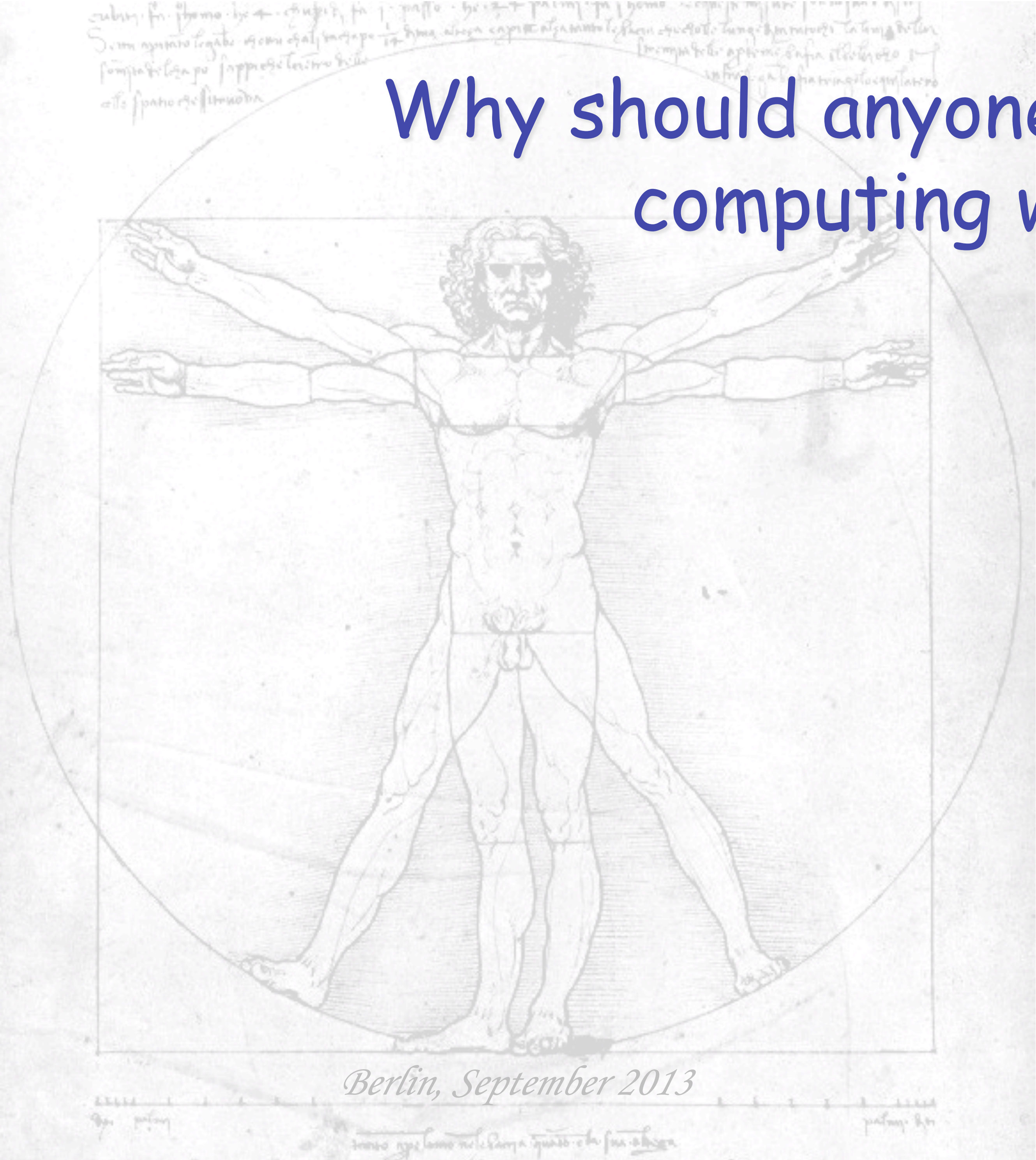


Why should anyone care about computing with anyons?

Jiannis K. Pachos

Toric Code and Quantum Doubles

Berlin, September 2013

EPSRC
Engineering and Physical Sciences Research Council

UNIVERSITY OF LEEDS

Anyons and Quantum Computation

•Error correction needs a **huge overhead**.

•Instead of performing active error correction let physics do the job.

•Perform QC in a physical medium that is **gapped** and **highly correlated**:

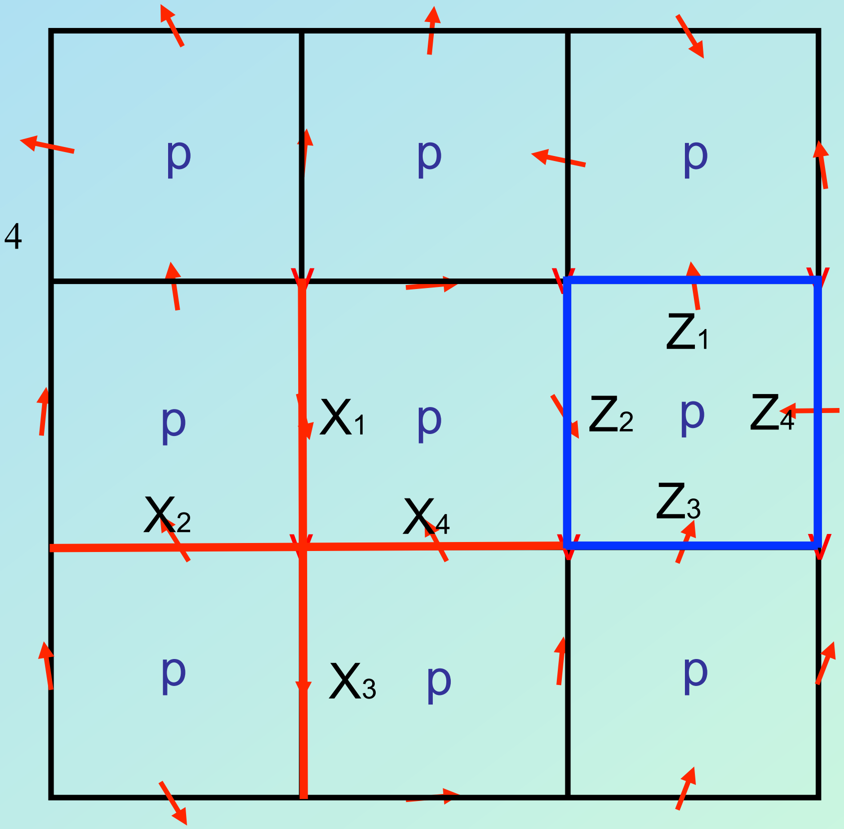
- •**Energy penalty for errors** (gapped).
- •**Make logical errors non-local** (very unlikely).

•Similar to **quantum error correction**, but without active control.

Toric Code: ECC

Consider the lattice Hamiltonian

$$H = -\sum_{p} Z_{p1}Z_{p2}Z_{p3}Z_{p4} - \sum_{v} X_{v1}X_{v2}X_{v3}X_{v4}$$



Spins on the edges.

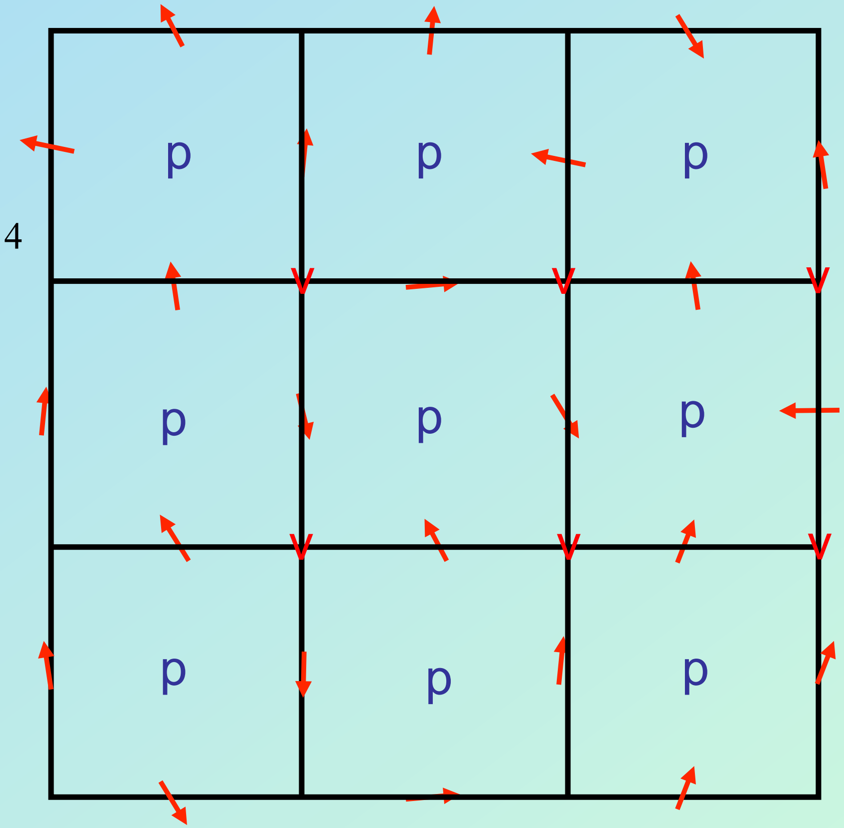
Two different types of interactions: ZZZZ or XXXX acting on plaquettes and vertices respectively.

