

meromorphic function which is real on the imaginary axis and satisfies the identities

$$\begin{aligned} f(\theta)f(-\theta) &= 1, \\ f(\theta) &= f(i\pi - \theta). \end{aligned} \quad (3.6)$$

Therefore the general solution having singularities on the imaginary axis only has the form

$$\sigma_2(\theta) = \left[\prod_{k=1}^L \frac{\text{sh } \theta + i \sin \alpha_k}{\text{sh } \theta - i \sin \alpha_k} \right] \sigma_2^{(0)}(\theta), \quad (3.7)$$

where α_k are real numbers and $\sigma_2^{(0)}$ is the “minimum” solution of eqs. (3.5) and (2.4a), i.e., the solution with the minimum set of singularities in the θ plane

$$\sigma_2^{(0)}(\theta) = Q(\theta)Q(i\pi - \theta), \quad (3.8)$$

where

$$Q(\theta) = \frac{\Gamma(\Delta - i\theta/2\pi) \Gamma(\frac{1}{2} - i\theta/2\pi)}{\Gamma(-i\theta/2\pi) \Gamma(\frac{1}{2} + \Delta - i\theta/2\pi)}, \quad (3.9)$$

and

$$\Delta = \frac{\lambda}{2\pi} = \frac{1}{N-2}. \quad (3.10)$$

In principle, all solutions (3.7) are permitted. However, the solution $\sigma_2 = \sigma_2^{(0)}$ is the only one which does not lead to an isospin degeneracy of the spectrum \star . This solution does not display any poles on the physical sheet of the s -plane, i.e., isovector particles cannot produce any bound states.

Note that in the case $N = 3$, i.e., $\Delta = 1$, the expression (3.8) is reduced to

$$\sigma_2^{(0)}(\theta) = \frac{\theta(i\pi - \theta)}{(2\pi i - \theta)(i\pi + \theta)}. \quad (3.11)$$

4. Comparison of the factorized S -matrix with the $1/N$ expansion of model (1.1)

It is convenient to develop the $1/N$ expansion of model (1.1) in the following way [13]. The generating functional for the Green functions of the $n_i(x)$ field can be written in the form $Z[J_i] = I[J_i]/I[0]$,

$$I[J_i] = \int \prod_x d\omega \prod_i dn_i \exp \{ i \int d^2x [\mathcal{L}'[n_i, \omega] + \sqrt{2g_0} J_i n_i] \}, \quad (4.1)$$

\star Another remarkable solution contains the single CDD pole $\alpha_1 = 2\pi\Delta$. Unlike $\sigma_2^{(0)}$, this solution corresponds to an attractive interaction and seems to be the exact S -matrix of the fundamental fermions of the Gross-Neveu model [14,18]. The arguments will be published elsewhere.

where

$$\mathcal{L}'[n_i, \omega] = \frac{1}{2g_0} [(\partial_\mu n_i)^2 - \omega n_i^2] + \frac{\omega(x)}{2g_0}. \quad (4.2)$$

The n_i integration in the integral (4.1) can be performed explicitly and leads to $Z[J_i] = I'[J_i]/I'[0]$,

$$I'[J_i] = \int \prod_x d\omega \exp \{i S_{\text{eff}}[\omega] + i \int J_i(x) J_i(x') G(x, x' | \omega) dx dx'\}, \quad (4.3)$$

where

$$S_{\text{eff}}[\omega] = i \frac{N}{2} \text{Tr} \ln (\partial_\mu^2 - \omega(x)) + \int \frac{\omega(x)}{2g_0} d^2x, \quad (4.4)$$

and $G(x, x' | \omega)$ is the Green function of the operator $\partial_\mu^2 - \omega(x)$. The perturbative calculation of the integral (4.3) leads to the $1/N$ expansion of model (1.1). The stationary-phase point of the integral (4.3), $\omega(x) = m^2 = \Lambda^2 \exp(-4\pi/Ng_0)$, should be taken into account, so the functionals S_{eff} and $G(x, x' | \omega)$ should be expanded in $\omega' = \omega - m^2$ rather than in ω .

