Anyons in one dimension

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Berlin School: Anyons with Ultracold Atoms, 24-28 Sep 2013

Physics in 1D is easy....

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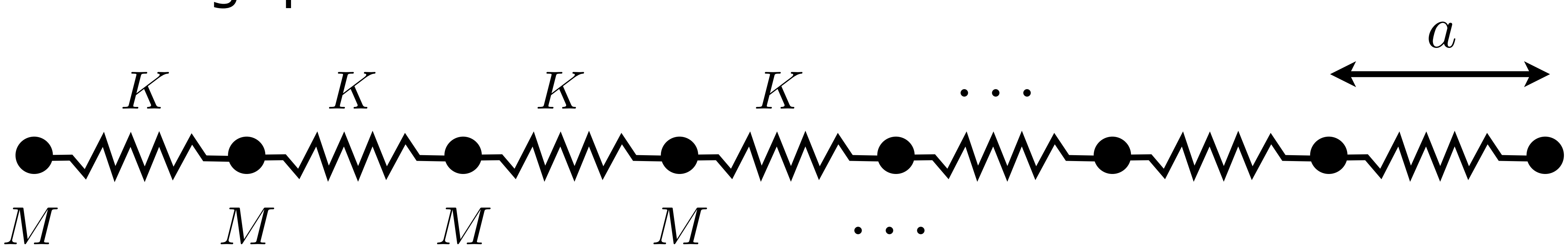
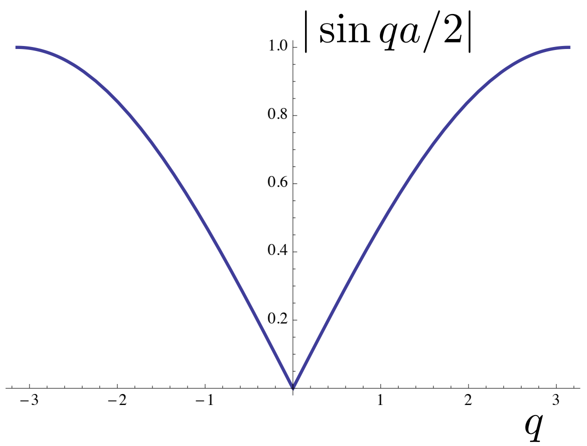
...or maybe not!

Strong quantum fluctuations in 1D!

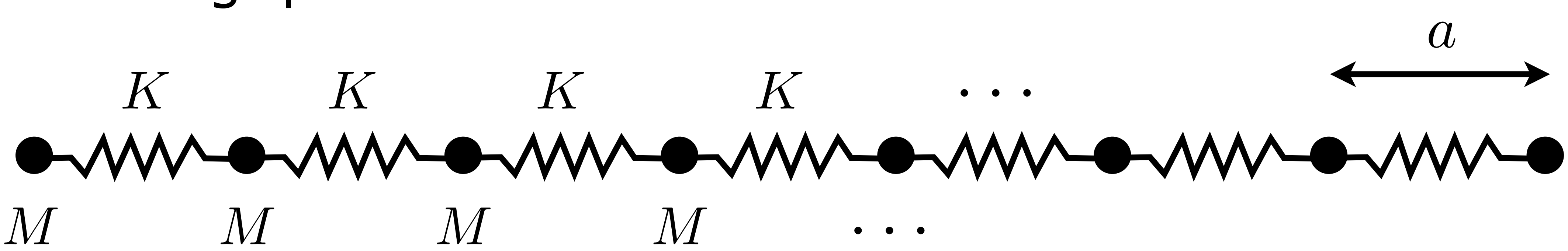




Phonon spectrum $\quad \omega(q) = 2\sqrt{\frac{K}{M}}|\sin qa/2| \approx cq$

Average displacement $\quad \langle u \rangle = 0$

Strong quantum fluctuations in 1D!



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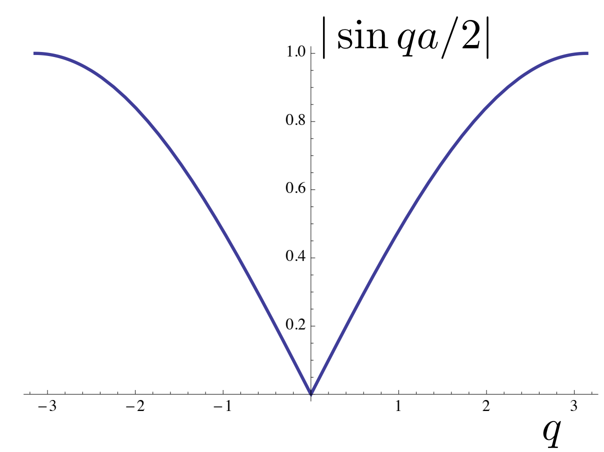


Average displacement $\langle u \rangle = 0$

For having a stable lattice $\langle u^2 \rangle \lesssim a^2$ (Lindemann)

Fluctuations at T=0 $\langle u^2 \rangle = \int \frac{d^d q}{(2\pi)^d} \frac{\hbar}{2M\omega(q)}$

IR divergence

Quantum melting!

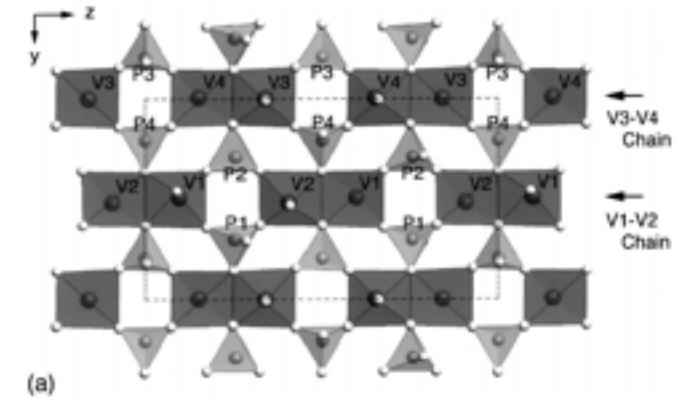
1D solids (crystals) necessarily lie in a higher dimensional support!

One dimensional systems

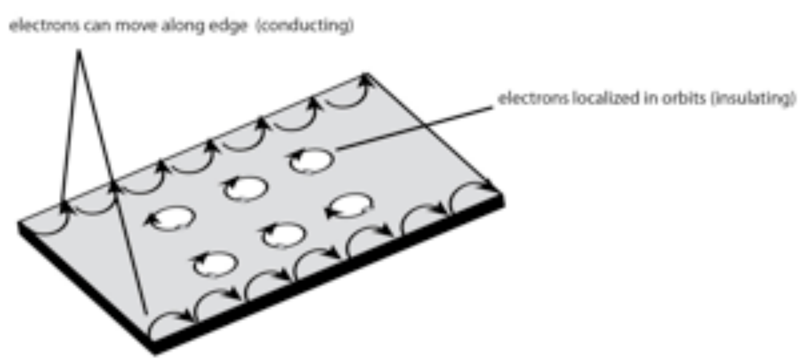
Transverse motion is frozen

One dimensional systems

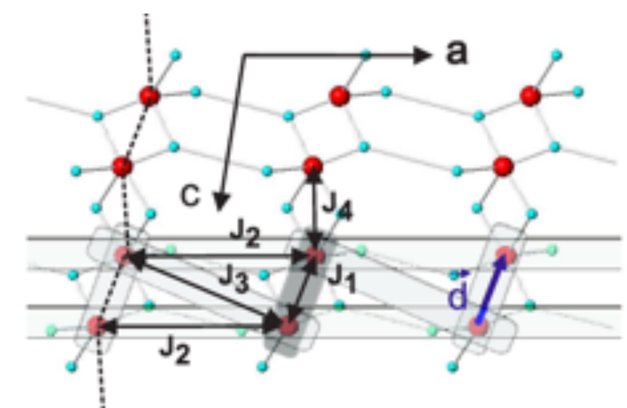
Transverse motion is frozen

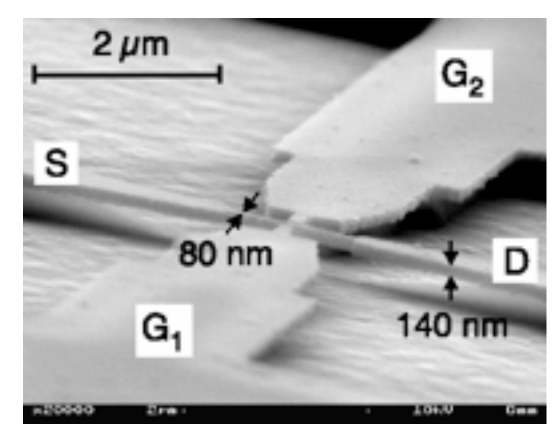


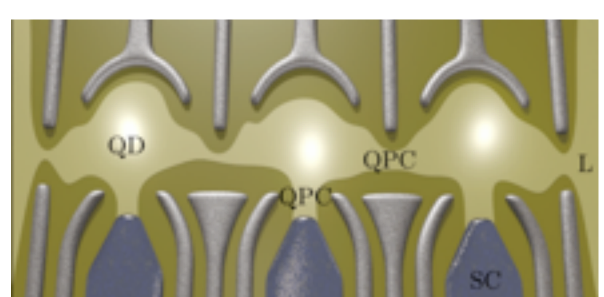
Condensed matter



- Spin chains and ladders
- Quantum wires
- FQHE edge states
- Organic conductors
- Nanotubes
- Quantum dot arrays







Quantum (analog) simulators

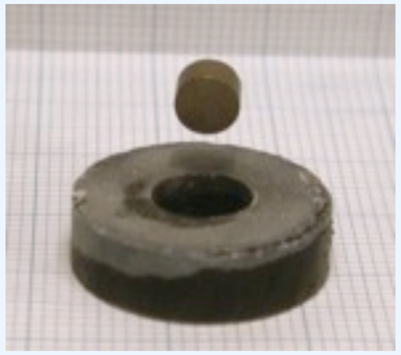
Quantum mechanical problems involving a large number of particles are typically hard to solve

Build a controllable “experimental” apparatus which emulates the model you want to solve, and extract information from it

Feynman 1982; Lloyd1996; Buluta and Nori 2006.

The route to knowedge

The route to knowedge

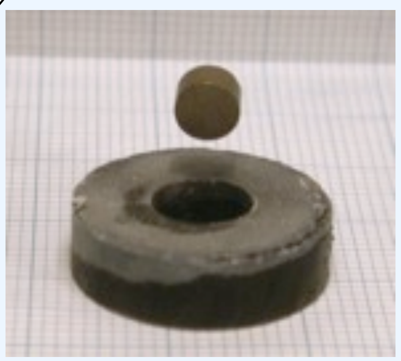


Challenging physical phenomenon (HTc superconductivity, strong coupling QCD,...)

The route to knowedge

Formulate a model for it
(usually, a
Hamiltonian)

$$\mathcal{H}, \dot{\rho}$$



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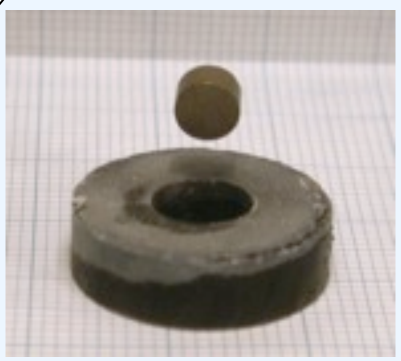
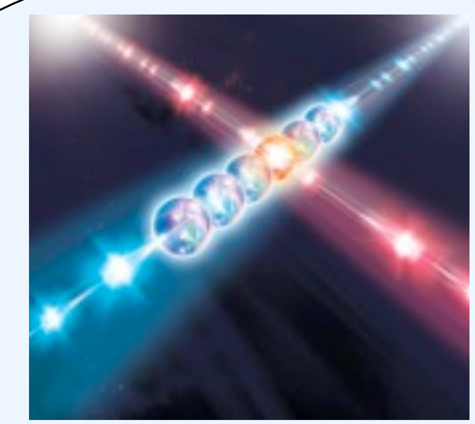
Build an experimental system which is exactly described by this model





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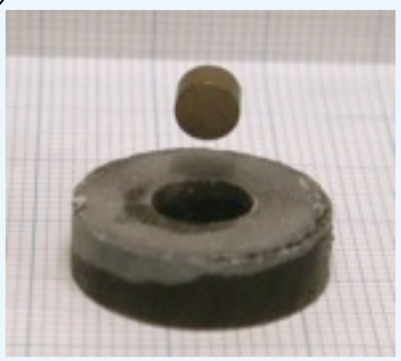

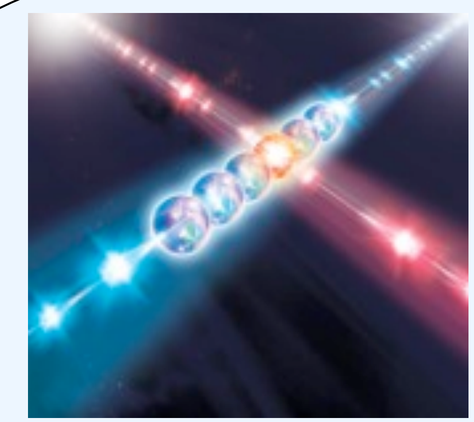

Challenging physical phenomenon (HTc superconductivity, strong coupling QCD,...)

Extract **physical information** from the **quantum simulator**

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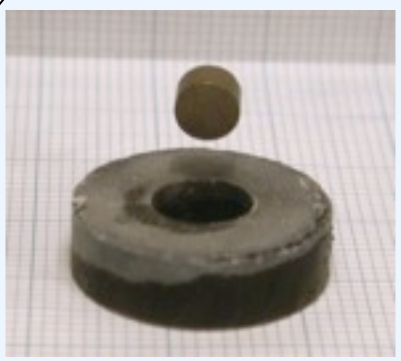
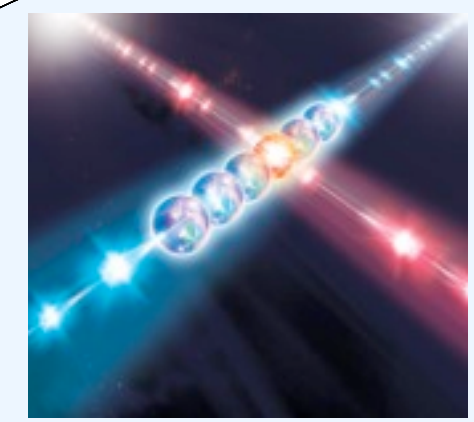


Compare the "simulation" results with the physical phenomenon!



Challenging physical phenomenon (HTc superconductivity, strong coupling QCD,...)

Extract **physical information** from the **quantum simulator**

Quantum simulators

Tunable interactions

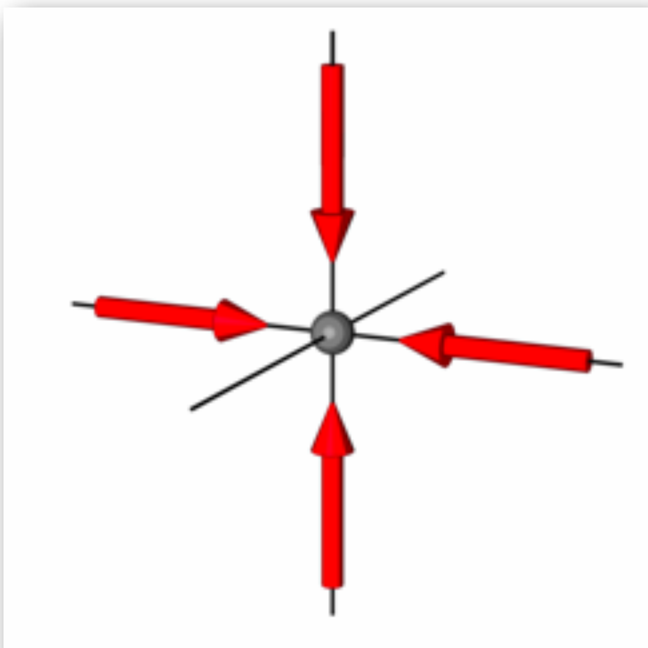
High isolation

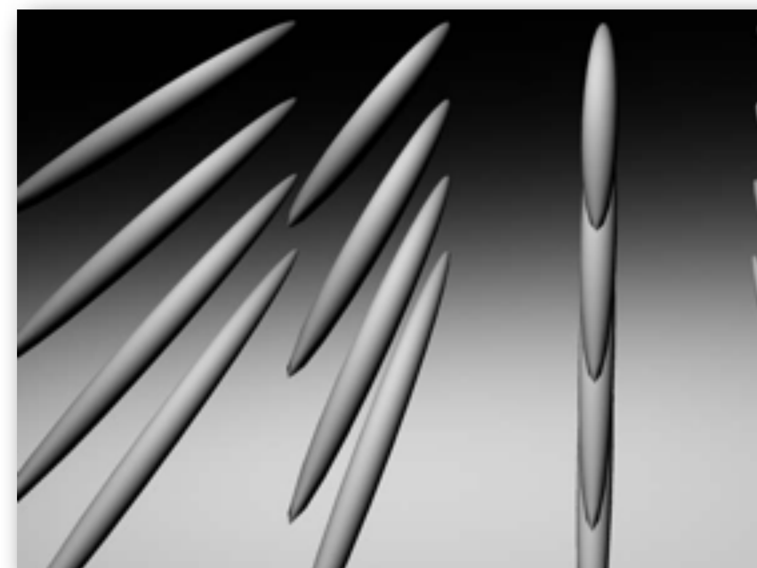
Different geometries

Artificial gauge fields

Controlled dynamics

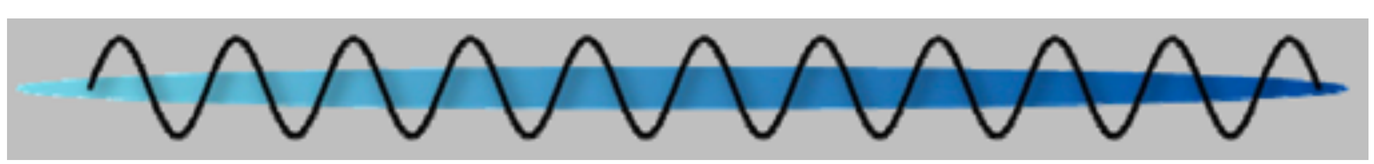
1D





2D array of 1D cigar-shaped potentials

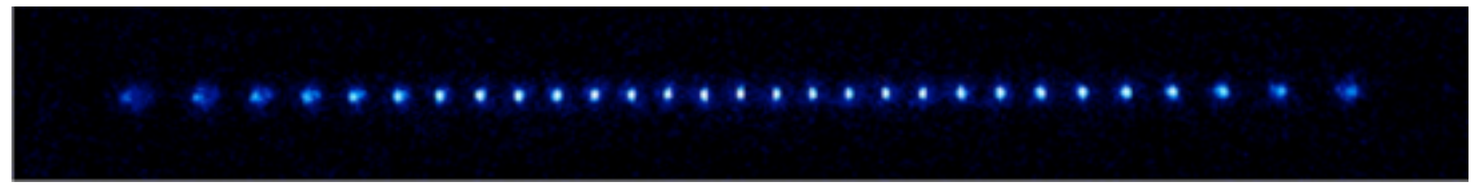
Lattices



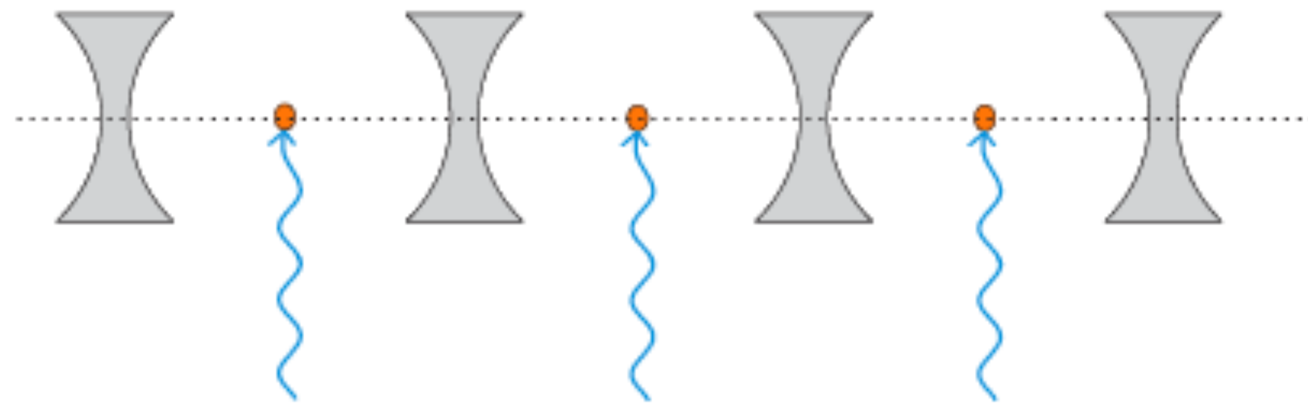
Confining potential → Non homogeneous phase

Not only ultracold atoms!

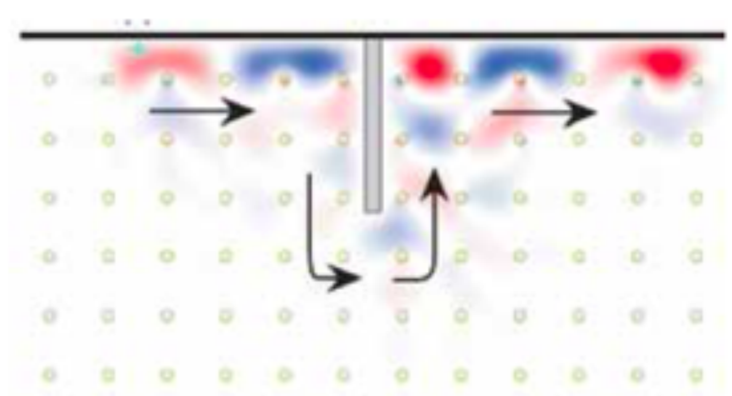
Ion traps



Coupled quantum cavities



Photonic gases in nonlinear media



Theorists' toolbox

Analytical methods

Exact solutions (Bethe-Ansatz, Integrability)

Effective field theories (Luttinger liquids, bosonization, nonlinear sigma models, ...)

Conformal field theories, scaling, renormalization

Variational methods

Mean Field

Approximate maps onto solvable models

Numerical methods

Exact diagonalization

DMRG, t-DMRG, MPS

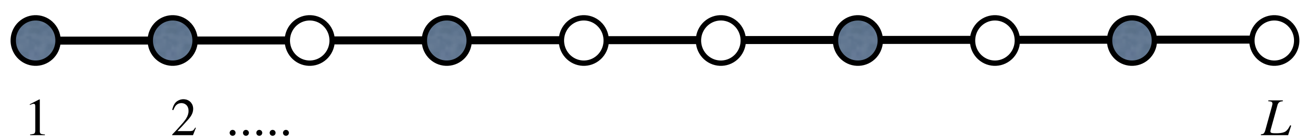
Quantum Monte Carlo

Anyway, there is no **universal** method!

