GROUND - STATE ENERGY OF CHARGED ANYON GASES

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Contents

Interacting anyons in 2D harmonic potential in the presence of external magnetic field: general setup (B. Abdullaev, et al., Phys. Rev. B 68, 165105 (2003))

Hamiltonian

$$\hat{H} = \frac{1}{2M}\sum_{k=1}^{N}\left(\left\{\vec{p}_k - \left(\vec{A}_\nu(\vec{r}_k) + e\vec{A}_{ext}(\vec{r}_k)/\boldsymbol{c}\right)\right\}^2 + M^2(\omega_0)^2|\vec{r}_k|^2\right) + \frac{1}{2}\sum_{k,j\neq k}^{N}\frac{e^2}{|\vec{r}_{kj}|}.$$

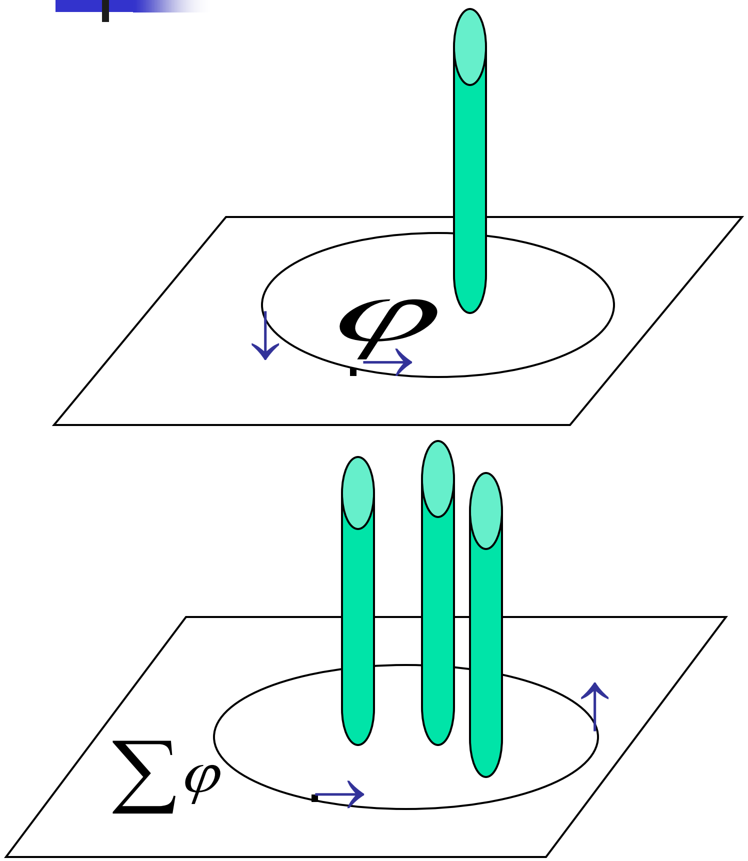
Anyon vector potential $\vec{A}_\nu(\vec{r}_k) = \hbar\nu\sum_{j\neq k}^{N}\frac{\vec{e}_z \times \vec{r}_{kj}}{|\vec{r}_{kj}|^2};$ Magnetic field $\vec{A}_{ext}(\vec{r}_k) = \frac{\vec{H}\times\vec{r}_k}{2}$

Minimization of energy $$E = \frac{\int \psi^*(\vec{R})\hat{H}\psi(\vec{R})d\vec{R}}{\int \psi^*(\vec{R})\psi(\vec{R})d\vec{R}}$$

with trail wave function $$\psi(\vec{R}) = \left(\frac{\alpha}{\pi}\right)^{N/2}\prod_{k=1}^{N} exp\left(-\alpha\frac{\left((x_k)^2 + (y_k)^2\right)}{2}\right)$$

Typically, $E = ReE + iImE$, however, for Gaussian $\psi(\vec{R})$, one has $ImE = 0$.

