Int. J. Engng Sci. Vol. 23, No. 9, pp. 927–935, 1985
Printed in Great Britain.

0020–7225/85 $3.00 + .00

TOWARDS A QUANTUM FIELD THEORY OF DEFECTS AND STRESSES—QUANTUM VORTEX DYNAMICS IN A FILM OF SUPERFLUID HELIUM

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Abstract—With the goal of developing a quantum field theory of defects under stress we first solve this problem for the simpler case of vortices in a thin film of superfluid ^{4}He. The theory we obtain, to be called *quantum vortex dynamics* (QVD), turns out to be what is known in quantum field theory as scalar quantum electrodynamics in 2 + 1 dimensions (also called scalar Abelian Higgs model).

1. INTRODUCTION

IN THE last seven years there has been considerable progress in understanding the interrelation between the two types of gauge fields in the equations of elasticity and plasticity. One of the two, the gauge field of elasticity, has been used for a long time by Maxwell, Beltrami, Schäfer, Kröner, and others [1] to find the stress field of arbitrary defect configurations. The fact that the plastic fields are also gauge fields has been discovered [2], physically interpreted [3, 4], and put to application [5] only recently. They can be used with great efficiency to calculate interaction energies between defect lines [6].

Within the Cosserat continuum of higher-gradient elasticity, the two gauge structures have their origin in the two mutually dual conservation laws, that of dislocations and disclinations:

$$\begin{aligned} \partial_i\theta_{ij} &= 0, \\ \partial_i\alpha_{ij} &= -\epsilon_{jkl}\theta_{kl}, \end{aligned} \tag{1}$$

and that of stresses and torque-stresses [7]

$$\begin{aligned} \partial_i\sigma_{ij} &= 0, \\ \partial_i\tau_{ij} &= -\epsilon_{jkl}\sigma_{kl}. \end{aligned} \tag{2}$$

It has repeatedly been noticed that the field equations of elasticity and plasticity show an asymmetry: While the equations of the elastic gauge fields in terms of any given conserved defect densities are determined by the elastic constants, the dual determination of the plastic gauge fields in terms of the stresses remained impossible due to our ignorance of the energy of the plastic gauge fields.

Several attempts were made to complete the equations. One is due to Schäfer [8], who tried to imitate Mie's way of completing the Maxwell-Lorentz theory of electrons and photons [9]. Another more recent attempt tries to take advantage of recent progress in the theory of elementary particles via non-abelian Yang-Mills gauge fields and postulates a similar Lagrangian for the defect gauge fields [10].‡

A third line of approach, which we think is the most physical one, starts from the observation that the defect gauge fields are really *discrete* objects [3–5]. As long as one was interested only in calculating stress fields for any given defect configurations, this observation was not relevant. For the opposite problem of finding the motion of defects in a stress field, however, the discreteness becomes extremely important.

† Supported in part by the Deutsche Forschungsgemeinschaft under Grant No. Kl 256/11-1.
‡ Something is physically wrong with this theory since the long-range stress field around a dislocation falls off exponentially. See p. 208 of [10].

2. STRUCTURAL CONSIDERATIONS

Faced with the unusual problem of finding equations of motion for integer-valued gauge fields we took advantage of the fact that there exist certain transformations of the partition function of these fields which carry the sum over an ensemble of defects into that over the fluctuations of a continuous complex disorder field [11, 12]. For this field it is then possible to derive equations of motion in the conventional way.

The structural basis of this transformation is rather simple. As stated above, the two gauge fields are a manifestation of the two sets of conservation laws. Conservation laws imply the existence of closed line configurations: The law $\partial_i \sigma_{ij} = 0$ implies closed lines of stress (associated with σ_{i1}, σ_{i2}, σ_{i3}), the law $\partial_i \theta_{ij}$ implies closed lines of disclinations (associated with θ_{i1}, θ_{i2}, θ_{i3}). In the continuum approximation, the two line structures are indistinguishable. Due to the discreteness of the crystalline structure, however, defect lines are discrete objects. They are "quantized" such that Burgers' and Frank vectors are integer multiples of the basis vectors and symmetry angles. Closed discrete lines appear in the theory of elementary particles. It is well–known that an ensemble of fluctuating particle orbits can be described by a quantum field or a fluctuating field. It is this dual relation between particle orbits and fields which formed the basis of our disorder field theory for ensembles of defect lines.

The situation became clearest when not directly considering defect lines in a solid but looking, instead, at the structurally similar but much simpler system of superfluid ^{4}He [11]. In that system the role of the stress energy is played by the hydrodynamic energy of superflow and that of defect lines by vortex lines.