Numerical methods

Quantum mechanics is linear → Eigenvalue problem

Generic state $$|\psi\rangle = \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} c_{n_1 \cdots n_M} |n_1, \ldots, n_M\rangle$$



Coupled 2-level systems (S=1/2)

$$dim\{\mathcal{H}\} \approx 2^L$$

Discrete symmetries can reduce dimension (not dramatically)

Typical Hamiltonians are sparse matrices

Numerical methods

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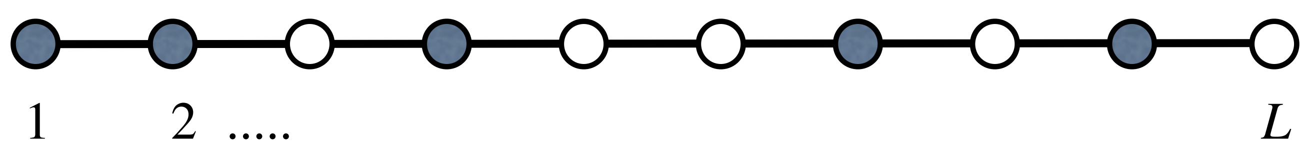

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$$dim\{\mathcal{H}\} \approx 2^L$$

Discrete symmetries can reduce dimension (not dramatically)

Typical Hamiltonians are sparse matrices

Lanczos algorithm: iterative procedure to reduce H in tridiagonal form and diagonalize it easily

Good: machine precision

Bad: long CPU time and few sites

Density matrix renormalization group (DMRG)

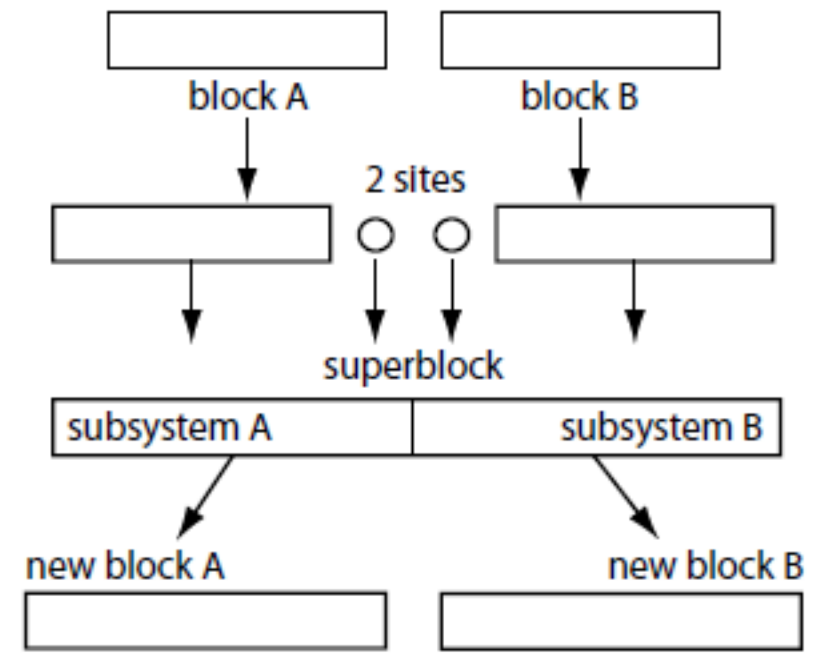
Allows to simulate bigger chains

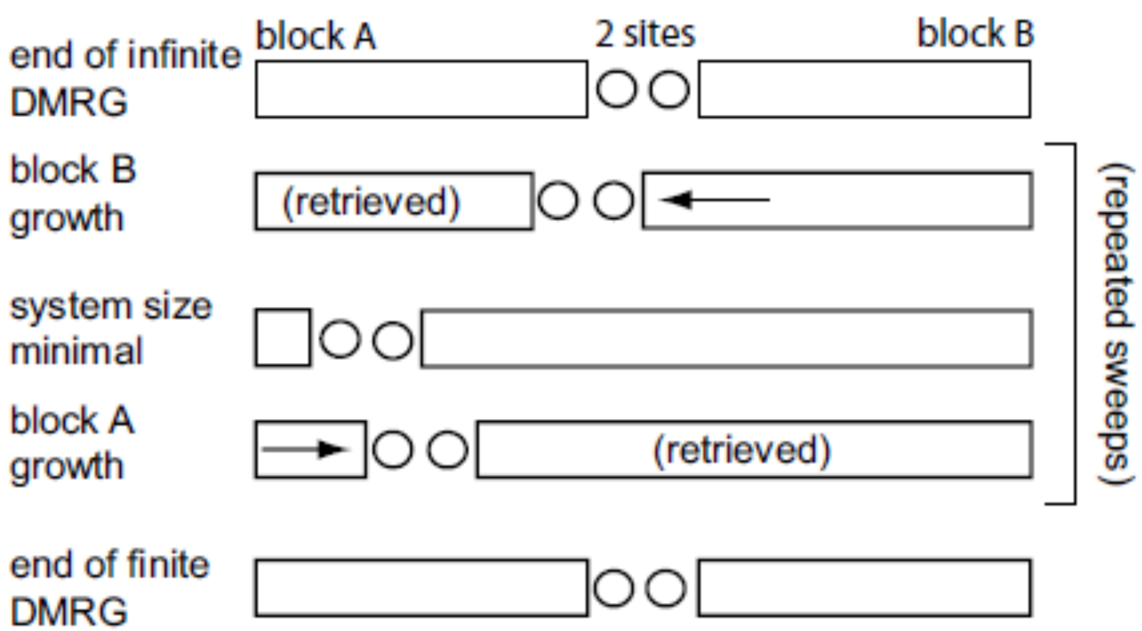
The Hilbert space of the block is truncated to m* states

The superblock is diagonalized with Lanczos

The truncation is performed retaining the m* highest weights of the density matrix

Works best when entanglement between blocks is bounded (gap)





Matrix product states (MPS)

$$|\Psi\rangle = \sum_{i_1,\dots i_M=1}^{d} \mathrm{Tr}[A^{[1],i_1} \dots A^{[M],i_M}] |i_1, \dots i_M\rangle$$

S.R. White, *Phys. Rev. Lett.* **69**, 2863 (1992).
U. Schollwoeck, *Rev. Mod. Phys.* **77**, 259 (2005).

Finite size scaling

Quantum many-body problems are "hard"

This is why we propose quantum simulators

Often the winning strategy is to use a combination of numerics and theory

Close to quantum phase transitions:

Scaling variable: $z = L^{1/\nu} t \sim \left(\dfrac{L}{\xi}\right)^{1/\nu}$

Regimes

$$\begin{cases} L >> \xi & \text{Thermodynamic limit} \\ L << \xi & \text{Critical} \end{cases}$$

Universality

- Details are not important close to the critical point
- The critical exponents depend only on:
  symmetries, *dimensionality* and *range of interactions*

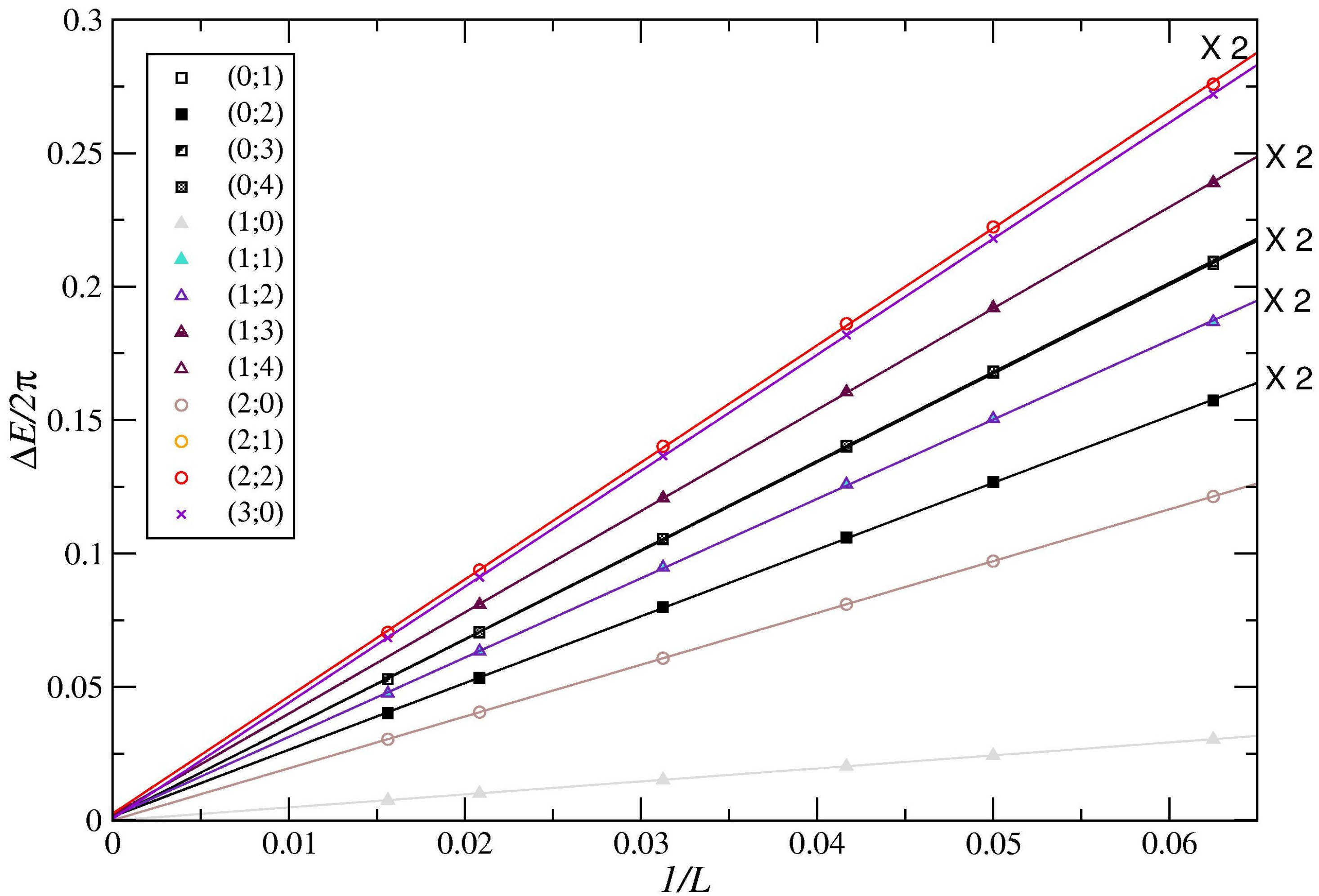
Finite size scaling (ii)

- CFT on a finite size chain of length L (PBC)

$$\frac{E_{GS}}{L} = e_\infty - \frac{\pi c v}{6L^2}$$

- The excited states are related to the dimensions

$$E_{mn} - E_{GS} = \frac{2\pi v}{L}\left(d_{mn} + r + \bar{r}\right) \qquad d_{mn} = \left(\frac{m^2}{4K} + n^2 K\right), \quad m,n \in Z$$



Anyons

$\boxed{\text{3D}}$ $\psi(\mathbf{x}_2, \mathbf{x}_1) = \pm\psi(\mathbf{x}_1, \mathbf{x}_2)$ Bosons or Fermions

[Leinaas and Myrheim, 1977]

[F. Wilczek, *Fractional Statistics and Anyon Superconductivity,*

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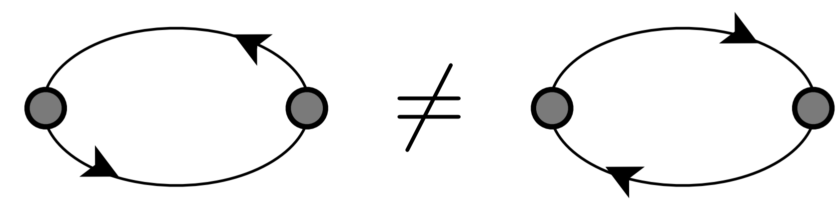
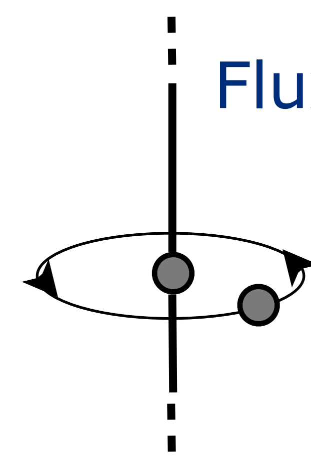
$\boxed{2D}$ $\psi(\mathbf{x}_2, \mathbf{x}_1) = e^{i\theta}\psi(\mathbf{x}_1, \mathbf{x}_2)$?



Transmutation of statistics

Adiabatic paths cannot intersect



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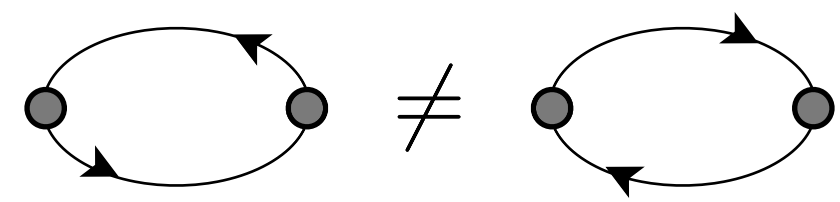

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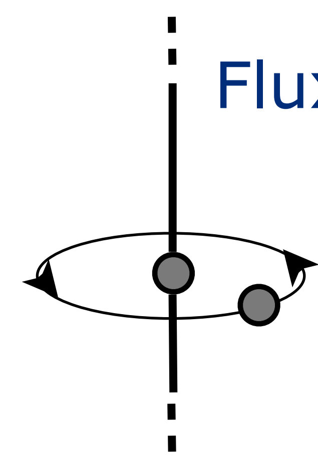

Flux tube (fictitious)

$\boxed{\text{1D}}$ How can particles exchange without touching?

Transmutation?

[Leinaas and Myrheim, 1977]

[F. Wilczek, *Fractional Statistics and Anyon Superconductivity,*

Transmutation from hard core bosons (spins) to fermions

spin-1/2 $\{\sigma_j^-, \sigma_j^+\} = \mathbb{I}$ $\quad [\sigma_j^-, \sigma_l^+] = 0 \quad j \neq l$ hard-core bosons

$[\sigma_j^-, \sigma_l^-] = 0$

$\sigma^\alpha =$ Pauli matrices

$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

fermions $\{c_j, c_l^\dagger\} = \delta_{jl}$

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Transformation $\quad$ ○ = ↓ $\quad$ ● = ↑

$$\sigma_j^+ = K_j c_j^\dagger, \quad \sigma_j^- = K_j^\dagger c_j, \quad \sigma_j^z = 2c_j^\dagger c_j - \mathbb{I}$$

[P. Jordan and E. Wigner, Z. Phys. **47**, 631 (1928)]

$$K_j = \prod_{l=1}^{j-1}(-\sigma_l^z) = \exp\left(i\pi \sum_{l=1}^{j-1} c_l^\dagger c_l\right)$$

parity operator

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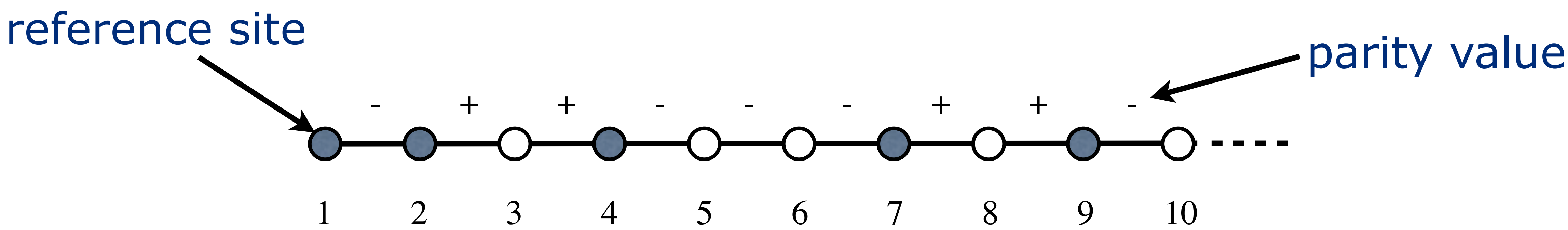
parity operator

Problem: verify that all the fermionic commutation relations are mapped onto the spin ones.

Jordan-Wigner transformation



local parity-conserving operators remain local

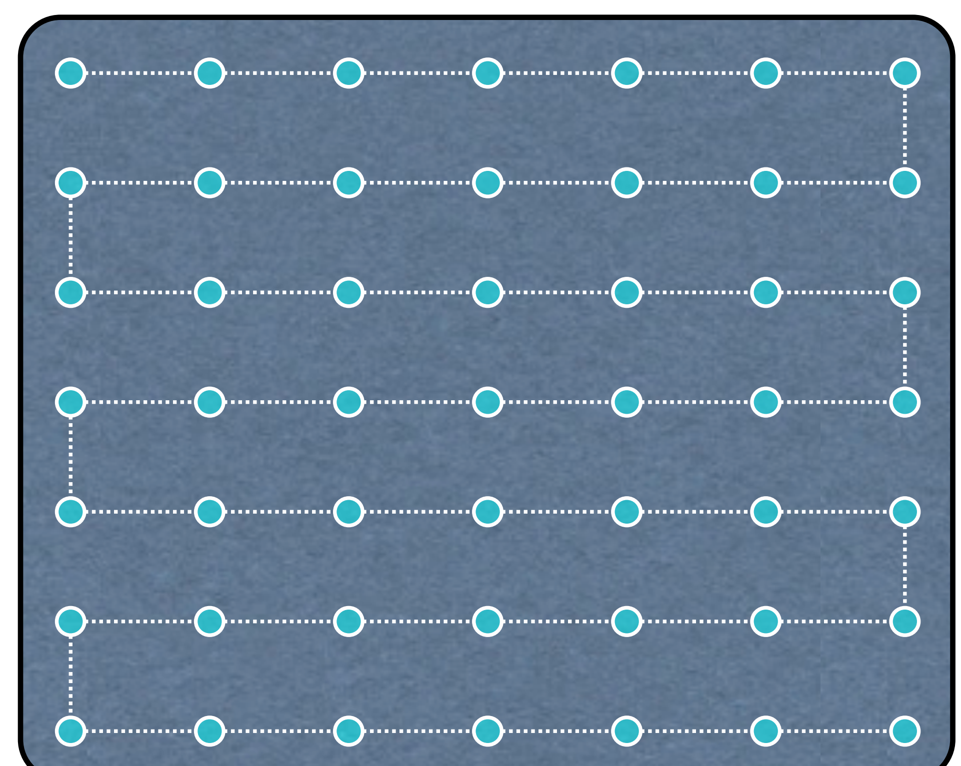
$$\sigma_i^+ \sigma_{i+1}^- = c_i^\dagger c_{i+1} \qquad \sigma_i^+ \sigma_{i+1}^+ = c_i^\dagger c_{i+1}^\dagger$$

But

$$\sigma_i^+ \sigma_{i+r}^- = c_i^\dagger e^{i\pi \sum_{j=i}^{i+r} n_j} c_{i+r}$$

many body operator

JW is useless in 2D



Quantum Ising model

transverse field

$$H = -\sum_{i=1}^{L} \left(J \sigma_i^x \sigma_{i+1}^x + h\sigma_i^z \right)$$

$$\boxed{\sigma_i^x = \sigma_i^+ + \sigma_i^-}$$

$\boxed{J = 1}$

$$H = -\sum_{i=1}^{L} \left[\left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^+ \sigma_{i+1}^+ + \mathrm{h.c.} \right) + h\sigma_i^z \right]$$

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$$\mathcal{N} = \sum_{i=1}^{L} c_i^\dagger c_i$$

total # particles

(JW)

$$H = -\sum_{i=1}^{L-1} \left(c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger + \text{h.c.} \right) - 2h \sum_{i=1}^{L} c_i^\dagger c_i + Lh$$

$$P^z = e^{i\pi\mathcal{N}}$$ parity is conserved

!