Anyons: many-body Aharonov - Bohm effect



Inside tube $\vec{H} \neq 0$, outside $\vec{H} = 0$ and $\vec{A} = (\phi/2\pi)\vec{\nabla}\varphi$, where ϕ is magnetic flux. If $\phi_0 = \pi\hbar c/|e|$ is elementary magnetic flux, then

$$\psi(\vec{r}) \rightarrow \psi(\vec{r})e^{i\nu\varphi}\ ; \quad \nu = \phi/\phi_0\ ;$$

$$\psi(\vec{r}_1, \vec{r}_2, \cdots) \rightarrow \psi(\vec{r}_1, \vec{r}_2, \cdots)e^{i\nu\sum\varphi}\ ;$$

$$e^{i\nu\sum\varphi} = \prod_{i\neq j}^{N} \frac{(z_i - z_j)^\nu}{|z_i - z_j|^\nu};\ \vec{A}_\nu(\vec{r}_k) = \hbar\nu \sum_{j\neq k}^{N} \frac{\vec{e}_z \times \vec{r}_{kj}}{|\vec{r}_{kj}|^2};$$

for $z = x + iy$. Thus Schrödinger equation is:

$$\frac{1}{2M}\sum_{i=1}^{N}\left(\vec{p}_i + \vec{A}_\nu(\vec{r}_i)\right)^2 \Phi\ (\vec{r}_1, \vec{r}_2, \cdots) = $$

$$= E\Phi\ (\vec{r}_1, \vec{r}_2, \cdots)$$

for bosonic representation of $\Phi\ (\vec{r}_1, \vec{r}_2, \cdots)$.

Non-interacting anyons in 2D harmonic well

Hamiltonian
$$\hat{H} = \frac{1}{2M}\sum_{k=1}^{N}\left(\left\{\vec{p}_k - \vec{A}_\nu(\vec{r}_k)\right\}^2 + M^2(\omega_0)^2|\vec{r}_k|^2\right).$$

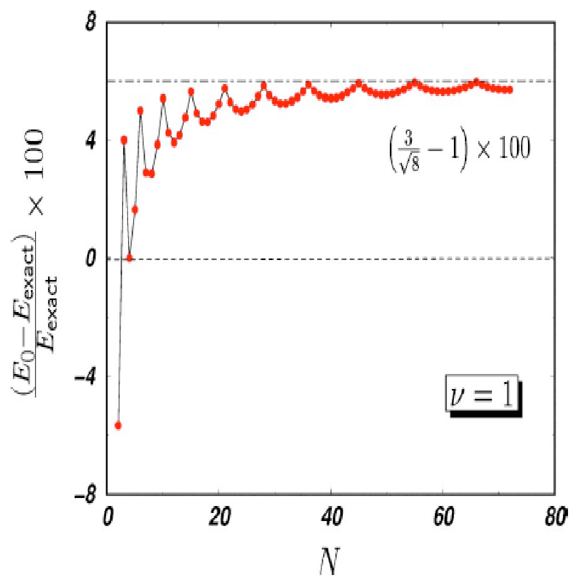


FIG. 1. Relative deviation (in percent) of the approximate ground state energy E_0 [Eq. (21)], from the exact ground state energy, E_{exact}, for up to $N=72$ noninteracting fermions ($\nu=1$) in a parabolic confining potential. The dash-dotted line indicates the asymptotic ($N\to\infty$) value.

Energy before minimization is $\frac{E}{\hbar\omega_0} = \frac{N}{2}\left(\mathcal{N}\alpha + \frac{1}{\alpha}\right)$, where $\mathcal{N} = 1 + \nu^2(N-1)\left[ln\left(\frac{1}{2\delta}\right) - 3^{1/2}ln\left(\frac{4}{3}\right)(N-2)\right]$. When the nearest distance between anyons $\delta \to 0$ then $\mathcal{N} \to \infty$. Origin of this divergence is three particle interaction term $\int \psi\left(\vec{R}\right)\frac{\vec{r}_{kj}\cdot\vec{r}_{kl}}{|\vec{r}_{kj}|^2|\vec{r}_{kl}|^2}\psi\left(\vec{R}\right)d\vec{R}$ for $k \neq j, k \neq l, j \neq l$. Minimization $\frac{dE}{d\alpha} = 0$ gives $\alpha_0 = \mathcal{N}^{-1/2}$, thus ground state energy is $E_0 = N\mathcal{N}^{1/2}$. Known from literature at $\nu \to 0$ limit energy is $E_{0l} \approx N + N(N-1)\nu/2$. Thus fitting at $\nu \to 0$ E_0 to E_{0l} (regularization !) one obtains expression $\mathcal{N} = 1 + \nu(N-1)$ and expression for δ.

