

Bose-Einstein Condensates in Toroidal Traps with Disorder

Bachelor Thesis by

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1 Introduction

Bose-Einstein condensation is a macroscopic quantum phenomenon occurring in Bose gases at very low temperatures. At a temperature scale close to absolute zero, where the de-Broglie wavelength of the particles is of the same order as the average distance between two particles, the ground state is occupied by a large fraction of the bosons. In this state, the entire many-body system of all the particles in the condensate can be described by a single wavefunction.

Bose-Einstein condensation was predicted theoretically in 1925 by Albert Einstein [1] based on work by Satyendranath Bose [2] on the statistical properties of indistinguishable bosons. Bose-Einstein condensates were produced and observed for the first time 70 years after their theoretical prediction in 1995 in rubidium-87 and sodium vapors by the research groups of Wolfgang Ketterle at MIT and Eric A. Cornell and Carl E. Wieman at JILA [3, 4]. Their discovery was rewarded with the Nobel prize in physics of 2001. Today, research on Bose-Einstein condensates is a very active field in physics.

Bose-Einstein condensates are closely related to the phenomenon of superfluidity, where the fluid loses its viscosity and can flow freely without any friction. This allows a superfluid to move unhindered through extremely fine cracks or capillaries. Macroscopic bodies can move through a superfluid without creating excited states in the fluid, as long as they move slower than a critical velocity predicted by Landau. At faster velocities, the body can cause excitations in the superfluid, leading to a breakdown of the non-dissipative flow. Superfluidity was observed in liquid helium by Pyotr L. Kapitsa, John F. Allen and Don Misener in 1937 [5].

In particular, recent years have shown a great interest in the observation of Bose-Einstein condensates in ring traps. This is motivated on the one hand by the possibility to gain a deeper understanding of superfluidity, e.g. through observation of a stable persistent superfluid flow [6] around the ring trap and on the other hand by applications such as the prospect of using Bose-Einstein condensates on a ring in interferometers and compact gyroscopes [7].

Another great research effort has been undertaken to solve the "dirty boson problem", where the Bose-Einstein condensate is subjected to a random continuous potential that is unknown and only described through its statistical properties. An important result was derived by Kerson Huang and Hsin-Fei Meng, who found that the density of the Bose-Einstein condensate in a weak disorder potential is depleted due to localization effects [8]. This result has been generalized for strong disorder potentials in Ref. [9] and for slowly moving condensates in Ref. [10].

The aim of this thesis is to determine how the presence of disorder in a toroidal trap affects the motion of a Bose-Einstein condensate around the ring. Intuitively, one expects that the motion of the Bose-Einstein condensate is hindered by the presence of disorder due to the formation of fragmented condensate droplets in local minima of the disorder potential and a resulting depletion of the superfluid density. Specifically, in analogy to the Landau critical velocity, we seek to investigate how fast the condensate can move relative to the disorder before the flow of the superfluid around the ring breaks down.

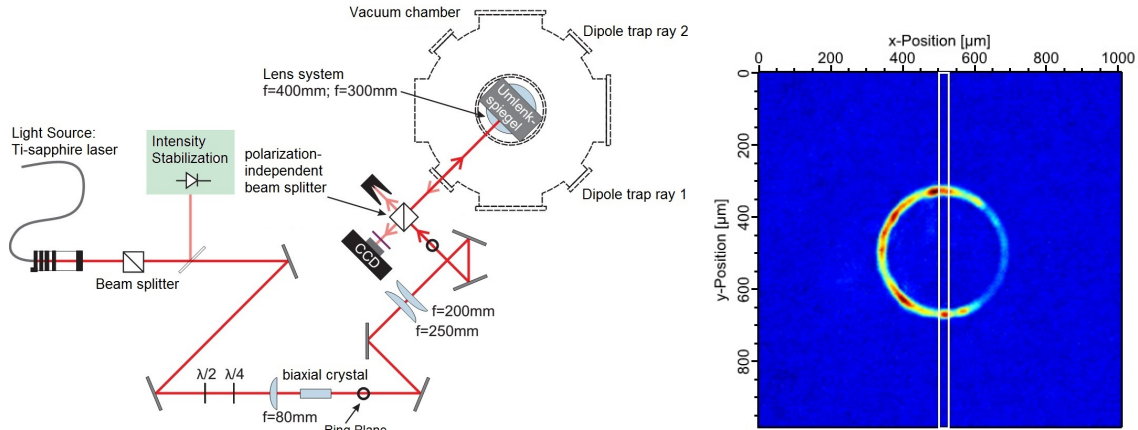


Figure 1: Ring trap using conical refraction in a biaxial crystal. Left: Experimental setup of such a ring trap. Translated and adapted from Ref. [12, p. 89]. Right: Density distribution of atoms loaded in the trap [12, p. 100].

1.1 Experimental Realizations of Toroidal Traps

We first briefly discuss some possible experimental realizations of a ring trap. There have been numerous methods of generating toroidal traps using optical potentials [11–16], magnetic fields [17–21] as well as combinations of both [6].

Optical Traps Contrary to the trapping of charged particles, electrostatic forces cannot be used to contain a gas of neutral atoms. Optical traps for neutral atoms therefore work with oscillating electrical dipole fields, usually generated by a laser. It can be shown that such an optical dipole field attracts a neutral atom when its frequency is below the atom’s resonance frequency (red detuning) and repels a neutral atom when its frequency is above the atom’s resonance frequency (blue detuning) [22].

Ring-shaped electrical dipole potentials have been realized with red-detuned Laguerre-Gaussian modes, which have a ring-shaped intensity profile [13].

More recently, another method to generate ring-traps uses the effect by which an unpolarized light beam that is lined up with one of the optic axes in an anisotropic biaxial crystal spreads as a cone [12]. Due to the spatially extended incoming light beam, a cross-section of this cone consists of two concentric rings, whose relative brightness can be tuned. The atoms can be trapped in the plane of such a cross-section. It is thereby possible to trap atoms either in a single red-detuned ring by making one ring much more intense than the other or in a potential minimum in between two blue-detuned rings. A schematic of the experimental apparatus for this type of trap and the density distribution of the loaded trap are shown in Figure 1.

A third possibility is the use of time-averaged optical potential, wherein a laser rapidly scans a raster of space points with varying intensity. If this scanning process is performed sufficiently fast, the atoms experience an effective time-averaged potential. With this technique, a variety of trap geometries including rings can be produced and even dynamically reshaped [16, 23].

Magnetic Traps The other major method to create ring traps involves magnetic fields. Historically, the first Bose-Einstein condensate observed in sodium atoms was trapped in a magnetic quadrupole that was "plugged" by shining a blue-detuned laser through its middle, thereby effectively leaving a toroidal configuration [3]. This technique of creating a repulsive optical potential at the center of a harmonic magnetic trap since has been used to create ring-traps for Bose-Einstein condensates [6]. As with optical traps, magnetic toroidal traps can also be generated through time-averaged potentials [18].

1.2 Fourier Series

We treat the BEC in a ring trap as a one-dimensional problem with periodic boundary conditions, where the major circumference of the torus is given by the length L . Due to the periodicity of this configuration, we will assume that all potentials and wavefunctions are periodic. We can therefore frequently use an ansatz in form of a discrete Fourier series with the definition

$$F(x) = \sum_{n=-\infty}^{\infty} f_n e^{ik_n x}, \quad (1)$$

where the respective Fourier coefficients are given by

$$f_n = \frac{1}{L} \int_{-L/2}^{L/2} dx F(x) e^{-ik_n x} \quad (2)$$

for some function $F(x)$, with the wavenumber defined as

$$k_n = \frac{2\pi}{L} n. \quad (3)$$

1.3 Disorder Potential

A disorder potential is introduced as a continuous, randomly fluctuating potential. We are not interested in the exact shape of the disorder realization, but require that it has the statistical properties

$$\langle U(x) \rangle = 0 \quad (4)$$

$$\langle U(x)U(x') \rangle = R(x - x'), \quad (5)$$

where $\langle \dots \rangle$ is the average value over an ensemble of many disorder realizations and $R(x)$ is the correlation function of the disorder. We will assume time-independence of the disorder and require that $R(x)$ is an even function, such that $R(-x) = R(x)$. It follows that the Fourier coefficients r_n have a corresponding symmetry such that $r_n = r_{-n}$. Furthermore, the disorder correlation is assumed to be a real-valued function, such that $r_{-n} = r_n^*$. From these relations, we immediately deduce that the Fourier coefficients are real valued.

Disorder potentials have been realized experimentally through laser speckles [24, 25] and in magnetic waveguides [26], where they occur naturally due to imperfections and

surface defects in the conductor.

