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Convergent Strong-Coupling Expansions from Divergent Weak-Coupling Perturbation Theory

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Divergent weak-coupling perturbation expansions for physical quantities can be converted into sequences of uniformly and exponentially fast converging approximations. This is possible with the help of an additional variational parameter to be optimized order by order. The uniformity of the convergence for any coupling strength allows us to take all expressions directly to the *strong-coupling* limit, yielding a simple calculation scheme for the coefficients of convergent strong-coupling expansions. As an example, we determine these coefficients for the ground state energy of the anharmonic oscillator up to 22nd order with a precision of about 20 digits.

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One of the important problems in the physics of strong interactions is the extraction of physically meaningful results from perturbation expansions which all diverge, even for small couplings, and become completely useless for strong couplings. The perturbation expansion for the energy eigenvalues of the quantum mechanical oscillator is often used to illustrate this difficulty. The potential is

$$V(x) = \frac{\omega^2}{2} x^2 + \frac{g}{4} x^4 \quad (\omega^2, g > 0), \tag{1}$$

and Rayleigh-Schrödinger perturbation theory for the ground-state energy yields a power-series expansion

$$E(g) = \omega \sum_{l=0}^{\infty} e_l \left(\frac{g/4}{\omega^3} \right)^l, \tag{2}$$

where e_l are rational numbers

$$\frac{1}{2}, \frac{3}{4}, -\frac{21}{8}, \frac{333}{16}, -\frac{30885}{128}, \dots . \tag{3}$$

With the help of the recursion relations found by Bender and Wu [1] it is easy to calculate a large number of these coefficients using some symbolic algebra program, for example, MAPLE. Nevertheless, as is well known, the series (2) cannot be used to find an accurate energy, since it has a zero radius of convergence caused by the factorial growth of the coefficients $e_l \approx -(1/\pi)\sqrt{6/\pi}\,(-3)^l l! l^{-1/2}$. Only for small couplings $g < 0.1$ it yields reasonable approximations if truncated at a finite order N, optimally at the integer closest to $3/4g$. For stronger couplings such as $g \approx 1$, the result becomes worse for increasing orders.

The purpose of this Letter is to point out a possible future remedy of this problem. It is based on a systematic extension of the Feynman-Kleinert variational approach to path integrals [2], which has recently been developed into a fully fledged convergent *variational perturbation theory* [3,4]. This theory converts ordinary divergent perturbation expansions into sequences of uniformly and exponentially fast converging approximations. In this Letter we show that the theory has a simple strong-coupling limit which can efficiently be used to calculate the coefficients of strong-coupling expansions. The latter are convergent for all $g > g_s$, where g_s may be quite small.

As an example, we consider the ground-state energy of the quantum mechanical anharmonic oscillator and find 23 strong-coupling expansion coefficients. The associated approximate expansion yields accurate energies for all $g \gtrsim 0.2$ (the convergence radius of the full series being $\approx$0.16). There is no problem in applying the same method to excited states.

The procedure goes as follows (see Sec. 5.13 of Ref. [4]). First, the harmonic term of the potential is split into a new harmonic term with a trial frequency Ω and a remainder:

$$\frac{\omega^2}{2}x^2 = \frac{\Omega^2}{2}x^2 + \left(\frac{\omega^2}{2} - \frac{\Omega^2}{2}\right)x^2. \quad (4)$$

After rewriting

$$V(x) = \frac{\Omega^2}{2}x^2 + V_{\rm int}(x), \quad (5)$$

with an interaction

$$V_{\rm int}(x) = \frac{g}{4}(rx^2 + x^4), \quad r = \frac{2}{g}(\omega^2 - \Omega^2), \quad (6)$$

we perform a perturbation expansion in powers of g at a fixed r:

$$E_N(g, r) = \Omega \sum_{l=0}^{N} e_l(r)\left(\frac{g/4}{\Omega^3}\right)^l. \quad (7)$$

The calculation of the new series up to a specific order N requires only little additional work, being easily obtained from the ordinary perturbation series (2) by replacing ω by $\sqrt{\Omega^2 + gr/2}$, and by reexpanding (2) in powers of g up to the Nth order. This yields the reexpansion coefficients

$$e_l(r) = \sum_{j=0}^{l} e_j \binom{(1-3j)/2}{l-j} (2r\Omega)^{l-j}. \quad (8)$$