PBC $\leftrightarrow$ ABC

for $P^z = 1$

PBC $\leftrightarrow$ PBC

for $P^z = -1$

Quantum Ising model (ii)

$$k = \frac{\pi\,(2n+1)}{L}$$ ABC

$$c_j = \frac{1}{\sqrt{L}} \sum_{k\in BZ} e^{ikj} c_k$$ Fourier space

$$H = 2\sum_{k\in BZ} \epsilon_k c_k^\dagger c_k + \sum_{k\in BZ}\left(W_k c_k^\dagger c_{-k}^\dagger + W_k^* c_{-k} c_k\right) + Lh$$

$$\boxed{\begin{aligned}\epsilon_k &= -\left(\cos k + h\right)\\ W_k &= -i\sin k\end{aligned}}$$

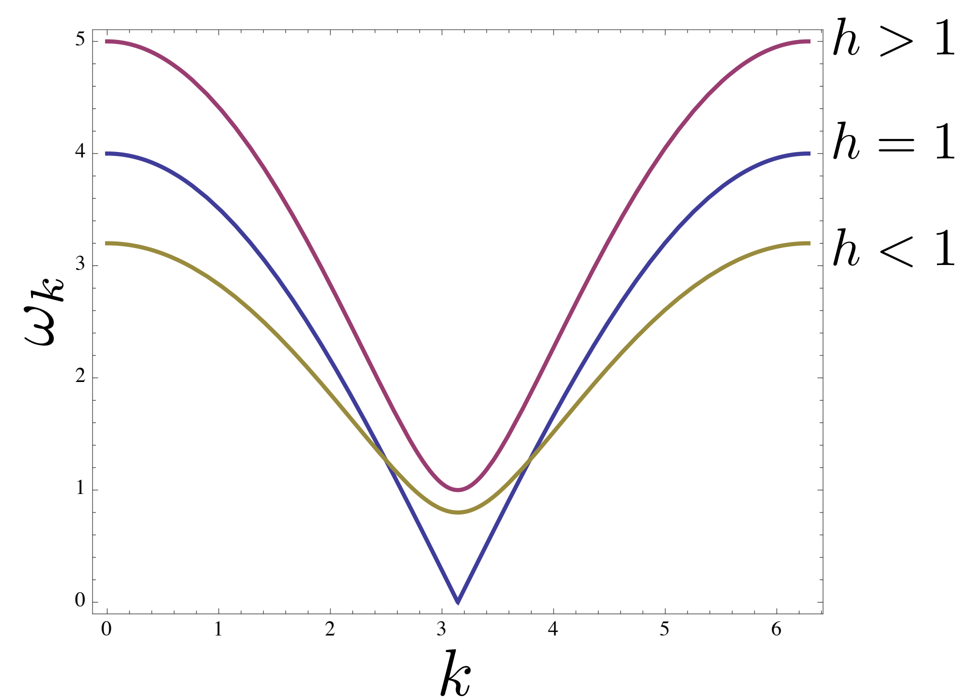
$$H = \sum_k \left(c_k^\dagger, c_{-k}\right)\begin{pmatrix}\epsilon_k & W_k\\ W_k^* & -\epsilon_k\end{pmatrix}\begin{pmatrix}c_k\\ c_{-k}^\dagger\end{pmatrix}$$

unitary transformation $$\begin{pmatrix}\eta_k\\ \eta_{-k}^\dagger\end{pmatrix} = R\begin{pmatrix}c_k\\ c_{-k}^\dagger\end{pmatrix}$$

η_k new fermionic operators

diagonalization $$H = \sum_k \omega_k\left(\eta_k^\dagger \eta_k - \frac{1}{2}\right)$$

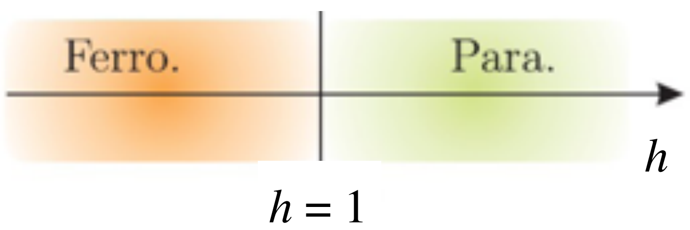
spectrum $$\omega_k = 2\sqrt{\left(\cos k + h\right)^2 + \sin^2 k}$$



Quantum Ising model (iii)

$$H = -\sum_{i=1}^{L} \left(\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z \right)$$

Phase diagram



$h << 1$ $\quad |\leftarrow \leftarrow \leftarrow \ldots \leftarrow \leftarrow\rangle \pm |\rightarrow \rightarrow \rightarrow \ldots \rightarrow \rightarrow\rangle$ $\quad P^z = \pm 1$

$h >> 1$ $\quad |\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\rangle$

$\boxed{h = 1}$ Critical point

$$\langle \sigma_i^+ \sigma_{i+r}^- \rangle = \langle c_i^\dagger e^{-i\pi \sum_{l=i}^{i+r-1} c_l^\dagger c_l} c_{i+r} \rangle$$

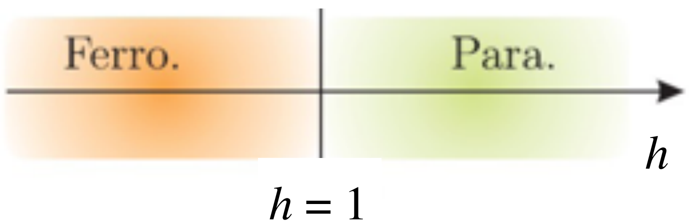
$\neq 0$ Ferro

$= 0$ Para

Quantum Ising model (iii)

$$\boxed{H = -\sum_{i=1}^{L} \left(\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z\right)}$$

Phase diagram



$h << 1$ $\quad |\leftarrow \leftarrow \leftarrow \ldots \leftarrow \leftarrow\rangle \pm |\rightarrow \rightarrow \rightarrow \ldots \rightarrow \rightarrow\rangle$ $\quad P^z = \pm 1$

$h >> 1$ $\quad |\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\rangle$

$\boxed{h = 1}$ Critical point

$$\langle \sigma_i^+ \sigma_{i+r}^- \rangle = \langle c_i^\dagger e^{-i\pi \sum_{l=i}^{i+r-1} c_l^\dagger c_l} c_{i+r} \rangle$$

$\neq 0$ Ferro

$= 0$ Para

Duality

$$\mu_j^z = \sigma_j^x \sigma_j^x$$

$$\mu_j^x = \prod_{k<j} \sigma_k^z$$

$$H = -\sum_i \left[h \mu_i^x \mu_{i+1}^x + \mu_i^z\right]$$

For h>1 $\quad \langle \mu_i^x \mu_{i+r}^x \rangle \xrightarrow[r\to\infty]{} \neq 0$

The model is *self-dual* at h=1

For $r \to \infty$ $\quad \langle e^{-i\pi \sum_{l=i}^{i+r-1} c_l^\dagger c_l} \rangle \to \neq 0$ $\quad$ Para

Topological phases and Majorana edge states

E. Majorana (1937):
- pure real solution to the Dirac equation
- particles are their own anti particles

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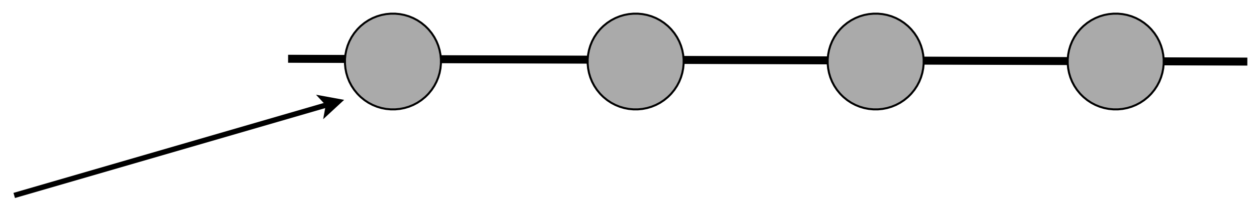

(complex) fermion

$$\{a_k^\dagger, a_l\} = \delta_{kl}$$

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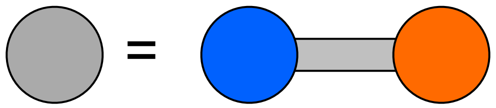

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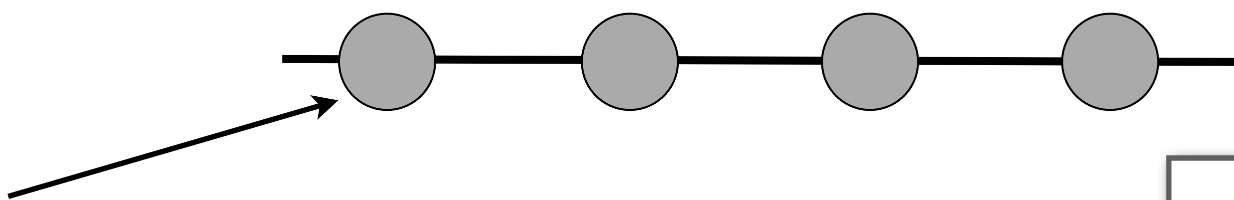

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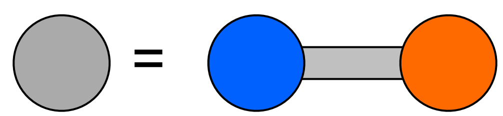

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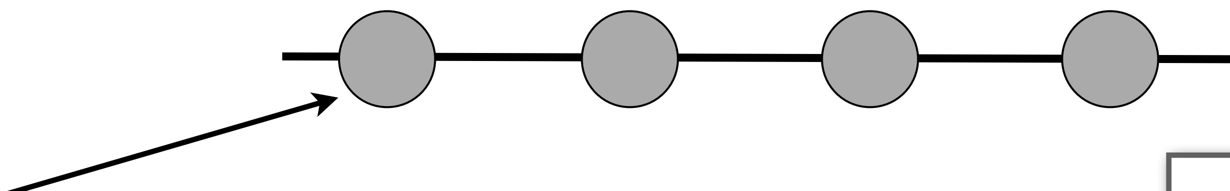

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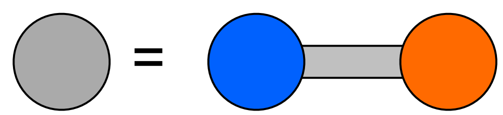

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Fundamental interest in complex constituents of matter

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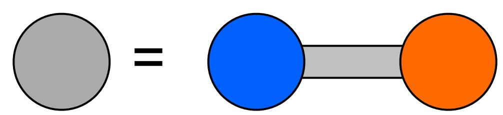

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Potential applications in topological quantum computation protocols

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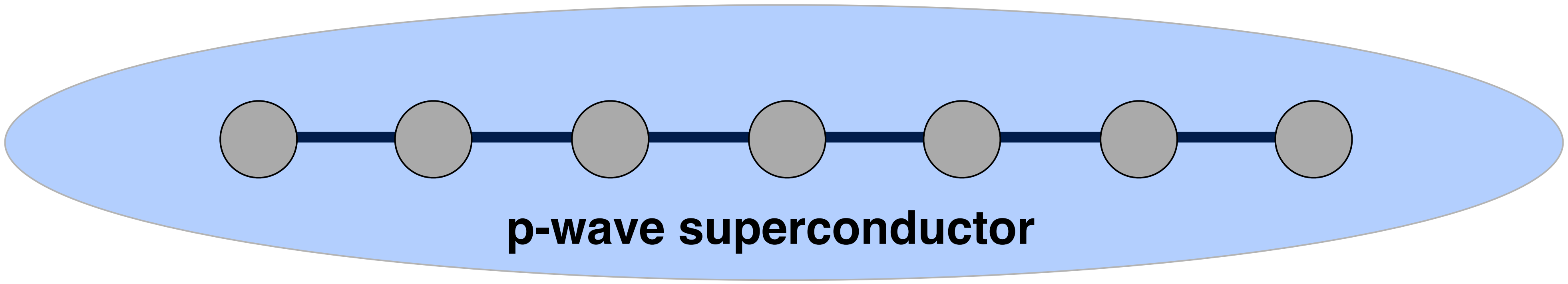
Majorana fermions emerge as edge states (fractionalization)



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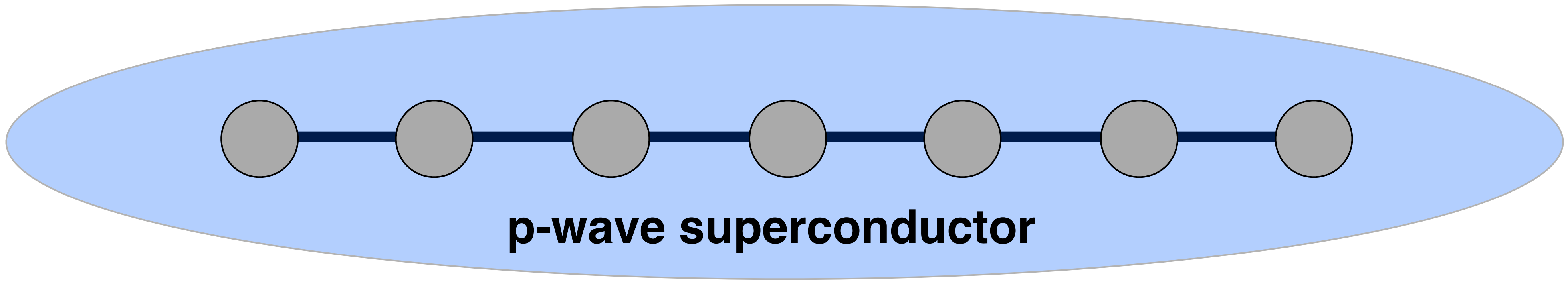

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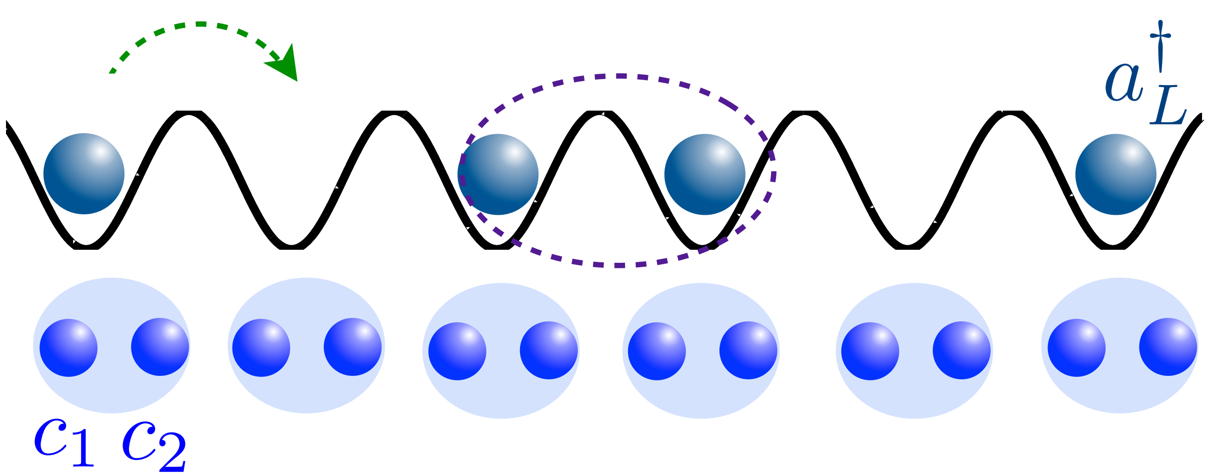

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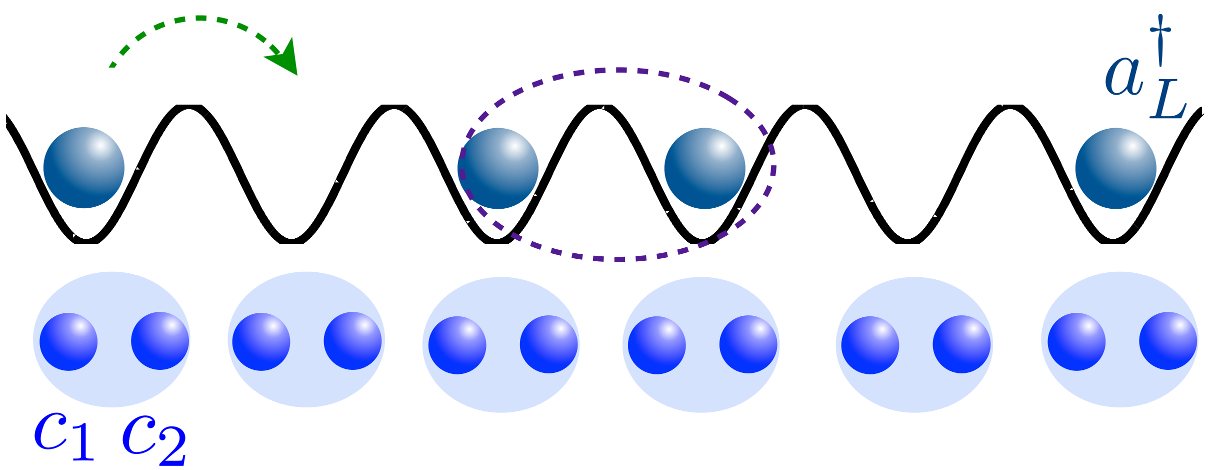

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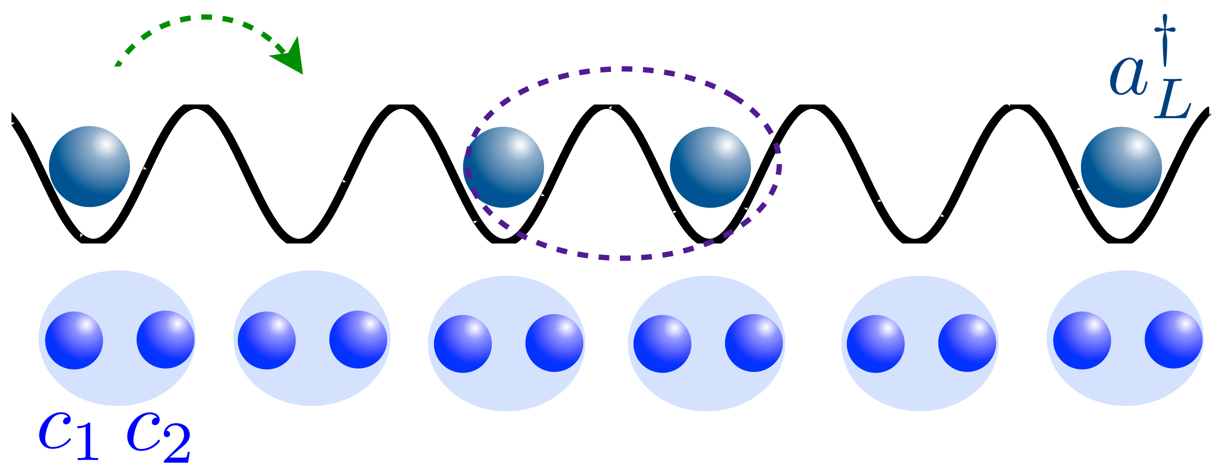

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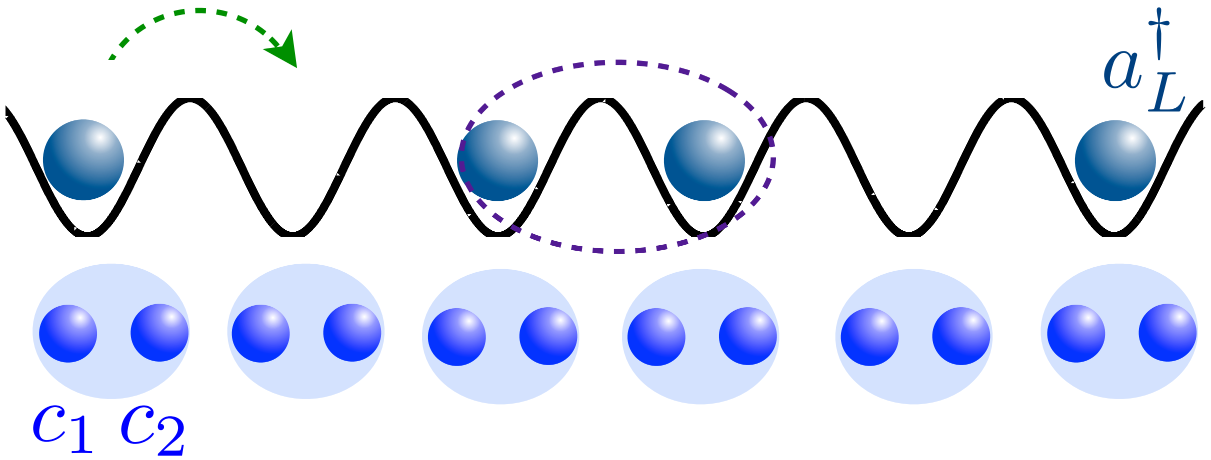

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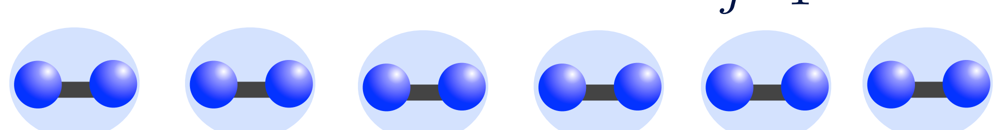

lattice completely full or empty

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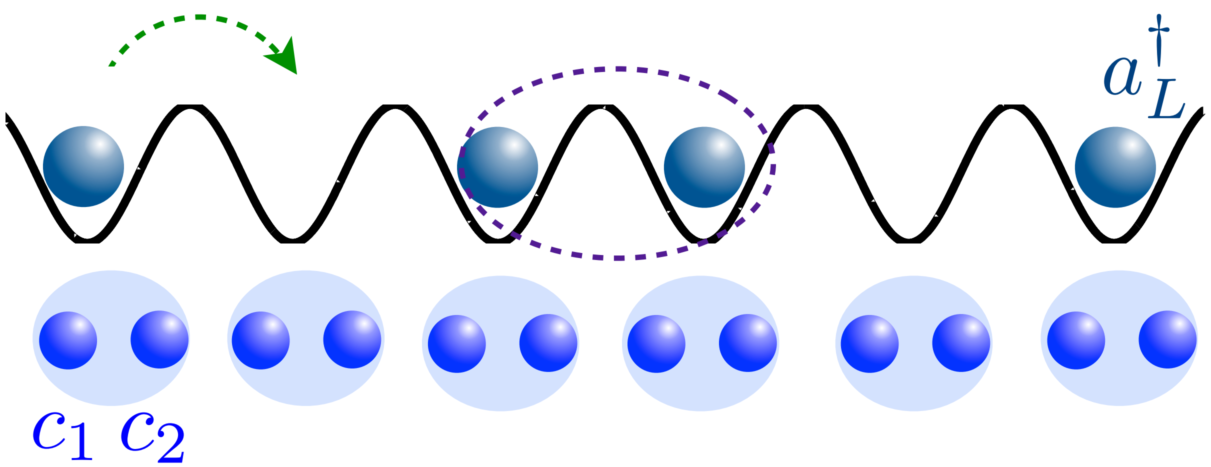

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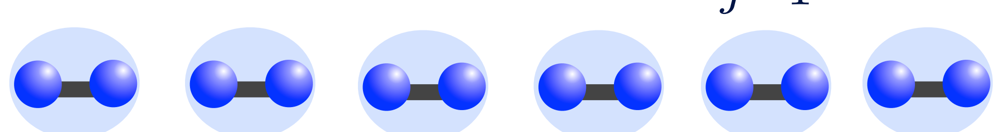

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Ideal case chain:

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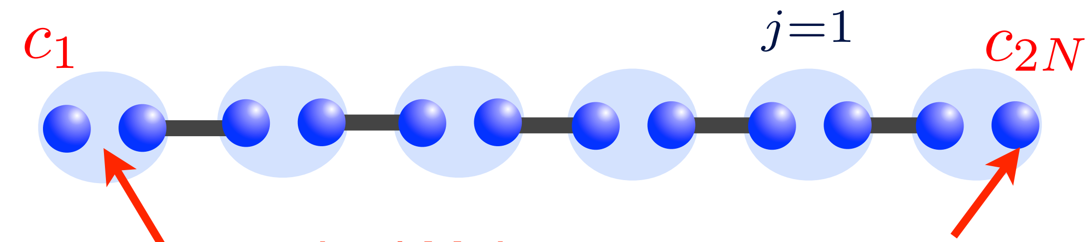

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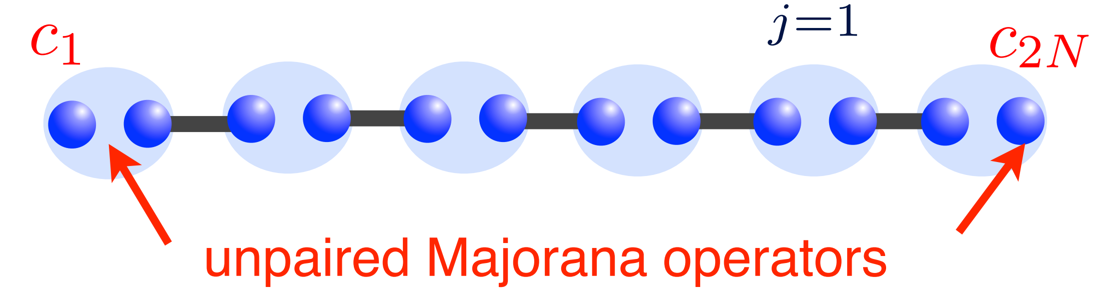

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Key feature: **parity symmetry**!

