Integralrechnung

Lösung Aufgabe 1:

(a) $\frac{a}{2}x^2 + bx + C$

(b) $\frac{1}{a}\ln(ax+b) + C$

(c) $-\frac{1}{4x^4} + C$

(d) $-\frac{1}{3}\cos 3x + C$

Lösung Aufgabe 2:

(a) $\int dx\, \frac{x^2}{x^3+1} = \frac{1}{3}\int dx\, g'(x)\frac{1}{g(x)} = \frac{1}{3}\ln g(x) + C = \frac{1}{3}\ln(x^3+1) + C$
mit $g(x) = x^3 + 1$ und $g'(x) = 3x^2$.

(b) $\int dx\, \frac{\cos x}{\sin x} = \int dx\, g'(x)\frac{1}{g(x)} = \ln g(x) + C = \ln \sin x + C$
mit $g(x) = \sin x$ und $g'(x) = \cos x$.

(c) $\int dx\, e^{\cos x}\sin x = e^{\cos x} + C$

(d) $\int \frac{dx}{\sinh x} = 2\int \frac{dx}{e^x - e^{-x}} = 2\int \frac{dy}{y}\frac{1}{y - 1/y} = 2\int \frac{dy}{y^2-1}$
$x = \ln y \quad \Rightarrow \quad x' = 1/y$
$\frac{1}{y^2-1} = \frac{A}{y-1} + \frac{B}{y+1} \quad \Rightarrow \quad A + B = 0,\ A - B = 1 \quad \Rightarrow \quad A = \frac{1}{2} = -B$, so dass
$\int \frac{dx}{\sinh x} = \int dy \left\{\frac{1}{y-1} - \frac{1}{y+1}\right\} = \ln\frac{y-1}{y+1} + C = \ln\frac{e^x-1}{e^x+1} + C = \ln(\tanh x/2) + C$

(e) $\int dx\, \tanh x = \int dx \frac{\sinh x}{\cosh x} = \ln(\cosh x) + C$

Lösung Aufgabe 3:

(a) $\int dx\, x^2 e^x = \int dx\, x^2 (e^x)' = x^2 e^x - \int dx\, 2xe^x = x^2e^x - 2\int dx\, x(e^x)'$
$= x^2e^x - 2xe^x + 2\int dx\, e^x \qquad \Rightarrow \qquad \int dx\, e^x \cos x = \frac{1}{2}e^x(\cos x + \sin x) + C$

(b) $\int dx\,\frac{\ln x}{x} = \int dx\,\left(\frac{d}{dx}\ln x\right)\ln x = \ln^2 x - \int dx\frac{\ln x}{x}$

$\Rightarrow \int dx\,\frac{\ln x}{x} = \frac{1}{2}\ln^2 x$

(c) $\int dx\,e^x \sin^2 x = \int dx\,e^x\frac{1}{2}[1-\cos 2x] = \frac{1}{2}e^x - \frac{1}{2}\int dx\,e^x\cos 2x,$

wobei

$\int dx\,e^x\cos 2x = e^x\cos 2x + 2\int dx\,e^x\sin 2x = e^x(\cos 2x + 2\sin 2x) - 4\int dxe^x\cos 2x$

$\Rightarrow \quad \int dx\,e^x\cos(2x) = \frac{1}{5}e^x(\cos 2x + 2\sin 2x)$, so dass

$\int dx\,e^x\sin^2 x = \frac{1}{2}e^x - \frac{1}{10}e^x(\cos 2x + 2\sin 2x)$

Lösung Aufgabe 4:

(a) $\int dx\,\frac{x^2}{x^2-a^2} = \int dx\frac{x^2+a^2-a^2}{x^2-a^2} = \int dx\,\left\{1-\frac{a^2}{x^2-a^2}\right\}$

$= x - a^2\int dx\frac{1}{x^2-a^2} = x - \frac{a}{2}\ln\frac{x-a}{x+a}$ (letzter Schritt: siehe Vorlesung)

(b) $\int dx\frac{x+7}{x^2=3x+2} = \int dx\frac{x+7}{(x-2)(x-1)} = \otimes$

NR: $x^2-3x+2=0 \quad\Rightarrow\quad x = \frac{3}{2}\pm\sqrt{\frac{9}{4}-2} = \frac{3}{2}\pm\frac{1}{2} \quad\Rightarrow\quad x=1$ oder $x=2$.