The result of our manipulations was a disorder field theory of the Ginzburg-Landau type, which is normally used in the theory of superconductivity. The difference lies only in the interpretation of the field quantities. In particular we observed a disorder version of the Meissner effect: At high temperature, the disorder field acquires a vacuum expectation value which screens the vector potential. This implied that, just as magnetic fields are expelled from an ordered state, the superflow cannot invade into the disordered state.

The possibility of writing down a local disorder field theory depended crucially on the three dimensions. Feynman diagrams are lines and picture directly the vortex lines [13, 14].

When generalizing the techniques to crystals we found again a field theory of the Ginzburg-Landau type, albeit a more complicated one. Again there was a disorder version of the Meissner effect, namely that transverse stress cannot invade into the disordered molten state [15].

The question arises whether one can find a similar field theory for *dynamically moving defects,* such that the partition functions can be used for a full *quantum statistical mechanics of defects and stresses.*

There is one immediate obstacle: Defects are lines in real space such that, in space-time, they form world sheets. Until now, in spite of many years of effort in field theory, nobody has come up with a satisfactory quantum field theory for fluctuating surfaces.

If we, however, set ourselves a more modest goal, namely that of finding a quantum field theory of defects and stresses in *two dimensions,* then it can be solved by means of the same methods developed before for classical fluctuations in three dimensions. In two dimensions, defects are point-like objects, such that in space-time they do form world lines and these again permit a representation in terms of a disorder quantum field theory.

3. QUANTUM PARTITION FUNCTION OF PHONONS

As a first step in developing a quantum field theory of defects and stresses in two dimensions we shall study again the simpler case of a two-dimensional superfluid, as represented by a film of ^{4}He below the λ point. Such a film is characterized by the phase variable $\gamma(\mathbf{x})$ of its condensate wave function. The action is similar to the elastic action of a crystal

$$A_0 = \frac{T_0}{2} \int \mathrm{d}t \mathrm{d}^2 x \left[\frac{1}{c^2} (\partial_t \gamma)^2 - (\partial_i \gamma)^2 \right], \tag{3}$$

where $T_0 = \rho(\hbar^2/M^2) \sim 1$ K and $a \sim 3.581$ Å is the interatomic spacing, $M \sim 4 \times 1.68 \times 10^{-24}$ g the mass of the ^{4}He atoms, $\rho \sim .145$ g/cm^3 the density, and c the sound velocity as given by the slope of the dispersion curve, $c \sim 18.2$ K/Å. The partition function of "elastic" quantum fluctuations can be formulated most specifically by making space-time discrete with spacing a and ϵ, respectively, and calculating the path integral on a simple cubic lattice:

$$Z_0 = \prod_{\mathbf{x},\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}\gamma(\mathbf{x}, \tau)}{\sqrt{(2\pi\epsilon c^2/Ta^2)}}\, \mathrm{e}^{-(T_0/2)(a^2/\epsilon c^2)\Sigma_{\mathbf{x},\tau=0,\ldots,1/T}[(\nabla_0\gamma)^2 + c^2(\epsilon^2/a^2)(\nabla_i\gamma)^2]} \tag{4a}$$

$$= \mathrm{e}^{(1/2)\Sigma_{n,k}\log(\Omega_n\bar{\Omega}_n + \epsilon^2(c^2/a^2)\mathbf{K}_i\mathbf{K}_i)}. \tag{4b}$$

The symbols ∇_0, ∇_i denote lattice gradients, $\nabla_0\gamma(\mathbf{x}, \tau) \equiv \gamma(\mathbf{x}, \tau + \epsilon) - \gamma(\mathbf{x}, \tau)$. $\nabla_i\gamma(\mathbf{x}, \tau) = \gamma(\mathbf{x} + \mathbf{i}a, \tau) - \gamma(\mathbf{x}, \tau)$, where $\mathbf{i}$ are the lattice vectors along the ith axis. The quantities $K_i = (\mathrm{e}^{ik_ia} - 1)/i = \bar{K}_i^*$, $\Omega_n = (\mathrm{e}^{i\omega_n\epsilon} - 1)/i = \bar{\Omega}_n^*$ are the eigenvalues of these lattice gradients, where

$$\omega_n = 2\pi \frac{n}{N_\tau} \frac{1}{\epsilon}$$

and

$$N_\tau = \frac{1}{T\epsilon}$$

is the number of time slices. In writing down the path integral we have followed the standard rules of quantum statistics by making the time variable imaginary $t \longrightarrow \tau = it$ and considered fields $\gamma(x, \tau)$ which are periodic in $\tau \in (0, 1/T)$.