Electron number is conserved modulo 2

Provided by the proximity effect to a superconductor

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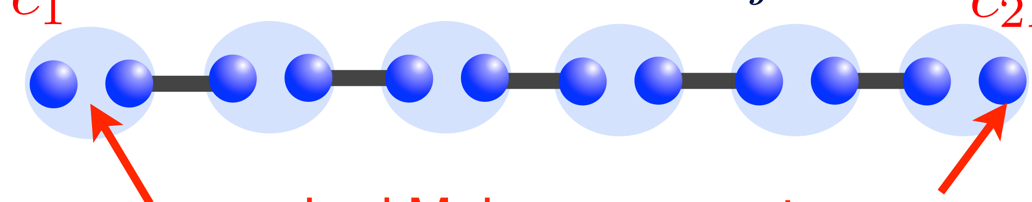

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Two-fold degenerate ground state (with opposite parities!)

Zero-energy modes, non local correlation between the edges

Localization of the excitation at the edges

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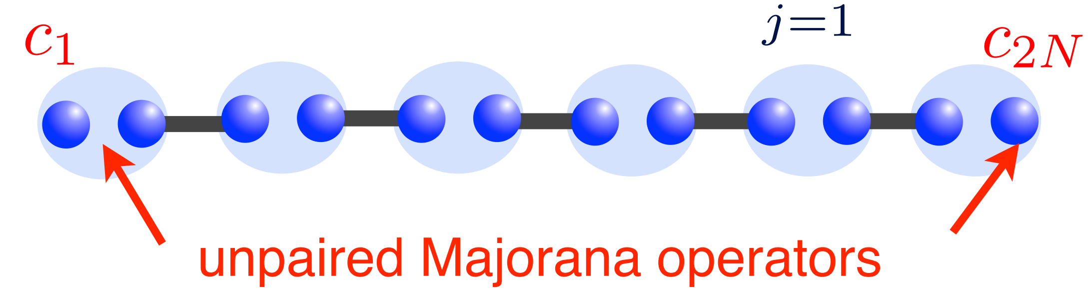

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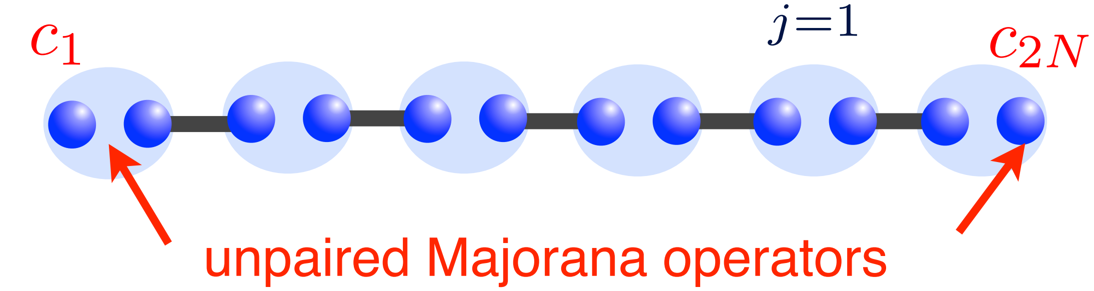

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Double degenerate entanglement spectrum in open and periodic chains

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Single ground state in periodic/anti-periodic chains (opposite parities)

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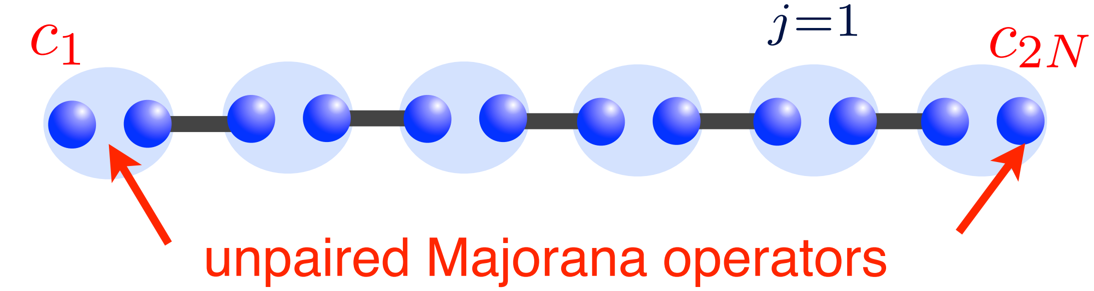

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Typical topological features

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Anyons in 1D lattice: commutation relations

$$a_j a_k^\dagger - e^{-i\theta\epsilon(j-k)} a_k^\dagger a_j = \delta_{jk}$$

$$a_j a_k - e^{i\theta\epsilon(j-k)} a_k a_j = 0$$

$$\epsilon(j-k) = \begin{cases} 1 & j > k \\ 0 & j = k \\ -1 & j < k \end{cases}$$

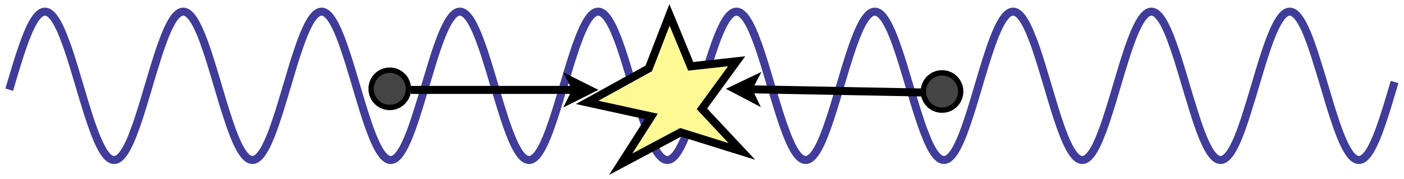
They behave like bosons on the same site

Commutation relations depend on ordering unless $\theta = 0, \pi$

Bosons

Pseudo-fermions

Exchanging particles in 1D?



Trying to exchange particles yields to collision

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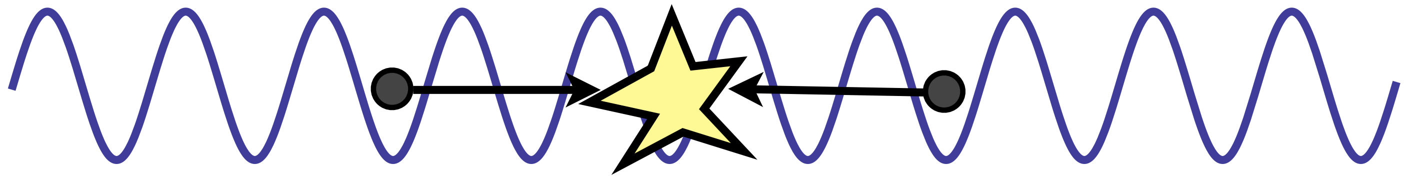

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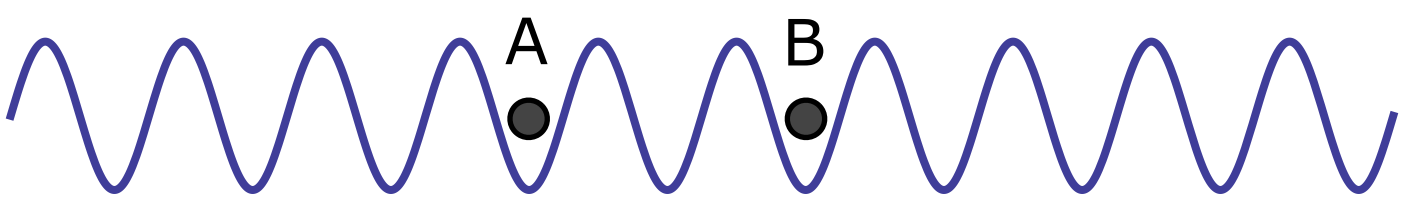
Exchange in
Fock space



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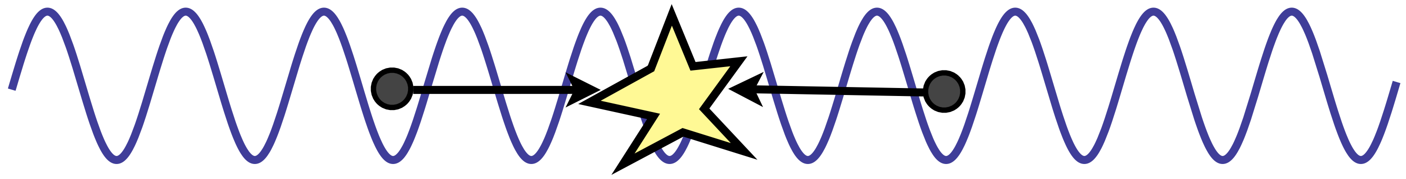

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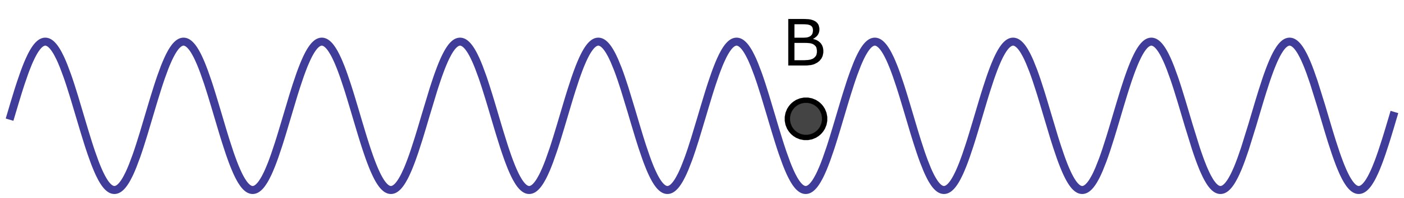

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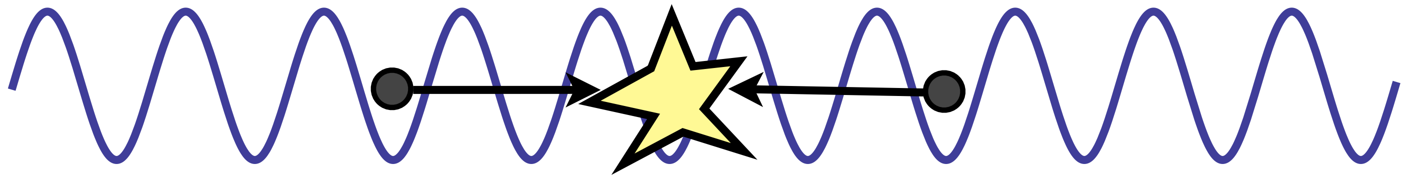

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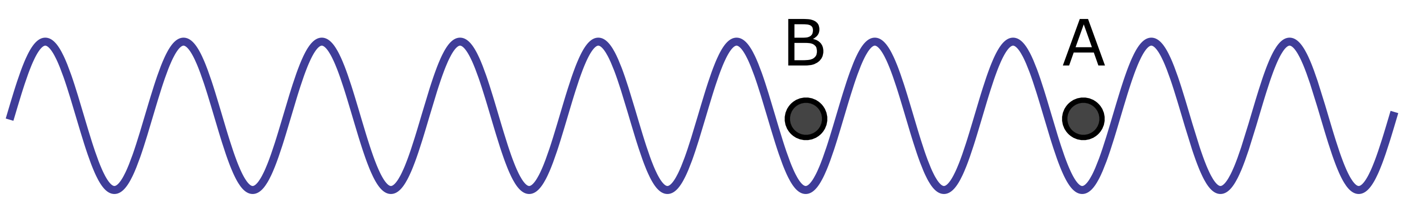

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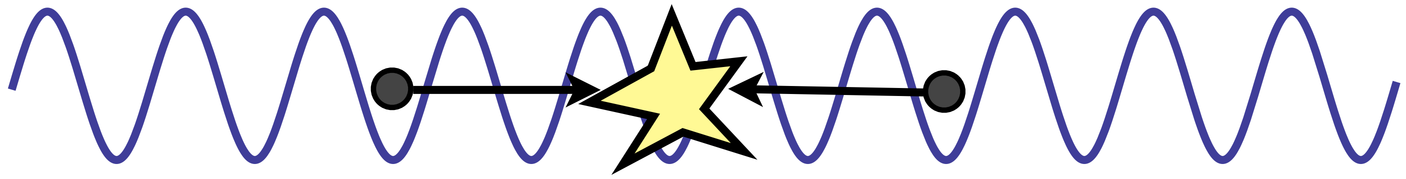

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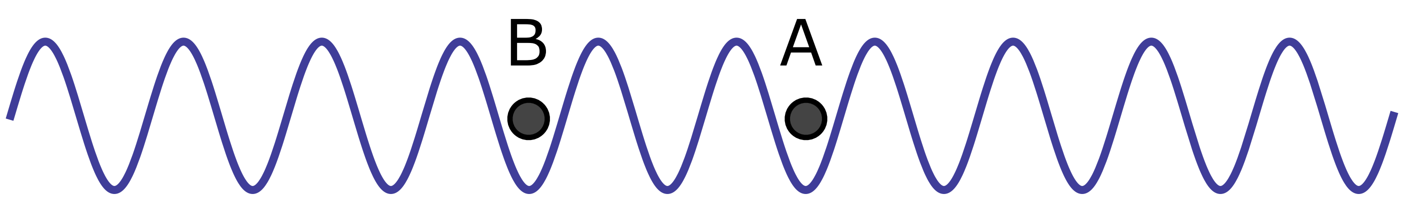

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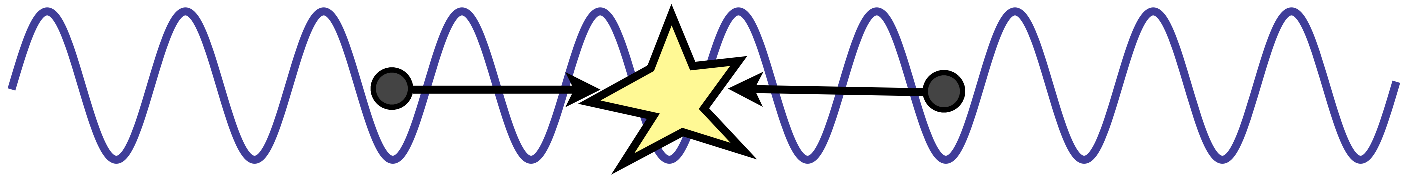

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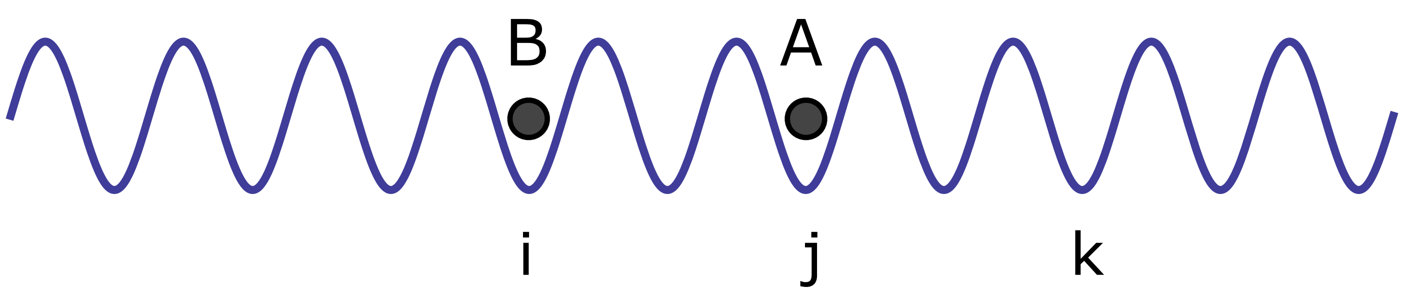

$$|\psi\rangle = a_j^\dagger a_i^\dagger |0\rangle$$

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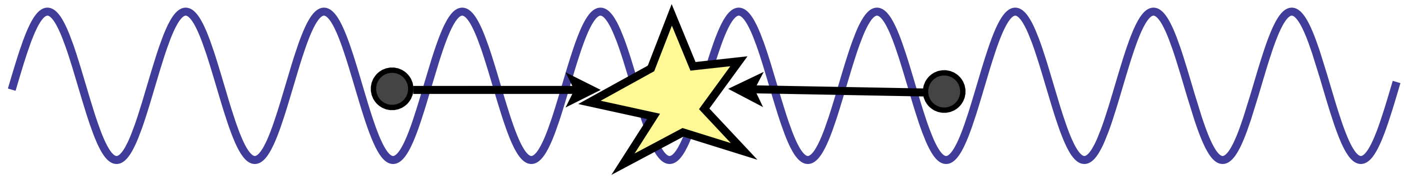

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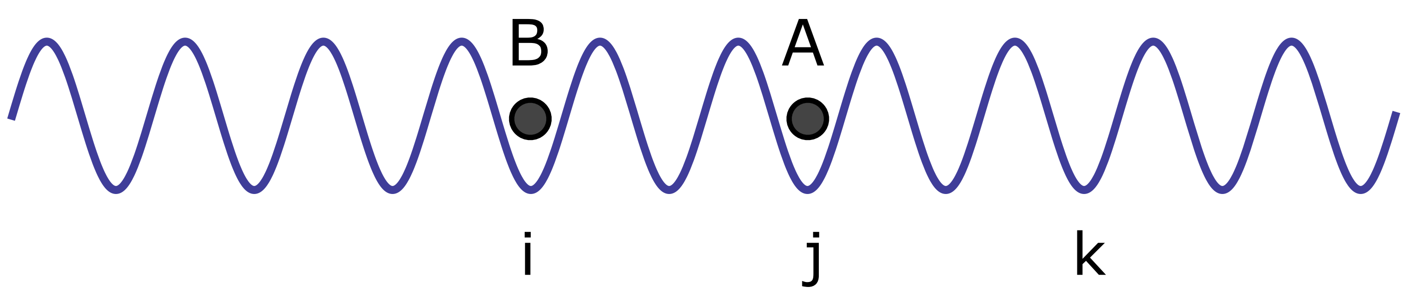

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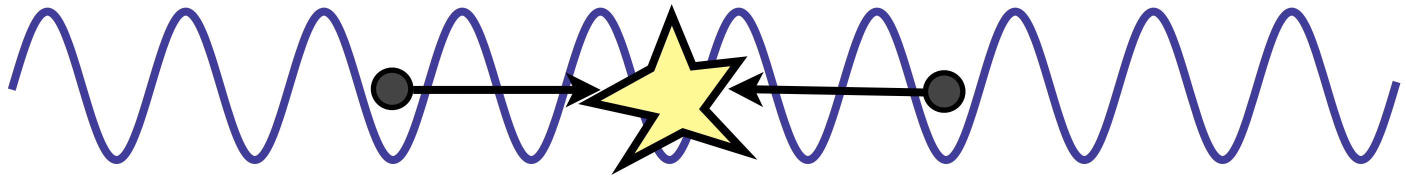

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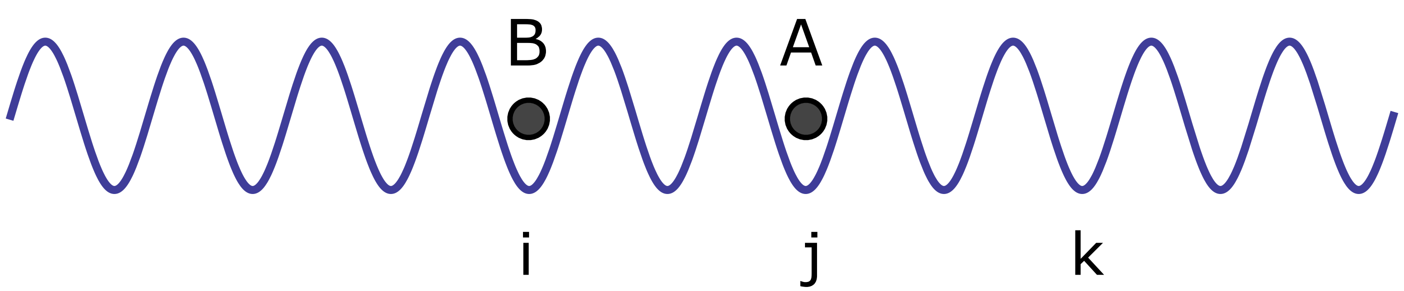

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The other way

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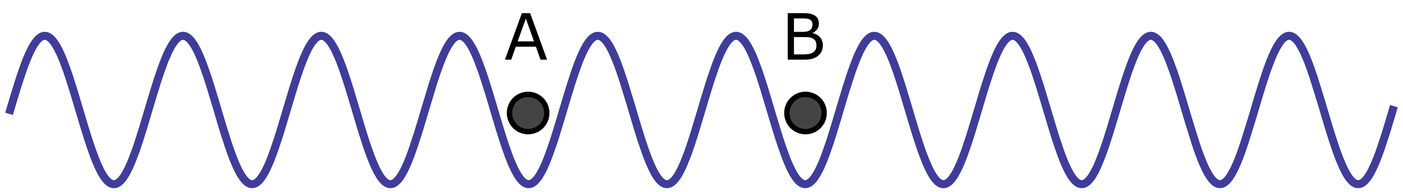
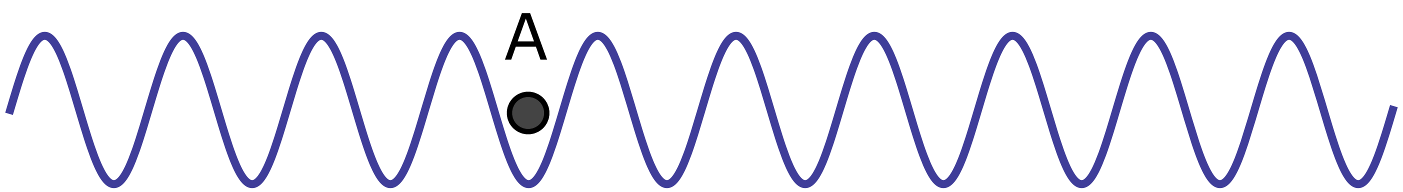