$\frac{x+7}{(x-2)(x-1)} = \frac{A}{x-2}+\frac{B}{x-1} \quad\Rightarrow\quad A+B=1,\ A+2B=-7$

$\Rightarrow \quad A=9,\ B=-8.$

$\otimes = \int dx\left\{\frac{9}{x-2}-\frac{8}{x-1}\right\} = 9\ln(x-2)-8\ln(x-1).$

(c) $\int dx\,\frac{1}{e^x-1} = \int dy\,\frac{1}{y}\frac{1}{y-1} = \otimes$

$x=\ln y \quad\Rightarrow\quad x' = \frac{1}{y}$

$\frac{1}{y(y-1)} = \frac{A}{y}+\frac{B}{y-1} \quad\Rightarrow\quad A+B=0,\ -A=1 \quad\Rightarrow\quad A=-B=-1.$

$\otimes = \int dy\left\{\frac{1}{y-1}-\frac{1}{y}\right\} = \ln\frac{y-1}{y} = \ln\frac{e^x-1}{e^x} = \ln(1-e^{-x})$

Lösung Aufgabe 5:

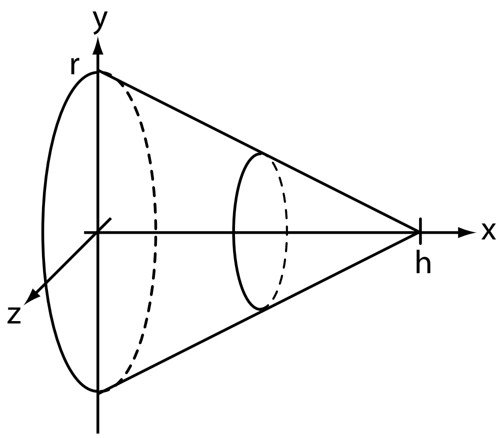
(a) $y = \pm\sqrt{b^2-\frac{b^2}{a^2}x^2} = \pm b\sqrt{1-\frac{x^2}{a^2}}$

$\Rightarrow \quad F = 2\int_{-a}^{a}dx\,b\sqrt{1-\frac{x^2}{a^2}} = 2b\int_{-a}^{a}dx\,\sqrt{1-\frac{x^2}{a^2}}$

$$x = a\,\sin\phi \quad \Rightarrow \quad x' = a\cos\phi$$
$$F = 2b\int_{-\pi/2}^{\pi/2} d\phi\, a\,\cos\phi\sqrt{1-\sin^2\phi} = 2ab\underbrace{\int_{-\pi/2}^{\pi/2} d\phi\,\cos^2\phi}_{\pi/2} = \pi ab.$$

(b) $V = \int_{-a}^{a} dx\,\pi b^2\left(1-\frac{x^2}{a^2}\right) = 2\pi ab^2 - \pi\frac{b^2}{a^2}\int_{-a}^{a} dx\,x^2 = 2\pi ab^2 - \frac{2\pi}{3}ab^2 = \frac{4\pi}{3}ab^2$

(c) $f(x) = r - \frac{r}{h}x$



$$V = \int_0^h dx\,\pi\left(r-\frac{r}{h}x\right)^2 = \pi r^2\int_0^h dx\,\left(1-\frac{2x}{h}+\frac{x^2}{h^2}\right)$$
$$= \pi r^2\left(h - h + \frac{h^3}{3h^2}\right) = \frac{\pi}{3}r^2h.$$