For the following discussion it will be useful to introduce the temperature $T_D = c/a$ which is equal to $1/\sqrt{(2\pi)} \sim .4$ times the proper Debye temperature θ_D of the superfluid. Then the partition function reads

$$Z_0 = \prod_{\mathbf{x},\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}\gamma(\mathbf{x}, \tau)}{\sqrt{(2\pi T_D^2/T_0 T N_\tau)}}\, \mathrm{e}^{-(1/2)(T_0/T)(1/N_\tau)\Sigma_{\mathbf{x},\tau}[N_\tau^2(T/T_D)^2(\nabla_0\gamma)^2 + (\nabla_i\gamma)^2]} \tag{4c}$$

Experimentally, it is known that the superfluid phase transition occurs at some critical temperature $T_c \sim 2.2$ K when T_0/T is almost $\frac{1}{2}$. For zero Debye temperature, $T_D = 0$, this would lie in the classical regime; for $T_D > T_0$ it lies in the quantum regime.† From the experimental value of c we have $T_D \sim 5$ K, $\theta_D \sim 13$ K such that $T_D \gg T_0$. In the limit, $\epsilon \longrightarrow 0$, the partition function (4b) becomes

$$Z_0 = \mathrm{e}^{-(N/2)\int[\mathrm{d}^2k/(2\pi)^2]\Sigma_n\log(\Omega_n\bar{\Omega}_n + c^2\epsilon^2\bar{\mathbf{K}}\mathbf{K}/a^2)}$$

$$\underset{\epsilon\longrightarrow 0}{\longrightarrow} \mathrm{e}^{-(N/2)(1/T)\int[\mathrm{d}^2k/(2\pi)^2][c\sqrt{(\mathbf{K}\bar{\mathbf{K}}/a^2)} + T\log(1 - \mathrm{e}^{-c\sqrt{(\bar{\mathbf{K}}\bar{\mathbf{K}}/a^2)}})]}$$

where N is the total number of lattice sites. This gives rise to a Debye spectrum of specific heat on a simple cubic lattice.

4. INTRODUCTION OF MOVING VORTICES

Let us now extend this theory to incorporate moving vortices. The first step is to introduce plastic gauge fields, the plastic "distortions" β_t, β_i and write the action in terms of the difference between total gradients $\partial_t\gamma$, $\partial_i\gamma$ and these plastic distortions:

$$A = T_0/2 \int \mathrm{d}t\,\mathrm{d}^2x\left[\frac{1}{c^2}(\partial_t\gamma - \beta_t)^2 - (\partial_i\gamma - \beta_i)^2\right]. \tag{5}$$

† The time slicing is, of course, a technical artifact and, at the end, we have to take the limit $\epsilon \longrightarrow 0$ i.e. $N_\tau \longrightarrow \infty$.

A single vortex line surrounding a Volterra surface S is given by

$$\beta_i = 2\pi\delta_i(S),$$
$$\beta_t = -2\pi v_k \delta_k(S), \tag{6}$$

where v_k is the speed with which the vortex line moves through space. This action is invariant under the defect gauge transformation

$$\beta_\mu \to \beta_\mu + \partial_\mu N,$$
$$\gamma \to \gamma + N, \tag{7}$$

where we have employed three-vector notation $x^\mu = (t, \mathbf{x})$, $\partial_\mu = (\partial_t, \partial_i)$, for brevity. This corresponds to shifting the Volterra cutting surface S of the vortex line to a new position S'. Indeed, doing this in (6) we find (7) with $N = 2\pi\delta(V)$, where V is the volume enclosed by $S' - S$. Obviously, this action is the superfluid version of the action which controls crystalline elastic and plastic deformations.