The four spin interactions involve spins of the same vertex/plaquette.

Toric Code: ECC

Consider the lattice Hamiltonian

$$H = -\sum_p Z_{p1}Z_{p2}Z_{p3}Z_{p4} - \sum_v X_{v1}X_{v2}X_{v3}X_{v4}$$



Good **quantum numbers**:

$$[H, Z_{p1}Z_{p2}Z_{p3}Z_{p4}] = 0$$

$$[H, X_{v1}X_{v2}X_{v3}X_{v4}] = 0$$

$$(X_{v1}X_{v2}X_{v3}X_{v4})^2 = 1$$

$$(Z_{p1}Z_{p2}Z_{p3}Z_{p4})^2 = 1$$

⇒eigenvalues of XXXX and ZZZZ: ±1

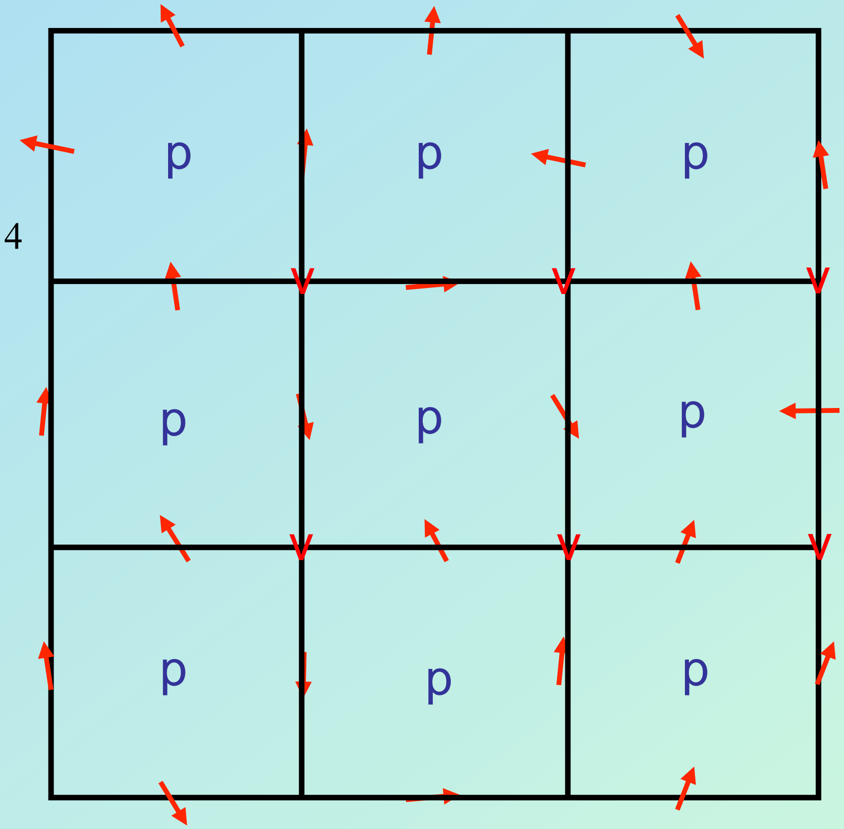
Also Hamiltonian **exactly solvable**:

$$[X_{v1}X_{v2}X_{v3}X_{v4}, Z_{p1}Z_{p2}Z_{p3}Z_{p4}] = 0$$

Toric Code: ECC

Consider the lattice Hamiltonian

$$H = -\sum_p Z_{p1} Z_{p2} Z_{p3} Z_{p4} - \sum_v X_{v1} X_{v2} X_{v3} X_{v4}$$



Indeed, the ground state is:

$$|\xi\rangle = \prod_v \frac{1}{\sqrt{2}} (I + X_{v1} X_{v2} X_{v3} X_{v4}) |00\ldots0\rangle$$

The |00…0> state is a ground state of the ZZZZ term.

The (I+XXXX) term projects that state to the ground state of the XXXX term.

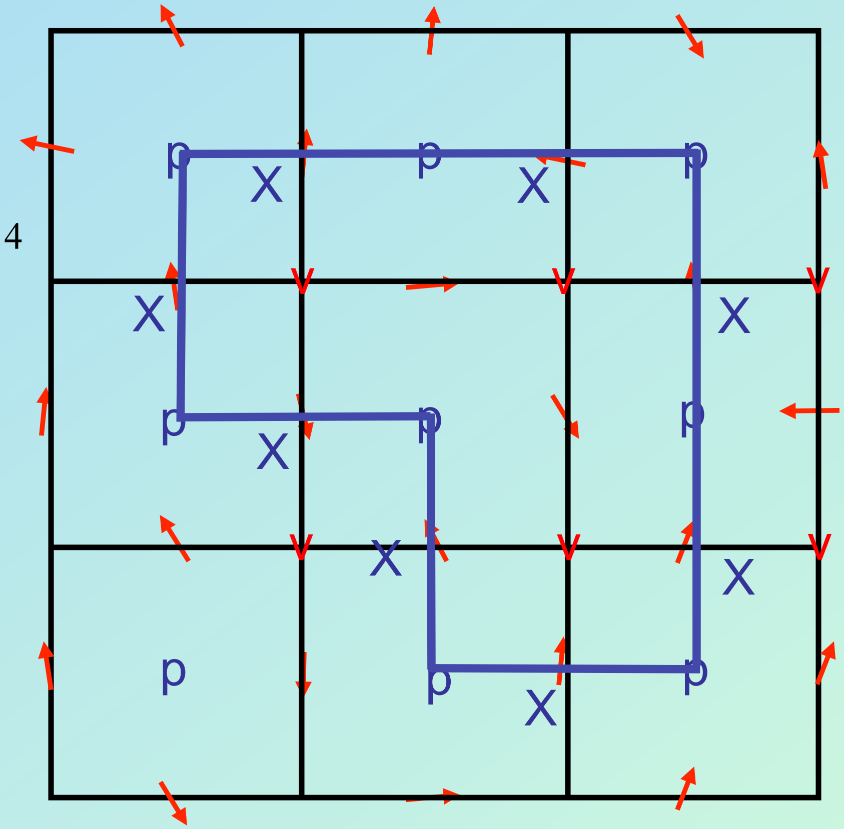
Toric Code: ECC

Consider the lattice Hamiltonian

$$H = -\sum_p Z_{p1} Z_{p2} Z_{p3} Z_{p4} - \sum_v X_{v1} X_{v2} X_{v3} X_{v4}$$

Indeed, the ground state is:

$$|\xi\rangle = \prod_v \frac{1}{\sqrt{2}} (\mathrm{I} + X_{v1} X_{v2} X_{v3} X_{v4}) |00...0\rangle$$



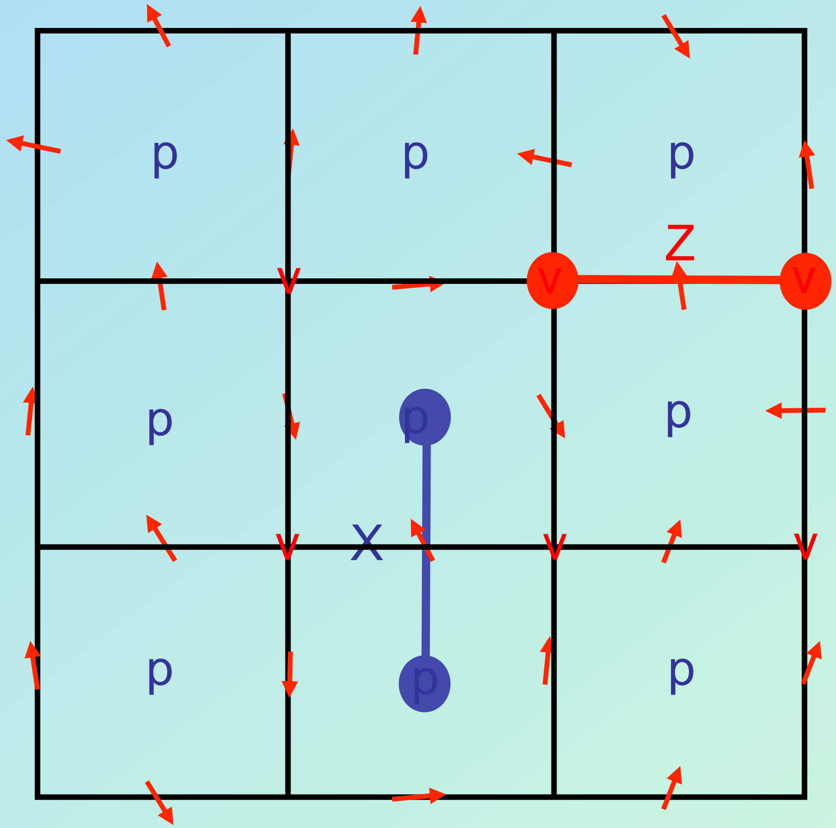
The ground state is a **superposition** of all X **loops**.