Hence, ground state energy of non-interacting anyons in 2D harmonic well is $\frac{E}{\hbar\omega_0} = N\left(1 + \nu(N-1)\right)^{1/2}$.

Coulomb-interacting anyons in 2D harmonic well

Hamiltonian of system $\hat{H} = \frac{1}{2M}\sum_{k=1}^{N}\left(\left\{\vec{p}_k - \vec{A}_\nu(\vec{r}_k)\right\}^2 + M^2(\omega_0)^2|\vec{r}_k|^2\right) + \frac{1}{2}\sum_{k,j\neq k}^{N}\frac{e^2}{|\vec{r}_{kj}|}.$

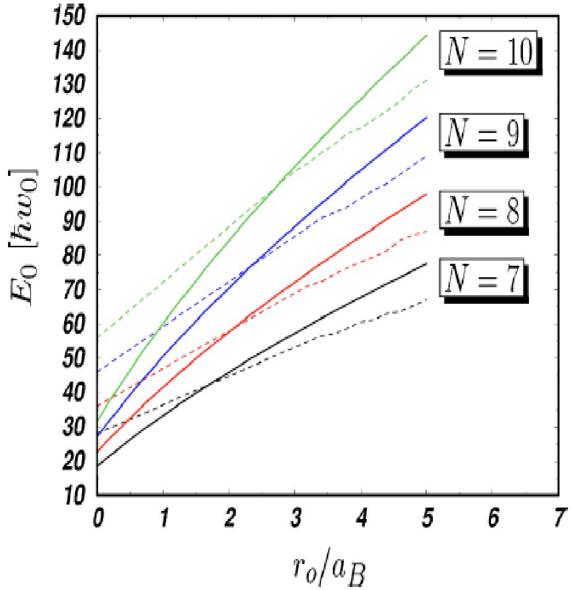


FIG. 3. Coulomb interaction parameter r_0/a_B dependence of the ground state energy for 7 – 10 electrons calculated by variational (Ref. 41) and fixed-node quantum Monte Carlo methods (Ref. 42) (dashed curves) (results of both calculations are indistinguishable in these curves) and by formula (44) (solid curves).

Expression for energy is $\frac{E}{\hbar\omega_0} = \frac{N}{2}\left(\mathcal{N}\alpha + \frac{1}{\alpha} + 2\mathcal{M}\alpha^{1/2}\right)$

with $\mathcal{M} = \left(\frac{\pi}{2}\right)^{1/2}\frac{r_{0(N-1)}}{2a_B}$ and $\mathcal{N} = 1 + \nu(N-1)$.

Minimization $\frac{dE}{d\alpha} = 0$ gives equation $X^4 - \mathcal{M}X - \mathcal{N} = 0$ for $X = 1/\alpha^{1/2}$ with solution:

$$X_0 = (A+B)^{1/2} + \left[-(A+B) + 2(A^2 - AB + B^2)^{1/2}\right]^{1/2},$$

where

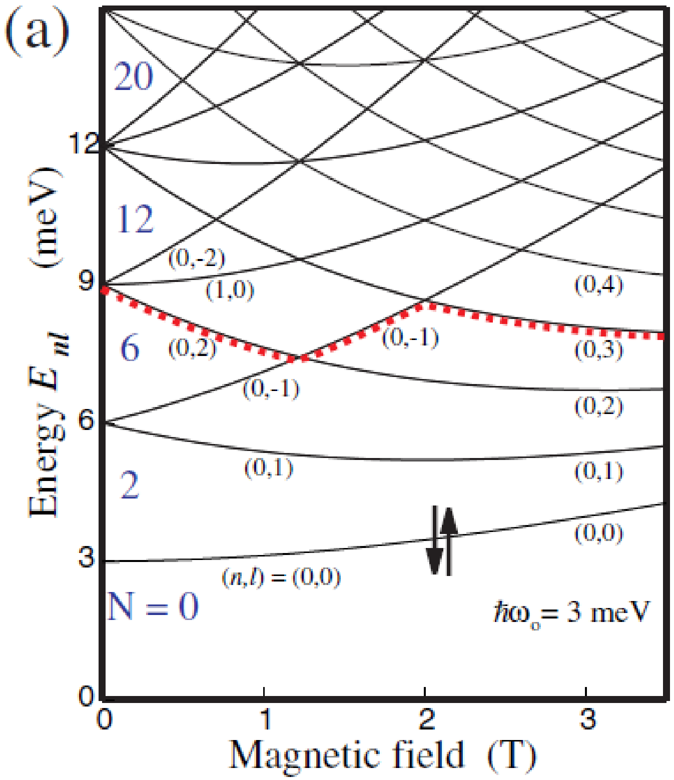
$$A = \left\{\frac{\mathcal{M}^2}{128} + \left[\left(\frac{\mathcal{N}}{12}\right)^3 + \left(\frac{\mathcal{M}^2}{128}\right)^2\right]^{1/2}\right\}^{1/3}$$

