

# EXACT S MATRIX OF GROSS-NEVEU "ELEMENTARY" FERMIONS

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The absence of multiple production and the factorization of the multiparticle  $S$  matrix of the Gross-Neveu model is demonstrated in the  $1/N$  expansion up to  $1/N^2$ . The exact  $S$  matrix for the "elementary" fermions of this model and some of their bound states are presented explicitly.

In ref. [1] Gross and Neveu studied the model of the two-dimensional  $N'$ -component Fermi field defined by the Lagrangian density

$$\mathcal{L} = i \sum_{\alpha=1}^{N'} \bar{\psi}_{\alpha} \hat{\partial} \psi_{\alpha} - g_0 \sigma(x) \sum_{\alpha=1}^{N'} \bar{\psi}_{\alpha} \psi_{\alpha} - \frac{1}{2} \sigma^2(x) \quad (1)$$

in the limit  $N' \rightarrow \infty$ . In this limit they found the spontaneous breakdown of discrete  $\gamma_5$  invariance (the Lagrange field  $\sigma$  acquires a nonzero vacuum expectation value  $\langle \sigma \rangle = m = \Lambda \exp(-\pi/N' g_0^2)$ ) and hence the dynamical mass transmutation.

Using a quasiclassical method Dashen, Hasslacher and Neveu [2] computed the particle spectrum of the model in the same  $N' \rightarrow \infty$  limit. The spectrum contains an  $O(N)$  vector multiplet<sup>†1</sup> of the "elementary particles" of the model (from here on we shall refer to them as  $A_i, i = 1, 2, \dots, N$ ) and a set of  $O(N)$  multiplets of particles which could be thought of as bound states of a number of "elementary" fermions.

The authors of ref. [2] managed to find explicitly a series of time-dependent solutions of the classical equations corresponding to model (1). Hardly this fact is a fortuity; most likely it indicates the presence of an infinite set of conservation laws such as in the sine-Gordon model [3].

In the present paper we argue in favour of the special features of scattering in the model (1), namely of the absence of multiple production and the factorization of the multiparticle  $S$  matrix. These features are characteristic for the models having an infinite number of conservation laws [3]. Our arguments are based on special cancellations taking place in the  $1/N$  expansion of the model (1), which are quite analogous to those found in ref. [4] in the  $1/N$  expansion of the two-dimensional  $O(N)$  symmetric nonlinear  $\sigma$ -model. The following considerations are mainly parallel to those of ref. [4].

The  $1/N$ -expansion technique developed in ref. [1] will be used here without explanations. Consider the amplitude of the process  $2 \rightarrow 4$ . In fig. 1 diagrams are drawn contributing to this amplitude in the order of  $1/N^2$  (for the sake of simplicity the case  $i \neq j \neq k \neq i$  is presented).

An arbitrary two-dimensional fermion loop can be calculated explicitly (see ref. [5]). In particular the triangle in fig. 1g is equal to the sum of three terms corresponding to three possible divisions of the loop through two lines (see fig. 2), each containing two tree diagrams on the right hand and on the left hand of the dashed line in fig. 2 multiplied by the factor

<sup>†1</sup> The Lagrangian (1) is invariant under orthogonal transformations on the  $N = 2N'$ -component vector  $\psi_{\alpha}^{(i)}$  ( $\alpha = 1, 2, \dots, N; i = 1, 2$ ;  
 $\psi_{\alpha} = \psi_{\alpha}^{(1)} + i\psi_{\alpha}^{(2)}$ ).

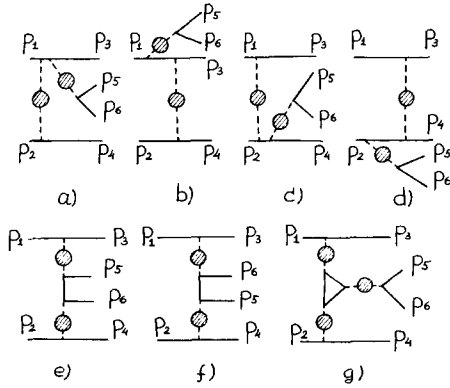
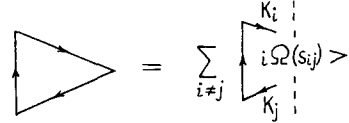
Fig. 1. Diagrams contributing to the  $2 \rightarrow 4$  amplitude.