In order to develop the desired quantum field theory we now have to insert the same type of plastic deformations into the lattice form (3) of the partition function. This becomes

$$Z = \sum_{\{n_\mu\}} \Phi[n_\mu] \prod_{\mathbf{x},\tau} \int_{-\infty}^{\infty} \frac{d\gamma(\mathbf{x}, \tau)}{\sqrt{(2\pi T_D^2/T_0 T N_\tau)}}$$
$$\times \exp -\frac{1}{2}\frac{T_0}{T}\frac{1}{N_\tau}\sum_{\mathbf{x},\tau}\left[N_\tau^2\left(\frac{T}{T_D}\right)^2(\nabla_0\gamma - 2\pi n_0)^2 + (\nabla_i\gamma - 2\pi n_i)^2\right]. \tag{8}$$

Here n_μ are integer numbers reflecting the fact that $\gamma(\mathbf{x}, \tau)$ is a phase variable of the condensate wave function such that γ and $\gamma + 2\pi n$ are physically indistinguishable. The defect gauge transformations are now integer-valued [3–5]:

$$n_\mu \to n_\mu + \nabla_\mu N,$$
$$\gamma \to \gamma + 2\pi N. \tag{9}$$

In order to avoid an infinite overall factor we have to fix the gauge in the sum over plastic gauge fields n_μ via, say, $\Phi[n_\mu] = \delta n_0$.

In the following it will be useful to abbreviate the quantity $c(\epsilon/a) = (T_D/T)(1/N_\tau)$ by $\hat{c}$. It is the sound velocity measured in units of a/ϵ. We further introduce $\beta \equiv T_0/T$ as the inverse temperature measured in units of $T_0^{-1} \approx 1\ \mathrm{K}^{-1}$.

In order to exhibit the world lines of vortices in Z we now introduce the (defect gauge invariant) superfluid current b^μ as the conjugate variable of $\nabla_\mu\gamma - 2\pi n_\mu$ and rewrite (7) as [12]

$$Z = \prod_{\mathbf{x},\tau,i} \int \frac{db^i(\mathbf{x}, \tau)}{\sqrt{(2\pi\beta/N_\tau)}} \prod_{\mathbf{x},\tau} \int \frac{db^0(\mathbf{x}, \tau)}{\sqrt{(2\pi\beta/\hat{c}^2 N_\tau)}} \prod_{\mathbf{x},\tau} \int \frac{d\gamma(\mathbf{x}, \tau)}{\sqrt{(2\pi\hat{c}^2 N_\tau/\beta)}}$$
$$\times \sum_{\{n_\mu\}} \delta_{n_0,0} \exp\left\{-\frac{N_\tau}{\beta}\sum_{\mathbf{x},\tau}(\hat{c}^2 b^{02} + b^{i2}) + i\sum_{\mathbf{x},\tau} b^\mu(\nabla_\mu\gamma - 2\pi n_\mu)\right\}. \tag{10}$$

Integrating out the γ field produces the conservation law of supercurrent

$$\bar{\nabla}_\mu b^\mu(\mathbf{x}, \tau) = 0, \tag{11}$$

where $\bar{\nabla}_0 f(\mathbf{x}, \tau) = f(\mathbf{x}, \tau) - f(\mathbf{x}, \tau - \epsilon)$, $\bar{\nabla}_i f(\mathbf{x}, \tau) = f(\mathbf{x}, \tau) - f(\mathbf{x} - \mathbf{i}a, \tau)$. This is fulfilled automatically by introducing the gauge field $A_\mu(x)$ via

$$b^{\mu}(x) = \epsilon^{\mu\nu\lambda}\bar{\nabla}_{\nu}A_{\lambda}(\mathbf{x} - \boldsymbol{\lambda}). \tag{12}$$

This leads to

$$Z = \det(-\bar{\nabla}_i\nabla_i)\frac{1}{\sqrt{(2\pi\beta/N_\tau)^{2N}}}\int d^3A_\mu\delta(\nabla_iA_i) \times \sum_{\{n_\mu\}}\delta n_{0,0}\exp\left\{-\frac{N_\tau}{2\beta}\sum_{\mathbf{x},\tau}[\hat{c}^2(\bar{\nabla}_iA_j)^2 + (\nabla_0A_j)^2 + (\nabla_iA_0)^2] + 2\pi i\sum_{\mathbf{x},\tau}A_\mu(\mathbf{x},\tau)l^\mu(\mathbf{x},\tau)\right\}. \tag{13}$$

The integers

$$l^{\mu}(\mathbf{x}, \tau) = \epsilon^{\mu\nu\lambda}\nabla_{\nu}n_{\lambda}(\mathbf{x} + \boldsymbol{\mu}) \tag{14}$$

satisfy $\bar{\nabla}_\mu l^\mu = 0$. As n_μ run through all integers in the gauge $n_0 = 0$, they describe an ensemble of closed nonbacktracking world lines [11, 12]. Integrating out the A_μ field gives

$$Z = Z_0\sum_{\{l^\mu\}}\delta_{\bar{\nabla}_\mu l^\mu,0}\,\mathrm{e}^{-(4\pi^2\beta/2N_\tau)\Sigma_{\mathbf{n},\mathbf{k}}[l^0(\mathbf{k})^*l^0(\mathbf{k})/K_i\bar{K}_i + (l^i(\mathbf{k})^*l^j(\mathbf{k})(\delta_{ij} - K_i\bar{K}_j/\bar{K}K)(K_0\bar{K}_0 + \hat{c}^2K_i\bar{K}_i)]} \tag{15}$$

where $K_0 \equiv \Omega_n$.