The truncated power series

$$W_N(g, \Omega) := E_N\left(g, \frac{2}{g}(\omega^2 - \Omega^2)\right) \quad (9)$$

is certainly independent of Ω in the limit $N \to \infty$. At any finite order, however, it *does* depend on Ω, the approximation having its fastest speed of convergence where it depends least on Ω. If we denote the order-dependent optimal value of Ω by Ω_N, the quantity $W_N(g, \Omega_N)$ is the new approximation to $E(g)$ [5]. In terms of the dimensionless constants

$$\hat{g} = g/\Omega^3, \quad \hat{\omega} = \omega/\Omega, \quad (10)$$

the approximation can be written as

$$W_N = (g/\hat{g})^{1/3} w_N(\hat{g}, \hat{\omega}^2). \quad (11)$$

From the approximate energies (9) it is easy to derive simple formulas for the coefficients of the strong-coupling expansion. We expand the function $w_N(\hat{g}, \hat{\omega}^2)$ in powers of $\hat{\omega}^2 = (g/\omega^3)^{-2/3}\hat{g}^{2/3}$ and find

$$W_N = (g/4)^{1/3}\left[\alpha_0 + \alpha_1\left(\frac{g/4}{\omega^3}\right)^{-2/3} + \alpha_2\left(\frac{g/4}{\omega^3}\right)^{-4/3} + \cdots\right], \quad (12)$$

with the coefficients

$$\alpha_n = \frac{1}{n!} w_N^{(n)}(\hat{g}, 0)\,(\hat{g}/4)^{(2n-1)/3}. \quad (13)$$

Here $w_N^{(n)}(\hat{g}, 0)$ denotes the nth derivatives of $w_N(\hat{g}, \hat{\omega}^2)$ with respect to $\hat{\omega}^2$ at $\hat{\omega}^2 = 0$. To calculate them, we note that $\hat{\omega}^2$ enters the reexpansion coefficients (8) in the form

$$e_l = \sum_{j=0}^{l} e_j \binom{(1-3j)/2}{l-j} [4(\hat{\omega}^2 - 1)/\hat{g}]^{l-j}. \quad (14)$$

The quantities $\frac{1}{n!} w_N^{(n)}(\hat{g}, 0)$ are therefore given by

$$\frac{1}{n!} w_N^{(n)}(\hat{g}, 0) = \sum_{l=0}^{N} \frac{1}{n!}\left(\frac{d}{d\hat{\omega}^2}\right)^n e_l \bigg|_{\hat{\omega}=0} (\hat{g}/4)^l = \sum_{l=0}^{N} (-1)^{l+n} \sum_{j=0}^{l-n} e_j \binom{(1-3j)/2}{l-j} \times \binom{l-j}{n} (-\hat{g}/4)^j. \quad (15)$$

The optimal value of Ω_N has the N dependence (see Ref. [4] and the third part of Ref. [3])

$$\Omega_N^3 = gcN\left(1 + 6.85/N^{2/3}\right), \quad (16)$$

where the coefficient c is found from a saddle point approximation to a dispersion relation for the reexpansion coefficients. By balancing two exponentially divergent terms, one obtains a simple transcendental equation yielding $c = 0.186\,047\,272\,987\,397\,512\,984\,554\,740\,462$; The constant 6.85 stems from an eyeball fit to the lower envelope of all extremal values of Ω. The corresponding N-dependent values of the dimensionless coupling constant $\hat{g} \equiv g/\Omega_N^3$ are inserted into (13) and produce the coefficients shown in Table I. Our results for α_0 agree to

TABLE I. Strong-coupling expansion coefficients α_n.

n	α_n
0	0.667 986 259 155 777 108 270 96
1	0.143 668 783 380 864 910 020 3
2	−0.008 627 565 680 802 279 128
3	0.000 818 208 905 756 349 543
4	−0.000 082 429 217 130 077 221
5	0.000 008 069 494 235 040 966
6	−0.000 000 727 977 005 945 775
7	0.000 000 056 145 997 222 354
8	−0.000 000 002 949 562 732 712
9	−0.000 000 000 064 215 331 954
10	0.000 000 000 048 214 263 787
11	−0.000 000 000 008 940 319 867
12	0.000 000 000 001 205 637 215
13	−0.000 000 000 000 130 347 650
14	0.000 000 000 000 010 760 089
15	−0.000 000 000 000 000 445 890 1
16	−0.000 000 000 000 000 058 989 8
17	0.000 000 000 000 000 019 196 00
18	−0.000 000 000 000 000 003 288 13
19	0.000 000 000 000 000 000 429 62
20	−0.000 000 000 000 000 000 044 438
21	0.000 000 000 000 000 000 003 230 5
22	−0.000 000 000 000 000 000 000 031 4