It is convenient to use in calculations the following diagrammatic technique which contains:

(i) the ω' field propagator

$$D(K^2) = \text{---}\text{---}\text{---}^K = \frac{i}{N\Phi(K^2)}, \quad (4.5)$$

(ii) the n_i propagator

$$G_{ij}(K^2) = \frac{i}{K^2} j = \frac{i\delta_{ij}}{K^2 - m^2 + i\epsilon}, \quad (4.6)$$

(iii) the vertices

$$\text{---}\text{---}\text{---}^i \text{---}\text{---}\text{---}^j = i\delta_{ij} \quad ; \quad \text{---}\text{---}\text{---} = N \text{---}\text{---}\text{---} ; \quad \dots \quad \text{---}\text{---}\text{---} = N \text{---}\text{---}\text{---} ; \quad \dots \quad (4.7)$$

In this technique the closed loops of n_i field lines should not be drawn (they are already taken into account in (4.7)), and

$$i\phi(k^2) = \frac{1}{(2\pi)^2} \int \frac{d^2p}{(p^2 - m^2 + i\epsilon)((p+k)^2 - m^2 + i\epsilon)}. \quad (4.8)$$

The calculation of loops in (4.7) can be made explicitly by means of the following “cutting rule” [19] *. The arbitrary loop is the sum of terms, each corresponding to

* An analogous result for the arbitrary fermion loop has been obtained in ref. [20].

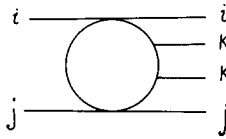


Fig. 2. The $2 \rightarrow 4$ amplitude.

any division of the loop through two lines.

$$\text{Diagram} = \sum \left[\text{Diagram}_1 \right] i\phi(s_{ij}) \left[\text{Diagram}_2 \right] ; \quad s_{ij} = (\kappa_i + \kappa_j)^2. \quad (4.9)$$

The momenta k_i and k_j are restricted by the condition $k_i^2 = k_j^2 = m^2$. The contribution of each division is equal to the product of the two trees separated by the dashed line in (4.9) by the function $i\phi(s_{ij})$. At s_{ij} fixed, the equations $k_i^2 = k_j^2 = m^2$ have two solutions ($k_i \leftrightarrow k_j$) both should be taken into account in rule (4.9).

Consider the $2 \rightarrow 4$ amplitude (fig. 2) to order $1/N^2$. For the sake of simplicity we shall concentrate on the case $i \neq j \neq k$. This amplitude is given by the sum of diagrams in fig. 3. Using the rule (4.9) one can replace the diagram in fig. 3g by the sum of the loop divisions.

Consider, for example, the division in fig. 4. Two solutions of $k_1^2 = k_2^2 = m^2$ are $k_1 = p_5, k_2 = p_6$ and $k_1 = p_6, k_2 = p_5$. The factor $i\phi(s_{56})$ in this division is the reciprocal wavy line with an opposite sign. Therefore the division in fig. 4 cancels out diagrams in fig. 3e and f. It is easy to check that other possible divisions of the triangle in fig. 4 cancel out diagrams in fig. 3a–d.

The cases $i = j, j = k$, and so on, contain more diagrams; however one can check that the same cancellation takes place in all these cases too.

Now let us turn to the process $3 \rightarrow 3$ (fig. 5) and consider again the case $i \neq j \neq k$. To order $1/N$ the matrix element contains disconnected diagrams only, the kinematics

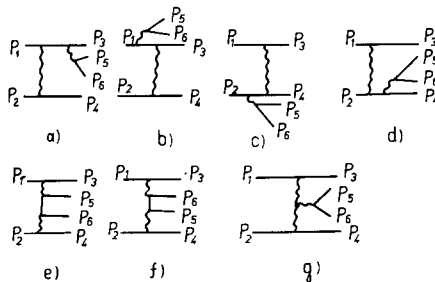


Fig. 3. Diagrams contributing to the amplitude in fig. 2 in the case $i \neq j \neq k$.

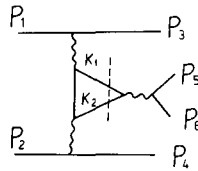


Fig. 4. Division of the three ω' vertex in the diagram in fig. 2g.

ensuring the conservation of the set of momenta. To order $1/N^2$ we have 7 connected diagrams listed in fig. 6.