$$B = \left\{\frac{\mathcal{M}^2}{128} - \left[\left(\frac{\mathcal{N}}{12}\right)^3 + \left(\frac{\mathcal{M}^2}{128}\right)^2\right]^{1/2}\right\}^{1/3}$$

and $\frac{E_0}{\hbar\omega_0} = \frac{N}{2}\left[\frac{\mathcal{N}}{(X_0)^2} + (X_0)^2 + \frac{2\mathcal{M}}{X_0}\right].$

Non-interacting anyons in 2D harmonic well and magnetic field

Hamiltonian $$\hat{H} = \frac{1}{2M}\sum_{k=1}^{N}\left(\left\{\vec{p}_k - \left(\vec{A}_\nu(\vec{r}_k) + e\vec{A}_{ext}(\vec{r}_k)/c\right)\right\}^2 + M^2(\omega_0)^2|\vec{r}_k|^2\right).$$



Fock-Darwin spectrum $E_{nl}(H)$.

Single electron Fock−Darwin spectrum

$$E_{nl} = P(2n + |l| + 1) + l\text{R},$$

where n and l are radial and angular quantum numbers,

$P = \hbar\left((\omega_0)^2 + (\omega_c/2)^2\right)^{1/2}$, $R = \hbar\omega_c/2$, $\omega_c = |e|H/mc$ with magnetic field $H = |\vec{H}|$.

Filling these states by N electrons one obtains ground state energy for lowest Landau levels at $\omega_c \geq \omega_0(N-2)/(N-1)^{1/2}$:

$$E = \frac{P}{2}N(N+1) - \frac{R}{2}N(N-1).$$

Calculation for ground state energy for anyons gives:

$$E_0 = PN\mathcal{N}^{1/2} - \frac{\nu R}{2}N(N-1).$$

For $\omega_c \to \infty$ it should be $E_0 \to NR$ for fermions $\nu = 1$ and bosons $\nu = 0$. For arbitrary large ω_c $E_0 \to NP$ for bosons. Thus

$$\mathcal{N}^{1/2} = 1 + \frac{\nu(N-1)}{2}.$$

Coulomb-interacting anyons in 2D harmonic well and magnetic field

Hamiltonian

$$\widehat{H} = \frac{1}{2M}\sum_{k=1}^{N}\left(\left\{\vec{p}_k - \left(\vec{A}_\nu(\vec{r}_k) + e\vec{A}_{ext}(\vec{r}_k)/c\right)\right\}^2 + M^2(\omega_0)^2|\vec{r}_k|^2\right) + \frac{1}{2}\sum_{k,j\neq k}^{N}\frac{e^2}{|\vec{r}_{kj}|}.$$

Calculated energy is $\frac{E_0}{\hbar\omega_0} = \frac{N}{2}\left[\frac{\mathcal{N}}{(X_0)^2} + \left(1 + \left(\frac{\omega_c}{2\omega_0}\right)^2\right)(X_0)^2 - \frac{\nu\omega_c}{2\omega_0}(N-1) + \frac{2\mathcal{M}}{X_0}\right].$

Expression for X_0 is the same but replacing $\mathcal{N} \to \mathcal{N}\left[1 + \left(\frac{\omega_c}{2\omega_0}\right)^2\right]^{-1}$ and $\mathcal{M} \to \mathcal{M}\left[1 + \left(\frac{\omega_c}{2\omega_0}\right)^2\right]^{-1}$.