Using the closedness of the world lines, $\bar{\nabla}_\mu l^\mu = 0$, we can replace $\bar{K}_jl^j$ by $\bar{K}_0l^0$ in the exponent and obtain

$$Z = Z_0\sum_{\{l^\mu\}}\delta_{\bar{\nabla}_\mu l^\mu,0}\,\mathrm{e}^{-(4\pi^2\beta/2N_\tau)\Sigma_{x,x'}l^\mu(x)v(x-x')l_\mu(x')}, \tag{16}$$

where we have introduced a metric $g_{\mu\nu} = \begin{pmatrix}\hat{c}^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ and a potential

$$v(x) = \frac{1}{N_\tau}\sum_n\int\frac{d^2k}{(2\pi)^2}\,\mathrm{e}^{-i(\omega_n\tau - \mathbf{kx})}\frac{1}{\Omega_n\bar{\Omega}_n + \hat{c}^2K_i\bar{K}_i}. \tag{17}$$

This potential is infinite due to an infrared devergence of the $n = 0$ term. This implies that only such $l^\mu(x, \tau)$ configurations can contribute which satisfy $\Sigma_{x,\tau}\, l^\mu(x, \tau) = 0$. For these, we can subtract

$$v(0) = \frac{1}{N_\tau}\sum_n\int\frac{d^2k}{(2\pi)^2}\frac{1}{\Omega_n\bar{\Omega}_n + \hat{c}^2K_i\bar{K}_i}, \tag{18}$$

define $v'(x) \equiv v(x) - v(0)$ and arrive at

$$Z = Z_0\sum_{\{l^\mu\}}\delta_{\bar{\nabla}_\mu l^\mu,0}\,\mathrm{e}^{-(4\pi^2\beta/2N_\tau)\Sigma_{x\neq x'}l^\mu(x)v'(x-x')l_\mu(x')} \tag{19}$$

This is the partition function of moving vortices in 2 + 1 dimensions with Biot-Savart-like interactions due to stress. Notice that the vortex part is a pure factor of the Debye partition function Z_0 of the phonons.

The asymptotic behavior of $v'(x)$ is given by

$$v'(x, \tau) \underset{|\mathbf{x}|\to\infty}{\longrightarrow} -\frac{1}{N_\tau}\frac{1}{\hat{c}^2}\frac{1}{2\pi}\log|\mathbf{x}| - \delta, \tag{20}$$

with

$$\delta = \frac{1}{N_\tau\hat{c}^2}\frac{1}{2\pi}\log(2\sqrt{2\mathrm{e}^\gamma})\ (\gamma \sim .577).$$

It is then useful to define a further subtracted potential $v''(x) \equiv v'(x) + \delta$ and rewrite as

$$Z = Z_0 \mathrm{e}^{-(4\pi^2\beta/2N_\tau)[\Sigma_x l^\mu(x) l_\mu(x) + \Sigma_{x \neq x'} l^\mu(x) v''(x-x') l_\mu(x')]}. \tag{21}$$

The potential $v''(x)$ has the Fourier transform $(\Omega_n \bar{\Omega}_n + \hat{c}^2 K_i \bar{K}_i)^{-1} - (2\pi)^2 \delta^{(2)}(k) \delta_{\omega_n,0}(v(0) - \delta)$. Going back the steps which led from (13) to (19) we can now reexpress the second term in (20) in terms of a fluctuating vector potential whose space-time independent part is modified (indicated by a double prime on the field energy $b^\mu b_\mu$) such as to correspond to this subtracted potential $v''(x)$. In this way we find

$$Z \propto \int \mathrm{d}^3 A_\mu \delta(\nabla_i A_i) \mathrm{e}^{-(N_\tau/2\beta)\Sigma_x b^\mu(x) b_\mu(x)} \times \sum_{\{n_\mu\}} \delta n_{0,0} \mathrm{e}^{-(4\pi^2\beta/2N_\tau)\delta \Sigma_x l^\mu l_\mu - 2\pi i \Sigma_x A_\mu l^\mu}. \tag{22}$$