In later calculations, we will require the disorder average of the product of two Fourier coefficients $\langle u_a u_b \rangle$:

$$\langle u_a u_b \rangle = L^{-2} \int_{-L/2}^{L/2} dx' \int_{-L/2}^{L/2} dx e^{-i(k_a x + k_b x')} \langle U(x) U(x') \rangle \quad (6)$$

$$= L^{-2} \int_{-L/2}^{L/2} dx' \int_{-L/2}^{L/2} dx e^{-i(k_a x + k_b x')} R(x - x') \quad (7)$$

$$= L^{-2} \int_{-L/2}^{L/2} dx e^{-ik_a x} \int_{-L/2}^{L/2} dx' e^{-ik_b x'} \left(\sum_{l=-\infty}^{\infty} r_l e^{ik_l(x-x')} \right) \quad (8)$$

$$= L^{-2} \sum_{l=-\infty}^{\infty} r_l \int_{-L/2}^{L/2} dx e^{i(k_l - k_a)x} \int_{-L/2}^{L/2} dx' e^{-i(k_b + k_l)x'} \quad (9)$$

$$= L^{-1} \sum_{l=-\infty}^{\infty} r_l \int_{-L/2}^{L/2} dx e^{i(k_l - k_a)x} \delta_{-b, l} = L^{-1} r_{-b} \int_{-L/2}^{L/2} dx e^{-i(k_a + k_b)x} = r_a \delta_{a, -b}. \quad (10)$$

1.4 Gross-Pitaevskii Equation

A dilute Bose-Einstein condensate at zero temperature can be described in a mean-field approximation by a Schrödinger equation with a nonlinear term for a two-particle interaction, called the Gross-Pitaevskii equation (GPE)[27, p. 124 ff]. The time-dependent Gross-Pitaevskii equation is

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x) + \int dx' V(x - x') \Psi^*(x', t) \Psi(x', t) \right) \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}, \quad (11)$$

where $U(x)$ is a term for an external potential and $V(x' - x)$ is the two-particle interaction, which is required to be an even function. In this thesis, $U(x)$ is a disorder potential with the properties described in section 1.3. Using a plane-wave ansatz

$$\Psi(x, t) = e^{-\frac{i\mu}{\hbar} t} \psi(x) \quad (12)$$

to separate the time dependence into a phase factor yields the time-independent Gross-Pitaevskii equation

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x) - \mu + \int dx' V(x - x') \psi^*(x') \psi(x') \right) \psi(x) = 0. \quad (13)$$

The solution $\Psi(x, t)$ of the GPE describes the entire condensate rather than one particle. Unlike the Schrödinger equation, it is therefore normalized to the number of particles

N in the condensate according to

$$N = \int dx \Psi^*(x, t) \Psi(x, t). \quad (14)$$

Due to this normalization, the quantity $\Psi^*(x, t) \Psi(x, t)$ is interpreted as a particle density.

1.5 Boosted Gross-Pitaevskii Equation

In this section, we derive a Gross-Pitaevskii equation describing a moving disorder potential. To this end, we will first consider a moving BEC in a ring trap with no disorder. We will then consider the case with disorder in a perturbative framework, where we formally introduce a moving disorder potential. We will then change coordinates to derive the appropriate GPE in the rest frame of the disorder. We obtain an equation which only depends on a relative velocity between the disorder and the superfluid.

Moving Solution of homogeneous GPE We will first derive a solution representing a moving BEC with no external potential, i.e. $U(x) = 0$. Given a solution $\Psi(x, t)$ of the time-dependent Gross-Pitaevskii equation (11) in the homogeneous case, i.e. without the potential $U(x)$, the function

$$\tilde{\Psi}(x, t) = \Psi(x - vt) e^{\frac{i}{\hbar} \left(mvx - \frac{mv^2}{2} t \right)} \quad (15)$$

is also a solution.

As an intermediate step to show this, we calculate the derivatives of $\tilde{\Psi}$:

$$\frac{\partial}{\partial t} \tilde{\Psi}(x, t) = \left[\frac{\partial}{\partial t} \Psi(x - vt, t) - i \frac{mv^2}{2\hbar} \Psi(x - vt, t) \right] e^{\frac{i}{\hbar} \left(mvx - \frac{mv^2}{2} t \right)} \quad (16)$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \tilde{\Psi}(x, t) = & \left[\frac{\partial^2}{\partial x^2} \Psi(x - vt, t) - \left(\frac{mv}{\hbar} \right)^2 \Psi(x - vt, t) \right. \\ & \left. + 2i \frac{mv}{\hbar} \left(\frac{\partial}{\partial x} \Psi(x - vt, t) \right) \right] e^{\frac{i}{\hbar} \left(mvx - \frac{mv^2}{2} t \right)} \end{aligned} \quad (17)$$

We then change coordinates to a new reference system (\tilde{x}, \tilde{t}) with

$$\tilde{x} = x - vt, \quad \tilde{t} = t, \quad \frac{\partial \Psi(\tilde{x}, t)}{\partial x} = \frac{\partial \Psi(\tilde{x}, t)}{\partial \tilde{x}}, \quad (18)$$

$$\frac{\partial \Psi(\tilde{x}, \tilde{t})}{\partial t} = \frac{\partial \Psi(\tilde{x}, \tilde{t})}{\partial \tilde{t}} + \frac{\partial \tilde{x}}{\partial \tilde{t}} \frac{\partial \Psi(\tilde{x}, t)}{\partial \tilde{x}} = \frac{\partial \Psi(\tilde{x}, \tilde{t})}{\partial \tilde{t}} - v \frac{\partial \Psi(\tilde{x}, t)}{\partial \tilde{x}}. \quad (19)$$

We then obtain

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \tilde{x}^2} \Psi(\tilde{x}, t) + \int d\tilde{x}' V(\tilde{x} - \tilde{x}') \Psi^*(\tilde{x}', \tilde{t}) \Psi(\tilde{x}', \tilde{t}) = i\hbar \frac{\partial \Psi(\tilde{x}, \tilde{t})}{\partial \tilde{t}}. \quad (20)$$

The terms proportional to v and v^2 emerging from the time- and space-derivatives cancel, leaving us with the original Gross Pitaevskii equation for $\Psi(\tilde{x}, \tilde{t})$ in the new coordinates, which is solved by Ψ by assumption.

Moving Disorder To consider the case of a moving disorder potential, we start with

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x - ut) + \int dx' V(x - x') \Psi^*(x', t) \Psi(x', t) \right] \Psi(x, t) = i\hbar \frac{\partial}{\partial t} \Psi(x, t), \quad (21)$$

which represents a Gross-Pitaevskii equation where we introduced a moving disorder potential. We search for a solution by perturbing the boosted homogeneous solution (15)

$$\Psi(x, t) = e^{\frac{i}{\hbar} \left[mvx - \left(\frac{mv^2}{2} + \mu \right) t \right]} \psi(x, t). \quad (22)$$

We now change to the rest frame of the disorder by introducing comoving coordinates

$$\tilde{x} = x - ut, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} - u \frac{\partial}{\partial \tilde{x}}, \quad (23)$$

Equation (21) then becomes

$$\begin{aligned} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \tilde{x}^2} + i\hbar u \frac{\partial}{\partial \tilde{x}} + U(\tilde{x}) + \int d\tilde{x}' V(\tilde{x} - \tilde{x}') \Psi^*(\tilde{x}' + ut, t) \Psi(\tilde{x}' + ut, t) \right] \Psi(\tilde{x} + ut, t) \\ = i\hbar \frac{\partial}{\partial t} \Psi(\tilde{x} + ut, t) \end{aligned} \quad (24)$$

Omitting the tilde for the new coordinate, the ansatz for the solution in the reference frame moving with the disorder becomes

$$\Psi(x, t) = e^{ik_S x} e^{-i \left[\frac{\hbar^2}{2m} (k_S^2 - 2k_U k_S) - \mu \right] t} \psi(x, t), \quad (25)$$

where we introduce wavenumbers k_U and k_S corresponding to disorder and superfluid velocity according to

$$k_U = \frac{m}{\hbar} u, \quad k_S = \frac{m}{\hbar} v. \quad (26)$$

Inserting this ansatz in equation (24), we get an equation for $\psi(x, t)$

$$\begin{aligned} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - i \frac{\hbar^2 K}{m} \frac{\partial}{\partial x} + U(x) - \mu + \int dx' V(x - x') \psi^*(x', t) \psi(x', t) \right] \psi(x, t) \\ = i\hbar \frac{\partial}{\partial t} \psi(x, t), \end{aligned} \quad (27)$$

where we have introduced a wavenumber K corresponding to the difference of the superfluid and the disorder velocity:

$$K = k_S - k_U \quad (28)$$

Using a similar argument as above, we can obtain a second equation for the complex conjugate ψ^* :

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + i \frac{\hbar^2 K}{m} \frac{\partial}{\partial x} + U(x) - \mu + \int dx' V(x-x') \psi^*(x,t) \psi(x,t) \right] \psi^*(x,t) = -i\hbar \frac{\partial}{\partial t} \psi^*(x,t) \quad (29)$$

2 Perturbative Approach

We now derive a perturbative solution of the Gross-Pitaevskii equations (27) and (29) representing a moving condensate in a trap with weak disorder, where the relative motion between the disorder and the condensate is given by K . To this end, we develop the wavefunction ψ from (25) in a power series in orders of the disorder strength

$$\psi(x) = \psi^{(0)} + \psi^{(1)} + \psi^{(2)} + \dots, \quad (30)$$

such that $\psi^{(i)} \propto U(x)^i$.

Since the Fourier transform of a derivative corresponds to a multiplication, we will solve the GPE in Fourier space by inserting the Fourier series defined in (1) into (27) and (29). The perturbative solution will, therefore, yield the Fourier coefficients of ψ up to first order in the disorder strength. Since we are not interested in any individual realizations of the disorder, we will directly derive disorder averages of the respective Fourier coefficients.