It is **stabilized** by the application of **all X loop operators**.

Equivalently for Z loops.

Toric Code: ECC

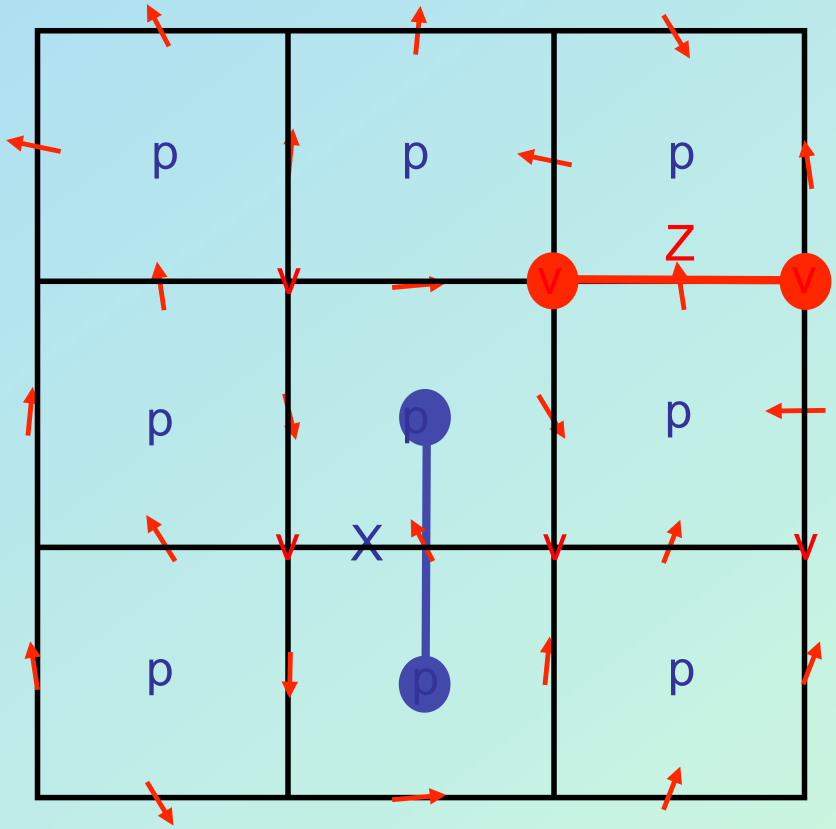
- **Excitations** are produced by Z or X rotations of one spin.
- These rotations **anticommute** with the X- or Z-part of the Hamiltonian, respectively.
- Z excitations on *v* vertices.
- X excitations on *p* plaquettes.



X and Z excitations behave as anyons with respect to each other.

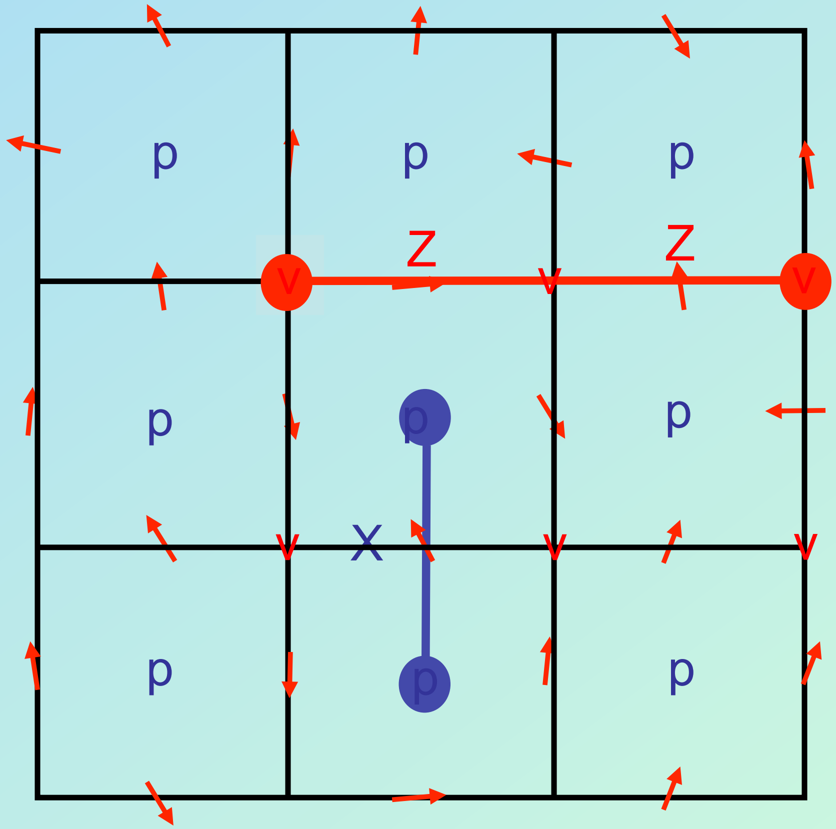
Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.
First create excitations with Z and X rotations.



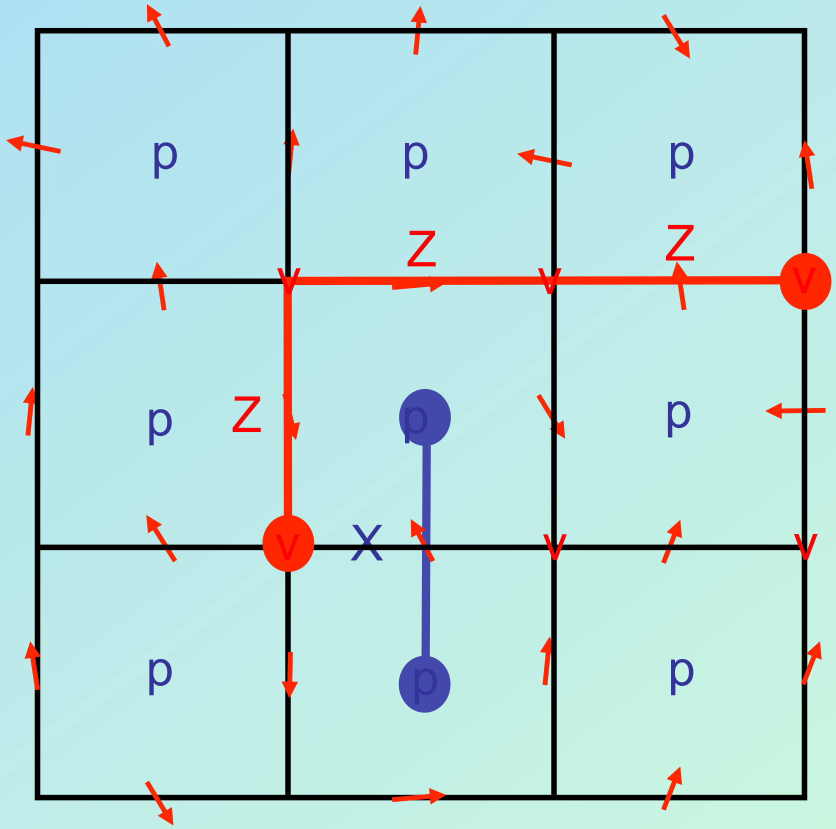
Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.
First create excitations with Z and X rotations.
Then rotate Z excitation around the X one.