This partition function displays the double-gauge nature of elasticity and plasticity. The fields of supercurrent b^μ and the vortices l^μ form both closed world lines $\bar{\nabla}_\mu b^\mu = 0$, $\bar{\nabla}_\mu l^\mu = 0$ [the analogue of the crystalline relations (2) and (1), respectively]. These are automatically fulfilled by the curl representation,

$$b^\mu = \epsilon^{\mu\nu\lambda} \bar{\nabla}_\nu A_\lambda(x - \lambda), \qquad l^\mu = \epsilon^{\mu\nu\lambda} \nabla_\nu n_\lambda(x + \mu)$$

The gauges of the stress gauge field A_λ and the defect gauge field n_λ are fixed by $\nabla_i A_i = 0$, $n_0 = 0$. The superflow energy is quadratic in b^μ. It generates a natural core energy of vortices which is quadratic in l^μ. The coupling is linear between the stress gauge field and the vortex current l^μ. Alternatively, we can also write $\sum_x A_\mu l^\mu = \sum_x n_\mu b^\mu$, i.e. the defect gauge field is coupled linearly to the supercurrent.

The important difference between the two fields lies in the fact that the stress gauge field is continuous while the defect gauge field is integer.† If this were not the case we could use the exponent at (22) as an action for the dynamics of superflow and vortices (stresses and defects). For integer fields, however, we would not know how to handle such equations. These difficulties are circumvented by representing the vortex ensemble by a continuous disorder field theory [12–15].

5. DISORDER FIELD THEORY

Let us return to the original loop sum (13) and set ourselves the goal of finding a field theory for the sum over all nonbacktracking world lines

$$\sum_{\{n_\mu\}} \delta n_{0,0}\, \mathrm{e}^{2\pi i \Sigma_x A_\mu(x) l^\mu(x)} = \sum_{\{l^\mu\}} \delta_{\bar{\nabla}_\mu l^\mu, 0}\, \mathrm{e}^{2\pi i \Sigma_x A_\mu(x) l^\mu(x)}; \tag{24}$$

for technical reasons we add, in the exponent, an infinitesimal core energy of the type produced by the stress fluctuations and consider

$$\sum_{\{l^\mu\}} \delta_{\bar{\nabla}_\mu l^\mu, 0}\, \mathrm{e}^{-(1/2)\tilde{\delta}\Sigma_x l_\mu l^\mu + 2\pi i \Sigma_x A_\mu(x) l^\mu(x)} \tag{25}$$

Later we shall take the limit $\tilde{\delta} \longrightarrow 0$. Forgetting a moment the A_μ fields we observe that this sum can be thought of as coming from an auxiliary partition function

$$\tilde{Z} = \sum_{\{\tilde{n}_\mu\}} \prod_x \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, \mathrm{e}^{-(1/2\tilde{\delta})\Sigma_x \{(1/\epsilon^2)(\nabla_0\theta - 2\pi n_0)^2 + (\nabla_i\theta - 2\pi n_i)^2\}} \tag{26}$$

where $\tilde{n}_\mu$ run over all integers. In order to verify this we merely introduce a conjugate variable of integration via a quadratic completion and write

† Notice that integer gauge fields have only three possibilities of fixing the gauge $n_0 = 0$, $n_1 = 0$ or $n_2 = 0$. We chose $n_0 = 0$. The transverse gauge $\nabla_i n_i = 0$ cannot be realized with integer fields n_μ!

$$\tilde{Z} = \frac{1}{2N}\prod_{x,\mu}\int_{-\infty}^{\infty}\frac{dl^{\mu}(x)}{\sqrt{2\pi/\tilde{\delta}}}\,\mathrm{e}^{-(\tilde{\delta}/2)\Sigma_x l^{\mu}(x)l_{\mu}(x)-2\pi i\Sigma_x l^{\mu}(\nabla_{\mu}\theta-2\pi\tilde{n}_{\mu})}. \tag{27}$$

Summing over all $\tilde{n}_\mu(x)$ squeezes $l^\mu(x)$ to integer values and integrating out the θ fields makes $l^\mu(x)$ divergenceless such that $\tilde{Z}$ becomes

$$\tilde{Z} \propto \sum_{\{l^\mu\}} \delta_{\bar{\nabla}_\mu l^\mu,0}\,\mathrm{e}^{-(\tilde{\delta}/2)\Sigma_x l^\mu(x)l_\mu(x)}. \tag{28}$$