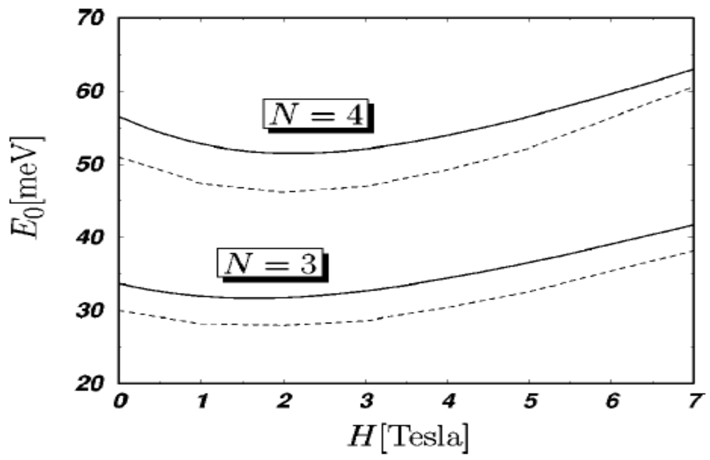


FIG. 4. Magnetic field H dependence of the ground state energy for $N=3$ and $N=4$ spin-polarized electrons in a harmonic potential calculated in Ref. 45 (the dashed curves), and using Eq. (50) (the solid curves). As in Ref. 45 we used $\hbar\omega_0 = 3.37$ meV ($r_0/a_B = \sqrt{H^*/(\hbar\omega_0)}$, where the effective Hartree H^* is equal to $H^* \simeq 11.86$ meV).

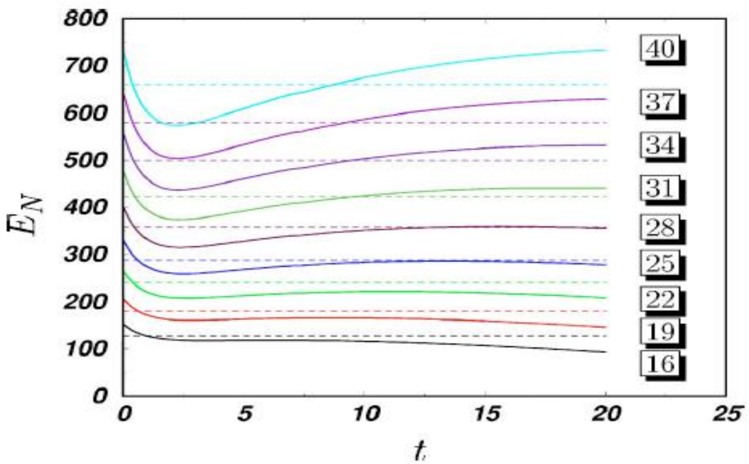


FIG. 5. Ground state energy $E_N = (E_0 - N\hbar\omega)/(\hbar\omega_0)$ for 16–40 electrons calculated using the expression Eq. (50) for $r_0/a_B = 1.911$, applying the expression for $\mathcal{N}$ Eq. (53) with $|\nu| = 1$ (solid curves), and energy for classical electrons (Ref. 48) (dashed lines). Here $\omega = (\omega_0^2 + \omega_c^2/4)^{1/2}$ and $t = \omega_c/\omega_0$.

Infinite Coulomb anyons gas

Hamiltonian $$\hat{H} = \frac{1}{2M}\sum_{k=1}^{N}\left[\left\{\vec{p}_k + \vec{A}_\nu\right\}^2 + M^2 (\omega_0)^2 |\vec{r}_k|^2 + \frac{1}{2}\left(\sum_{k,j\neq k}^{N}\frac{e^2}{|\vec{r}_{kj}|} + V(\vec{r}_k)\right)\right]$$

with $V(\vec{r}_k) = -\rho\int\frac{e^2 d^2 r}{|\vec{r}_k - \vec{r}|}$.

