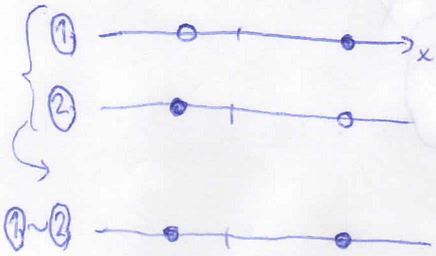
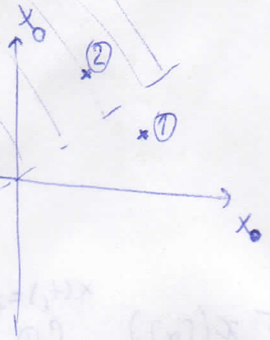
Configuration spaces of identical particles

2 particles in $\mathbb{R}^1$:

Leinaas, Myrheim (1977)





Config. space of N particles (in D dimensions)

- distinguishable: $(\mathbb{R}^D)^N \setminus \Delta$
- indistinguishable (identical): $\left[(\mathbb{R}^D)^N \setminus \Delta\right] / S_N$

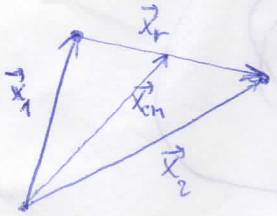
where coincidence set $\Delta = \{(\vec{x}_1, \dots, \vec{x}_N) \mid \vec{x}_i = \vec{x}_j$ for some $i \neq \dots$

, permutations $\pi \in S_N$

identification of elements $(\vec{x}_1, \dots, \vec{x}_N) \sim (\vec{x}_{\pi(1)}, \dots, \vec{x} \dots$

indistinguishable particles from now on

Two particles in D dimensions



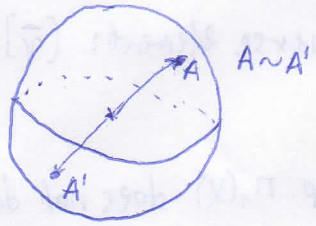
$$\vec{x}_{cm} = \frac{\vec{x}_1 + \vec{x}_2}{2}$$

$$\vec{x}_r = \vec{x}_2 - \vec{x}_1 \quad \rightarrow \Delta = \{(\vec{x}_{cm}, \vec{x}_r) \mid \vec{x}_r \neq \vec{0}\}$$

config. space: $\left[(\underset{\vec{x}_1}{\mathbb{R}^D} \times \underset{\vec{x}_2}{\mathbb{R}^D}) \setminus \Delta\right] / S_2 = \underbrace{\mathbb{R}^D}_{\vec{x}_{cm}} \times \overbrace{(0, +\infty) \times (S^{D-1}/S_2)}^{\vec{x}_r}$

S^{D-1}/S_2: sphere with antipodal identification (or projective space $\mathbb{R}P^{D-1}$)

nontrivial topology!



Path integration in multiply connected spaces

X simply connected

X multiply (non-simply) connected

$$\langle x_b, t_b | x_a, t_a \rangle = \sum_{[\gamma] \in \pi_1(X)} \chi([\gamma]) \int_{\substack{x(t_a)=x_a \\ x(t) \in [\gamma]}}^{x(t_b)=x_b} \mathcal{D}x(t)\; e^{\frac{i}{\hbar} S[x(t)]}$$

$\pi_1(X)$.. fundamental group of the configuration space X

$\chi(\cdot)$.. unitary representation of $\pi_1(X)$

($=1$ for bosons, ± 1 for fermions, $e^{i\alpha n}$ for anyons, ...)