Now we realize that the coupling to the A_μ field can be incorporated by adding in the exponent of (27) the expression $-2\pi i A_\mu$ which, in turn, can be obtained from (26) by adding there $-2\pi A_\mu$ to the gradients $\nabla_\mu\theta$. Thus we see that [16]

$$\sum_{\{l^\mu\}} \delta_{\bar{\nabla}_\mu l^\mu,0}\mathrm{e}^{2\pi i\Sigma_x A_\mu l^\mu} = \lim_{\tilde{\delta}\to 0}\sum_{\{\tilde{n}_\mu\}}\prod_x\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta(x)}{2\pi}\,\mathrm{e}^{-(1/2\tilde{\delta})\Sigma_x\{(1/\hat{c}^2)(\nabla_0\theta-2\pi A_0-2\pi\tilde{n}_0)^2+(\nabla_i\theta-2\pi A_i-2\pi n_i)^2\}} \tag{29}$$

Since $\tilde{\delta}$ is so small, the fluctuations of θ are squeezed completely into the periodic potential valleys. For this reason one can replace the exponent by a periodic function with the same valleys and write

$$\sum_{\{l^\mu\}} \delta_{\bar{\nabla}_\mu l^\mu,0}\mathrm{e}^{2\pi i\Sigma_x A_\mu l^\mu} \propto \lim_{\tilde{\delta}\to 0}\prod_x\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta(x)}{2\pi}\,\mathrm{e}^{(1/\tilde{\delta})\Sigma_x\{(1/\hat{c}^2)\cos(\nabla_0\theta-2\pi A_0)+\cos(\nabla_i\theta-2\pi A_i)\}}. \tag{30}$$

This is the form which we can manipulate directly into the desired disorder field theory [11, 12]. For this we consider first the case $A_\mu = 0$ and write $\cos\nabla_\mu\theta = \cos\theta\cos\theta(x+\mu) + \sin\theta\sin\theta(x+\mu)$ or, with the two-vector fields $U_a(x) = (\cos\theta(x), \sin\theta(x))$,

$$\cos\nabla_\mu\theta = U_a\nabla_\mu U_a + U_a^2. \tag{31}$$

Then we observe that

$$\begin{aligned}\sum_x \cos\nabla_\mu\theta &= \tfrac{1}{2}\sum_x\{U_a(\nabla_\mu U_a) + (\nabla_\mu U_a)U_a + 2U_a^2\}\\ &= \tfrac{1}{2}\sum_x U_a(\nabla_\mu - \bar{\nabla}_\mu)U_a + 2U_a^2,\end{aligned} \tag{32}$$

and further, since

$$\nabla_\mu - \bar{\nabla}_\mu = \bar{\nabla}_\mu\nabla_\mu = \nabla_\mu\bar{\nabla}_\mu, \tag{33}$$

also

$$\sum_x \cos\nabla_\mu\theta = \sum_x U_a\left(1 + \frac{\bar{\nabla}_\mu\nabla_\mu}{2}\right)U_a. \tag{34}$$

Thus the exponent in (30) becomes

$$\mathrm{e}^{(\lambda/\tilde{\delta})\Sigma_x\{U_a(1+(1/2\lambda)\bar{\nabla}_\mu\nabla^\mu)U_a-(1-(1/\hat{c}^2+2)/\lambda)\}}, \tag{35}$$

where λ is an arbitrary x dependent parameter† which we shall take constant, for simplicity.

Equivalently, we can use a complex field $U = U_1 + iU_2$ and write U^+U instead of U_aU_a. We now introduce the complex disorder field φ by rewriting the exponential as follows:

† It always drops out since $U_a^2 = 1$.

$$\prod_x \int \frac{d\varphi d\varphi^+}{4\pi\sqrt{\delta}}\, e^{-(\tilde{\delta}/4\lambda)\Sigma_x|\varphi|^2+(1/2)\Sigma_x(\tilde{\varphi}^+U+cc)}, \tag{36}$$

where $\tilde{\varphi}$ is short for‡

$$\tilde{\varphi} \equiv \sqrt{\left(1 + \frac{\bar{\nabla}_\mu \nabla^\mu}{2\lambda}\right)} \tag{37}$$

and $\int d\varphi d\varphi^+ \equiv \int_{-\infty}^{\infty} d\mathrm{Re}\varphi\, d\mathrm{Im}\varphi$. Inserting (36) into (30) we can integrate out the angular variables θ, and the sum over defect loops for $A_\mu = 0$ becomes

$$\sum_{\{l^\mu\}} \delta_{\bar{\nabla}_\mu l^\mu,0} \propto \lim_{\tilde{\delta}\to 0} \prod_x \int \frac{d\varphi d\varphi^+}{4\pi/\tilde{\delta}} \exp -(\tilde{\delta}/4\lambda)\Sigma_x|\varphi|^2+\log I_0(|\tilde{\varphi}|). \tag{38}$$