From here on, v_n does not represent the unperturbed superfluid velocity, but instead the Fourier coefficients of the interaction potential $V(x)$. Similarly, u_n represents the Fourier coefficients of the disorder potential $U(x)$ and not the disorder velocity.

2.1 Zeroth Order

Due to the homogeneity of the problem, we assume a constant zeroth order solution. Then, the derivative terms vanish and we are left with the algebraic equation

$$-\mu \psi^{(0)} + \left(\psi^{(0)} \right)^2 \int_{-L/2}^{L/2} dx' V(x-x') = 0, \quad (31)$$

which is solved by

$$\psi^{(0)} = \sqrt{\frac{\mu}{Lv_0}}. \quad (32)$$

The equation for $\psi^{*(0)}$ gives the same result.

2.2 Time-independence of Higher orders

We will now show that the solutions of equations (27) and (29) are time-independent. Assuming that ψ is time-independent up to i -th order, we will show that the higher order term $\psi^{(i+1)}$ is also time-independent, following an approach described in Ref. [28].

To this end, we consider the Fourier-transform of equations (27) and (29) by inserting the Fourier ansatz from (1) and additionally performing a Fourier transformation with respect to time.

Since the total order of the terms from the interaction must be equal to $i + 1$, we conclude that there will be three interesting, potentially time-dependent interaction terms:

$$L \left(\psi^{(0)} \right)^2 v_0 \psi_n^{(i+1)}, \quad L \left(\psi^{(0)} \right)^2 v_n \psi_n^{(i+1)}, \quad L \left(\psi^{(0)} \right)^2 v_n \psi_n^{*(i+1)}. \quad (33)$$

Here, we derived the Fourier transform of the interaction term, which has the form of a convolution. We obtain

$$\begin{aligned} \int_{-L/2}^{L/2} dx' V(x-x') \psi^{(i+1)}(x') &= \int_{-L/2}^{L/2} dx' \left(\sum_{n=-\infty}^{\infty} v_n e^{ik_n(x-x')} \right) \left(\sum_{m=-\infty}^{\infty} \psi_m^{(i+1)} e^{ik_m x'} \right) \\ &= \sum_{n=-\infty}^{\infty} v_n e^{ik_n x} \sum_{m=-\infty}^{\infty} \psi_m^{(i+1)} \int dx' e^{i(k_m - k_n)x'} = L \sum_{n=-\infty}^{\infty} v_n \psi_n^{(i+1)} e^{ik_n x}, \end{aligned} \quad (34)$$

and correspondingly

$$\begin{aligned} \int_{-L/2}^{L/2} dx' V(x-x') \psi^{(i+1)}(x) &= \int_{-L/2}^{L/2} dx' \left(\sum_{m=-\infty}^{\infty} v_m e^{ik_m(x-x')} \right) \left(\sum_{n=-\infty}^{\infty} \psi_n^{(i+1)} e^{ik_n x} \right) \\ &= \sum_{m=-\infty}^{\infty} v_m \int dx' e^{-ik_m x'} \sum_{n=-\infty}^{\infty} \psi_n^{(i+1)} e^{i(k_n + k_m)x} = L \sum_{n=-\infty}^{\infty} v_0 \psi_n^{(i+1)} e^{ik_n x}. \end{aligned} \quad (35)$$

The first of the terms in (33) cancels with the chemical potential according to (32). All other terms from the interaction will only involve lower orders of ψ , which are assumed as time-independent.

The disorder term can only introduce terms involving ψ^i . All disorder and interaction terms including ψ up to i -th order will be collected in an auxiliary quantity f , since they are time-independent by assumption and play no further role.

We then obtain the equation

$$\int_{-\infty}^{\infty} dt \left[\left(\epsilon_n + c_n + \frac{\hbar^2 k_n K}{m} \right) \psi_n^{(i+1)} + c_n \psi_n^{*(i+1)} - i\hbar \frac{\partial \psi^{(i+1)}}{\partial t} \right] e^{-i\omega t} \quad (36)$$

$$= f \left(\psi^{(0)}, \dots, \psi^{(i)} \right) \int_{-\infty}^{\infty} dt e^{-i\omega t} = 2\pi \delta(\omega) f \left(\psi^{(0)}, \psi^{(1)}, \dots, \psi^{(i)} \right) \quad (37)$$

where we use the abbreviations

$$\epsilon_n = \frac{\hbar^2 k_n^2}{2m} \quad (38)$$

and

$$c_n = L \left(\psi^{(0)} \right)^2 v_n. \quad (39)$$

Integrating the time derivative in (36) by parts, we get

$$\int_{-\infty}^{\infty} dt \left[(\epsilon_n + c_n + \frac{\hbar^2 k_n K}{m} - \hbar\omega) \psi_n^{(i+1)} + c_n \psi_n^{*(i+1)} \right] e^{-i\omega t} = 2\pi\delta(\omega) f(\psi^{(0)}, \psi^{(1)}, \dots, \psi^{(i)}). \quad (40)$$

Since the Fourier transform of a constant yields a delta function, we conclude that

$$\left(\epsilon_n + c_n + \frac{\hbar^2 k_n K}{m} - \hbar\omega \right) \psi_n^{(i+1)} + c_n \psi_n^{*(i+1)} = C \quad (41)$$

with some term C , which is constant in time.

With a similar argument for $\psi_n^{*(i+1)}$, we find that

$$\left(\epsilon_n + c_n - \frac{\hbar^2 k_n K}{m} + \hbar\omega \right) \psi_n^{*(i+1)} + c_n \psi_n^{(i+1)} = C' \quad (42)$$

with some other constant term C' .

By solving (42) for $\psi_n^{*(i+1)}$ and substituting it in (41), we find that $\psi_n^{(i+1)}$ and $\psi_n^{*(i+1)}$ are time-independent. Using $\psi^{(0)}$ as a base case, we find that ψ is time-independent in all orders of U and, therefore, derive $\psi(x)$ from the time-independent equations

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - i \frac{\hbar^2 K}{m} \frac{\partial}{\partial x} + U(x) - \mu + \int dx' V(x-x') \psi^*(x') \psi(x') \right] \psi(x) = 0, \quad (43)$$

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + i \frac{\hbar^2 K}{m} \frac{\partial}{\partial x} + U(x) - \mu + \int dx' V(x-x') \psi^*(x') \psi(x') \right] \psi^*(x) = 0. \quad (44)$$

2.3 First Order

In first order, (43) becomes

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - i \frac{\hbar^2 K}{m} \frac{\partial}{\partial x} - \mu \right) \psi^{(1)}(x) + U(x) \psi^{(0)} + \left(\psi^{(0)} \right)^2 \int dx' V(x-x') \left[\psi^{(1)}(x) + \psi^{*(1)}(x') + \psi^{(1)}(x') \right] = 0. \quad (45)$$

We use the Fourier series ansatz for $\psi^{(1)}(x)$, $V(x)$ and $U(x)$ as defined in equations (1)-(3) to obtain an algebraic equation for the Fourier coefficients $\psi_n^{(1)}$:

$$\sum_{n=-\infty}^{\infty} \left\{ \left[\frac{\hbar^2 k_n^2}{2m} + \frac{\hbar^2 k_n}{m} K + L \left(\psi^{(0)} \right)^2 v_n \right] \psi_n^{(1)} + L \left(\psi^{(0)} \right)^2 v_n \psi_n^{*(1)} + u_n \psi^{(0)} \right\} e^{ik_n x} = 0$$

From this, we conclude

$$\left[\epsilon_n \left(1 + \frac{2K}{k_n} \right) + c_n \right] \psi_n^{(1)} + c_n \psi_n^{*(1)} + u_n \psi^{(0)} = 0, \quad (46)$$

where we again use ϵ_n and c_n as defined in (38) and (39). The chemical potential cancels with the term $L \left(\psi^{(0)} \right)^2 v_0 \psi_n^{(1)}$ from the interaction.

Correspondingly, from equation (44) for the complex conjugate ψ^* , we get

$$\left[\epsilon_n \left(1 - \frac{2K}{k_n} \right) + c_n \right] \psi_n^{*(1)} + c_n \psi_n^{(1)} + u_n \psi^{(0)} = 0. \quad (47)$$

We now have two linear equations with the solution

$$\psi_n^{(1)} = - \frac{\left(1 - \frac{2K}{k_n} \right) u_n \psi^{(0)}}{E_n}, \quad (48)$$

$$\psi_n^{*(1)} = - \frac{\left(1 + \frac{2K}{k_n} \right) u_n \psi^{(0)}}{E_n}, \quad (49)$$

where we introduce the abbreviation

$$E_n = \epsilon_n \left[1 - \left(\frac{2K}{k_n} \right)^2 \right] + 2c_n. \quad (50)$$

We find that the disorder average vanishes in first order, since

$$\langle \psi^{(1)}(x) \rangle = \sum_{n=-\infty}^{\infty} \frac{\left(1 - \frac{2K}{k_n} \right) \psi^{(0)}}{E_n} \langle u_n \rangle e^{ik_n x} = 0, \quad (51)$$

according to (4). The same holds for $\langle \psi^{*(1)} \rangle$.