Let us consider no interacting case and $\nu = 1$ then at $N\to\infty$ $E_0 = \hbar\omega_0 N^{3/2}$ since $E_0 = \hbar\omega_0 N\mathcal{N}^{1/2}$and $\mathcal{N} = 1 + \nu(N-1)$. Ground state energy of 2D electron gas with no interaction is $E_{0eg} = N\hbar^2/m(r_0)^2$, where r_0 is mean distance between electrons. From $E_0 = E_{0eg}$ one gets $\hbar\omega_0 = \hbar^2/\left(m(r_0)^2 N^{1/2}\right)$ (harmonic potential regularization with vanishing confinement at $N\to\infty$!).

From $\frac{E_0}{\hbar\omega_0} = \frac{N}{2}\left[\frac{\mathcal{N}}{(X_0)^2} + (X_0)^2 + \frac{2\mathcal{M}}{X_0}\right]$ one obtains energy per particle (in Rydberg $Ry = e^2/2a_B$ units, where a_B is Bohr radius):

$$\frac{E_0}{N} = \frac{2f(\nu, r_s)}{(r_s)^2}\left[\frac{\nu}{2(K_X)^2} + \frac{(K_X)^2}{2} - \frac{K}{K_X}\right]$$

$$K_X = (K_A + K_B)^{1/2} + \left[-(K_A + K_B) + 2\left((K_A)^2 - K_A K_B + (K_B)^2\right)^{1/2}\right]^{1/2}$$

Ground state energy of infinite Coulomb anyon gas

$$K_A = \left\{\frac{K^2}{128} + \left[\left(\frac{\nu}{12}\right)^3 + \left(\frac{K^2}{128}\right)^2\right]^{1/2}\right\}^{1/3}$$

$$K_B = \left\{\frac{K^2}{128} - \left[\left(\frac{\nu}{12}\right)^3 + \left(\frac{K^2}{128}\right)^2\right]^{1/2}\right\}^{1/3},$$

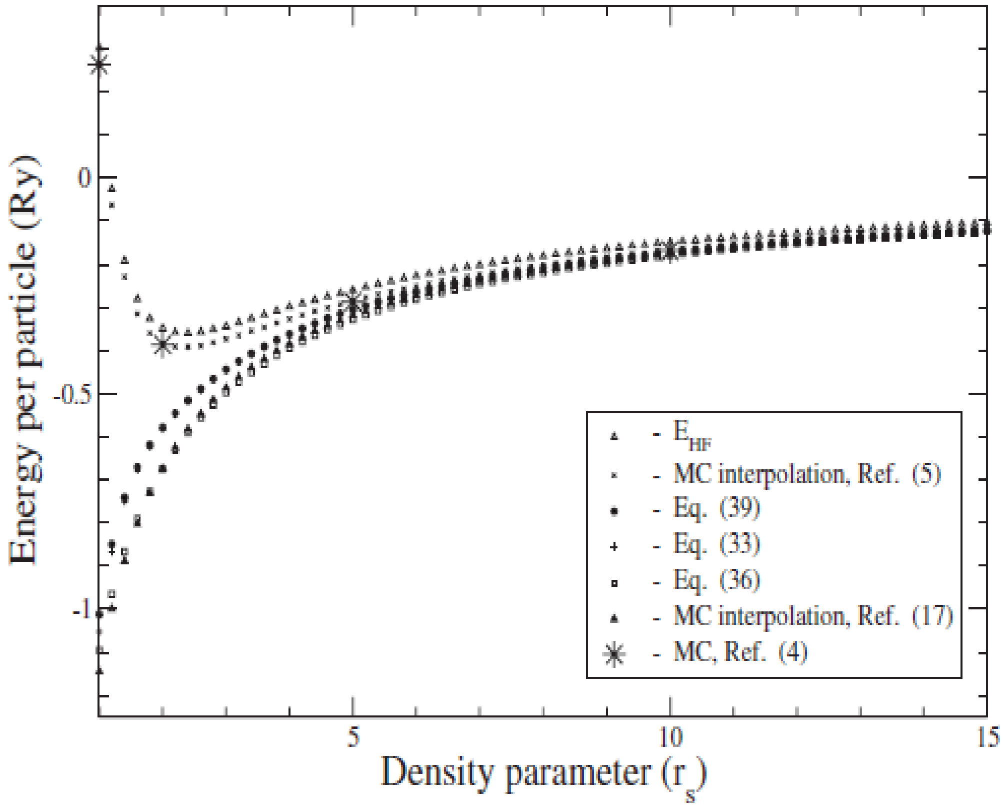
where one used

$\mathcal{N} = \nu N,$

$\mathcal{M} = N^{3/4}K$ and

$K = c_{WC} r_s / f^{1/2}(\nu, r_s)$

with $(c_{WC})^{2/3} = 2.2122$ taken from classical Wigner crystal energy.