For small and smooth fields $\varphi(x)$, the exponent has the Landau expansion

$$-\frac{1}{8\lambda}\sum_x \nabla_\mu\varphi\nabla^\mu\varphi - \frac{1}{4}\left(\frac{\tilde{\delta}}{\lambda} - 1\right)\sum_x |\varphi|^2 - \frac{1}{64}\sum_x |\varphi|^4 - \cdots. \tag{39}$$

We are now ready to introduce the coupling to the gauge field A_μ. For this we realize that with the complex fields $U(x)$, the exponent in (30) reads $\mathrm{Re}U^+(x)U(x + \mu)e^{-2\pi i A_\mu(x)}$. This suggests introducing covariant derivatives on the lattice

$$\begin{aligned} D_\mu U(x) &\equiv U(x+\mu)e^{-2\pi i A_\mu(x)} - U(x), \\ \bar{D}_\mu U(x) &\equiv U(x) - U(x-\mu)e^{2\pi i A_\mu(x-\mu)}, \end{aligned} \tag{40}$$

for which

$$\begin{aligned} \mathrm{Re}\sum_x U^+(x)U(x+\mu)e^{-2\pi i A_\mu(x)} &= \tfrac{1}{2}\sum_x (U^+D_\mu U + (D_\mu U)^+U + 2U^+U] \\ &= \tfrac{1}{2}\sum_x (U^+(D_\mu - \bar{D}_\mu)U + 2U^+U] \\ &= \sum_x U^+\left(1 + \frac{\bar{D}_\mu D_\mu}{2}\right)U. \end{aligned} \tag{41}$$

When we go again to the complex disorder fields φ we see that, instead of (38), we now have

$$\sum_{\{l^\mu\}} \delta_{\bar{\nabla}_\mu l^\mu,0}\, e^{2\pi i \Sigma_x A_\mu l^\mu} \propto \lim_{\tilde{\delta}\to 0} \prod_x \int d\varphi(x)d\varphi^+(x) \exp -(\tilde{\delta}/4\lambda)\Sigma_x|\varphi|^2+\log I_0|\overset{A}{\tilde{\varphi}}|)$$

where $\overset{A}{\tilde{\varphi}}$ is given by the same equation as (37) except with ∇_μ replaced by D_μ.

For small and smooth fields we can write again a Landau expansion where D_μ is replaced by the usual covariant derivative

$$D_\mu \to \partial_\mu - i2\pi A_\mu \tag{42}$$

and arrive at the partition function of quantum vortex dynamics (QVD)

$$\begin{aligned} Z \propto &\prod_{x,\mu} \int dA_\mu \delta(\nabla_i A_i) e^{-(N_\tau/2\beta)\Sigma_x b^\mu b_\mu} \\ &\times \prod_x \int d\varphi d\varphi^+ \exp -\Sigma_x\{(1/8\lambda)D^\mu\varphi D_\mu\varphi+(1/4)(\tilde{\delta}/\lambda)-1)|\varphi|^2+(1/64)|\varphi|^4+\cdots\}. \end{aligned} \tag{43}$$

‡ It does not matter which branch of the square root is taken since the result (38) depends only on $|\tilde{\varphi}|^2$.

Since we have no control over the neglected higher terms the coupling constants are really free parameters.

When continued back to ordinary time, the action in this partition function is recognized as the $2 + 1$ dimensional scalar electrodynamics, also known as the abelian Higgs model. As such it has been studied in detail in the field theoretic literature. In particular, the structure of the Hilbert space is well understood [17].

5. CONCLUSION

We have seen that the integer nature of the defect gauge fields naturally gives rise to a continuous complex disorder field for the world lines of vortices. The "stress" energy between the vortices is automatically taken into account by a minimal coupling to the stress gauge field. The resulting gauge theory is well known to field theorists. The insights gained in the general field-theoretic studies should contribute to our understanding of the dynamical and quantum-statistical behavior of vortex ensembles.

The generalization to crystalline defects in two dimensions is left to a separate publication.

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(*Received* 5 *June* 1984)