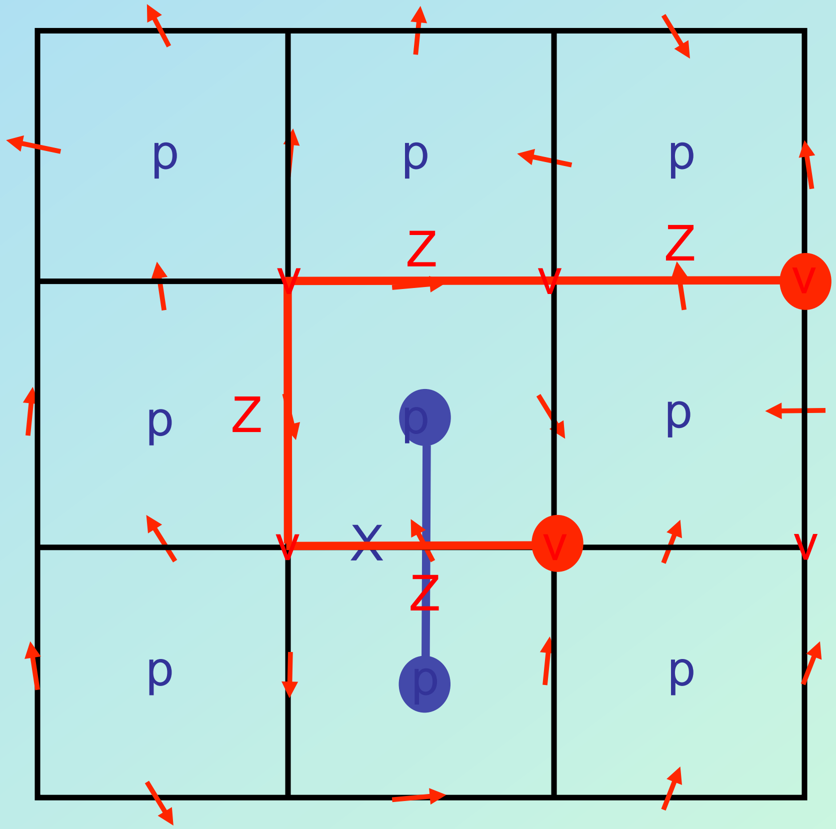
Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.
First create excitations with Z and X rotations.
Then rotate Z excitation around the X one.



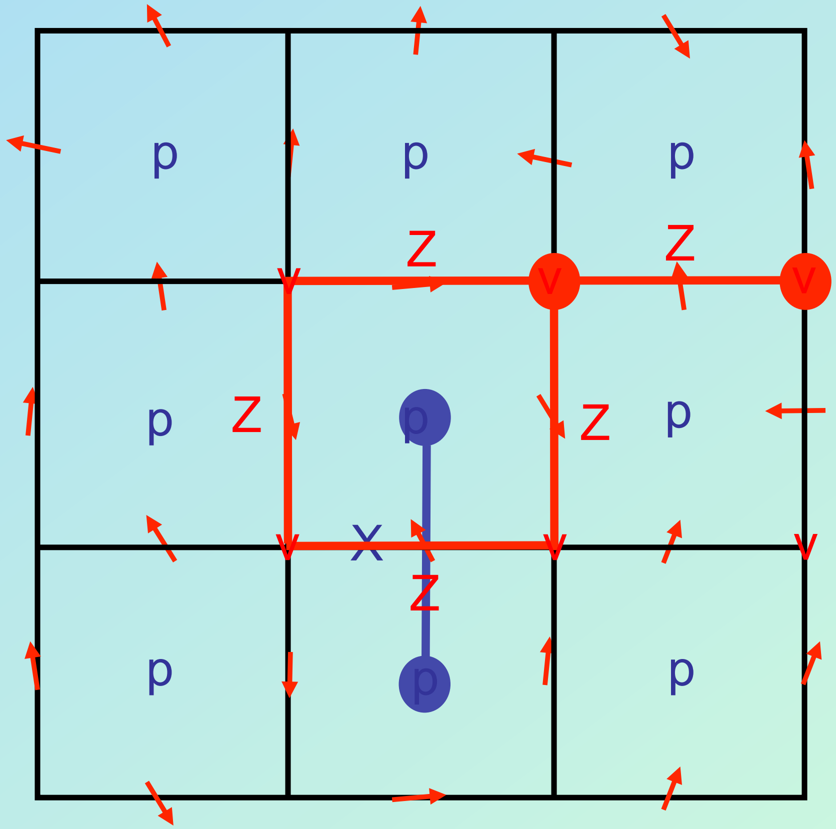
Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.
First create excitations
with Z and X rotations.
Then rotate Z excitation around the X one.



Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.
First create excitations with Z and X rotations.
Then rotate Z excitation around the X one.



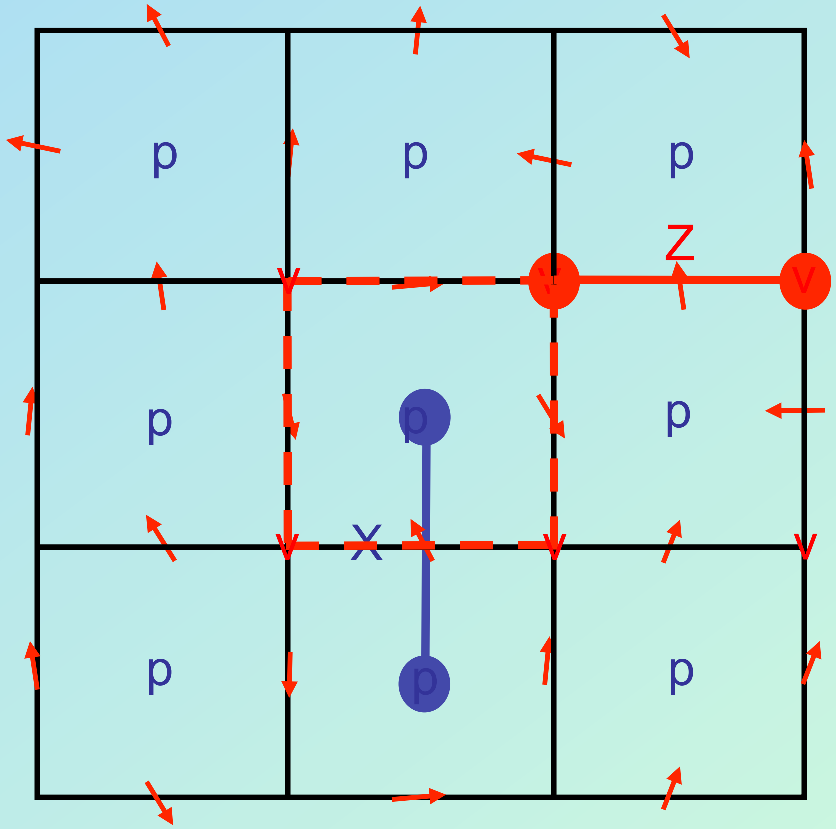
Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.

First create excitations with Z and X rotations.

Then rotate Z excitation around the X one.

This results in plaquette operator detecting the X excitation. Gives -1



$$|Final\rangle = X_4X_3X_2X_1|Z\rangle = (X_4X_3X_2X_1)Z_3|\xi\rangle$$

$$-Z_3\,(X_4X_3X_2X_1)|\xi\rangle = -|Initial\rangle$$

Toric Code: ECC

$$|Final\rangle = X_4X_3X_2X_1|Z\rangle = (X_4X_3X_2X_1)Z_1|\xi\rangle$$

$$-Z_1(X_4X_3X_2X_1)|\xi\rangle = -|Initial\rangle$$

Anyonic statistics

After a complete rotation of an X anyon around a Z anyon (two successive exchanges) the resulting state gets a phase π (a minus sign): hence **ANYONS** with statistical angle $\pi/2$