2.4 Contact Interaction

In the following sections, we will assume that the interaction between atoms is treated in a mean-field framework as a delta-interaction. Since we require a periodic potential function for the interaction, we assume a Dirac comb potential, given by a term of the form

$$V(x) = g \sum_{n=-\infty}^{\infty} \delta(x - nL), \quad (52)$$

with the Fourier series

$$V(x) = \frac{g}{L} \sum_{n=-\infty}^{\infty} e^{ik_n x}. \quad (53)$$

The value of g , which determines the effective interaction strength, does not coincide with the usual three-dimensional interaction constant $g = 4\pi\hbar^2 a_s/m$ obtained from scattering theory, which can be found in Ref. [27, sec. 5.2]. Our simplification of the trap geometry to a one-dimensional system requires a more involved calculation to obtain a value which can be used in our one-dimensional model. An exemplary derivation for the potential of the trap used in Ref. [6] can be found in Ref. [7, pp. 101-108], with the result that the effective 1-D interaction strength is the usual 3-D value multiplied by a constant involving various trap parameters. For our purposes, it should therefore be sufficient to assume that the interaction strength is given by some constant g .

We therefore obtain

$$\psi^{(0)} = \sqrt{\frac{\mu}{g}}. \quad (54)$$

In the first order solutions (48)-(49), the interaction Fourier coefficients v_n do not appear explicitly. However, they are contained in the value of E_n in (50), which changes due to (39), (53), (54) to

$$E_n = \epsilon_n \left[1 - \left(\frac{2K}{k_n} \right)^2 \right] + 2\mu. \quad (55)$$

2.5 Condensate Depletion

We are now deriving the depletion of the global condensate due to the disorder. In the presence of disorder, local minima are introduced on a short length scale, in which parts of the condensate are localized. In terms of a description of the global condensate, this corresponds to a loss of particles. We begin our discussion of the condensate depletion by briefly discussing the particle and the condensate density.

Particle Density

The particle density is defined as the total number of ground-state particles in the trap. As described in section 1.4, the squared modulus of the wavefunction can be interpreted as the particle density due to the normalization of the wavefunction in the Gross-Pitaevskii equation.

In our perturbative ansatz, ψ is a complex field with a modulus and a phase. However, in our derivation of the particle and condensate density, we have to use the modulus of the wavefunction Ψ and therefore introduce it as

$$a = \sqrt{\psi^* \psi} \quad (56)$$

$$= \psi^{(0)} \left(1 + \frac{\psi^{*(1)} + \psi^{(1)}}{2\psi^{(0)}} + \frac{\psi^{*(1)}\psi^{(1)}}{2(\psi^{(0)})^2} + \frac{\psi^{*(2)} + \psi^{(2)}}{2\psi^{(0)}} - \frac{(\psi^{*(1)} + \psi^{(1)})^2}{8(\psi^{(0)})^2} + \dots \right), \quad (57)$$

where we use the Taylor series $(1+x)^\alpha = 1 + \alpha x + \frac{1}{2}\alpha^2 x^2 + \dots$ in the last line, keeping terms up to second order in the disorder strength.

The disorder-averaged particle density is then given by the expression

$$n = \langle a^2 \rangle = (\psi^{(0)})^2 \left(1 + \frac{\langle \psi^{*(1)}\psi^{(1)} \rangle}{(\psi^{(0)})^2} + \frac{\langle \psi^{*(2)} + \psi^{(2)} \rangle}{\psi^{(0)}} + \dots \right). \quad (58)$$

We note that in zeroth order of the disorder strength, we can write

$$n = (\psi^{(0)})^2 = \frac{\mu}{g}. \quad (59)$$

Condensate Density

The onset of the BEC phase is characterized by the occurrence of long-range correlations in the Bose gas. Therefore the condensate density n_0 is commonly defined as the order parameter of the BEC phase transition and taken as the far off-diagonal elements of the density matrix. We obtain for the disorder-averaged condensate density

$$n_0 = \lim_{|x-x'|\rightarrow\infty} \langle \Psi^*(x)\Psi(x') \rangle = \langle a \rangle^2 \quad (60)$$

In the limit where $|x' - x|$ is large compared to a characteristic correlation length of the disorder, spatial averaging can be exchanged with ensemble averaging. [28, p. 7]

Substituting a from (57), we get

$$n_0 = \langle \Psi^*(x)\Psi(x') \rangle = \langle a \rangle^2 \quad (61)$$

$$= (\psi^{(0)})^2 \left(1 + \frac{\langle \psi^{*(1)}\psi^{(1)} \rangle}{(\psi^{(0)})^2} + \frac{\langle \psi^{*(2)} + \psi^{(2)} \rangle}{\psi^{(0)}} - \frac{\langle (\psi^{*(1)} + \psi^{(1)})^2 \rangle}{4(\psi^{(0)})^2} + \dots \right). \quad (62)$$

Condensate Depletion

We define the condensate depletion as the difference

$$n - n_0 = \langle a^2 \rangle - \langle a \rangle^2 = \frac{1}{4} \left\langle \left(\psi^{(1)} + \psi^{*(1)} \right)^2 \right\rangle, \quad (63)$$

where we have used (58) and (62). Inserting the first-order solution (48) and (49), and using (10) and (59) yields:

$$n - n_0 = \frac{1}{4} \left(\langle \psi^{(1)}\psi^{(1)} \rangle + 2\langle \psi^{(1)}\psi^{*(1)} \rangle + \langle \psi^{*(1)}\psi^{*(1)} \rangle \right) \quad (64)$$

$$= \frac{(\psi^{(0)})^2}{4} \sum_{n,m=-\infty}^{\infty} \frac{\langle u_n u_m \rangle}{E_n E_m} \left[\left(1 - \frac{2K}{k_n} \right) \left(1 - \frac{2K}{k_m} \right) + 2 \left(1 - \frac{2K}{k_n} \right) \left(1 + \frac{2K}{k_m} \right) + \left(1 + \frac{2K}{k_n} \right) \left(1 + \frac{2K}{k_m} \right) \right] e^{i(k_n+k_m)x} \quad (65)$$

$$= \frac{n}{4} \sum_{n=-\infty}^{\infty} \frac{r_n}{E_n^2} \left[2 \left(1 - \frac{4K^2}{k_n^2} \right) + \left(1 - \frac{2K}{k_n} \right)^2 + \left(1 + \frac{2K}{k_n} \right)^2 \right] = n \sum_{n=-\infty}^{\infty} \frac{r_n}{E_n^2} \quad (66)$$

$$= n \sum_{n=-\infty}^{\infty} \frac{r_n}{\left[\frac{\hbar^2}{2m} (k_n^2 - 4K^2) + 2gn \right]^2} \quad (67)$$

To evaluate the sum in (67), we consider the limit of a large ring trap. As $L \rightarrow \infty$, we can move from a discrete variable k_n to a continuous variable k . The sum then has the form of a Riemann sum, which can be evaluated as an integral by identifying

$$k = k_n = \frac{2\pi n}{L}, \quad (68)$$

$$dk = \frac{2\pi}{L}. \quad (69)$$

We further assume a delta-correlated potential with the correlation function

$$R(x) = r \sum_{n=-\infty}^{\infty} \delta(x - nL), \quad (70)$$

$$= \frac{r}{L} \sum_{n=-\infty}^{\infty} e^{ik_n x}. \quad (71)$$

We obtain

$$n - n_0 = n \frac{r}{L} \left(\frac{2m}{\hbar^2} \right)^2 \frac{L}{2\pi} \int_{k=-\infty}^{\infty} dk \frac{1}{\left(k^2 - 4K^2 + \frac{4mgn}{\hbar^2} \right)^2}. \quad (72)$$

The solution of the integral is

$$\int_{-\infty}^{\infty} dk \frac{1}{(k^2 + c)^2} = \lim_{a, b \rightarrow \infty} \left[\frac{k}{2c(k^2 + c)} + \frac{\arctan\left(\frac{k}{\sqrt{c}}\right)}{2c^{3/2}} \right]_{-a}^b = \frac{1}{2} \frac{1}{c^{3/2}} \pi. \quad (73)$$

Substituting this result in (72) with $c = \frac{4mgn}{\hbar^2} - 4K^2$ yields

$$n - n_0 = \left(\frac{2m}{\hbar^2} \right)^2 \frac{nr}{2\pi} \frac{\pi}{4\sqrt{2}} \left(\frac{\hbar^2}{2mgn - 2\hbar^2 K^2} \right)^{3/2} \quad (74)$$

$$= \frac{m^2}{\hbar} \frac{nr}{8} \left(\frac{1}{mgn - \hbar^2 K^2} \right)^{3/2}. \quad (75)$$

We find that the denominator in (75) tends to zero as $K \rightarrow \sqrt{\frac{mgn}{\hbar^2}}$. This wavenumber, which we will refer to as K_{sound} , corresponds to the Bogoliubov speed of sound [27, p. 189], which is the velocity at which excitations propagate through the condensate, given by the expression

$$v_{\text{sound}} = \sqrt{\frac{gn}{m}} = \frac{\hbar}{m} K_{\text{sound}}. \quad (76)$$

We now consider a critical velocity K_{crit} , at which the entire condensate is depleted into local condensate fragments. K_{crit} can be determined by setting $(n - n_0)|_{K=K_{\text{crit}}} = n$:

$$n = \frac{m^2}{\hbar} \frac{nr}{8} \left(\frac{1}{mgn - \hbar^2 K_{\text{crit}}^2} \right)^{3/2}, \quad (77)$$

$$K_{\text{crit}} = K_{\text{sound}} \sqrt{1 - \frac{1}{mgn} \left(\frac{m^2}{8\hbar} r \right)^{2/3}}. \quad (78)$$

Although the accuracy of our perturbative results is debatable as we approach this value due to our assumption of a weak disorder, it is nevertheless interesting to investigate this quantity. We find that it is always less than the sound velocity and decreases monotonically with increasing disorder strength.