all 23 digits with the most accurate value for α_0 available in the literature [6]:

$$\alpha_0 = 0.667\,986\,259\,155\,777\,108\,270\,962\,016\,919\,860\,199\,430\,404\,936\,984\,060\,455\,976\,663\,80\,. \qquad (17)$$

For α_1 to α_{11}, our results are consistent with, but considerably more accurate than, previous results in Ref. [7] (e.g., $\alpha_1 = 0.143\,668\,783\,380\,865$, $\alpha_2 = -0.008\,627\,565\,680\,803$).

As a further check we have evaluated our strong-coupling series at $g/4 = 0.1$, 0.3, 0.5, 1, 2 (setting $\omega = 1$) and compared the numbers with the very precise lower and upper bounds of Vinette and Čížek [6]. Table II shows that the energies are accurate to about 20 digits for all couplings $g/4 \geq 1$. The accuracy is limited by the precision of the α_n. Note that our strong-coupling expansion gives very good energies down to very small couplings g—even at $g/4 = 0.1$ the so-obtained energy agrees to seven digits with the value in Ref. [6].

In Fig. 1 we show the approach of α_n to the asymptotic value given in Table I by plotting

$$\Delta_N \equiv |(\alpha_n)_N - \alpha_n| \qquad (18)$$

on a logarithmic scale. The periodic structure in the data is caused by an oscillatory approach of $(\alpha_n)_N - \alpha_n$

TABLE II. Ground-state energies from strong-coupling series expansion. The lines labeled "lb" and "ub" are the lower and upper bounds of Ref. [6].

$g/4$	$n_{\max}$	E_0
0.1	15	0.559 146 597 503 562 187 0
	20	0.559 146 201 201 805 544 6
	22	0.559 146 344 373 873 126 9
	lb	0.559 146 327 183 519 576 3
	ub	0.559 146 327 183 519 576 7
0.3	15	0.637 991 783 178 536 025 3
	20	0.637 991 783 171 236 149 3
	22	0.637 991 783 171 280 381 8
	lb	0.637 991 783 171 278 528 3
	ub	0.637 991 783 171 278 529 6
0.5	15	0.696 175 820 765 191 516 9
	20	0.696 175 820 765 145 887 5
	22	0.696 175 820 765 145 928 8
	lp	0.696 175 820 765 145 925 1
	ub	0.696 175 820 765 145 928 5
1.0	15	0.803 770 651 234 273 812 047 6
	20	0.803 770 651 234 273 769 350 9
	22	0.803 770 651 234 273 769 354 1
	lb	0.803 770 651 234 273 756
	ub	0.803 770 651 234 273 786
2.0	15	0.951 568 472 729 500 011 184 213 69
	20	0.951 568 472 729 500 011 146 930 27
	22	0.951 568 472 729 500 011 146 930 52
	lb	0.951 568 472 729 499 9
	ub	0.951 568 472 729 500 1