The other way



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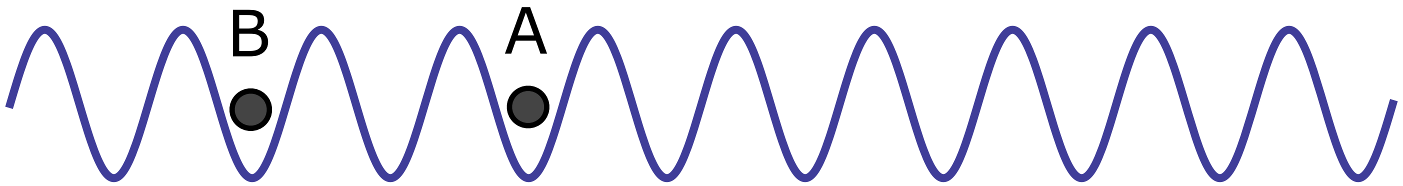

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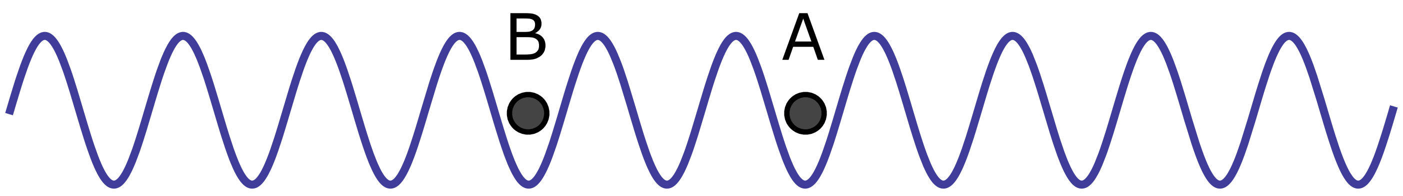

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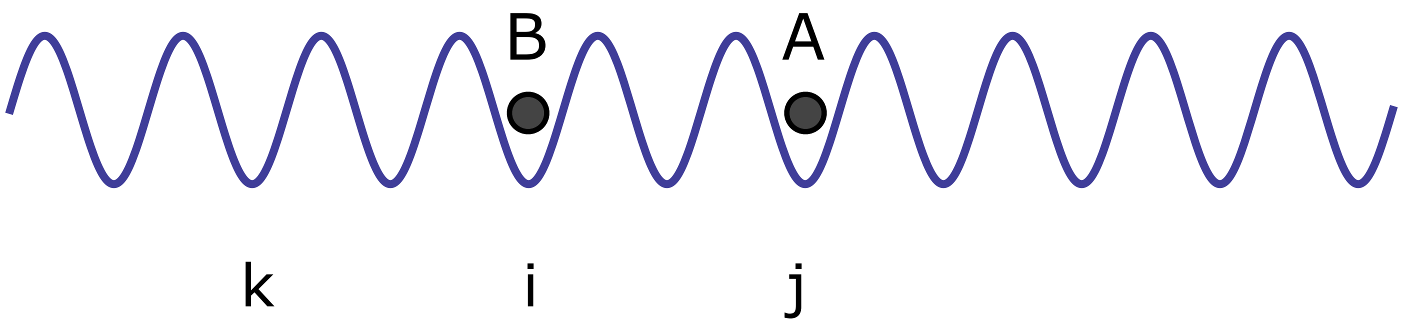
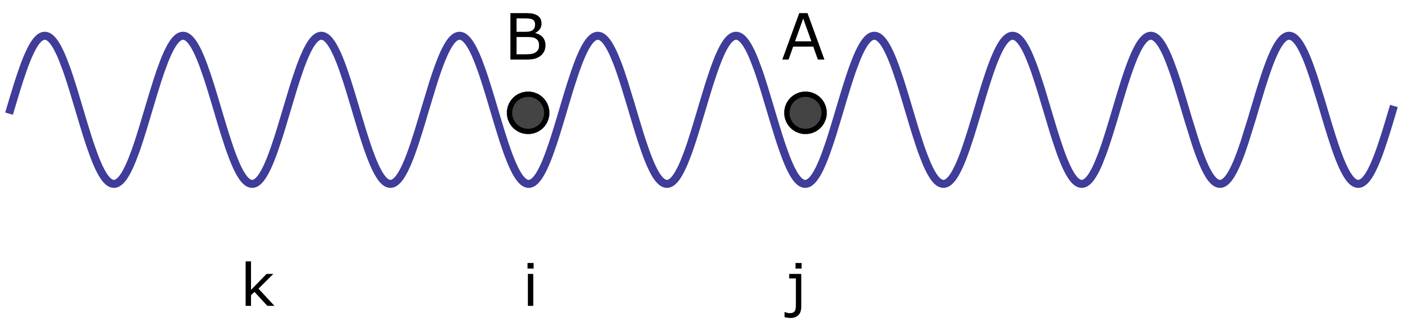


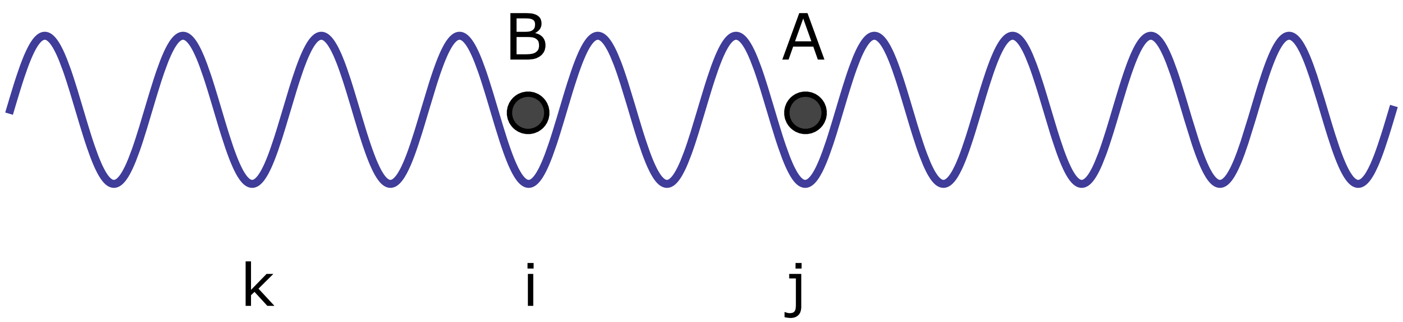
$$|\psi\rangle = a_j^\dagger a_i^\dagger |0\rangle$$

The other way



$$|\psi\rangle = a_j^\dagger a_i^\dagger |0\rangle \xrightarrow{\text{exchange}} |\psi\rangle = a_k^\dagger a_j a_j^\dagger a_i^\dagger |0\rangle = a_k^\dagger a_i^\dagger a_j a_j^\dagger |0\rangle = e^{-i\theta} a_i^\dagger a_k^\dagger |0\rangle$$

The other way



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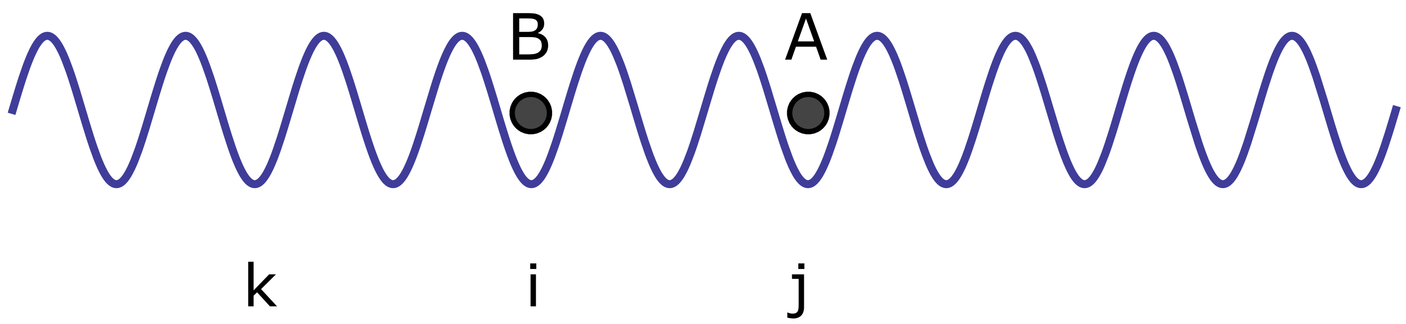
$$\xrightarrow{\text{translate}} e^{-i\theta} a_j^\dagger a_i^\dagger |0\rangle = e^{-i\theta} |\psi\rangle$$

The other way



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$$\xrightarrow{\text{translate}} e^{-i\theta} a_j^\dagger a_i^\dagger |0\rangle = e^{-i\theta} |\psi\rangle$$

Which object in 1D plays the role of the flux tube in 2D?

Generalized Jordan-Wigner transformation

$$a_j = b_j \exp\left(i\theta \sum_{i=1}^{j-1} n_i\right) \qquad a_j^\dagger = \exp\left(-i\theta \sum_{i=1}^{j-1} n_i\right) b_j^\dagger$$

Bosonic variable

[T. Keilmann, S. Lanzmich, I. McCulloch & M R, Nature Communications **2**, 361 (2011).]

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Bosonic variable

$$n_i = a_i^\dagger a_i = b_i^\dagger b_i$$

On-site quantities remain the same

$$a_j^\dagger a_{j+1} \longrightarrow b_j^\dagger b_{j+1} e^{i\theta n_j}$$

Hopping anyons are mapped onto bosonic correlated hopping or conditional hopping processes

[T. Keilmann, S. Lanzmich, I. McCulloch & M R, Nature Communications **2**, 361 (2011).]

Correlated hopping?

Electron correlations in narrow energy bands

By J. Hubbard

Theoretical Physics Division, A.E.R.E., Harwell, Didcot, Berks

(*Communicated by B. H. Flowers, F.R.S.—Received* 23 *April* 1963)

It is pointed out that one of the main effects of correlation phenomena in *d*- and *f*-bands is to give rise to behaviour characteristic of the atomic or Heitler–London model. To investigate this situation a simple, approximate model for the interaction of electrons in narrow energy bands is introduced. The results of applying the Hartree–Fock approximation

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Cold Bosonic Atoms in Optical Lattices

D. Jaksch,[1,2] C. Bruder,[1,3] J. I. Cirac,[1,2] C. W. Gardiner,[1,4] and P. Zoller[1,2]
[1]*Institute for Theoretical Physics, University of Santa Barbara, Santa Barbara, California 93106-4030*
[2]*Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria*
[3]*Institut für Theoretische Festkörperphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany*
[4]*School of Chemical and Physical Sciences, Victoria University, Wellington, New Zealand*
(Received 26 May 1998)

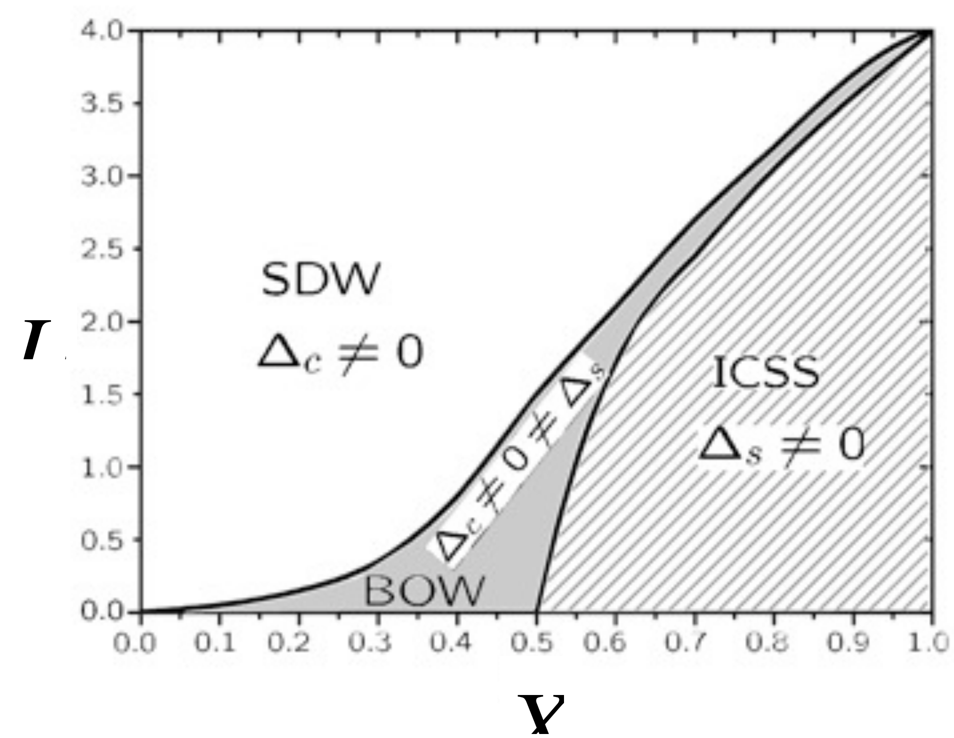
The dynamics of an ultracold dilute gas of bosonic atoms in an optical lattice can be described by a Bose-Hubbard model where the system parameters are controlled by laser light. We study the

Correlated hopping in condensed matter

Application of the Hubbard model to materials with **extended orbitals**: the charge localized in the bonds affects the screening of the effective potential between the valence electrons, the extension of the Wannier orbitals and the hopping between them. Relevant for **hole superconductivity** [Hirsch and co-workers, 1989].

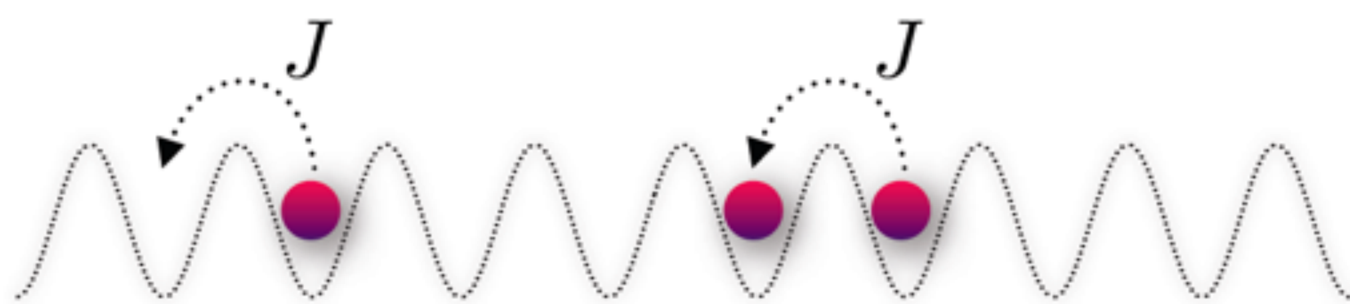
Hubbard model with correlated hopping

$$\mathcal{H} = -\sum_{i\sigma}\left[1 - X\left(n_{i\bar{\sigma}} + n_{i+1\bar{\sigma}}\right)\right]\left(c^{\dagger}_{i\sigma}c_{i+1\sigma} + \mathrm{H.c.}\right)$$
$$+U\sum_{i} n_{i\uparrow}n_{i\downarrow}$$



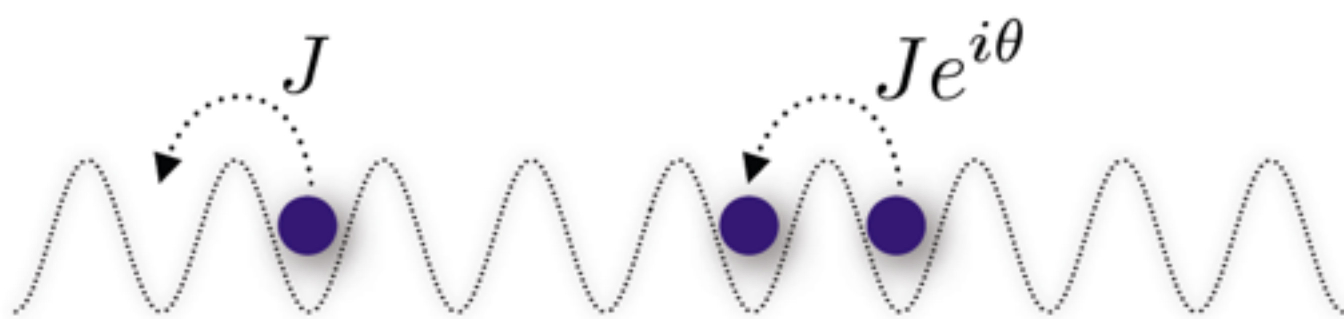
Anyon-Hubbard model

Anyons



$$H^a = -J \sum_j^L (a_j^\dagger a_{j+1} + \text{h.c.}) + \frac{U}{2} \sum_j^L n_j(n_j - 1)$$

Conditional-hopping bosons



$$H^b = -J \sum_j^L (b_j^\dagger b_{j+1} e^{i\theta n_j} + \text{h.c.}) + \frac{U}{2} \sum_j^L n_j(n_j - 1)$$

In the hopping process the phase term depends only on the occupation in the left side.

As expected, anyons brake parity and time reversal, except for $\theta = 0, \pi$

In the hopping process the phase term depends only on the occupation in the left side.

As expected, anyons brake parity and time reversal, except for $\theta = 0, \pi$

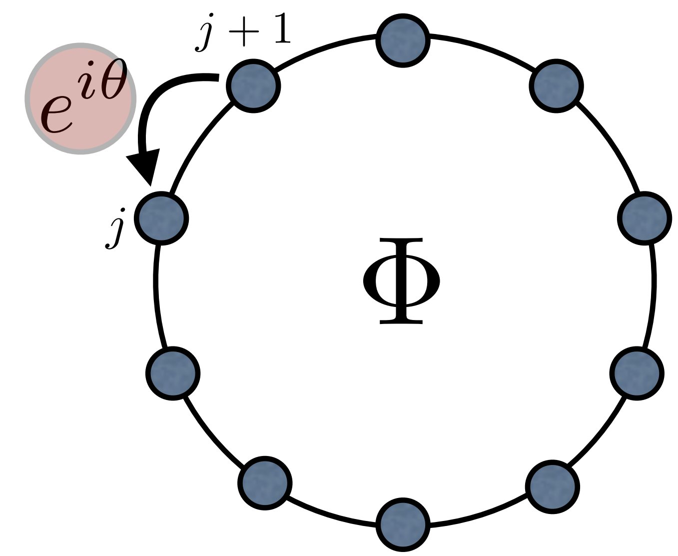
Artificial gauge fields $\Rightarrow$ **Artificial particles**

Anyons $\xrightarrow{\text{(map)}}$ Bosons $\longrightarrow$ Boson properties $\xrightarrow{\text{(map)}^{-1}}$ Anyon properties

Artificial magnetic field

$$H = \frac{1}{2m}\left[\mathbf{p} - q\mathbf{A}\right]^2$$

Peierls phase



Phase acquired from j+1 to j

$$\psi(x_j) \rightarrow \exp\left(\frac{1}{\hbar}\int_{j+1}^{j} q\mathbf{A}\cdot d\mathbf{l}\right)\psi(x_j)$$

$$\Phi = \int \mathbf{B}\cdot d\mathbf{S}$$

Magnetic flux

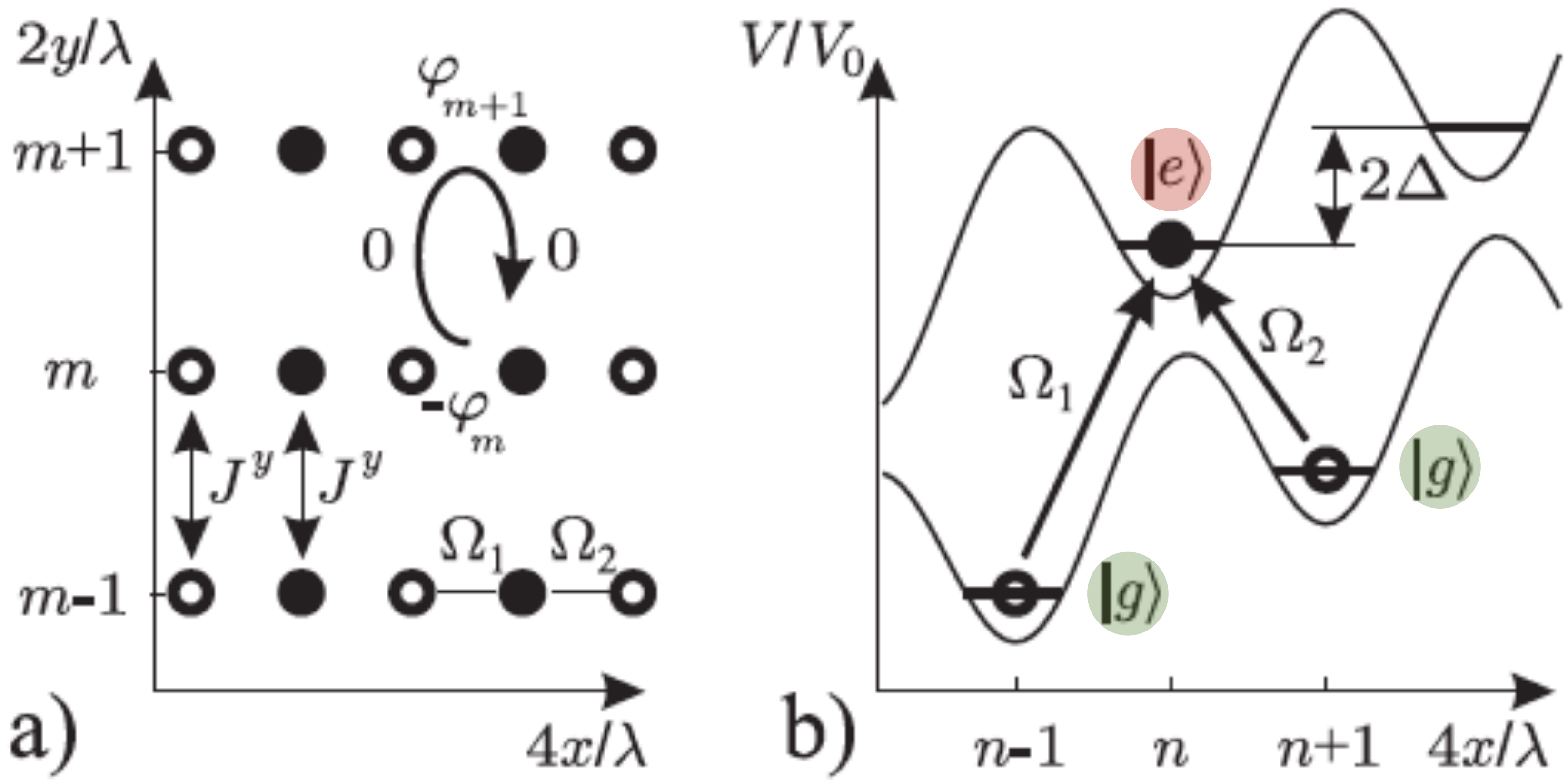
$$H = -J\sum_j (b_j^\dagger b_{j+1} e^{i\theta} + \mathrm{h.c.})$$

$$\theta = \frac{1}{\hbar}\int_{j+1}^{j} q\mathbf{A}\cdot d\mathbf{l}$$

In 1D θ can be “gauged away” to the border (Aharonov-Bohm)

Jaksch-Zoller proposal

[Jaksch, D. & Zoller, P., New J. Phys. 5, 56 (2003)]



No hopping along z

Normal tunneling along y

Tilted deep lattice along x

Two Raman lasers between internal states |g> and |e> induce hopping in x

State dependent OL: |e> is halfway between two adjacent |g>

Phase difference between Rabi frequencies Ω_1, Ω_2 gives a Peierls term

Peierls phases depend on y, creating a net flux per plaquette

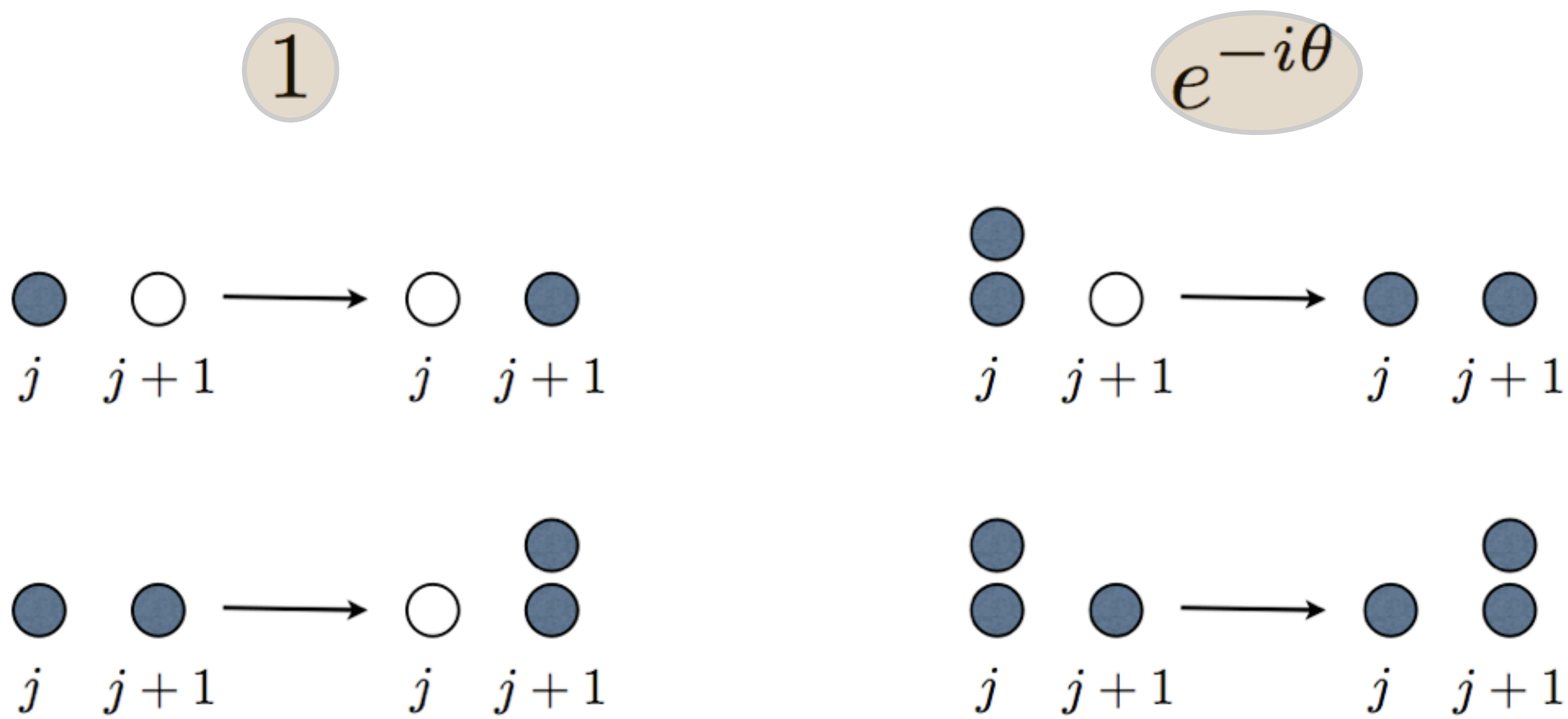
Density-dependent hopping phase

$$H^b = -J \sum_{j}^{L} (b_j^\dagger b_{j+1} e^{i\theta n_j} + e^{-i\theta n_j} b_{j+1}^\dagger b_j)$$

Truncation of local Hilbert space

$n_j = 0, 1$ No phase terms: same spectrum as hard core bosons (free fermions)

$n_j = 0, 1, 2$ Non-trivial interference effects



Photon assisted Raman tunneling

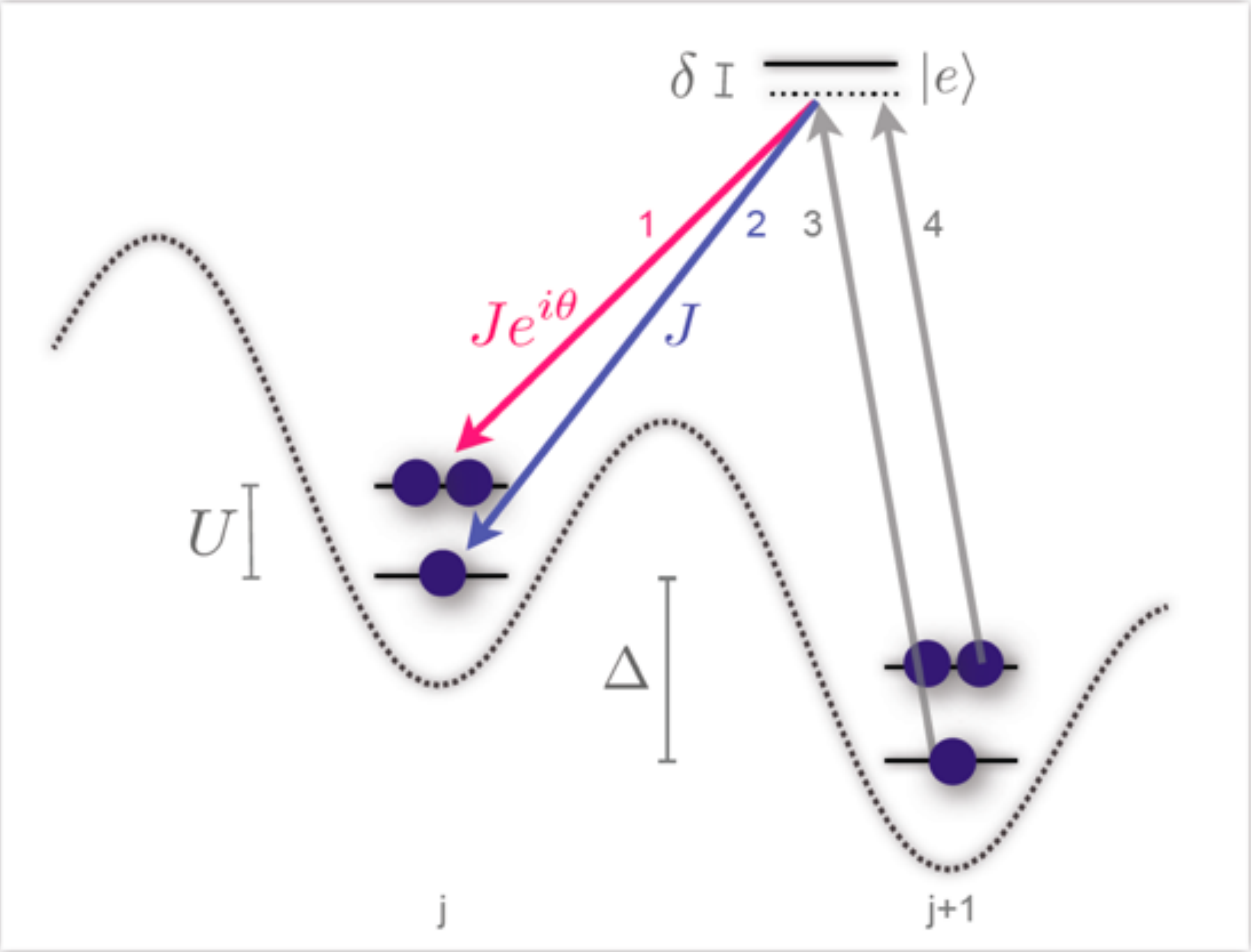
We distinguish energetically different occupation numbers by interaction U

4-dimensional GS manifold

Various way of implementation:
e.g. spin-dependent lattices

|g> $F = 1, m_F = -1$

|e> $F = 1, m_F = 0$

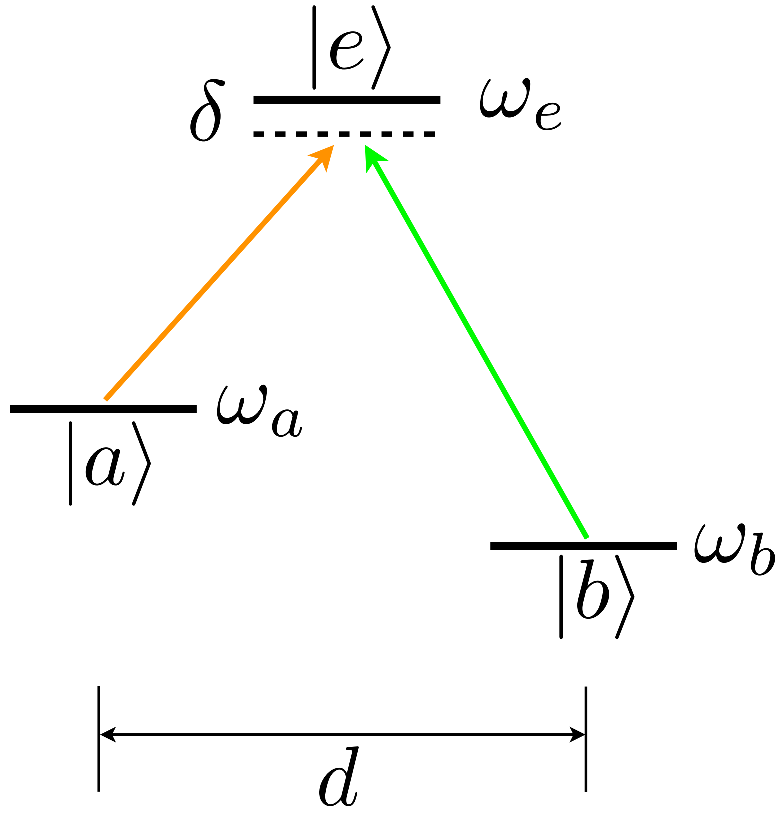


We want
$$J_{23} = J_{24} = J$$
$$J_{13} = J_{14} = Je^{i\theta}$$

For each tunneling rate we define a Λ-scheme: we need 4 different lasers

Photon assisted Raman tunneling (ii)

Let us focus on two states $|a\rangle$, $|b\rangle$



$$H = \sum_{i=a,b,e} \hbar\omega_i |i\rangle\langle i| + \frac{\hbar}{2}\left(\gamma_a|e\rangle\langle a| + \gamma_b|e\rangle\langle b| + \text{h.c.}\right)$$

$$\gamma_{a(b)} = \Omega^e_{a(b)} W^e_{a(b)} e^{-i(\omega_e - \omega_{a(b)} - \delta)t}$$

off-diagonal terms

$$W^e_a = e^{ik_a x_a} \int w^*_e(x + x_e) e^{ik_a x} w_a(x) dx$$

$$W^e_b = e^{ik_b(x_a+d)} \int w^*_e(x + x_e) e^{ik_b x} w_b(x + d) dx$$

superposition integrals (sizable)

Ω^e_a, Ω^e_b Rabi frequencies

$w(x)$ Wannier functions

$k_{a(b)}$ x-component of laser field $\quad |\mathbf{k}_{a(b)}| = (\omega_e - \omega_{a(b)} - \delta)/c$

$\gamma_{a(b)} \in \mathbb{C}$ modulus and phase tuned by choosing the appropriate intensity, polarization and direction of the driving fields.

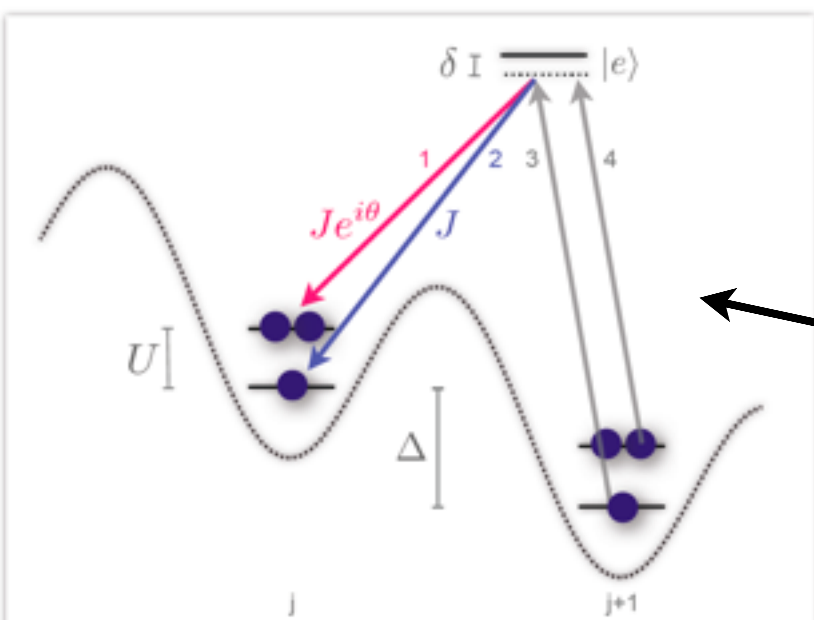
Photon assisted Raman tunneling (iii)

For sufficiently large δ, the level |e> is not populated and can be adiabatically eliminated and in the RWA

$$H_{\text{eff}} = -\frac{\hbar}{4\delta}\begin{pmatrix} |\tilde{\gamma}_a|^2 & \tilde{\gamma}_a^*\tilde{\gamma}_b \\ \tilde{\gamma}_b^*\tilde{\gamma}_a & |\tilde{\gamma}_b|^2 \end{pmatrix}$$

effective Hamiltonian for |a>, |b>

$\tilde{\gamma}$ = non-rotating γ



$$J_{ab} = \tilde{\gamma}_a^*\tilde{\gamma}_b/2\delta$$

$$\boxed{\tilde{\gamma}_1 = \tilde{\gamma}_2 e^{i\theta}}$$

But $|\tilde{\gamma}_a| = |\tilde{\gamma}_b|$ implies that D and U vanish → "free" anyons

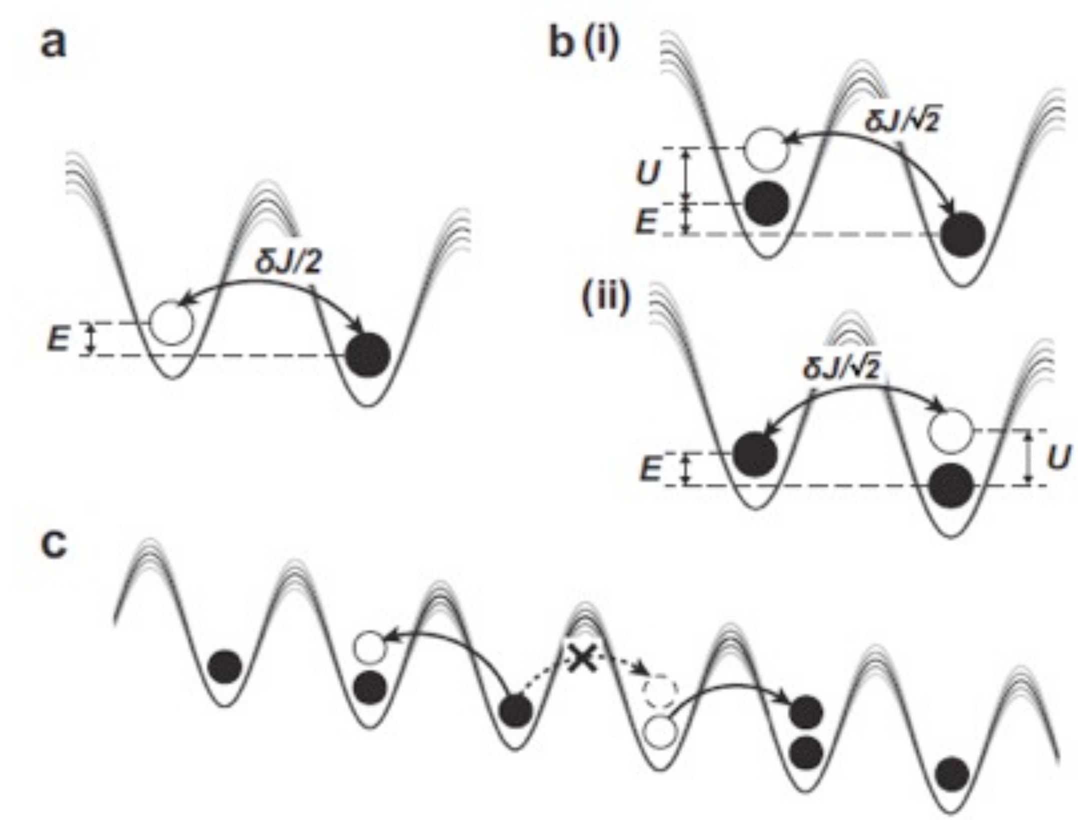
Alternative proposals for correlated hopping

AC-driven laser potentials

Lattice shaking

(Floquet analysis)

rescaled hopping



R.Ma et al., PRL 107, 095301 (2011)

[Y.-A. Chen *et al.*, arXiv:1104.1833]

Eckart et al., PRL **95**, 260404 (2005),

Struck J, et al., Phys. Rev. Lett. **108,** 225304 (2012)

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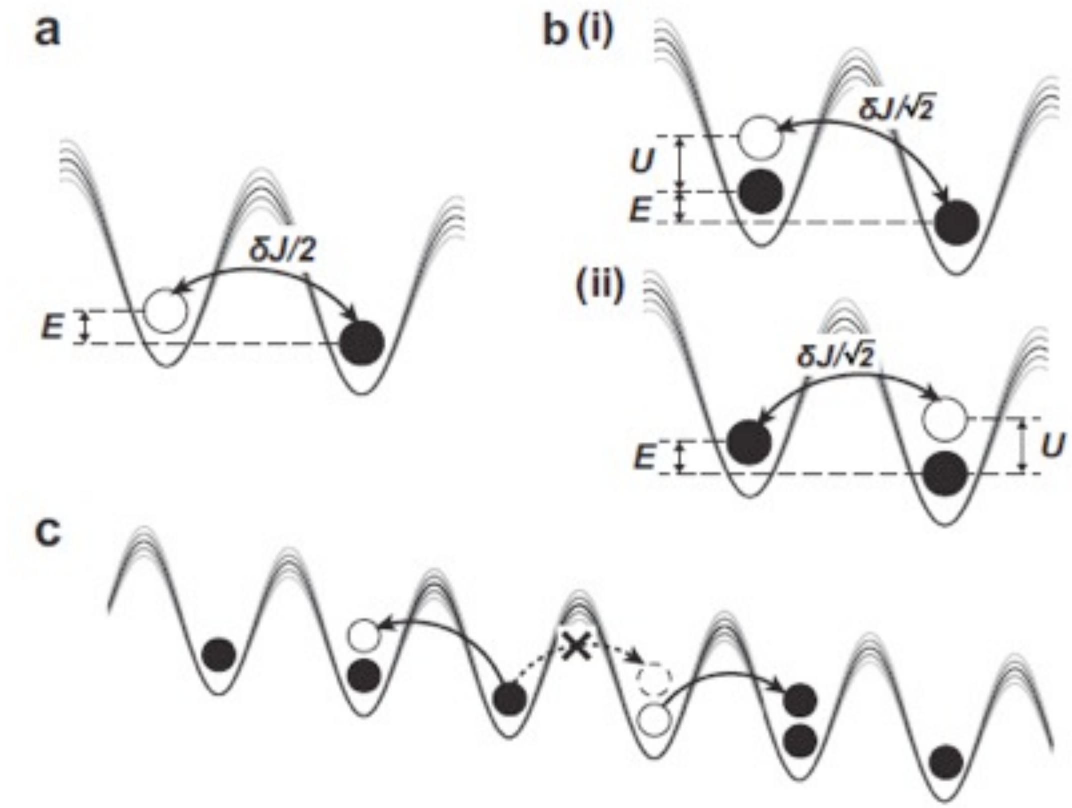
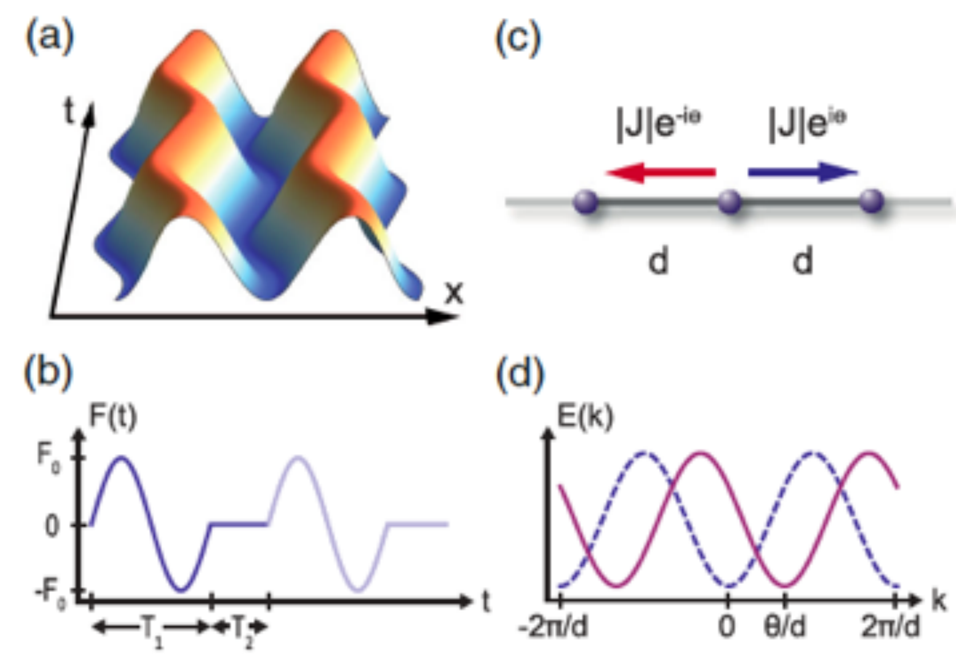


Non sinusoidal driving

complex phase factor



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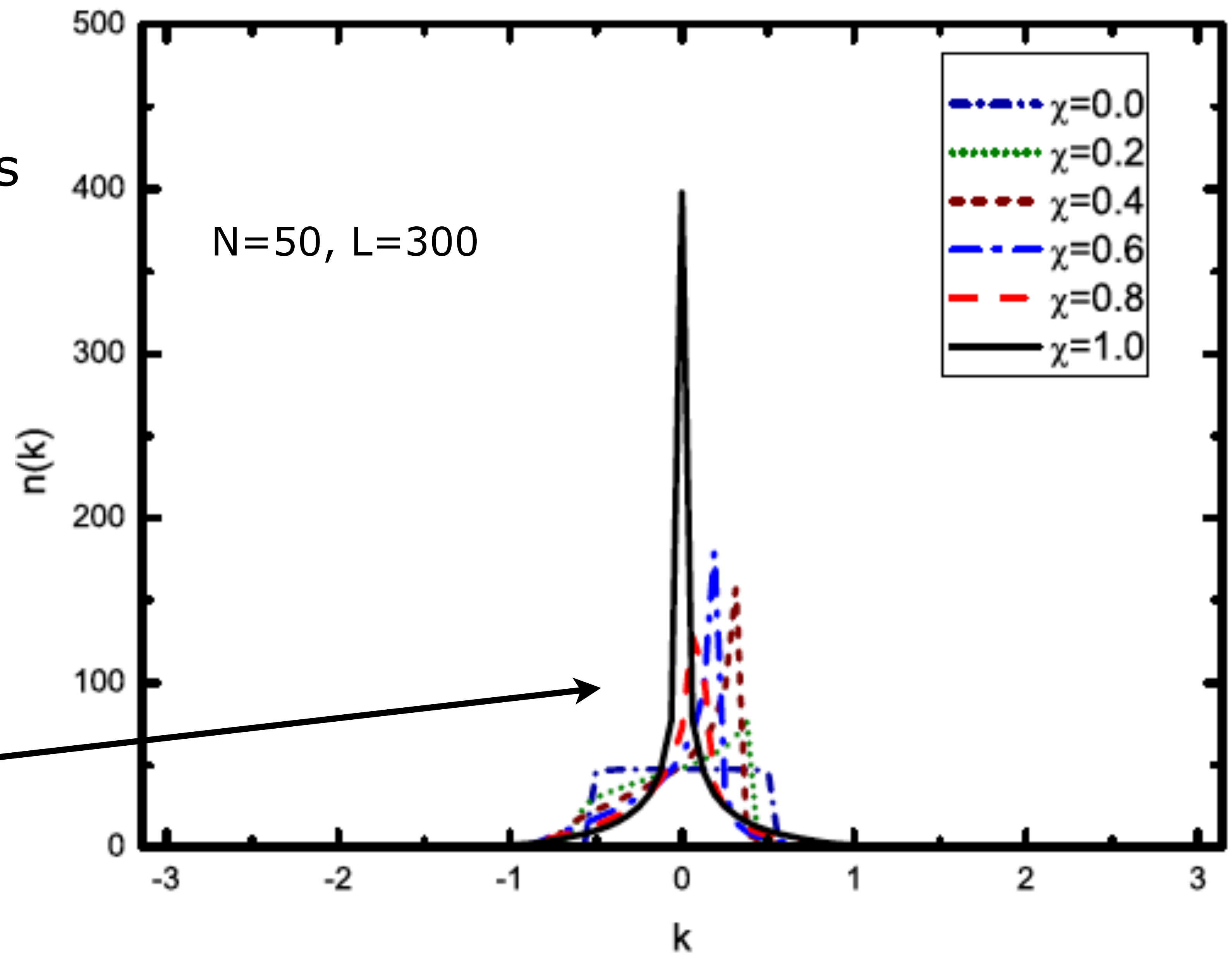
Hard-core limit case $(n_j = 0, 1)$

$$H = -J \sum_j \left(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right)$$

- Phase terms disappear
- Same spectrum as fermions
- Same density distribution in trap

But...

Density distribution in momentum space is asymmetrical



[Hao, Y. et al., PRA **79**, 043633 (2009).]

DMRG

$$k = \frac{2\pi}{L}m \qquad m = 0, 1, \ldots, L-1$$

quantization of momenta

local basis truncation

$$n_j \leq 3$$

$$\langle n_k \rangle = \frac{1}{L} \sum_{ij} e^{ik(x_i - x_j)} \langle b_i^\dagger b_j \rangle$$

observed in TOF experiments

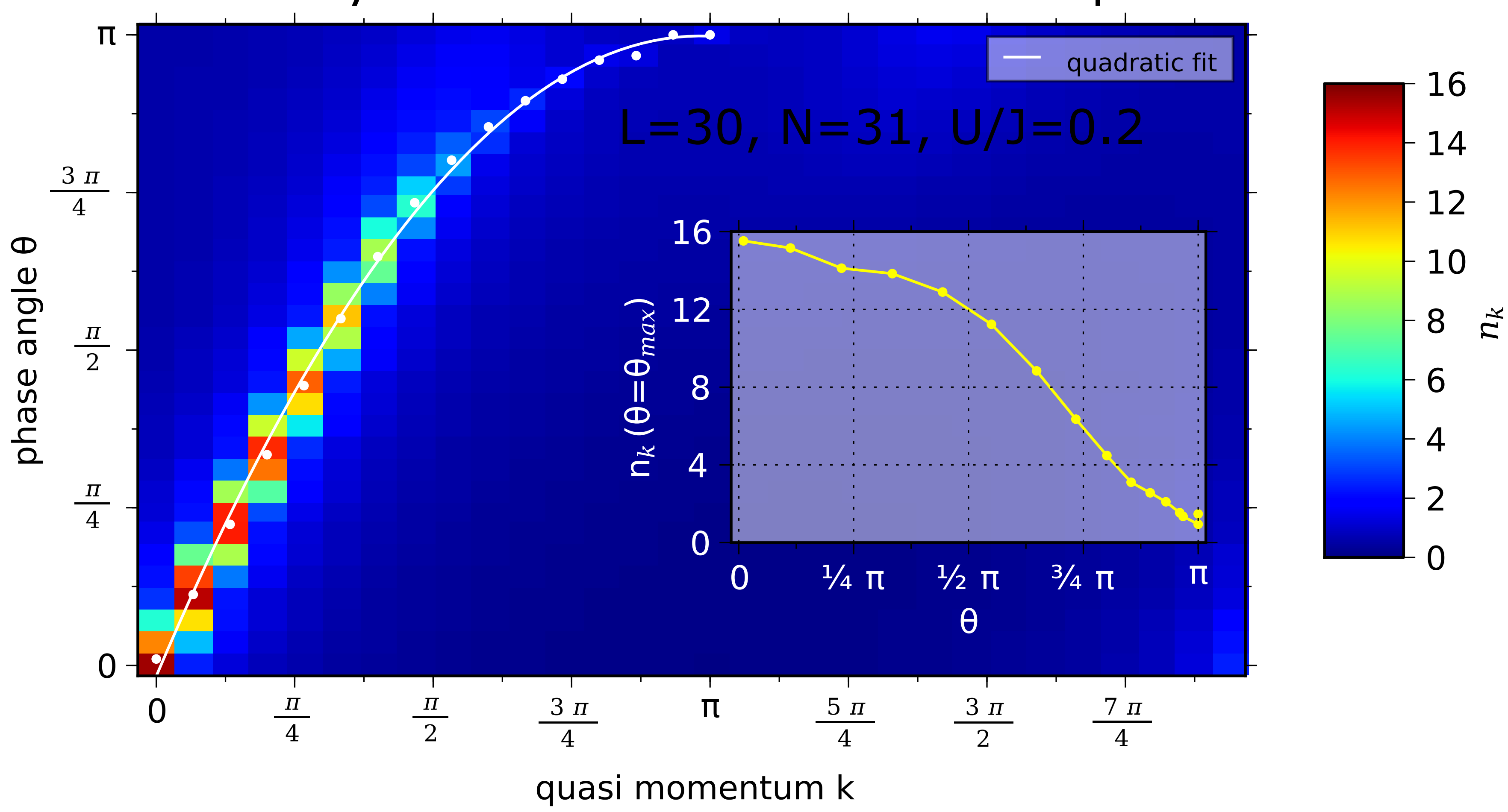
The peak decays with increasing θ (decoherence)

The peak shifts quadratically

$$\theta_{max}(k) = \alpha(k - k_0)^2 + \beta$$

$$\beta = k_0 = \pi, \quad \alpha \approx -1/\pi$$

Density distribution in momentum space

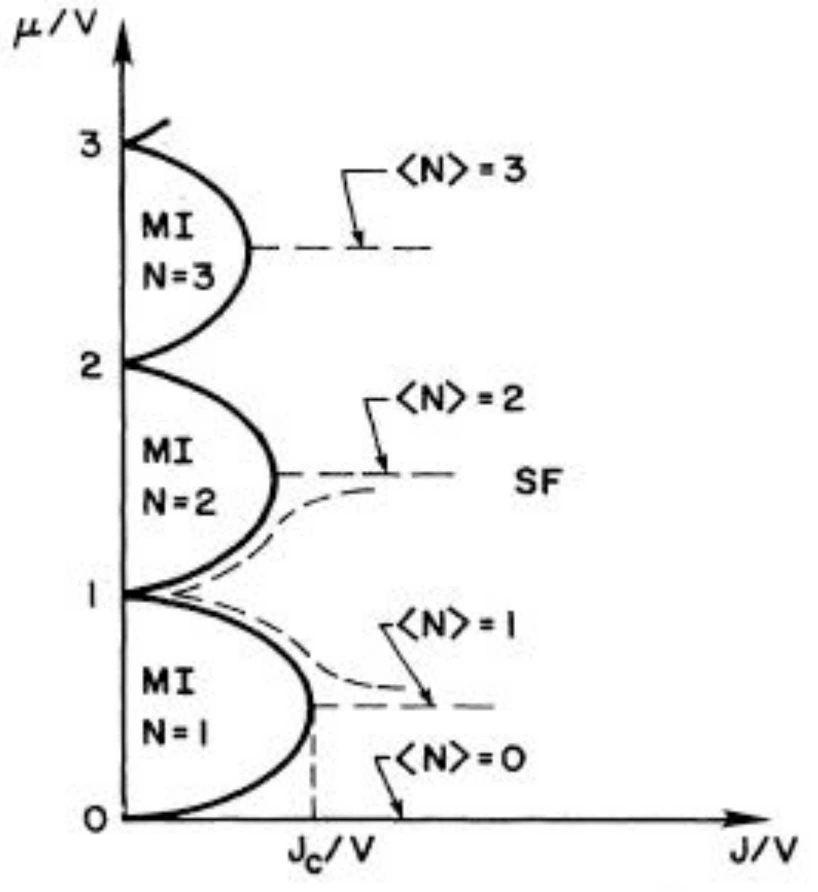


Phase diagram

$\theta = 0$ → Bose-Hubbard model

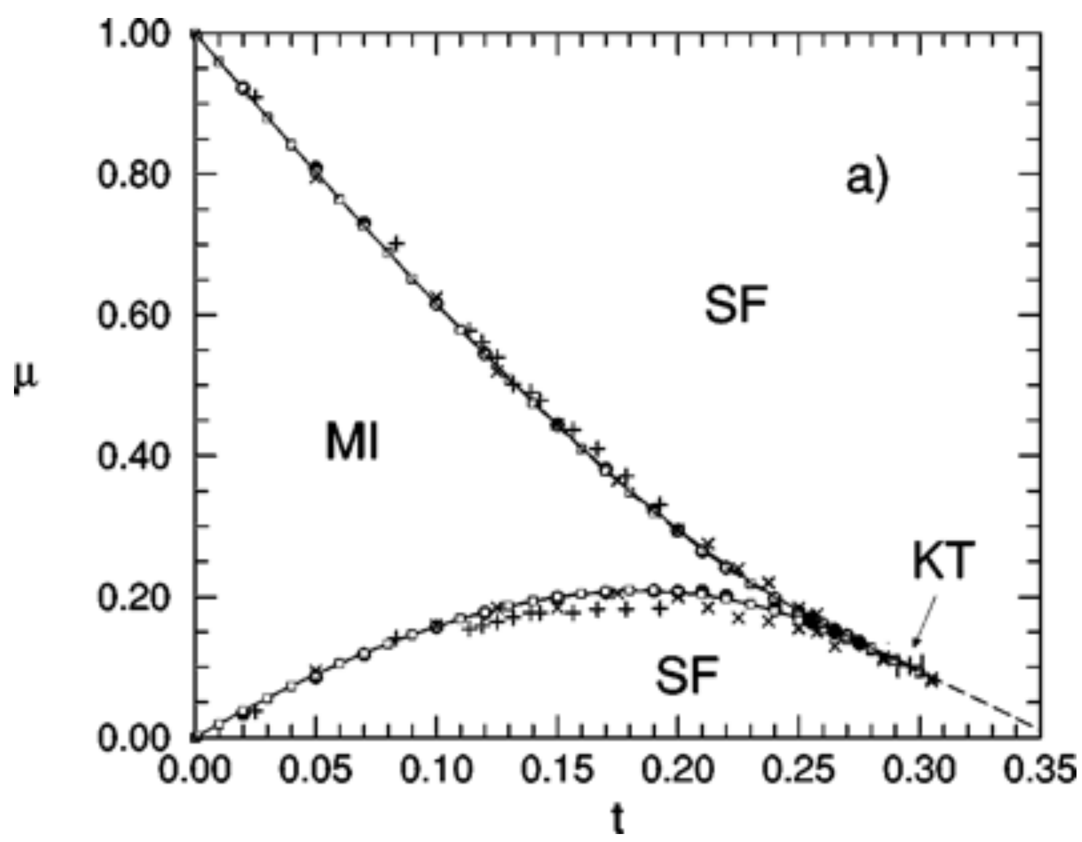
$$H = -J\sum_j \left(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j\right) + \frac{U}{2}\sum_j n_j(n_j-1) - \mu\sum_j n_j$$

Mean field



[M. P. A. Fisher et al., PRB **40** 546 (1989)]

Numerical (DMRG)



[Kühner, T. D., S. R. White, and H. Monien, PRB **61**, 12474 (2000)]

DMRG on the anyon-Hubbard

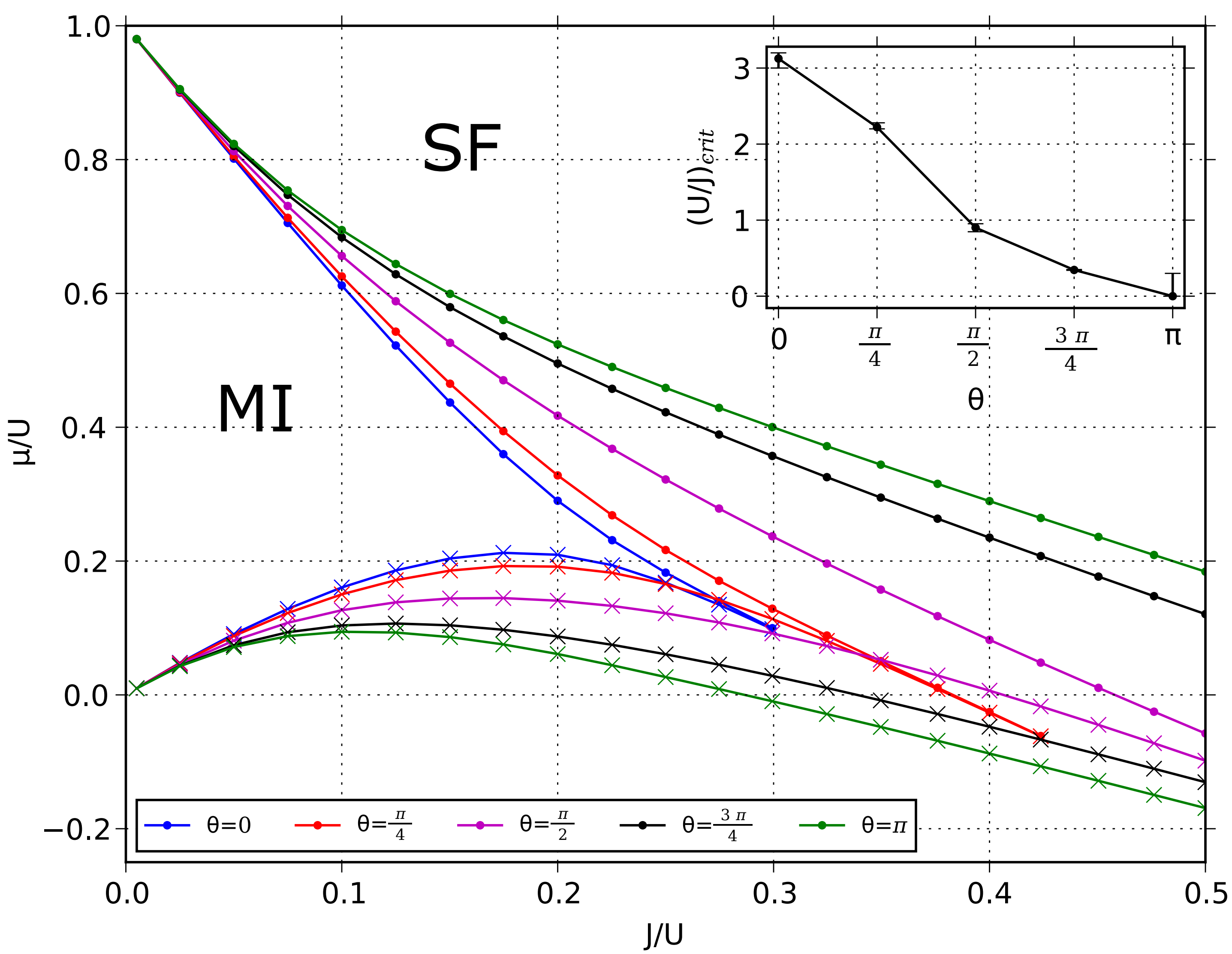
$E_0^{(n)}$ GS energy at filling n → $E_+^{(n)}$ +1 particle; $E_-^{(n)}$ - 1 particle

$n = N/L$
filling

MI-SF phase transition points $\mu_+^{(n)} = E_+^{(n)} - E_0^{(n)}$, $\mu_-^{(n)} = E_0^{(n)} - E_-^{(n)}$,