Furthermore, by setting $K = 0$, our result reproduces the results by Huang and Meng [8]:

$$(n - n_0)|_{K=0} = \frac{1}{8\hbar g} \sqrt{\frac{m}{gn}} r. \quad (79)$$

It is now useful to introduce dimensionless quantities according to

$$\tilde{x} = \frac{x}{L} \quad \tilde{K} = KL \quad \tilde{g} = \frac{Lg}{\hbar\omega_0} \quad (80)$$

$$\tilde{p} = \frac{L}{\hbar} p \quad \tilde{n} = nL \quad \tilde{r} = \frac{r}{(\hbar\omega_0)^2} \quad (81)$$

with the trap circumference L and the characteristic frequency

$$\omega_0 = \frac{\hbar}{2mL^2}. \quad (82)$$

We then get a dimensionless condensate depletion

$$\frac{n - n_0}{n} = \frac{\tilde{r}}{32 \left(\frac{\tilde{g}\tilde{n}}{2} - \tilde{K}^2 \right)^{3/2}}. \quad (83)$$

For the critical velocity in units of the sound velocity, we get the dimensionless expression

$$\frac{K_{\text{crit}}}{K_{\text{sound}}} = \sqrt{1 - \frac{2}{\tilde{g}\tilde{n}} \left(\frac{\tilde{r}}{32} \right)^{2/3}}, \quad (84)$$

which is plotted in figure 2.

A comparison with the results by Huang and Meng is shown in Fig. 3. Plots of the depletion (83) for various values of r and g are shown in Fig. 4-7.

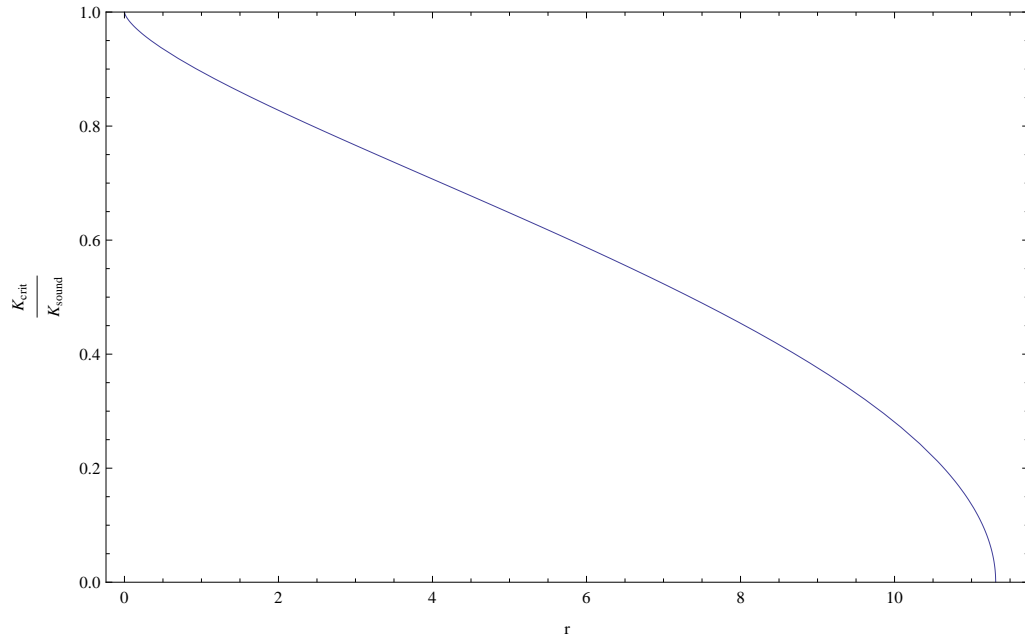


Figure 2: A plot of the critical velocity K_{crit} as a function of the dimensionless disorder strength in units of K_{sound} .

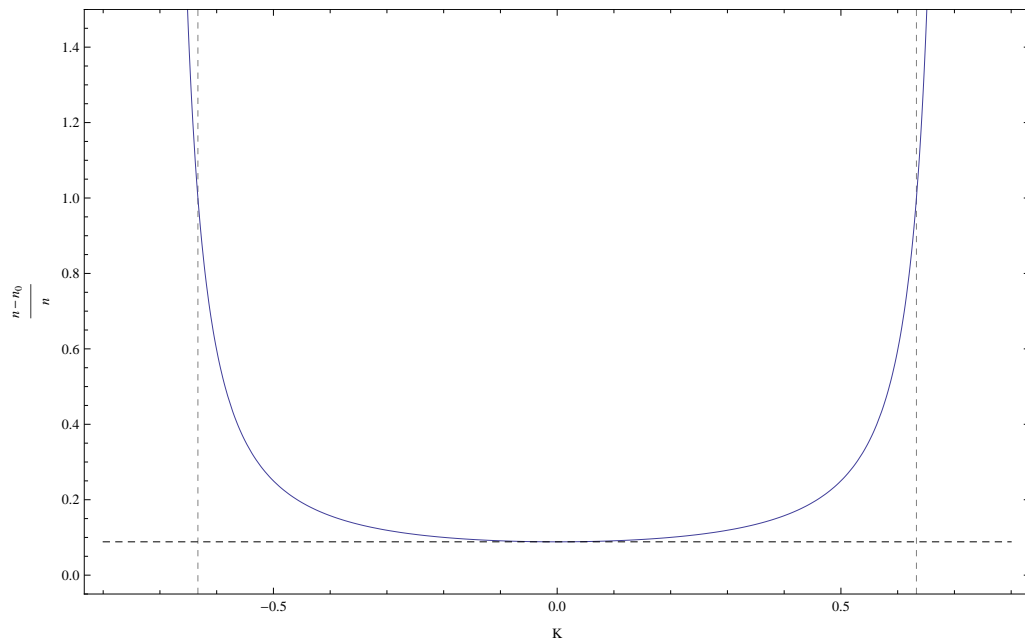


Figure 3: A plot of the condensate depletion as a function of \tilde{K} . Horizontal dashed line: Result by Huang and Meng, vertical dashed lines: values of the critical velocity \tilde{K}_{crit} .

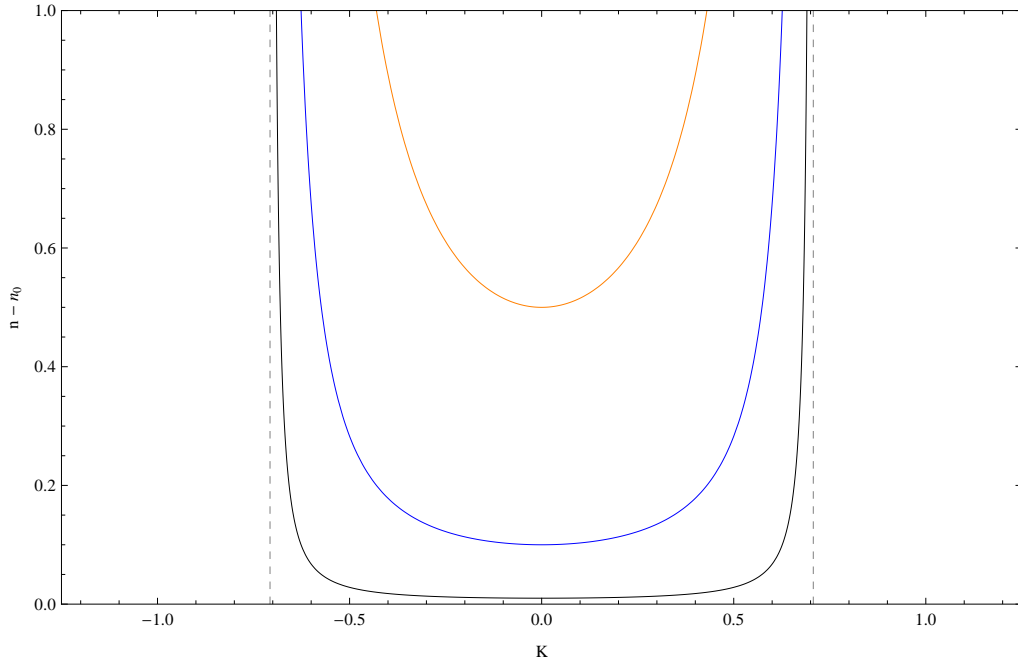


Figure 4: A plot of the condensate depletion as a function of \tilde{K} for different values of the disorder strength \tilde{r} , with $\tilde{n} = 1$, $\tilde{g} = 1$. Black: $\tilde{r} = 0.01$, Blue: $\tilde{r} = 0.1$, Orange: $\tilde{r} = 0.5$. The dashed line indicates the sound velocity \tilde{K}_{sound} .