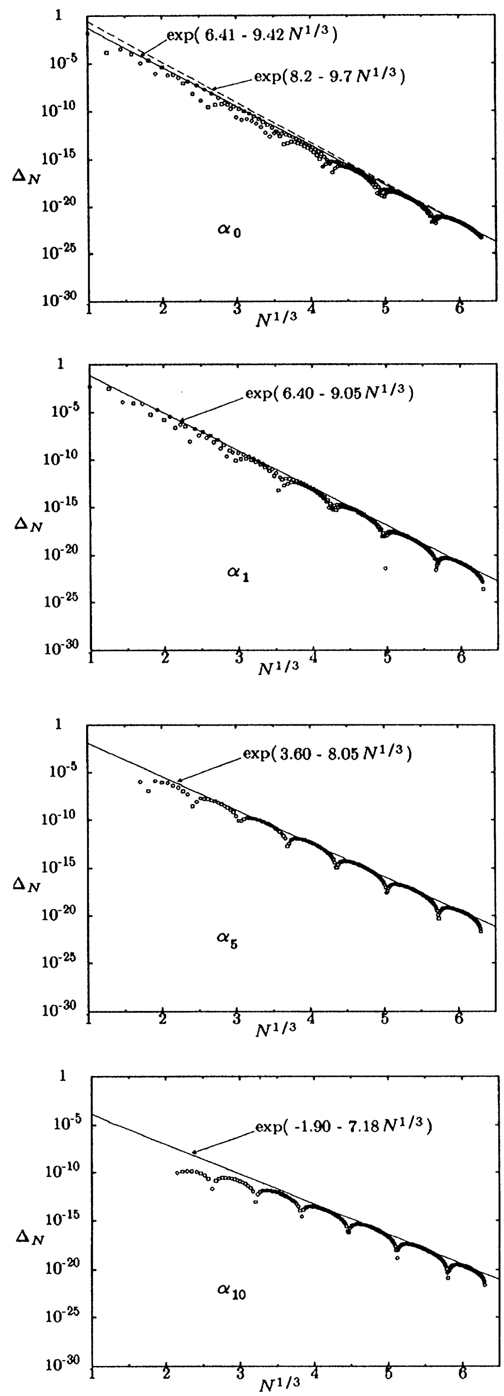

FIG. 1. Convergence of the variational perturbation expansion for the strong-coupling coefficients $\alpha_0, \alpha_1, \alpha_5$, and α_{10}. The solid straight lines are the best eyeball fits to the envelope of the data. The dashed line in the α_0 data has a slope equal to the theoretically expected value 9.7 [4].

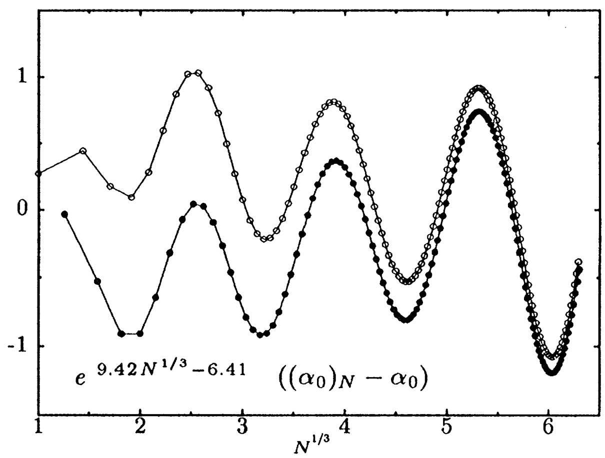

FIG. 2. Oscillatory behavior around the exponential approach to the limiting value of α_0.

to zero (see Fig. 2). The envelope of these oscillations follows the curve

$$\Delta_N = \exp\left(-\kappa_0 - \kappa_1 N^{1/3}\right). \tag{19}$$

For the leading term α_0 of the strong-coupling expansion, the behavior (16) suggests a coefficient $\kappa_1 \approx 9.7$ [4]. From the eyeball fits shown in Fig. 1 we extract $\kappa_1 \approx 9.42, 9.05, 8.05$, and 7.18 for α_n with $n = 0, 1, 5$ and 10, respectively. These estimates should be taken with some care, however, since we do not know how close we are to the asymptotic regime for $N \approx 65, \ldots, 251$, where the natural expansion parameter is $1/N^{1/3} \approx 0.25, \ldots, 0.16$. The N dependence of the coefficient α_0, for instance, is just as well fitted by the dashed line with $\kappa_1 = 9.7$.

It is worth mentioning that variational perturbation theory also permits calculating the imaginary parts of energies on the left-hand cut in the complex coupling constant plane [8]. The results not only improve the semiclassical values [9] in the tunneling regime, but also determine the imaginary part at large negative g [8,10], where the decay proceeds by sliding rather than tunneling.

Until now it has been unclear how the same method converges for other potentials, such as x^6 and x^8. Once this is better understood, it may be possible to adapt the calculation scheme to field theories and learn more about strong interactions from perturbation expansions.

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Note added.—While this paper was being reviewed, the exponential convergence behavior with superimposed oscillations in Figs. 1 and 2 was quantitatively explained by H. Kleinert and W. Janke, Phys. Lett. A **206**, 283 (1995). The exponential convergence was rigorously proven by R. Guida, K. Konishi, and H. Suzuk, University of Genoa, Report No. th/9505084, 1995.