A property we used is that $X_4X_3X_2X_1|\xi\rangle = |\xi\rangle$

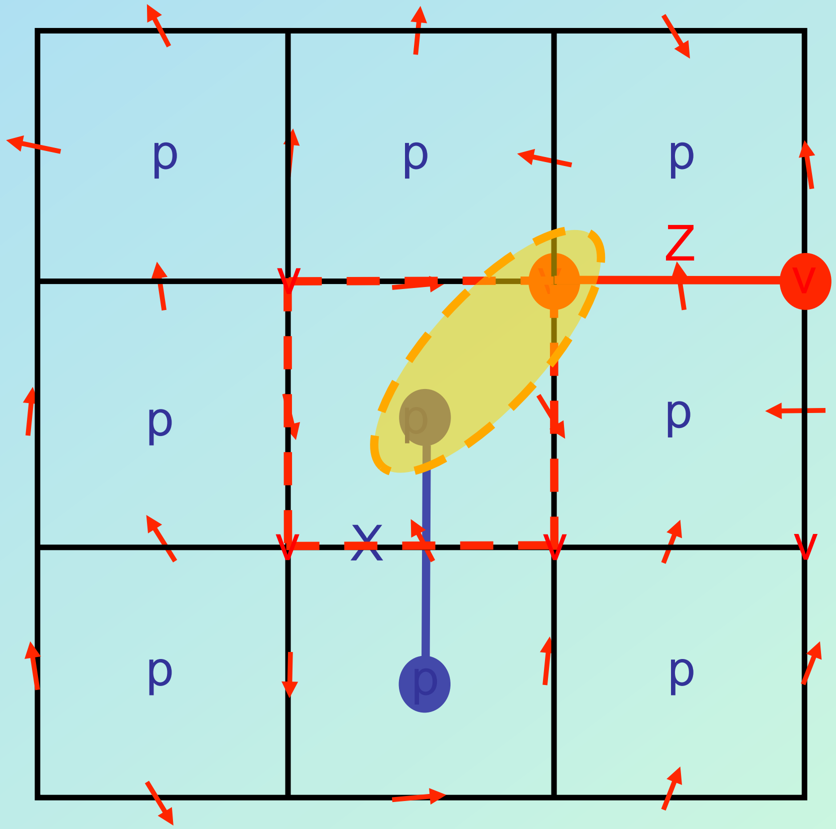
Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.

First create excitations with Z and X rotations.

Then rotate Z excitation around the X one.

This results in plaquette operator detecting the X excitation. Gives -1



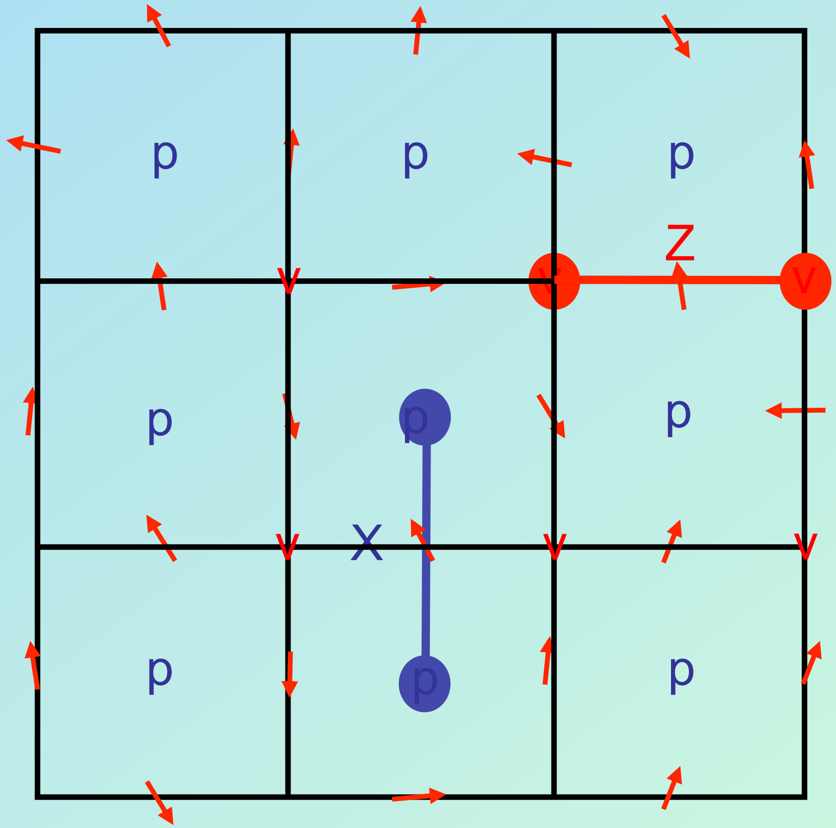
$$|Final\rangle = X_4X_3X_2X_1|Z\rangle = (X_4X_3X_2X_1)Z_3|\xi\rangle$$

$$-Z_3(X_4X_3X_2X_1)|\xi\rangle = -|Initial\rangle$$

Toric Code: Anyons

Hence Toric Code has particles:

$$1, e\,(X), m\,(Z), \varepsilon\,(\text{fermion})$$



Fusion rules:

$$e \times e = 1, m \times m = 1, \varepsilon \times \varepsilon = 1$$

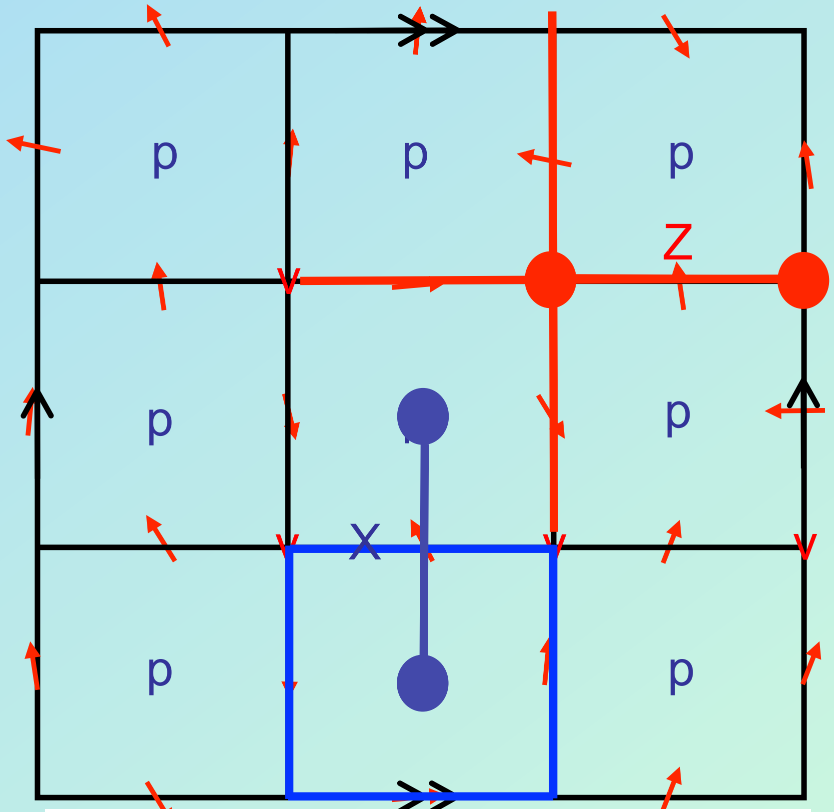
$$e \times m = \varepsilon, e \times \varepsilon = m, m \times \varepsilon = e$$

Fusion moves: F are trivial

Braiding moves R: $\quad R^{\varepsilon}_{em} = i, R^{1}_{\varepsilon\varepsilon} = -1$

Toric Code: Encoding

Toric code as a **quantum error correcting code.**

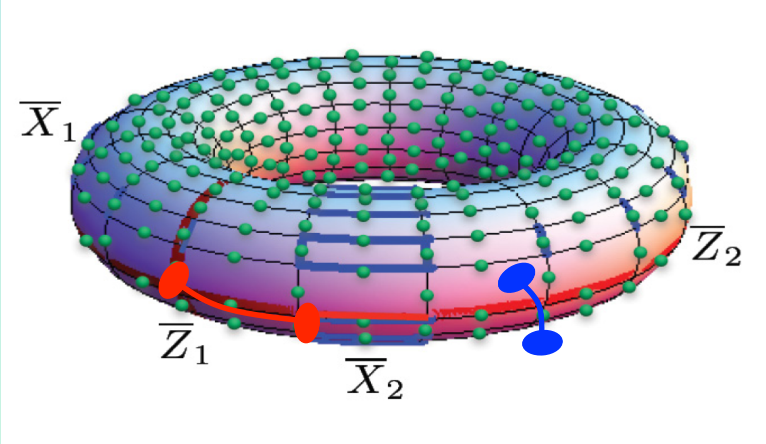


Consider periodic boundary conditions: TORUS of size L

Syndrome: Anyons

Error correction: detect anyons/errors and connect shortest distance between the same type of anyons.

$$\prod_v X_{v1} X_{v2} X_{v3} X_{v4} = \prod_p Z_{p1} Z_{p2} Z_{p3} Z_{p4} = 1$$