It can easily be checked, that if all the intermediate propagators in the diagrams in figs. 6a–f are non-singular, different divisions of the diagram in fig. 6g cancel the other diagrams in the same manner, as in the previous example. Mass-shell singularities of diagrams in fig. 6a–f require more detailed analysis. For example, if $p'_1 \rightarrow p_3$, $p'_2 \rightarrow p_1$, $p'_3 \rightarrow p_2$, the diagrams in fig. 6c, d and f acquire mass-shell poles. It can be shown, however, that the principal parts of these three diagrams cancel each other, and one remains with some regular function and terms with mass-shell δ -functions. The diagram in fig. 6g cannot cancel the latter, being non-singular in this region (all the momentum transfers are space-like). Finally we are left with δ -function terms only, the δ -functions ensuring the factorized structure of the S -matrix element in fig. 5.

Using the technique (4.5), (4.6) and (4.7), one can calculate two-particle S -matrix elements. To order $1/N$ they are

$$\mathcal{S}_2(\theta) = \frac{P_1}{P_2} \frac{P_1}{P_2} + \frac{P_1}{P_2} \frac{P_1}{P_2} = 1 - i \frac{2\pi}{NSh\theta}, \quad (4.10a)$$

$$\mathcal{S}_3(\theta) = \frac{P_1}{P_2} \frac{P_2}{P_1} = -i \frac{2\pi}{N\theta}, \quad (4.10b)$$

$$\mathcal{S}_1(\theta) = \frac{P_1}{P_2} \frac{P_1}{P_2} = -i \frac{2\pi}{N(i\pi - \theta)}. \quad (4.10c)$$

The expressions (4.10a–c) indeed coincide with the first terms in the $1/N$ expansion of the solution (3.8), (3.2) and (3.3).

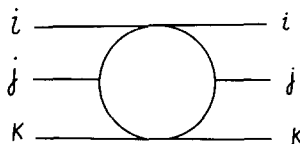
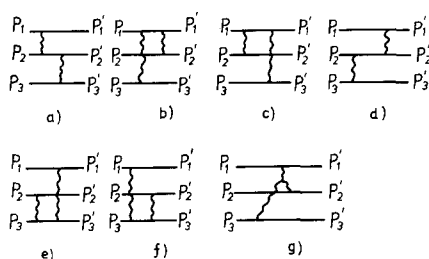


Fig. 5. The $3 \rightarrow 3$ amplitude.

Fig. 6. Diagrams contributing to the amplitude in fig. 5 in the case $i \neq j \neq k$.

Another possible expansion check of the S -matrix obtained is worth mentioning. Assuming the S -matrix (3.8), (3.2) and (3.3) to correspond to some renormalizable and asymptotically free field theory, one can expand matrix elements which are functions of

$$\ln \frac{s}{m^2} = \ln \frac{s}{\mu^2} + \int \frac{g(\mu)}{\beta(g)} dg, \quad (4.11)$$

in asymptotic series in powers of $g(\mu)$. Taking the first term [10] of the Gell-Mann–Low function of model (1.1)

$$\beta(g) = -\frac{N-2}{2\pi} g^2 + O(g^2), \quad (4.12)$$

one gets (up to $g^2(g = g(\mu))$)

$$\begin{aligned} \sigma_2(s) &= 1 - \frac{ig^2}{8} + O(g^3), \\ \sigma_3(s) &= -\frac{ig}{2} + i\frac{N-2}{2\pi} g^2 \ln \frac{s}{\mu^2} + O(g^3), \\ \sigma_1(s) &= i\frac{g}{2} - i\frac{N-2}{2\pi} g^2 \ln \frac{s}{\mu^2} - \frac{N-2}{8} g^2 + O(g^3). \end{aligned} \quad (4.13)$$

In eqs. (4.13) the asymptotic behaviour $s \rightarrow \infty$ is written down and power terms in s are dropped.

The usual g -perturbations of model (1.1) are based on the Goldstone vacuum and therefore lead in two dimensions to infrared divergences. However, one can obtain the asymptotic behaviour of the scattering amplitudes by calculating the ultraviolet logarithms of the scattering amplitudes of the Goldstone particles (to circumvent the infrared difficulties one can formally assume the mass of the Goldstone particle). Calculations are straightforward and the result coincides with (4.13).

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