Fig. 2. Calculation of the triangle.

$$i\Omega(S_{ij}) = -\frac{N}{(2\pi)^2} \text{tr} \int \frac{d^2 p}{(\hat{p} - m + i\epsilon)(\hat{p} - \hat{k}_i + \hat{k}_j - m + i\epsilon)} + \frac{\ln(\Lambda^2/m^2)}{2\pi}. \quad (2)$$

In eq. (2)  $S_{ij} = (k_i - k_j)^2$ ,  $m$  is the mass of the fermion in the loop and momenta  $k_i$  and  $k_j$  are fixed by the condition  $k_i^2 = k_j^2 = m^2$ . At the same time the  $\sigma$ -field propagator is equal to  $-\Omega^{-1}(k)$ . Therefore different divisions of the triangle in fig. 1g cancel all the other diagrams in fig. 1 completely.

One can verify in the same manner that all diagrams contributing to the  $3 \rightarrow 3$  amplitude in the order  $1/N^2$  are cancelled in their principal parts leaving a factorized amplitude only (for details see ref. [4]).

Of course the considerations presented do not prove rigorously the factorization of the  $S$  matrix of the model (1). We shall suppose, however, the factorization of the total  $S$  matrix to be the exact property of this model, and construct the total  $S$  matrix of "elementary" particles  $A_i$ . This  $S$  matrix will be used to derive the spectrum of bound states and to compute the  $S$ -matrix elements of their scattering.

The total factorized  $S$  matrix of particles  $A_i$  can be described by means of a simple algebraic construction [4]: introduce  $N$  types of noncommutative symbols  $A_i(\theta)$  and identify the asymptotic states of scattering theory with ordered products of these symbols, each symbol  $A_i(\theta_a)$  corresponding to a particle  $A_i$  in the state having momentum  $p_a^\mu$ ;  $p_a^0 = m \cosh \theta_a$ ,  $p_a^1 = m \sinh \theta_a$ ,  $m$  is the mass of the "elementary" fermion. Identify in(out)-states with products arranged in the order of increasing (decreasing)  $\theta$ . One can rearrange any in-state into a superposition of out-states (obtaining the  $S$  matrix elements) by means of the commutation rule:

$$A_i(\theta_1)A_j(\theta_2) = \delta_{ij}\sigma_1(\theta_{12}) \sum_{k=1}^N A_k(\theta_2)A_k(\theta_1) + \sigma_2(\theta_{12})A_j(\theta_2)A_i(\theta_1) + \sigma_3(\theta_{12})A_i(\theta_2)A_j(\theta_1). \quad (3)$$

In eq. (3)  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  are two-particle matrix elements and  $\theta_{12} = \theta_1 - \theta_2$ .

Unitarity, analyticity and factorization equations (the latter are the self-consistency conditions for commutation rules (3)) turn out to have a unique solution for functions  $\sigma(\theta)$  (there is certainly an arbitrariness of CDD type):

$$\sigma_3(\theta) = -i\frac{\lambda}{\theta}\sigma_2(\theta), \quad \sigma_1(\theta) = -i\frac{\lambda}{i\pi - \theta}\sigma_2(\theta), \quad \sigma_2(\theta) = \sigma_2^{(0)}(\theta) \prod_{k=1}^L \frac{\sinh \theta + i \sin \alpha_k}{\sinh \theta - i \sin \alpha_k}, \quad (4)$$

where  $\lambda = 2\pi/(N-2)$ ,  $\alpha_k$  are real numbers and

$$\sigma_2^{(0)}(\theta) = Q(\theta)Q(i\pi - \theta), \quad Q(\theta) = \frac{\Gamma(\Delta - i\theta/2\pi)\Gamma(\frac{1}{2} - i\theta/2\pi)}{\Gamma(-i\theta/2\pi)\Gamma(\frac{1}{2} + \Delta - i\theta/2\pi)}; \quad \Delta = \lambda/2\pi. \quad (5)$$

In ref. [4] the solution  $\sigma_2 = \sigma_2^{(0)}$  was suggested to describe the total  $S$  matrix of the  $O(N)$  chiral model. For the

model (1) we suggest the solution (4) with  $L = 1$  and  $\alpha_1 = \lambda$ . This solution corresponds to attractive interaction between two "elementary" particles which form bound states of the same mass in the isoscalar and antisymmetric tensor channels (in agreement with the result of ref. [2]).