Toric Code: Encoding

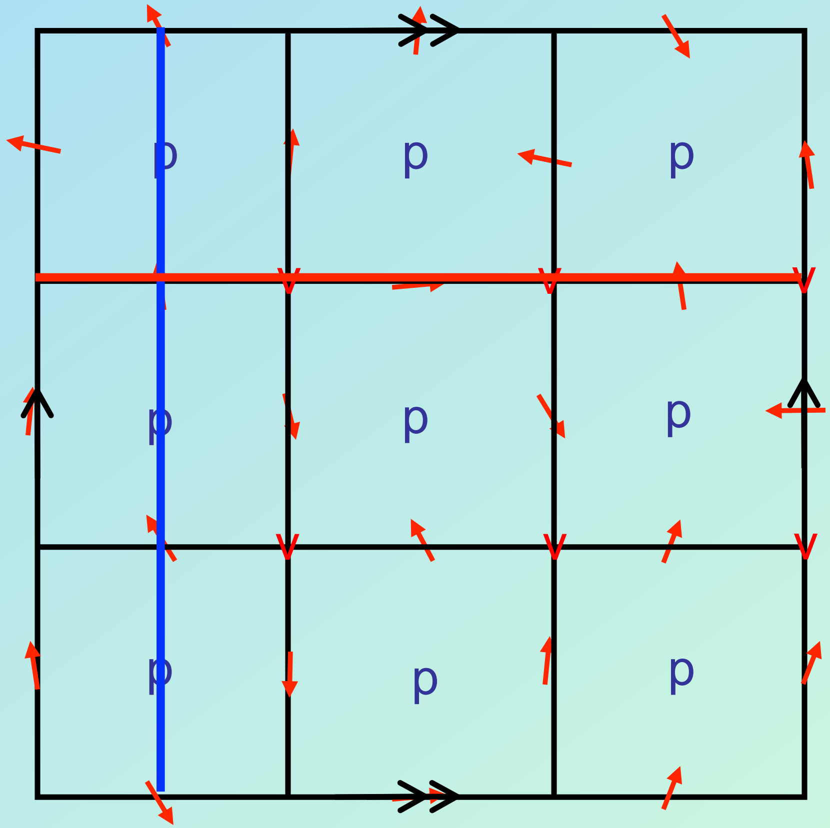
Toric code as a **quantum error correcting code.**

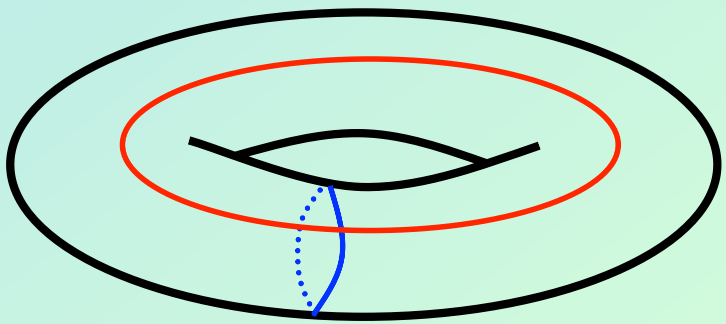
Consider periodic boundary conditions: TORUS of size L

Syndrome: Anyons

Error correction: detect anyons/errors and connect shortest distance between the same type of anyons.

Logical Gates: non-trivial loops





Toric Code: Encoding

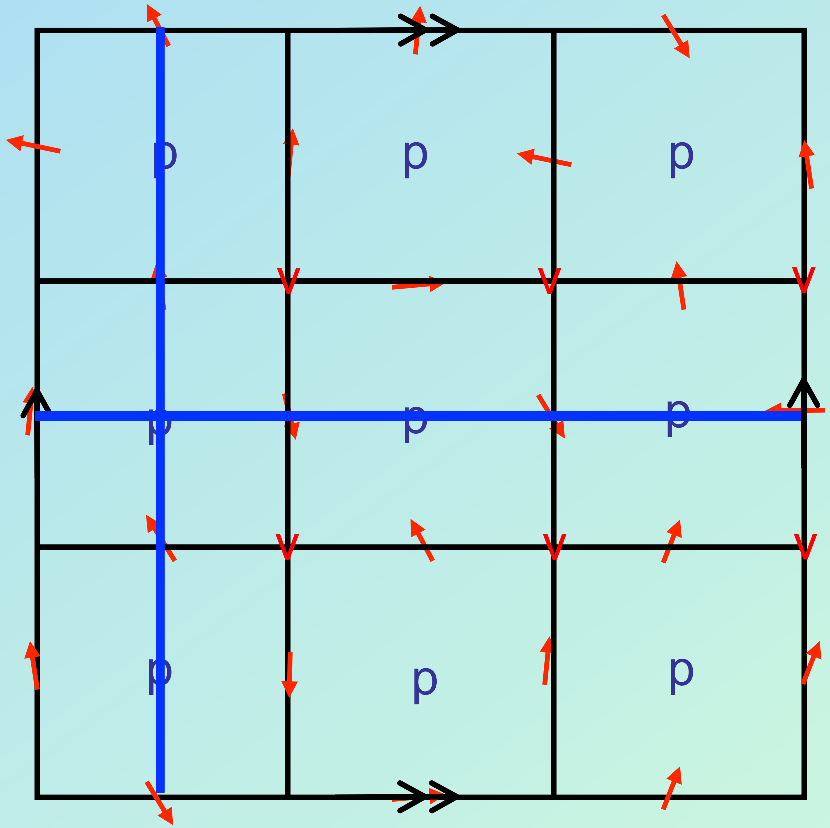
Logical Space and Gates

$$|\Psi_1\rangle$$

$$|\Psi_2\rangle = C_X^1 |\Psi_1\rangle$$

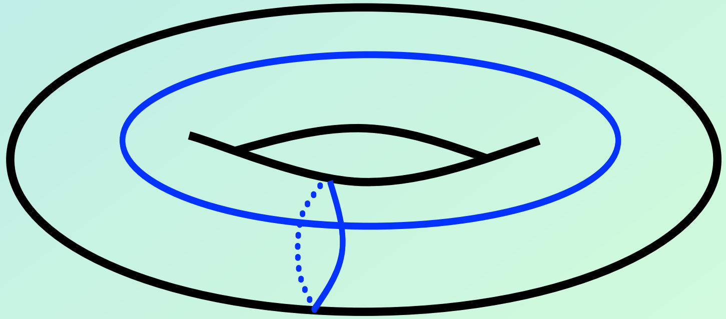
$$|\Psi_3\rangle = C_X^2 |\Psi_1\rangle$$

$$|\Psi_4\rangle = C_X^2 C_X^1 |\Psi_1\rangle$$



Can store two qubits and perform Clifford group operations!

Higher genus, g, stores 2g qubits.



Quantum Double Models

Toric Code is an example of **quantum double models**.

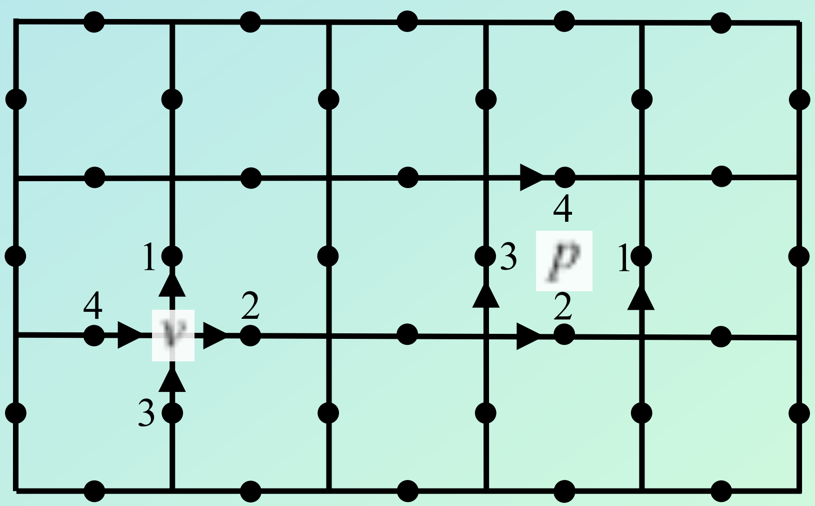
Corresponding group $Z_2 = \{1, e; e^2=1\}$ that gives rise to qubit states $|1\rangle, |e\rangle$.

Imagine a general finite **group** $G=\{g_1, g_2, \ldots, g_d\}$ and the corresponding qudit with states $|g_i\rangle$, $i=1,\ldots,d$.

Consider a **qudit** positioned at each **edge** of a square lattice.