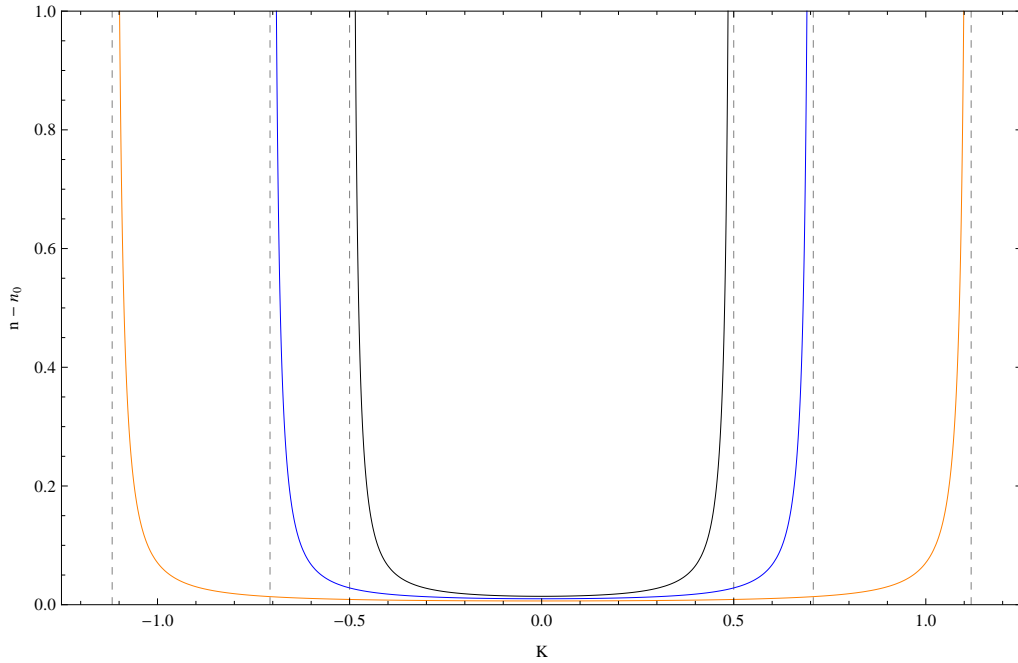


Figure 5: A plot of the condensate depletion as a function of \tilde{K} for different values of the particle density \tilde{n} , with $\tilde{g} = 1$, $\tilde{r} = 0.01$. Black: $\tilde{n} = 0.5$, Blue: $\tilde{n} = 1.0$, Orange: $\tilde{n} = 2.5$. The dashed lines indicate the respective sound velocities \tilde{K}_{sound} .

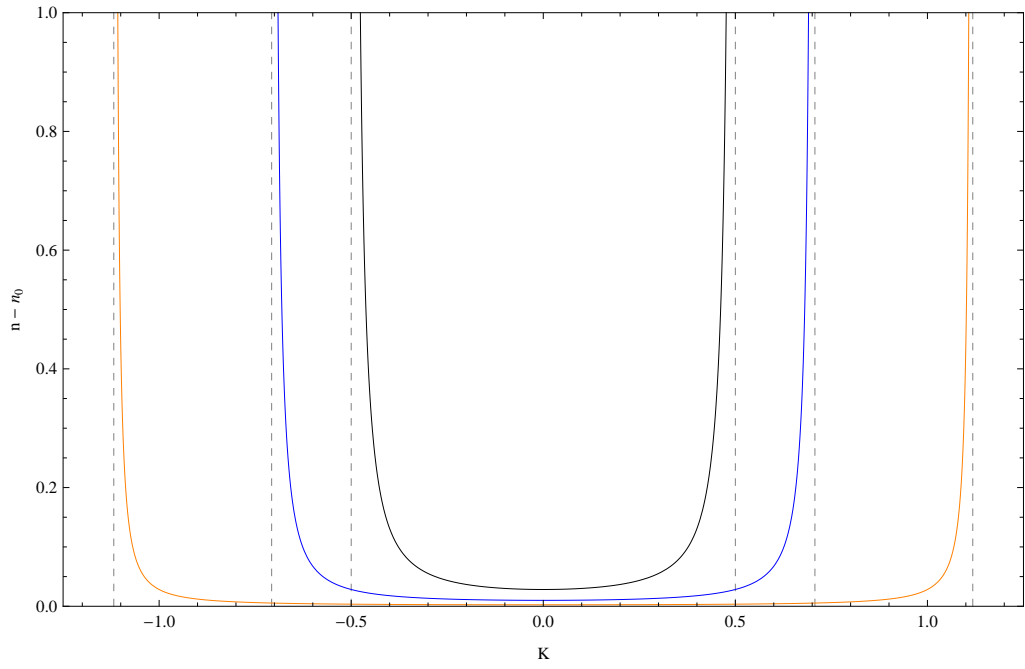


Figure 6: A plot of the condensate depletion as a function of \tilde{K} for different values of the interaction strength \tilde{g} , with $\tilde{n} = 1$, $\tilde{r} = 0.01$. Black: $\tilde{g} = 0.5$, Blue: $\tilde{g} = 1.0$, Orange: $\tilde{g} = 2.5$. The dashed lines indicate the respective sound velocities \tilde{K}_{sound} .

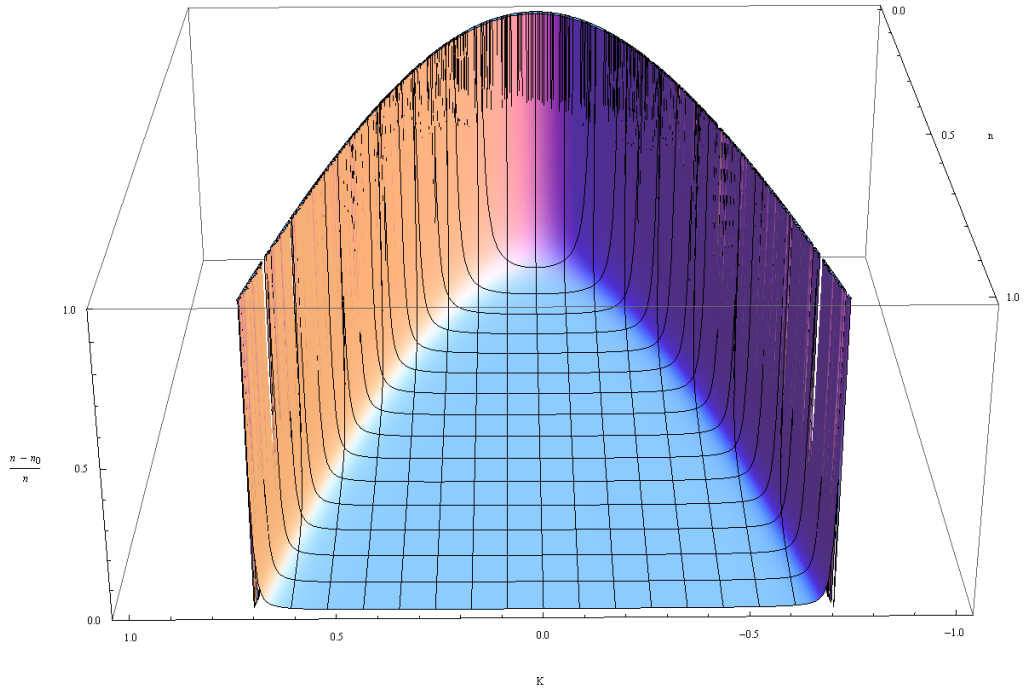


Figure 7: A 3D plot of the condensate depletion as a function of \tilde{K} and \tilde{n} with $\tilde{g} = 1$, $\tilde{r} = 0.01$.

2.6 Momentum

We now derive the expectation value for the momentum $p(n, K)$ of the superfluid. Applying the momentum operator $\hat{p} = -i\hbar\nabla$ to the wavefunction (25), we find that

$$p = \langle \Psi^* | -i\hbar\nabla | \Psi \rangle = \hbar k_s \langle \psi^* \psi \rangle - i\hbar \left(\langle \psi^{*(1)} \nabla \psi^{(1)} \rangle + \psi^{(0)} \langle \nabla \psi^{(2)} \rangle \right) \quad (85)$$

$$= \hbar k_s n - \hbar n \sum_{n=-\infty}^{\infty} r_n \frac{k_n \left(1 + \frac{2K}{k_n}\right)^2}{\left[\epsilon_n \left(1 - \frac{4K^2}{k_n^2}\right) + 2gn\right]^2}. \quad (86)$$

For a Delta-correlated disorder in the limit of a large ring, where we can once again move from summation to integration, we find

$$p \approx \hbar k_s n - \hbar n \frac{r}{L} \int_{k=-\infty}^{\infty} \frac{dk}{2\pi/L} \frac{(k + 2K)^2}{k \left[\epsilon_n \left(1 - \frac{4K^2}{k^2}\right) + 2gn\right]^2} \quad (87)$$

$$= \hbar k_s n - \frac{\hbar n r K}{2} \left(\frac{m}{\hbar^2}\right)^2 \left(\frac{\hbar^2}{mgn - \hbar^2 K^2}\right)^{3/2} \quad (88)$$

$$= \hbar k_s n - \frac{K n r m^2}{2} \left(\frac{1}{mgn - \hbar^2 K^2}\right)^{3/2}. \quad (89)$$

Although equations (43) and (44) only depend on the relative velocity K , we find that the result for the momentum of the superfluid depends on the unperturbed superfluid velocity k_s . The case of a superfluid flowing around the stationary ring corresponds to $k_s = K$, so that we obtain

$$p(K, n) = \hbar K n \left[1 - \frac{r m^2}{2\hbar} \left(\frac{1}{mgn - \hbar^2 K^2}\right)^{3/2} \right]. \quad (90)$$

As with the condensate depletion, we find that the momentum diverges as K approaches the sound velocity K_{sound} .

Substituting dimensionless quantities in (90), we obtain a dimensionless momentum function \tilde{p} :

$$\tilde{p}(\tilde{K}, \tilde{n}) = \tilde{n} \tilde{K} \left[1 - \frac{\tilde{r}}{8 \left(\frac{\tilde{g}\tilde{n}}{2} - \tilde{K}^2\right)^{3/2}} \right] \quad (91)$$

A plot of this function is shown in figures 9 and 11, where we vary the particle density and the disorder strength, respectively. Figure 12 shows the dependence of the momentum on the velocity of the superfluid. We find a linear behaviour of p for small K , followed by a steep decline as the nonlinear term in (91) becomes dominant.

Even before the relative velocity reaches the critical value, the momentum becomes negative, as shown in Fig. 8. This behaviour is counterintuitive and indicates that our model is no longer valid near K_{crit} .

With increasing disorder strength, we find that the nonlinear behaviour of the momentum emerges at lower velocities, confirming the intuitive guess that the disorder hinders

the motion of the superfluid.

We also find that increasing the interaction strength significantly raises the velocity at which the momentum drops off, corresponding to a more "robust" motion of the condensate around the ring.

We can use the linear behaviour of the function near the origin to deduce a superfluid density, by using the ansatz

$$p \approx \hbar n_S K, \quad (92)$$

for small K .

We get the result

$$n_S = \left. \frac{1}{\hbar} \frac{\partial p}{\partial K} \right|_{K=0} = n \left[1 - \frac{rm^2}{2\hbar (mgn)^{3/2}} \right]. \quad (93)$$

Rearranging for $n - n_S$, we recover the Huang-Meng result [8]

$$n - n_S = \frac{1}{2hg} \sqrt{\frac{m}{gn}} r \quad (94)$$

In the result by Huang and Meng, the depletion of the superfluid in D dimensions can be derived according to the expression

$$n - n_S = \frac{4rn}{D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left(\frac{\hbar^2}{2m} k^2 + 2gn\right)^2} = \frac{1}{2hg} \sqrt{\frac{m}{gn}} r. \quad (95)$$

As is expected, we obtain the following relation between condensate and superfluid depletion:

$$n - n_0 = \frac{1}{4}(n - n_S) \quad (96)$$

Our result for the condensate depletion is therefore consistent with the result by Huang and Meng.

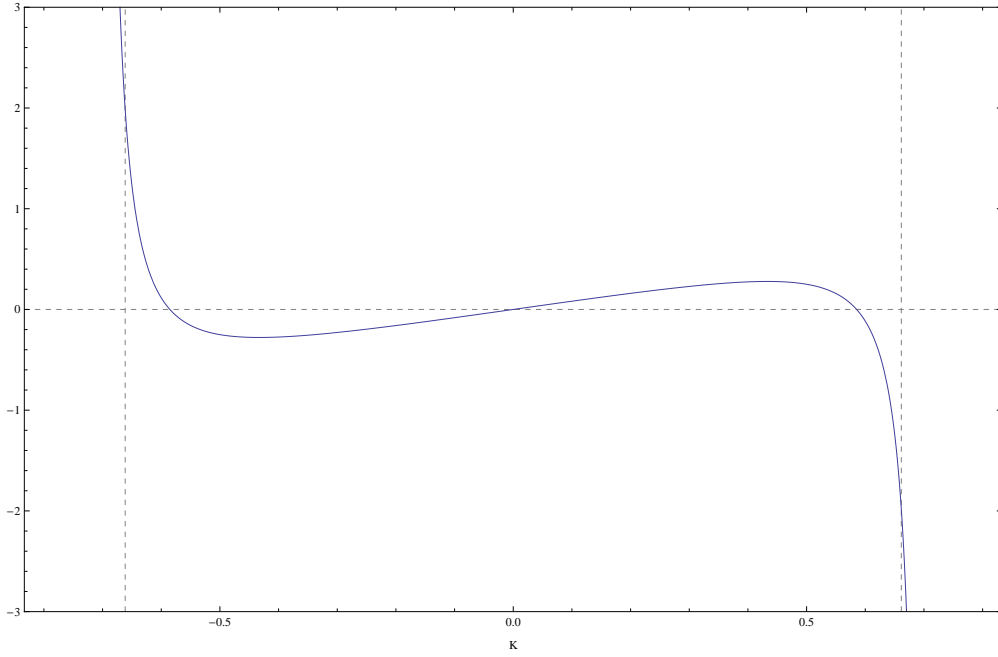


Figure 8: A plot of the momentum \tilde{p} as a function of \tilde{K} with $\tilde{g} = 1$, $\tilde{r} = 0.5$, $n = 1$. The dashed lines indicate the critical velocity \tilde{K}_{crit} .

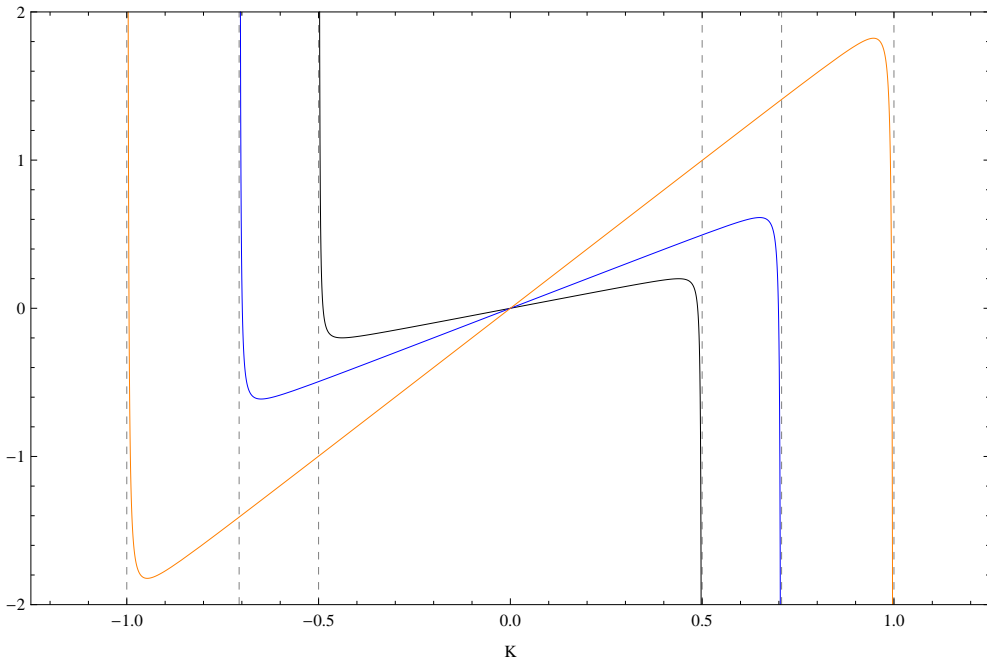


Figure 9: A plot of the momentum \tilde{p} as a function of \tilde{K} for different values of the particle number \tilde{n} , with $\tilde{g} = 1$, $\tilde{r} = 0.01$. Black: $\tilde{n} = 0.5$, Blue: $\tilde{n} = 1.0$, Orange: $\tilde{n} = 2$. The dashed lines indicate the respective sound velocities \tilde{K}_{sound} .

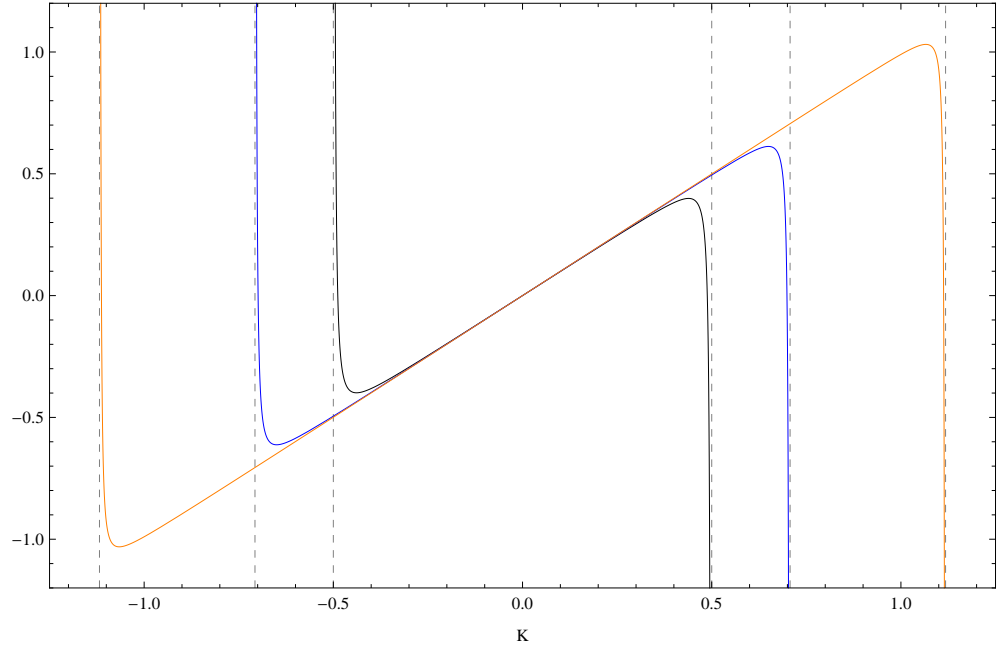


Figure 10: A plot of the momentum \tilde{p} as a function of \tilde{K} for different values of the interaction strength \tilde{g} , with $\tilde{n} = 1$, $\tilde{r} = 0.01$. Black: $\tilde{g} = 0.5$, Blue: $\tilde{g} = 1.0$, Orange: $\tilde{g} = 2.5$. The dashed lines indicate the respective sound velocities \tilde{K}_{sound} .