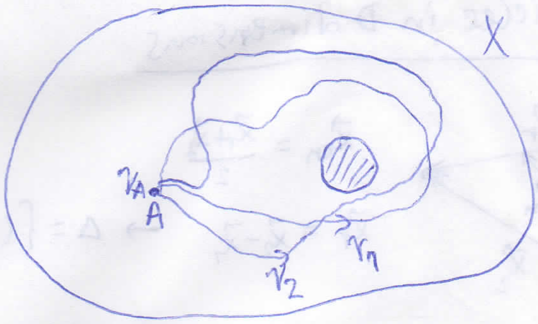
Fundamental group - an introduction

X.. connected topological space

γ.. closed oriented loop in X

$\gamma_1 \overset{\text{homotopic to}}{\sim} \gamma_2 \Leftrightarrow \gamma_2$ = continuous deformation (homotopy) of γ_1

$\downarrow$



eg.: $\gamma_1 \sim \gamma_2$, but $\gamma_1 \not\sim \gamma_A$
(i.e. $[\gamma_1] = [\gamma_2]$, but $[\gamma_1] \neq [\gamma_A]$)

equivalence classes $[\gamma] \in \pi_1(X)$.. fundamental group of X

operation: $\gamma_1 * \gamma_2$ .. γ_1 followed by γ_2

identity element: $[\gamma_A]$ .. $\overset{\gamma_A}{\dot{A}}$

inverse element: $[\bar{\gamma}] = [\gamma]^{-1}$

opposite orientation

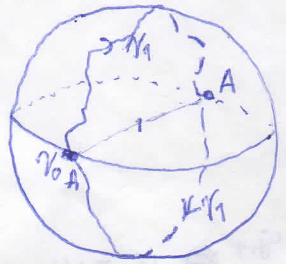
group $\pi_1(X)$ does not depend on the choice of point A.

Topology of two-particle configuration space and representations of the fundamental group

$$\underbrace{\underbrace{\mathbb{R}^D}_{\vec{x}_{cm}} \times \underbrace{(0,+\infty)}_{|\vec{x}_r|}}_{\substack{\text{simply connected}\\ \text{(trivial topology)}}} \times \underbrace{S^{D-1}/S_2}_{\text{angular part of } \vec{x}_r=\vec{x}_2-\vec{x}_1}$$

$\to \pi_1(S^{D-1}/S_2) = ?$:

$D \geq 3$



$\gamma_1 \not\sim \gamma_0$

$\gamma_1 * \gamma_1 = \gamma_0$

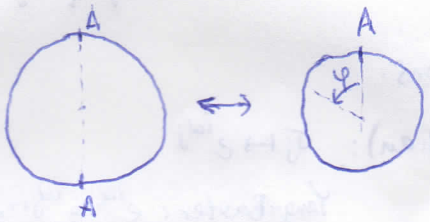
$\pi_1 = \mathbb{Z}_2 = (\{0,1\}, + \bmod 2)$

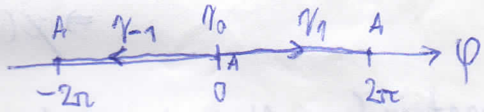
representations (scalar unitary)

1) $\chi_{BOS}(0) = \chi_{BOS}(1) = +1$

2) $\chi_{FER}(0) = +1$, $\chi_{FER}(1) = -1$

$D=2$





$\pi_1 = \mathbb{Z} = (\mathbb{Z}, +)$

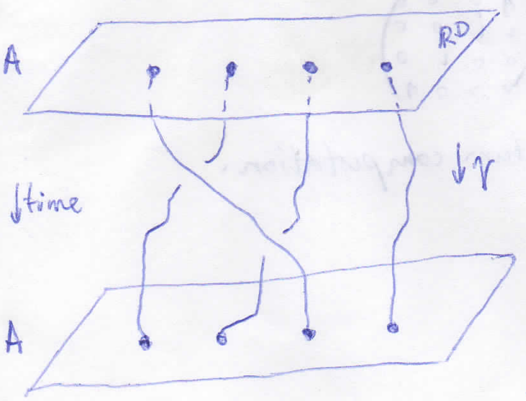
repres.: $\chi_{ANY}(n) = e^{i\alpha n}$, $\alpha \in \mathbb{R}$ arbitrary

$D=1$

$\dot{A}$ $\dot{A}$

π_1 trivial

N particles in D dimensions



inequal permutations are not homotopic

$\pi \neq \pi' \in S_N \quad \Rightarrow [\gamma_\pi] \neq [\gamma_{\pi'}] \quad (\gamma_\pi \not\sim \gamma_{\pi'})$

way of crossing:

in D=2 matters $\not\sim$

in D≥3 does not matter $\sim$ ("colour dimension")