Define **orientation** on the lattice:

Upwards and Rightwards



Quantum Double Models

Define operators:

$$L_+^g|z\rangle = |gz\rangle, \quad L_-^g|z\rangle = |zg^{-1}\rangle, \quad T_+^h|z\rangle = \delta_{h,z}|z\rangle, \quad T_-^h|z\rangle = \delta_{h^{-1},z}|z\rangle$$

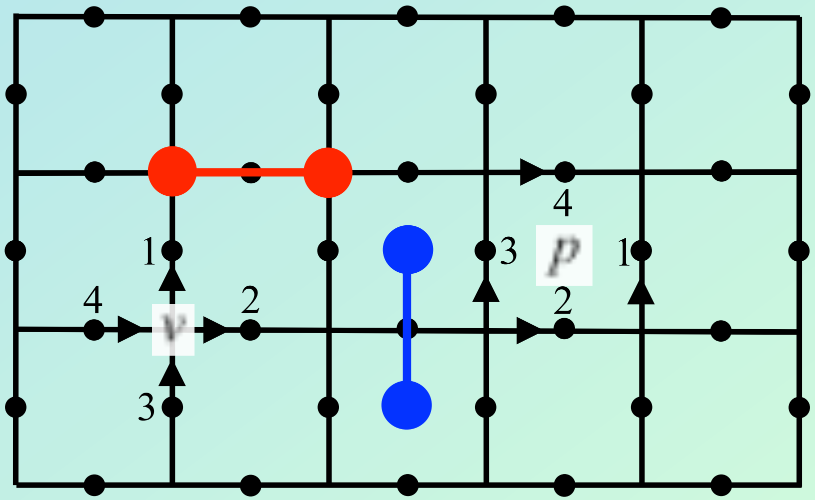
$$A(v) = \frac{1}{|G|}\sum_{g\in G} L_+^g(e_1)L_+^g(e_2)L_-^g(e_3)L_-^g(e_4), \quad B(p) = \sum_{h_1\ldots h_4=1} T_-^{h_1}(e_1)T_-^{h_2}(e_2)T_+^{h_3}(e_3)T_+^{h_4}(e_4)$$

Hamiltonian and ground state:

$$H = -\sum_v A(v) - \sum_p B(p)$$

$$A(v)|\xi\rangle = |\xi\rangle$$

$$B(p)|\xi\rangle = |\xi\rangle$$



Quantum Double Models

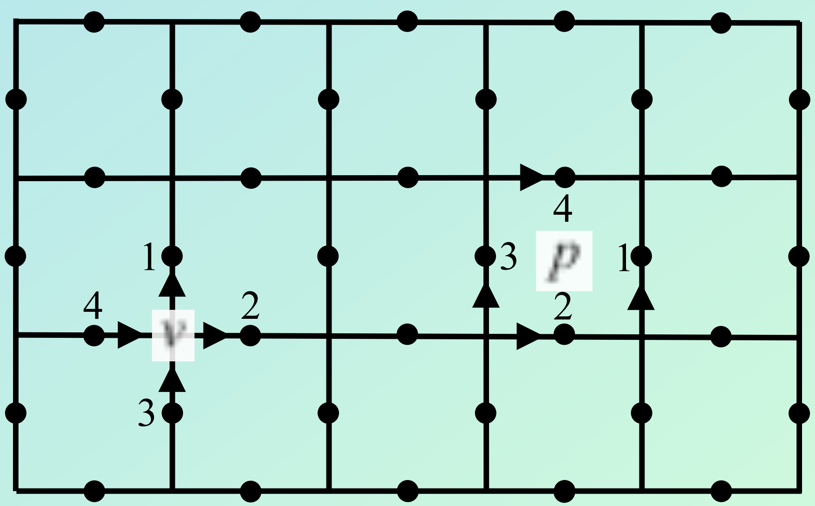
This is also an error correcting code defined from the stabilizer formalism.

The syndromes are **anyons**, Abelian or non-Abelian, with the corresponding fusion rules, B and F matrices.

These properties can be explicitly determined.

Examples: $D(Z_2)$, $D(Z_2 x Z_2)$, $D(S_3)$

$S_3=\{1,x,y,y^2,xy,xy^2;\ x^2=1, y^3=1\}$

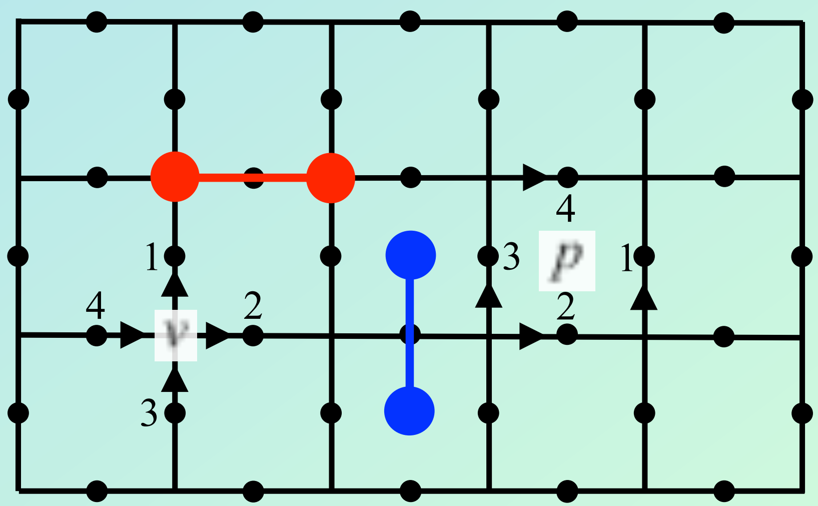


Quantum Double Models

This is also an error correcting code defined from the stabilizer formalism.

The syndromes are **anyons**, Abelian or non-Abelian, with the corresponding fusion rules, B and F matrices.

Information can be encoded in the fusion space of non-Abelian anyons and manipulated by braiding them.

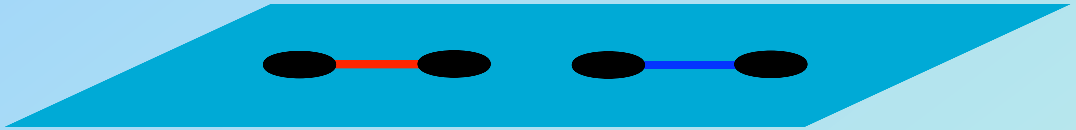


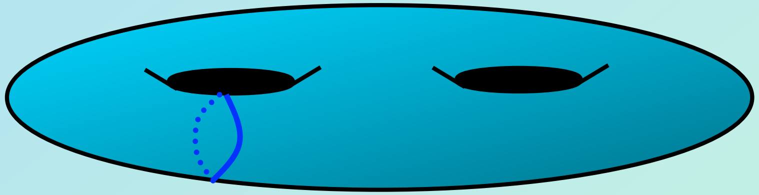
Realizations:

Josephson junctions, photons, optical lattices,...

From Abelion to Nonabelion

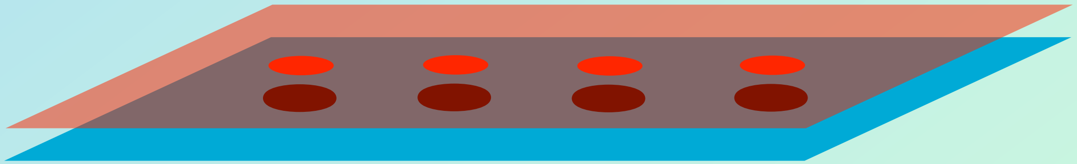
Consider the toric code with higher **genus** surface or with **punctures**: encoding Hilbert space becomes larger.