The investigation of the multiparticle  $S$ -matrix elements of particles  $A_i$  (the analysis is quite the same as in ref. [6]) shows that the pole  $\theta = i\lambda$  leads to the whole spectrum obtained in ref. [2]:

$$m_n = m \frac{\sin [\pi n/(N-2)]}{\sin [\pi/(N-2)]}; \quad n = 1, 2, \dots < \frac{1}{2}(N-2). \quad (6)$$

The strong isospin degeneracy of the spectrum obtained in ref. [2] quasiclassically turns out to be exact, the masses coincide with those of ref. [2] where  $N$  is replaced by  $N-2$ <sup>†2</sup>.

One can include these bound states into the  $S$  matrix calculating residues in their poles in the corresponding multiparticle  $S$ -matrix elements. We have not been able to derive a general form of the matrix elements and present the  $S$  matrix including two-particle bound states only (we shall denote them by  $B$  and  $B_{ij}$ ). Corresponding algebraic symbols can be introduced formally by:

$$A_i(\theta - \frac{1}{2}i\lambda)A_j(\theta + \frac{1}{2}i\lambda) = \delta_{ij}B(\theta) + \sqrt{N-4}B_{ij}(\theta). \quad (7)$$

Substituting eq. (7) into products  $A_i(\theta_1)A_j(\theta_2)A_k(\theta_3)$  one obtains

$$\begin{aligned} A_i(\theta_1)B(\theta_2) &= \left\{ \left[ 1 - \frac{\lambda^2}{(\theta - \frac{1}{2}i\lambda)(i\pi - \theta - \frac{1}{2}i\lambda)} \right] B(\theta_2)A_i(\theta_1) \right. \\ &\quad \left. - \frac{\sqrt{N-4}\lambda^2}{(\theta - \frac{1}{2}i\lambda)(i\pi - \theta - \frac{1}{2}i\lambda)} B_{il}(\theta_2)A_l(\theta_1) \right\} \sigma_2(\theta + \frac{1}{2}i\lambda)\sigma_2(\theta - \frac{1}{2}i\lambda), \\ A_i(\theta_1)B_{kl}(\theta_2) &= \left\{ - \frac{\sqrt{N-4}\lambda^2}{(\theta - \frac{1}{2}i\lambda)(i\pi - \theta - \frac{1}{2}i\lambda)} [\frac{1}{2}\delta_{ik}B(\theta_2)A_l(\theta_1) - \frac{1}{2}\delta_{il}B(\theta_2)A_k(\theta_1)] + \right. \\ &\quad + B_{kl}(\theta_2)A_i(\theta_1) + \frac{i\lambda}{i\pi - \theta - \frac{1}{2}i\lambda} [\delta_{ik}B_{lm}(\theta_2)A_m(\theta_1) - \delta_{il}B_{km}(\theta_2)A_m(\theta_1)] \\ &\quad \left. + \frac{i\lambda}{\theta - \frac{1}{2}i\lambda} [B_{ik}(\theta_2)A_l(\theta_1) - B_{il}(\theta_2)A_k(\theta_1)] \right\} \sigma_2(\theta + \frac{1}{2}i\lambda)\sigma_2(\theta - \frac{1}{2}i\lambda). \end{aligned} \quad (8)$$

The spectrum of the model (1) contains particles which cannot be thought of as bound states of any finite number of "elementary" ones. These are so-called Callan, Coleman, Gross and Zee kinks [2]. Therefore the  $S$  matrix of the "elementary" particles and all their bound states is not a total one of the model. The total  $S$  matrix should include elements of the kink-kink and kink-particle scattering. These elements cannot be obtained using the method described above. They could probably be derived as a solution of unitarity, analyticity and factorization equations for the kink-kink  $S$  matrix. This task is beyond the scope of this paper.

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<sup>†2</sup> Some arguments in favour of this replacement are presented in ref. [2].

- [1] D. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235.
- [2] R. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D12 (1975) 2443.
- [3] V.E. Korepin and L.D. Faddeev, Teor. Mat. Fiz. 25 (1975) 147.
- [4] A.B. Zamolodchikov and A.B. Zamolodchikov, Preprint JINR E2-10857 (1977).
- [5] B. Berg, Preprint FUB/HEP 77/6 (1977).
- [6] B. Schroer, T.T. Truong and P.H. Weisz, Phys. Lett. 63B (1976) 422.