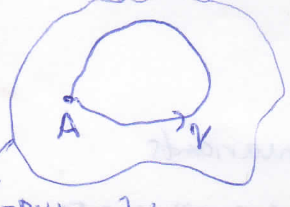
D≥3: π_1 = permutation group S_N

scalar repres.: $\chi_{BOS}([\gamma_\pi]) = 1$, $\chi_{FER}([\gamma_\pi]) = \operatorname{sgn} \pi$

matrix repres. ⇒ parastatistics

D=2: π_1 = braid group B_N

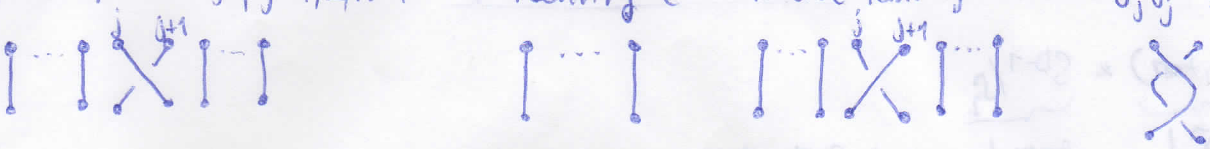
schematically: $\updownarrow$



$[(\mathbb{R}^D)^N \setminus \Delta]/S_N$

Braid group and its representations

generators: σ_j, $j = 1, \dots, N-1$ identity e inverse elem. σ_j^{-1} $\sigma_j \sigma_j^{-1} = e$



group operation: concatenation (~~staggering~~)

relations: 1) $\sigma_j \sigma_k = \sigma_k \sigma_j$ if $|j-k| \geq 2$ 2) (Yang-Baxter) $\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}$

unitary representations:

/ scalar (Abelian): $\sigma_j \mapsto e^{i\alpha_j}$

Yang-Baxter: $e^{i\alpha_j} e^{i\alpha_{j+1}} e^{i\alpha_j} = e^{i\alpha_{j+1}} e^{i\alpha_j} e^{i\alpha_{j+1}} \Rightarrow \alpha_j = \alpha_{j+1} \; \forall j$

$\Rightarrow \chi(\sigma_j) = e^{i\alpha}$, $\alpha \in \mathbb{R}$

\ matrix (non-Abelian) example: $\sigma_j \mapsto B_j$, $m \equiv \frac{N}{2} - 1$ (N even) $\leftarrow SU(2)_2$ anyons

$B_1 = R \otimes \underbrace{\mathbb{1}_2 \otimes \dots \otimes \mathbb{1}_2}_{m-1}$, $B_{N-1} = \mathbb{1}_2 \otimes \dots \otimes \mathbb{1}_2 \otimes R$ ($2^m \times 2^m$ matrices)

$B_{2k} = \mathbb{1}_2 \otimes \dots \otimes \mathbb{1}_2 \otimes \underset{\uparrow k}{B} \otimes \mathbb{1}_2 \otimes \dots \otimes \mathbb{1}_2$, $B_{2k+1} = \mathbb{1}_2 \otimes \dots \otimes \mathbb{1}_2 \otimes \underset{\uparrow k}{A} \otimes \mathbb{1}_2 \otimes \dots \otimes \mathbb{1}$

where: $R = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, $B = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Non-Abelian anyons could be used for quantum computation.
(See the lectures of Jiannis Pachos.)

Fragments from the knot theory

braid closure $\langle \dots \rangle$  $\rightarrow$ knot or link

e.g.: $\langle \sigma_1^2 \rangle$ ... Hopf link

$\langle \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \rangle$ ... Borromean rings

Traces of products of braid matrices are related to knot or link invariants

$\rightarrow$ Jones polynomial, Kauffman bracket 1) $\langle \bigcirc \rangle = 1$ 2) $\langle \bigcirc \cup L \rangle = (-A^2 - A^{-2}) \langle L \rangle$ 3) …