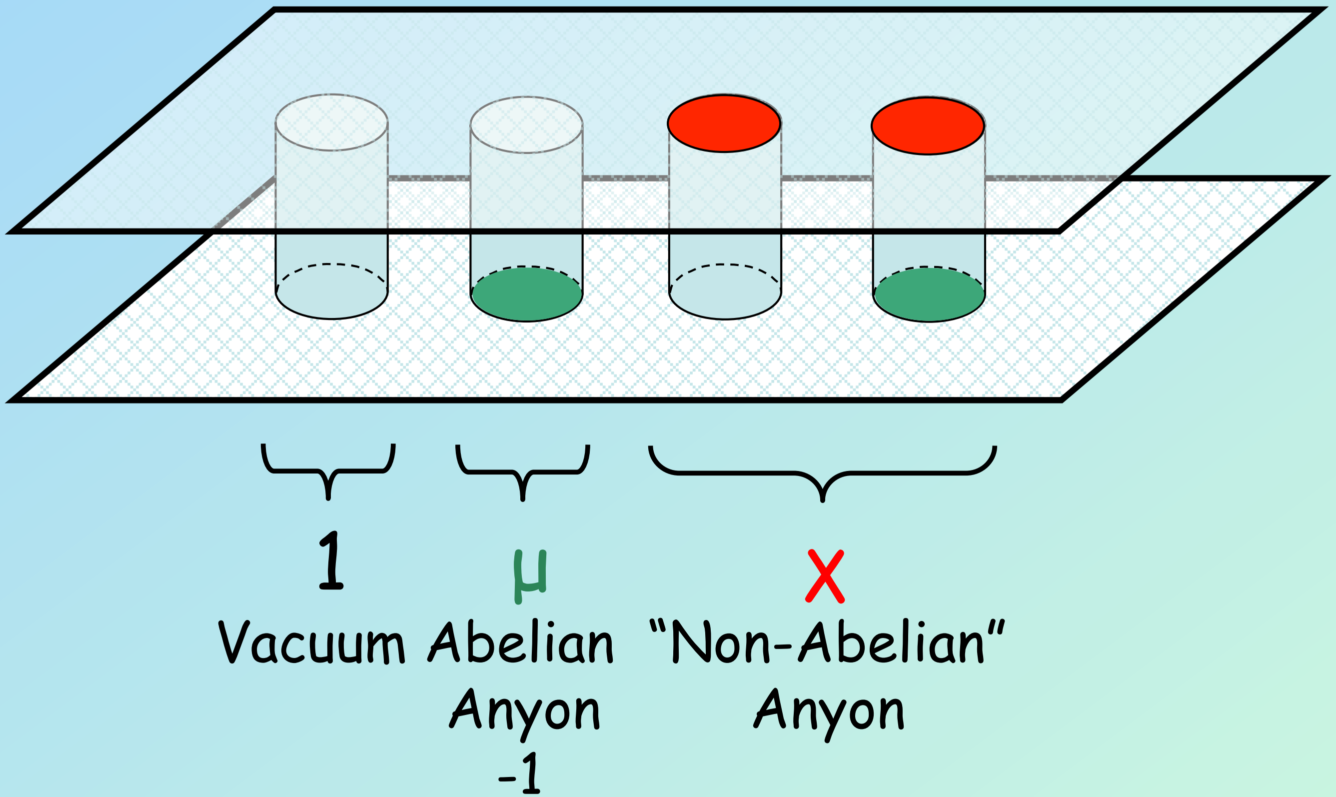
Punctures are *better* for storing and manipulating info.

If you do not like punctures:



From Abelion to Nonabelion

- The scheme: $D(Z_2 x Z_2)$ [or $D(S_3)$] similar to two toric codes



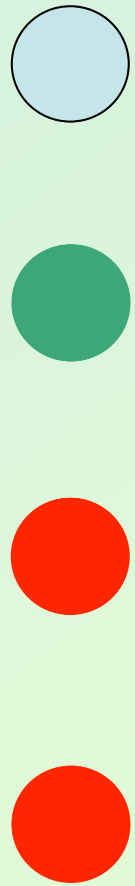
Hamiltonian bird's eye view:

Empty

Lower only

Upper only

Upper and Lower



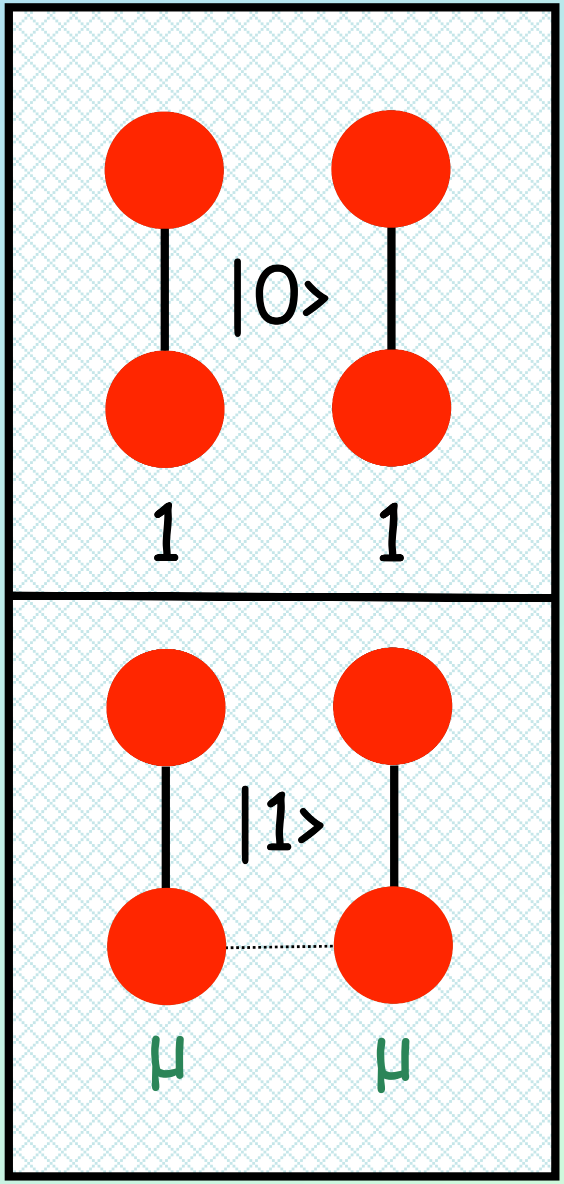
From Abelion to Nonabelion

- Encode information in **fusion channels**:

  $\mu \times \mu = 1, \qquad \chi \times \chi = 1 + \mu$

- Qubit needs four anyons
- Logical **|0>** when each pair fuses to the **vacuum 1**
- Logical **|1 >** when each pair fuses to **μ**
- 1, μ indistinguishable to local operations when dressed with χ

- ***Measurement by fusion***

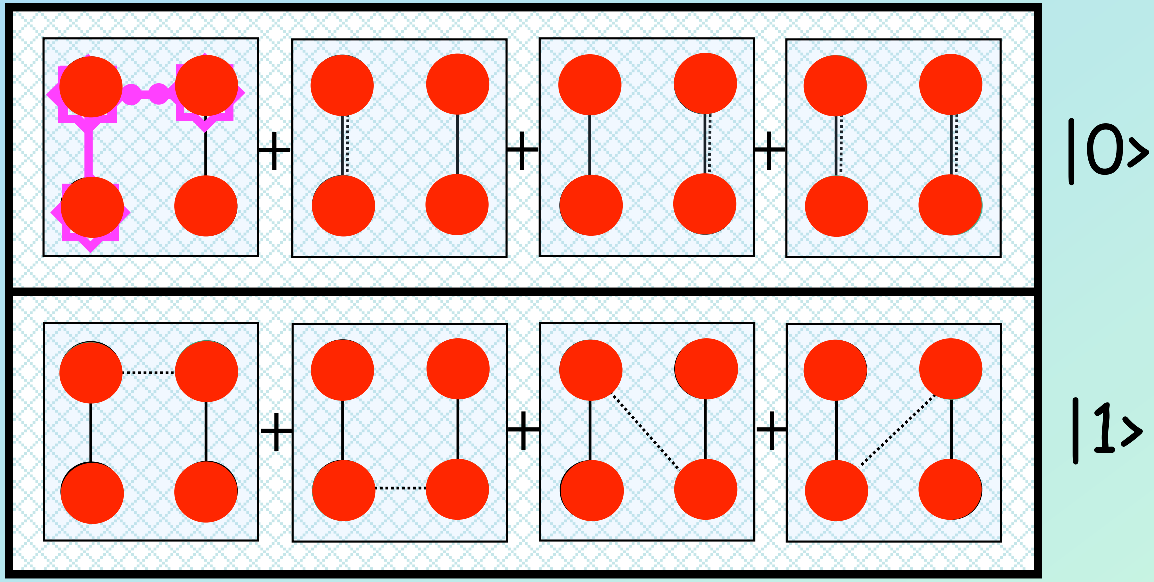


From Abelion to Nonabelion

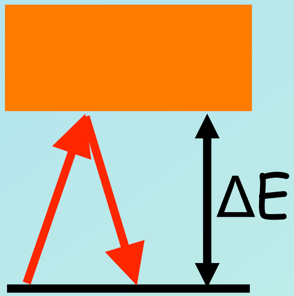
- Fault-tolerance
  - Phase flips
  - Bit flips

  by **non-local** operators only

  → topo. protection



Energy gap present even during gate operations



- **Redundancy** and non-locality protects against **virtual transitions**
- Braiding is only Abelian.

Summary

- **Quantum Double models:**
  - Toric Code
  - Abelian encoding and quantum computation
  - Non-Abelian models
- **Degenerate** encoding states
- **Energy gap** above encoding space
- **Manipulations** of code space: