Some Open Problems Related With Non-interacting Anyon Gas

I list few open (unsolved) problems related with the Statistical mechanics of an ideal anyon gas. Note that while such a problem was solved for ideal Fermi and ideal Bose gases right after the introduction of these statistics, even after the path breaking paper of Leinaas and Myrheim of 1977, it is still an unsolved problem in 2013!

1. Not even one energy level is known in the problem of $N(\geq 3)$ anyons experiencing external oscillator potential with energy being nonlinear function of the anyon parameter α (with $\alpha = 0(1)$ corresponding to the bosons (fermions)). Numerically it is well known by now that such energy levels exist for $N \geq 3$. I might add here that it is these energy levels (with nonlinear dependence on α) which give nontrivial contribution to the third virial coefficient (and perhaps even for the higher virial coefficients) of an ideal anyon gas. I believe that even knowing one such energy level analytically will help immensely in our understanding.

2. Why is N anyon interaction around the N fermion ground state repulsive when $N = 3, 6, 10, 15$ (i.e. when fermion shells are closed) but is attractive for other values of N when the shells are open, i.e. when $N = 2, 4, 5, 7, ...$? Understanding this fact is crucial in having reliable approximation schemes like mean field theory. Note that, on the other hand, the N anyon interaction around the N boson ground state is always repulsive, no matter what N is. A related fact is that the ground state energy of 3 anyons is *not* maximum at the fermionic end ($\alpha = 1$) but at $\alpha = 0.71$?. From here I speculate that while the N anyon ground state energy is maximum at $\alpha = 1$ in case fermionic shells are not closed (i.e. $N = 2, 4, 5, ...$), it is not maximum at $\alpha = 1$ but at some other value of α $(0 < \alpha < 1)$ in case the fermionic shells are closed. It would be nice to prove or disprove this conjecture.

3. Does an ideal anyon gas exhibit Bose-Einstein condensation? Note that there is no BE condensation for an ideal Bose gas in 2-dimensions, a celebrated theorem due to Mermin-Wagner-Berezinskii. In fact these people also proved that there is no phase transition for an interacting Bose gas so long as the interactions are two-body, short ranged interactions. However, we know that these assumptions are no more valid in the case of an ideal anyon gas since an ideal anyon gas is equivalent to Bose

gas with long ranged, vector, two body and three-body interactions. My hunch is that even an ideal anyon gas does not exhibit BE condensation but only a rigorous proof will settle the issue.

4. Does virial expansion exists for an ideal anyon gas? A necessary condition for the existence of the virial expansion in the thermodynamic limit is: if particles interact by a two body potential in d-dimensions and if the potential decreases as r^{-n} as $r --- > \infty$, then $n > d$. But as we know, a non-interacting anyon gas is equivalent to Bose gas with long ranged, vector, two body and three body interactions. So it is not obvious that virial expansion will exist for an ideal anyon gas. In this context, I might add that the virial expansion *does not* hold good in the case of long ranged scalar interaction. In particular, if one adds $H_{int}^{rel} = g^2/mr^2$ to the anyon Hamiltonian then the second virial coefficient $a_2(\alpha, g)$ turns out to be divergent (see Page 127 of the second edition of my Book). The fact that the second virial coefficient $a_2(\alpha)$ of an ideal anyon gas, on the other hand, is finite is therefore a nontrivial statement. This suggests that perhaps the virial expansion exists for an ideal anyon gas. But it would be nice if either one rigorously proves the existence or at least proves that the third virial coefficient $a_3(\alpha)$ of an ideal anyon gas is finite. Note that only for the third and the higher virial coefficients, the nontrivial brading effects manifest themselves.

5. To date we do not have any result about an ideal anyon gas at high density ρ and low temperature T whereas at low density ρ and high temperature T, we at least know the second virial coefficient, $a_2(\alpha)$, for an ideal anyon gas. It would be nice if at least some exact result can be obtained for an ideal anyon gas at high density and low temperature. A somewhat related question is, how many rigorous theorems derived in Statistical Mechanics are still valid for an ideal anyon gas. Note that the theorems in statistical mechanics have been derived assuming short range, two-body, interactions while in the anyon case one has long ranged, vector, two body and three body interactions.

6. We now know that to $O(\alpha^2)$, the 4th, 5th and 6th virial coefficients $a_4(\alpha), a_5(\alpha), a_6(\alpha)$ all have lots of logarithms in them. Is this an indication that an ideal anyon gas exhibits KT transition?

7. Is the semi-classical approximation exact for the virial coefficients of an ideal anyon gas? The semi-classical approximation is, in fact, known to be exact for the second virial coefficient $a_2(\alpha)$? This

is understandable since anyons do not introduce any scale in the problem so the only scale in the problem is the thermal wavelength $\lambda = (\frac{2\pi\hbar^2}{mkT})^{1/2}$. Thus dimensionally, the n'th virial coefficient must have the form $a_n = C(\lambda)^{2(n-1)}$ where C is a dimensionless constant. And since by construction the semi-classical approximation must be exact as $T - -- > \infty$, hence it follows that the semi-classical approximation must be exact for the virial coefficients at all T. It would be nice if one can prove or disprove this argument. One possible way is to evaluate $a_3(\alpha)$ using the semi-classical approximation. However, as we have seen (see page 136 of the second edition of my book), we do not know how to sum over l_ρ and l_η so as to exclude contributions from the antisymmetric and the mixed symmetry states. If one can find a prescription to do that, one can then compute $a_3(\alpha)$ within the semi-classical approximation and see if it satisfies the well known exact results on $a_3(\alpha)$.

8. Now a general open problem. As far as I know, the full classification of the higher dimensional representations of the braid group is still an unsolved problem. Note that these higher dimensional representations are of importance in the context of non-abelian anyons which are now playing important role in the context of fractional quantum Hall effect, ultra-cold atomic gases and topological quantum computation. Besides these are also useful in the context of links, knots and Jones polynomials.

9. For the mathematically inclined readers, in addition I will point out two open problems. Firstly, what is so special about the semion point, i.e. $\alpha = 1/2$? For example, in the case of two anyons in an oscillator potentials, one finds that a la bosons and fermions, even in the semion case the spectrum is equispaced, except that while in the fermion and the boson case, the spacing is $2\hbar\omega$, it is only $\hbar\omega$ in the semion case. What is the extra symmetry in the semion case (unlike for other values of $0 < \alpha < 1$) because of which the spectrum is equispaced? Similarly, in the case of three anyons in an oscillator potential, most of the semion energy levels are two-fold degenerate. As a result, one also has the remarkable result $a_3(\alpha) = a_3(1-\alpha)$. What is the origin of this extra symmetry in the semion case? Is there such an extra symmetry even in the case of $N > 3$ anyon spectrum in an oscillator potential?

10. On Page 76 of my book (second edition), one has an identity for triple summation involving Jacobi polynomials if one uses Eqs. (3.138) and Eqs. (3.140). It would be nice if some one can prove this identity.