Ground state energy as function of Coulomb density parameter. From B. Abdullaev, U. Roessler, M. Musakhanov, Phys. Rev. B 76, 075403 (2007).

Explicit derivation of ground state energy formulas by taking into account short range correlations in wave function

Replacing trial wave function $\psi(\vec{R}) \rightarrow \prod_{i \neq j} |\vec{r}_{ij}|^{\nu} \psi(\vec{R})$, one derives explicitly (with no logarithmic divergence regularization procedure !):

- $\frac{E}{\hbar\omega_0} = N\mathcal{N}^{1/2}$ with $\mathcal{N} = 1 + \nu(N-1)$ (Abdullaev, C.-H. Park, and M. M. Musakhanov, Physica C **471**, 486 (2011)) ;
- $\frac{E_0}{\hbar\omega_0} = \frac{N}{2}\left[\frac{\mathcal{N}}{(X_0)^2} + (X_0)^2 + \frac{2\mathcal{M}}{X_0}\right]$ with $\mathcal{N} = 1 + \nu(N-1)$ (unpublished);
- $E_0 = PN\mathcal{N}^{1/2} - \frac{\nu R}{2} N(N-1)$ with $\mathcal{N} = 1 + \nu(N-1)$ (unpublished);
- $\frac{E_0}{\hbar\omega_0} = \frac{N}{2}\left[\frac{\mathcal{N}}{(X_0)^2} + \left(1 + \left(\frac{\omega_c}{2\omega_0}\right)^2\right)(X_0)^2 - \frac{\nu\omega_c}{2\omega_0}(N-1) + \frac{2\mathcal{M}}{X_0}\right]$ with $\mathcal{N} = 1 + \nu(N-1)$ (unpublished).

Do anyons and fermions exist in the ground state of 2D in concept of anyons?

Introducing the Zeeman term $\frac{\hbar}{m}\sum_{k=1}^{N}\hat{\vec{s}}\cdot\vec{b}_k$ with anyon (statistical) magnetic field:

$$\vec{b}_k = -2\pi\hbar\nu\vec{e}_z \sum_{j(k\neq j)}^{N}\delta^{(2)}(\vec{r}_k - \vec{r}_j) \quad \text{and} \quad s_z = \hbar/2$$

one obtains for Schrödinger equation

$$\frac{1}{2m}\sum_{k=1}^{N}[(\vec{p}_k + \vec{A}_k)^2 + \frac{\hbar}{m}\hat{\vec{s}}\cdot\vec{b}_k]\Phi(r_1, r_2, \ldots) = E\Phi(r_1, r_2, \ldots)$$

with $\Phi(\vec{r}_1, \vec{r}_2, \ldots) \Rightarrow \prod_{i\neq j}|\vec{r}_{ij}|^{\nu}\Phi(\vec{r}_1, \vec{r}_2, \ldots)$

term connected with statistics $\pi\nu\frac{\hbar^2}{m}\sum_{j(k\neq j)}^{N}\delta^{(2)}(\vec{r}_k - \vec{r}_j)$

and the Zeeman term $-\pi\nu\frac{\hbar^2}{m}\sum_{j(k\neq j)}^{N}\delta^{(2)}(\vec{r}_k - \vec{r}_j)$.

Conclusion

1. Approximate expression for ground state energy of Coulomb interacting anyons in 2D harmonic potential in the presence of external magnetic field has derived;
2. Approximate expression for ground state energy of Coulomb interacting infinite anyon gas has derived;
3. Exact cancellation of statistics and Zeeman terms in the anyon Hamiltonian has found.

Thanks for attention.