[1] C. M. Bender and T. T. Wu, Phys. Rev. **184**, 1231 (1969); Phys. Rev. D **7**, 1620 (1973).

[2] R. P. Feynman and H. Kleinert, Phys. Rev. A **34**, 5080 (1986). See also the independent development by R. Giachetti and V. Tognetti, Phys. Rev. Lett. **55**, 912 (1985); Int. J. Magn. Mater. **54–57**, 861 (1986); R. Giachetti, V. Tognetti, and R. Vaia, Phys. Rev. B **33**, 7647 (1986).

[3] The variational perturbation theory was developed for path integrals in H. Kleinert, Phys. Lett. A **173**, 332 (1993); H. Kleinert and H. Meyer, Phys. Lett. A **184**, 319 (1994); For energy eigenvalues, related expansions have been proposed by R. Seznec and J. Zinn-Justin, J. Math. Phys. **20**, 1398 (1979); T. Barnes and G. I. Ghandour, Phys. Rev. D **22**, 924 (1980); B. S. Shaverdyan and A. G. Usherveridze, Phys. Lett. B **123**, 316 (1983); K. Yamazaki, J. Phys. A **17**, 345 (1984); P. M. Stevenson, Phys. Rev. D **30**, 1712 (1985); **32**, 1389 (1985); P. M. Stevenson and R. Tarrach, Phys. Lett. B **176**, 436 (1986); A. Okopinska, Phys. Rev. D **35**, 1835 (1987); **36**, 2415 (1987); W. Namgung, P. M. Stevenson, and J. F. Reed, Z. Phys. C **45**, 47 (1989); U. Ritschel, Phys. Lett. B **227**, 44 (1989); Z. Phys. C **51**, 469 (1991); M. H. Thoma, Z. Phys. C **44**, 343 (1991); I. Stancu and P. M. Stevenson, Phys. Rev. D **42**, 2710 (1991); R. Tarrach, Phys. Lett. B **262**, 294 (1991); H. Haugerud and F. Raunda, Phys. Rev. D **43**, 2736 (1991); A. N. Sissakian, I. L. Solivtosv, and O. Y. Sheychenko, Phys. Lett. B **313**, 367 (1993); First convergence proofs were given by I. R. C. Buckley, A. Duncan, and H. F. Jones, Phys. Rev. D **47**, 2554 (1993); C. M. Bender, A. Duncan, and H. F. Jones, Phys. Rev. D **49**, 4219 (1994); A. Duncan and H. F. Jones, Phys. Rev. D **47**, 2560 (1993); R. Guida, K. Konishi, and H. Suzuki, University of Genoa Report No. hep-th/9407037, 1995; C. Arvanitis, H. F. Jones, and C. S. Parker, Imperial College London Report No. hep-th/9502386, 1995. These proofs do not, however, account for the exponentially fast convergence at strong couplings found in the present work (for this see note added in proof).

[4] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics* (World Scientific, Singapore, 1995), 2nd ed. German readers may consult the German edition published with (B. I.-Wissenschaftsverlag, Mannheim, 1993).

[5] The N dependence of the variational parameter Ω_N is crucial for the convergence of the approximations.

[6] The most accurate energies have been calculated by F. Vinette and J. Čížek, J. Math. Phys. **32**, 3392 (1991).

[7] F. M. Fernández and R. Guardiola, J. Phys. A **26**, 7169 (1993); F. M. Fernández, Phys. Lett. A **166**, 173 (1992); R. Guardiola, M. A. Solís, and J. Ros, Nuovo Cimento B **107**, 713 (1992). See also the earlier less accurate results by A. V. Turbiner and A. G. Ushveridze, J. Math. Phys. **29**, 2053 (1988); B. Bonnier, M. Hontebeyrie,

and E.H. Ticembal, J. Math. Phys. **26**, 3048 (1985); M.A. Smondyrev, J.O.I.N.T. Report Dubna, 1984, F.T. Hioe and E.W. Montroll, J. Math. Phys. **16**, 1945 (1975).

[8] H. Kleinert, Phys. Lett. B **300**, 261 (1993).

[9] A.I. Vainshtein, "Decaying Systems and the Divergence of Perturbation Series," Novosibirsk report, 1964, in Russian (unpublished); J.S. Langer, Ann. Phys. **41**, 108 (1967).

[10] R. Karrlein and H. Kleinert, Phys. Lett. A **187**, 133 (1994).