Extended
Mott lobes

MI-SF phase
transition
with θ



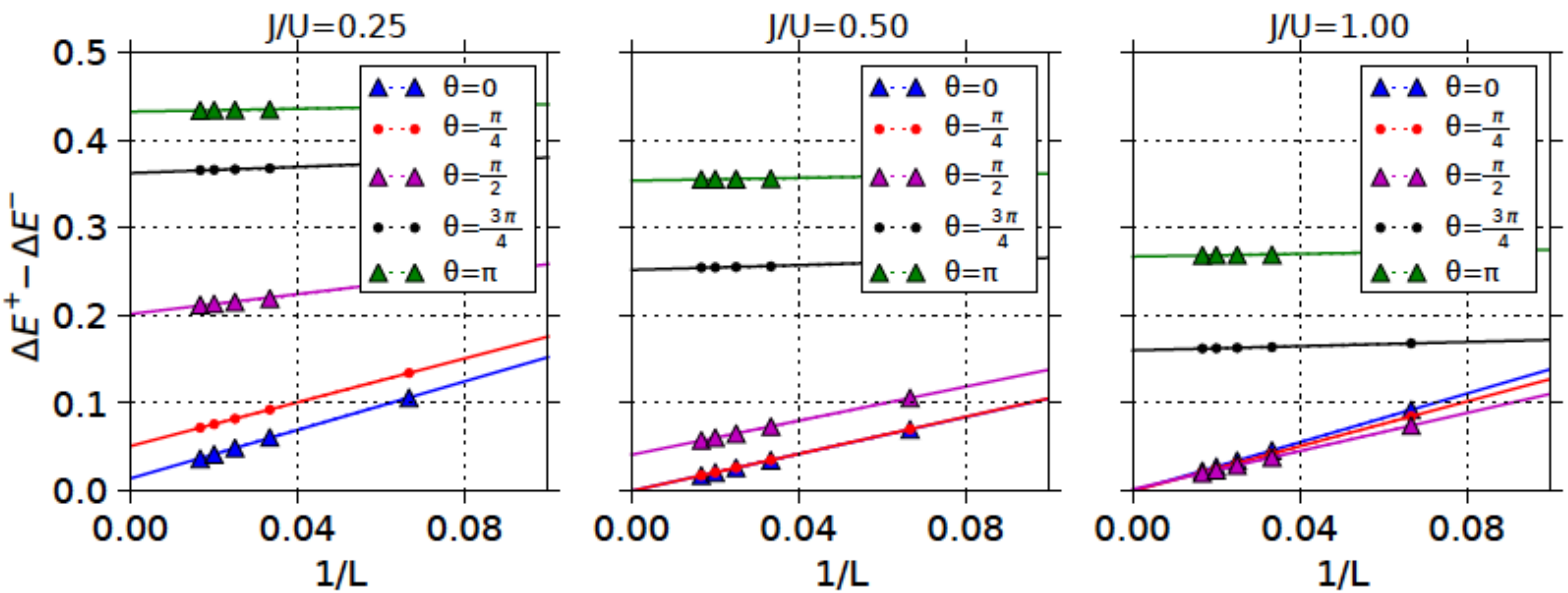
local basis
truncation

$n_j \leq 3$

Finite size scaling

The gap are calculated at different lattice sizes   L = 15, 30, 40, 50, 60

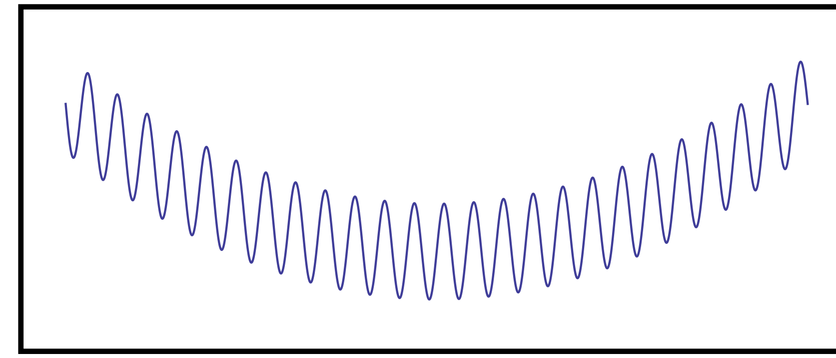
Then extrapolated to $L \to \infty$

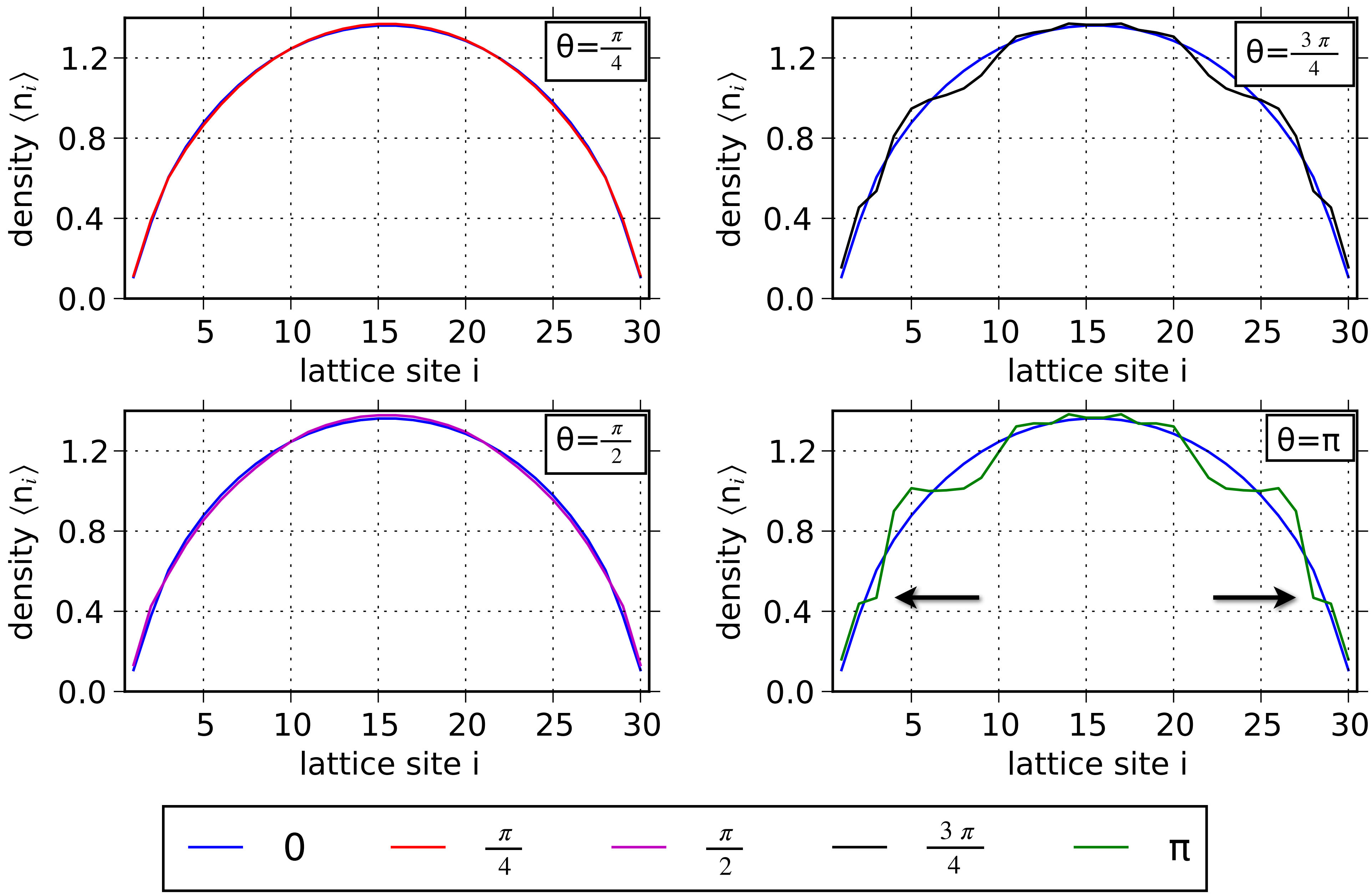


The gaps go to zero in the superfluid phase

Trap potential

$$H^b_{tr} = H^b + V \sum_i ((L+1)/2 - i)^2 n_i$$





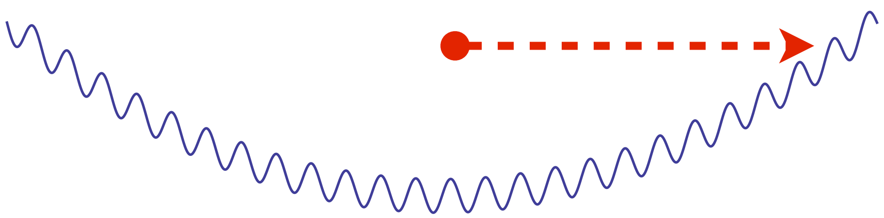
Appearance of fractional plateaus?

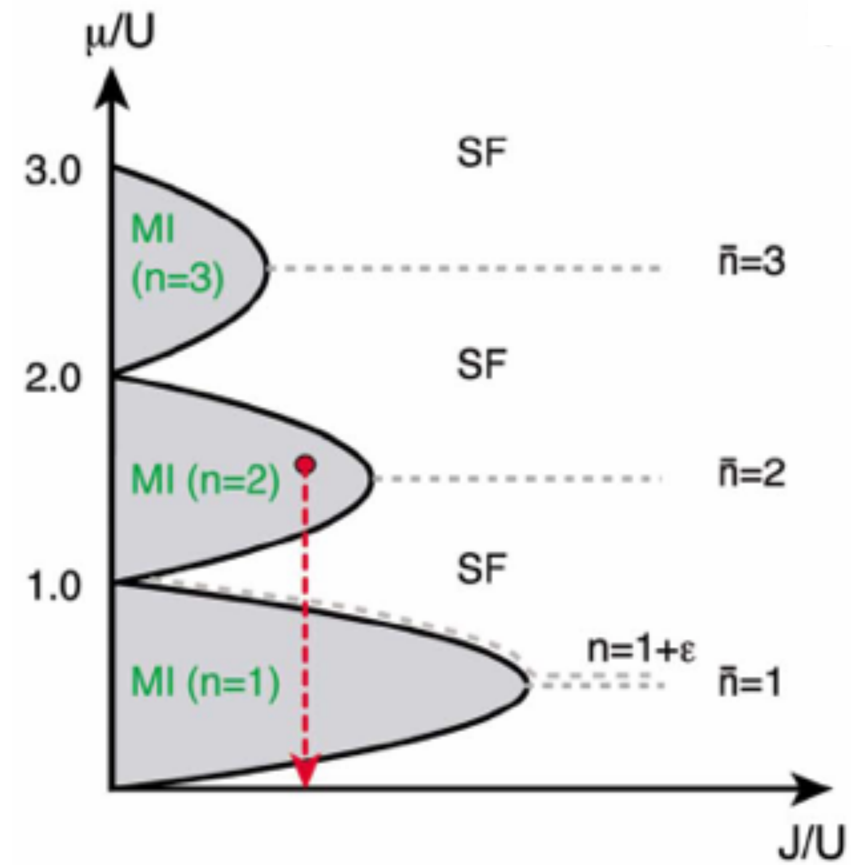
N=L=30, J/U=0.5, V/U=0.01

Local density approximation

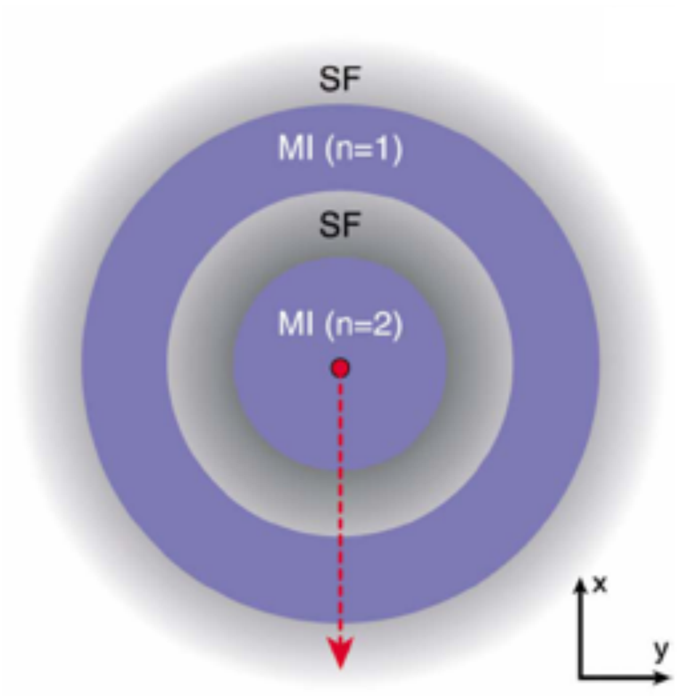
The trap is like a site-dependent chemical potential

Spatial correlations are neglected

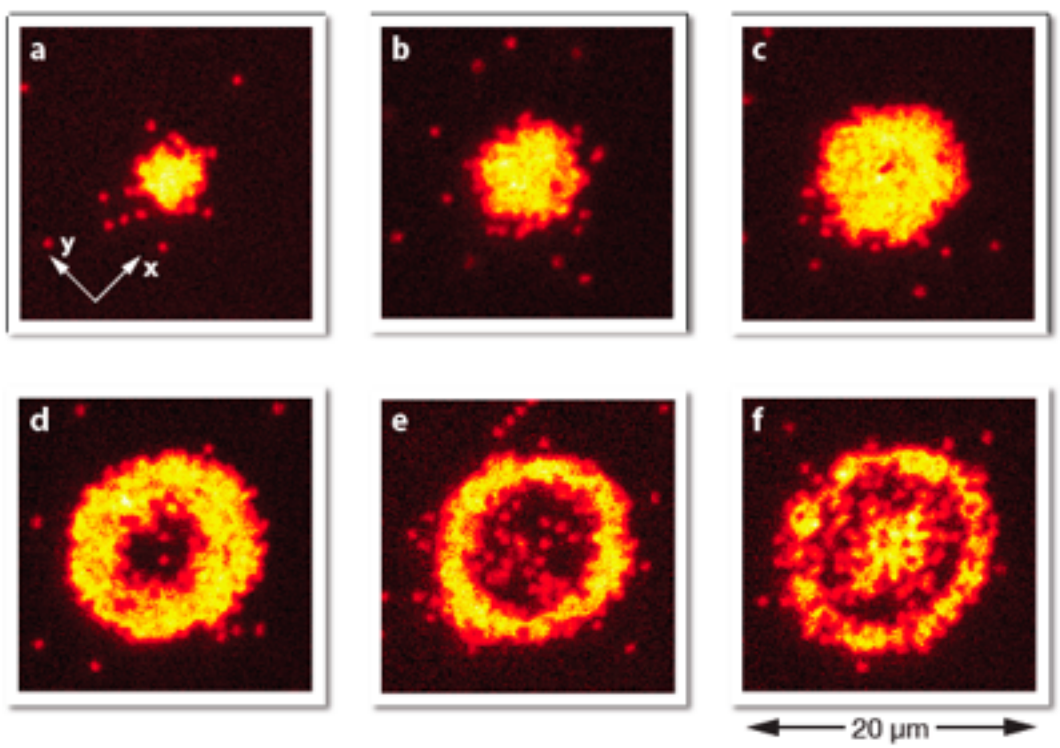




In 2D



wedding cake structure



In-situ imaging

[J.F. Sherson et al., Nature **467**, 68 (2010)]

Mean field solution (i)

$$H = \sum_j \left[\frac{1}{2} n_j (n_j - 1) - \mu n_j - J(c_j^\dagger b_{j+1} + b_{j+1}^\dagger c_j) \right]$$

anyon-Hubbard

$$\boxed{c_j = e^{-i\theta n_j} b_j}$$

$\boxed{\text{J=0}}$ $\quad |\Psi_0\rangle = |\psi\rangle^{\otimes L} \qquad |\psi\rangle = \sum_{\nu=0}^{\infty} c_\nu \frac{(b^\dagger)^\nu}{\sqrt{\nu!}} |0\rangle$ Gutzwiller

$\nu = N/L$
filling

local energies $\quad \epsilon(\nu) = \frac{1}{2}\nu(\nu - 1) - \mu\nu$

ground state $\Rightarrow$ ν particles in $\mu_-^{(\nu)} < \mu < \mu_+^{(\nu)}$ $\Rightarrow$ $\nu - 1 < \mu < \nu$

local gaps $\quad \epsilon(\nu + 1) - \epsilon(\nu) = \nu - \mu, \qquad \epsilon(\nu - 1) - \epsilon(\nu) = -(\nu - 1) + \mu$

[M. P. A. Fisher, et al., Phys. Rev. B 40 546 (1989)]

Mean field solution (ii)

$$c_j^\dagger b_{j+1} \approx -\alpha_2^* \alpha_1 + \alpha_2^* b_{j+1} + \alpha_1 c_j^\dagger$$ decoupling hopping term

$$\alpha_1 = \langle b_j \rangle \qquad \alpha_2 = \langle c_j \rangle$$ MF parameters

MF Hamiltonian $$H = \sum_j H_j + LJ(\alpha_2^* \alpha_1 + \alpha_1^* \alpha_2)$$

$$H_j = \frac{1}{2} n_j (n_j - 1) - \mu n_j - J(\alpha_2 b_j^\dagger + \alpha_2^* b_j + \alpha_1 c_j^\dagger + \alpha_1^* c_j)$$

trivial solution $\alpha_1 = \alpha_2 = 0$ MI phase

$\alpha_l \neq 0$ ➡ instability towards SF phase

Self-consistency map $$\boxed{\alpha_l = \Lambda_{ll'} \alpha_{l'}}$$

instability occurs when maximal eigenvalue of Λ is >1

Mean field solution (iii)

perturbative theory $H = H_0 + H_J$

$$|\psi\rangle = |\psi^{(0)}\rangle + |\psi^{(1)}\rangle \qquad |\psi^{(0)}\rangle = |\nu\rangle$$

$$|\psi^{(1)}\rangle = \sum_{\nu'} \frac{\langle\nu'|H_J|\nu\rangle}{\epsilon(\nu)-\epsilon(\nu')}|\nu'\rangle = J\frac{\sqrt{\nu}(\alpha_2^* + \alpha_1^* e^{-i\theta(\nu-1)})}{\mu-\nu+1}|\nu-1\rangle + J\frac{\sqrt{\nu+1}(\alpha_2+\alpha_1 e^{i\theta\nu})}{\nu-\mu}|\nu+1\rangle$$

self-consistency relations $\alpha_1 = \langle\psi|b_j|\psi\rangle \qquad \alpha_2 = \langle\psi|c_j|\psi\rangle$ give

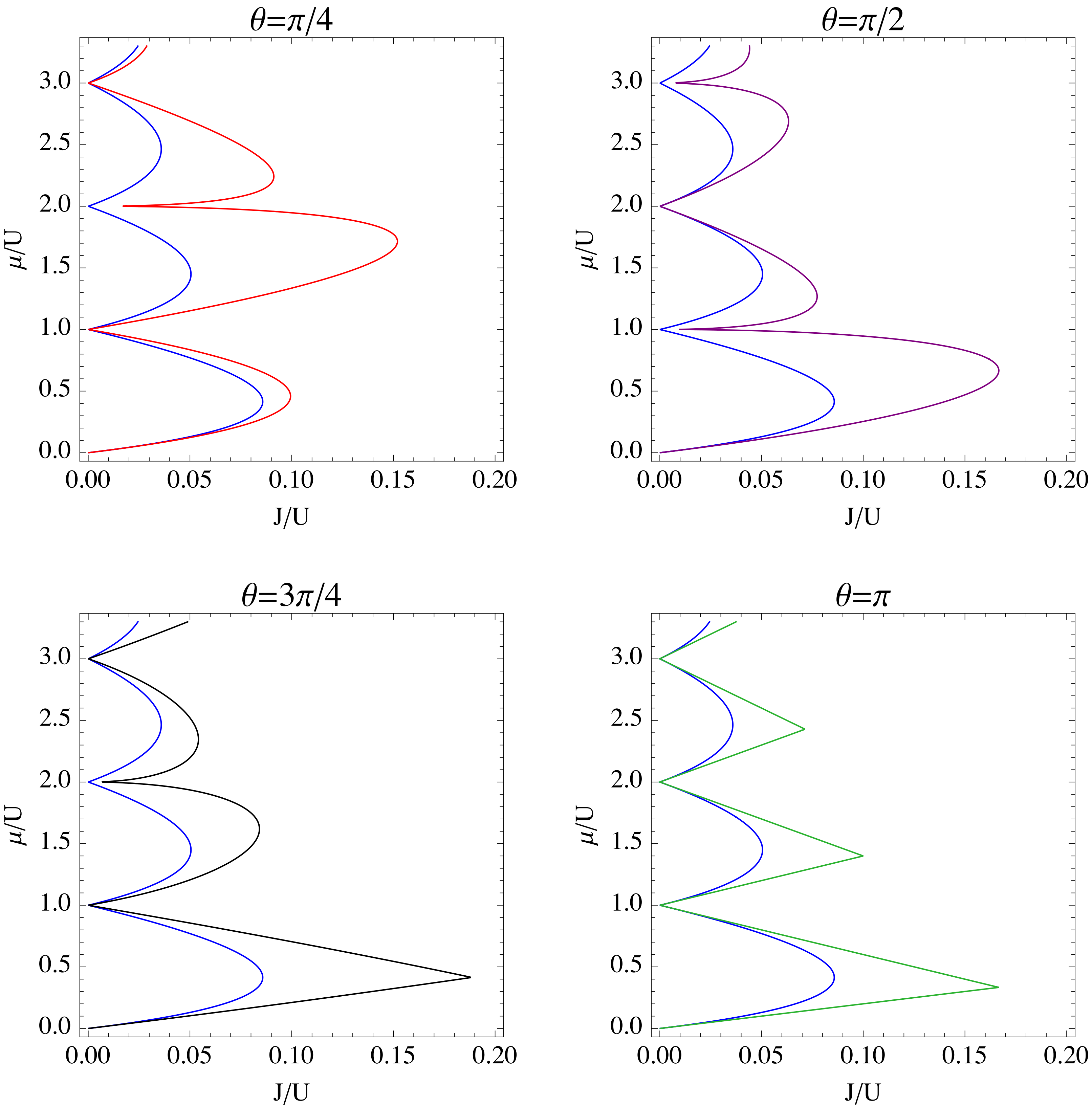
$$\Lambda = J\begin{pmatrix} f(\theta) & A \\ A & f(-\theta) \end{pmatrix}, \qquad f(\theta) = e^{i\theta\nu}\left[A + (e^{-i\theta}-1)B\right]$$

$$A = \frac{\mu+1}{(\mu-[\mu])([\mu]-\mu+1)} \qquad B = \frac{[\mu]+1}{\mu-[\mu]}$$

lobe labeling
$\nu = [\mu]+1$

eigenvalues $$\lambda_\pm = \frac{J}{2}\left[f(\theta)+f(-\theta) \pm \sqrt{4A^2 + (f(\theta)-f(-\theta))^2}\right]$$

Mean field solution (iv)



Bosons in the 1D continuum

$$g_{1\mathrm{D}} = \hbar^2 c/m$$

Interacting gas in 1D

$$H = \frac{\hbar^2}{2m} \int_0^L \mathrm{d}x\, \partial\Psi^\dagger(x)\partial\Psi(x) + \frac{1}{2}\, g_{1\mathrm{D}} \int_0^L \mathrm{d}x\, \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)$$

Dimensionless coupling $\gamma = c/n$

Bosons ⇨ Lieb-Liniger (Bethe-Ansatz)

For $\gamma \gg 1$ ⇨ Tonks-Girardeau gas

$$\Psi_B(x_1, \ldots, x_N) = \prod_{i<j} |\sin[\pi(x_j - x_i)/L]|$$

Anyons in the 1D continuum

Anyons

$$\Psi(x_1)\Psi^\dagger(x_2) = \mathrm{e}^{-\mathrm{i}\kappa w(x_1,x_2)}\Psi^\dagger(x_2)\Psi(x_1) + \delta(x_1 - x_2)$$
$$\Psi(x_1)\Psi(x_2) = \mathrm{e}^{\mathrm{i}\kappa w(x_1,x_2)}\Psi(x_2)\Psi(x_1),$$
$$\Psi^\dagger(x_1)\Psi^\dagger(x_2) = \mathrm{e}^{\mathrm{i}\kappa w(x_1,x_2)}\Psi^\dagger(x_2)\Psi^\dagger(x_1).$$

$$\begin{cases} w(x_1,x_2) = -w(x_2,x_1) = 1 & \text{for} \quad x_1 > x_2, \\ w(x,x) = 0 \end{cases}$$

$$\kappa = 0, \pi$$

Bosons

Pseudo-fermions

Bethe Ansatz

$$|\Phi\rangle = \int_0^L \mathrm{d}x^N \exp\left(-\mathrm{i}\frac{\kappa N}{2}\right)\chi(x_1\cdots x_N)\Psi^\dagger(x_1)\cdots\Psi^\dagger(x_N)|0\rangle$$

$$\chi(x_1\cdots x_N) = \exp\left(-\frac{\mathrm{i}\kappa}{2}\sum_{x_i<x_j}^N w(x_i,x_j)\right)\sum_P A(k_{P1}\cdots k_{PN})\mathrm{e}^{\mathrm{i}(k_{P1}x_1+\ldots+k_{PN}x_N)}$$

M. T. Batchelor , X.-W. Guan. and J.-S. He, JSTAT P03007 (2007)

Relations between coefficients

$$A(\dots k_j, k_i \dots) = \frac{k_j - k_i + \mathrm{i}c'}{k_j - k_i - \mathrm{i}c'} A(\dots k_i, k_j \dots)$$

Bethe-Ansatz equations

$$\mathrm{e}^{\mathrm{i}k_j L} = -\mathrm{e}^{\mathrm{i}\kappa(N-1)} \prod_{\ell=1}^{N} \frac{k_j - k_\ell + \mathrm{i}\,c'}{k_j - k_\ell - \mathrm{i}\,c'}$$

$$E = \sum_{j=1}^{N} k_j^2$$

Effective interaction

$$\boxed{c' = c/\cos(\kappa/2)}$$

$0 \le \kappa \le \pi$ repulsive

$\pi \le \kappa \le 2\pi$ attractive

TG gas limit $\gamma \gg 1$

Energy $$E_0/L \approx \frac{1}{3} n^3 \pi^2 \left(1 - \frac{4\cos(\kappa/2)}{\gamma}\right),$$

$$\mu \approx n^2 \pi^2 \left(1 - \frac{16\cos(\kappa/2)}{3\gamma}\right),$$ Chem. pot.

Pressure $$P_0 \approx \frac{2}{3} n^3 \pi^2 \left(1 - \frac{6\cos(\kappa/2)}{\gamma}\right),$$

$$Q \approx n\pi \left(1 - \frac{2\cos(\kappa/2)}{\gamma}\right).$$ Cut-off momentum

Thank you