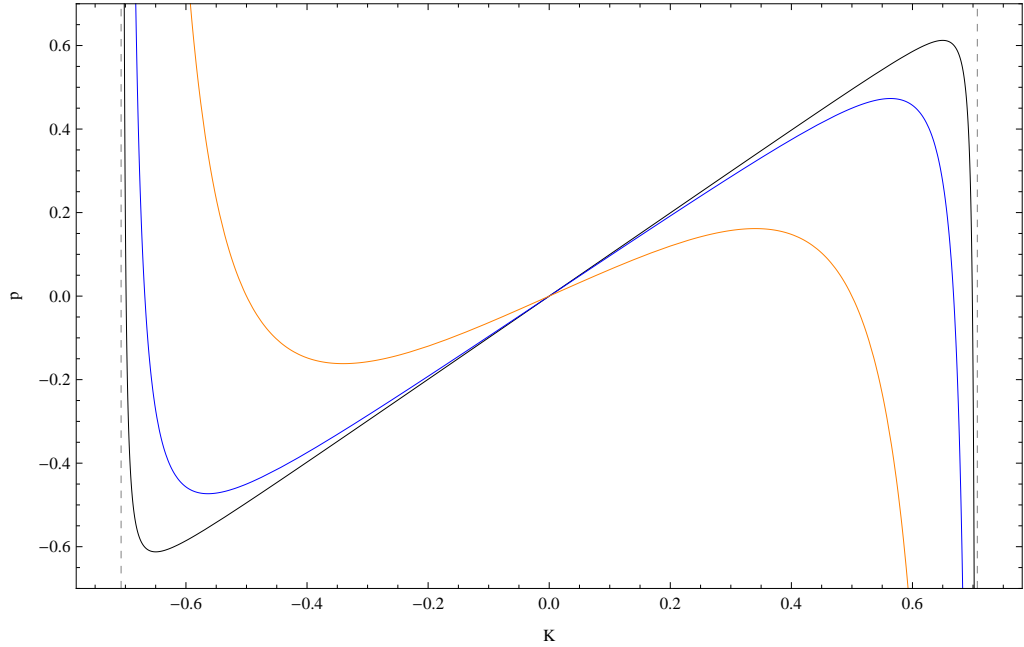


Figure 11: A plot of the momentum \tilde{p} as a function of \tilde{K} for different values of the disorder strength \tilde{r} , with $\tilde{n} = 1$, $\tilde{g} = 1$. Black: $\tilde{r} = 0.01$, Blue: $\tilde{r} = 0.1$, Orange: $\tilde{r} = 1$. The dashed line indicates the sound velocity \tilde{K}_{sound} .

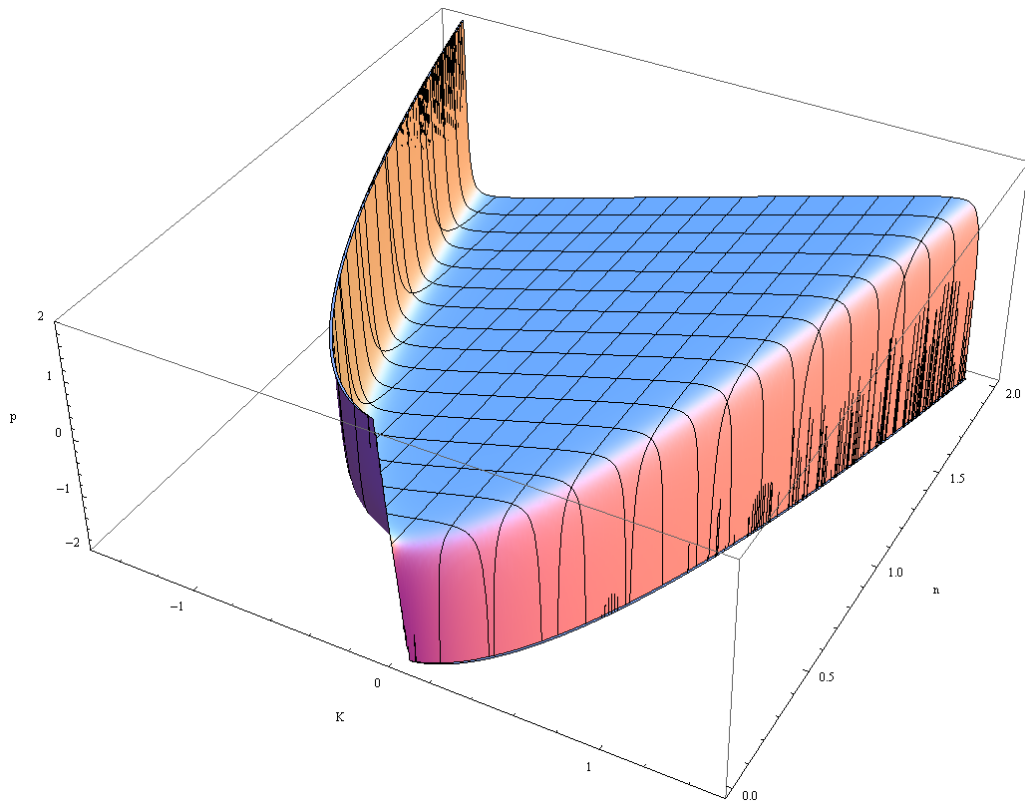


Figure 12: A 3D plot of the momentum \tilde{p} as a function of \tilde{K} and \tilde{n} with $\tilde{g} = 1$, $\tilde{r} = 0.01$.

3 Numerical Analysis

To numerically solve the Gross-Pitaevskii equation, we use an existing codebase designed to find stationary solutions of the one-dimensional GPE with a harmonic trapping potential with the split-step Crank-Nicolson method [29, 30]. We base the numerical software for the following work on the source code of the `imagtime1d` program in Ref. [30], including modifications to allow for the inclusion of a Delta-correlated disorder potential and to account for the additional term in the Gross-Pitaevskii equation due to the relative motion of the disorder. All numerical calculations are performed with Python.

3.1 Disorder Generation

We now present a method to generate periodic delta-correlated disorder potentials. We consider the disorder as the partial sum of a Fourier series

$$U(x) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} [A_n \cos(k_n x) + B_n \sin(k_n x)]. \quad (97)$$

For this ansatz, some important results are derived in Ref. [31], which we now briefly discuss.

Inserting this ansatz in the statistical constraints of the disorder (4) and its correlation function (5), we obtain

$$\langle U(x) \rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} [\langle A_n \rangle \cos(k_n x) + \langle B_n \rangle \sin(k_n x)] = 0, \quad (98)$$

$$\langle U(x)U(0) \rangle = \frac{1}{N} \sum_{n,m=0}^{N-1} [\langle A_n A_m \rangle \cos(k_n x) + \langle B_n A_m \rangle \sin(k_n x)] = R(x). \quad (99)$$

In our case, a periodic potential is assumed. Therefore, the wavenumbers k_n are given by (3). If the k_n are sampled from a gaussian distribution, a Gaussian correlated disorder is obtained. This case is discussed in depth in Ref. [31].

We are interested in generating a delta-correlated disorder potential. We can use the Fourier representation of a delta-correlated disorder (71) to write

$$R(x) = \sum_{n=0}^{\infty} \frac{2r}{L} \cos(k_n x). \quad (100)$$

The coefficients of the sine terms of the Fourier series vanish since the correlation is an even function.

From (98), we immediately find

$$\langle A_n \rangle = \langle B_n \rangle = 0, \quad (101)$$

and by comparing coefficients of (99) and (100), we obtain:

$$\sum_{m=0}^{N-1} \langle A_n B_m \rangle = 0, \quad (102)$$

$$\sum_{m=0}^{N-1} \langle A_n A_m \rangle = \frac{2rN}{L}. \quad (103)$$

These constraints are met if

$$\langle A_n B_m \rangle = 0, \quad (104)$$

$$\langle A_n A_m \rangle = \frac{2rN}{L} \delta_{n,m}. \quad (105)$$

We therefore draw the coefficients A_n from a normal distribution with standard deviation σ and mean μ according to

$$\sigma = \sqrt{\frac{2rN}{L}}, \quad (106)$$

$$\mu = 0. \quad (107)$$

A sample potential generated by this method is shown in figure 13, while the convergence of the correlation to the representation of the delta comb is shown in figure 14.

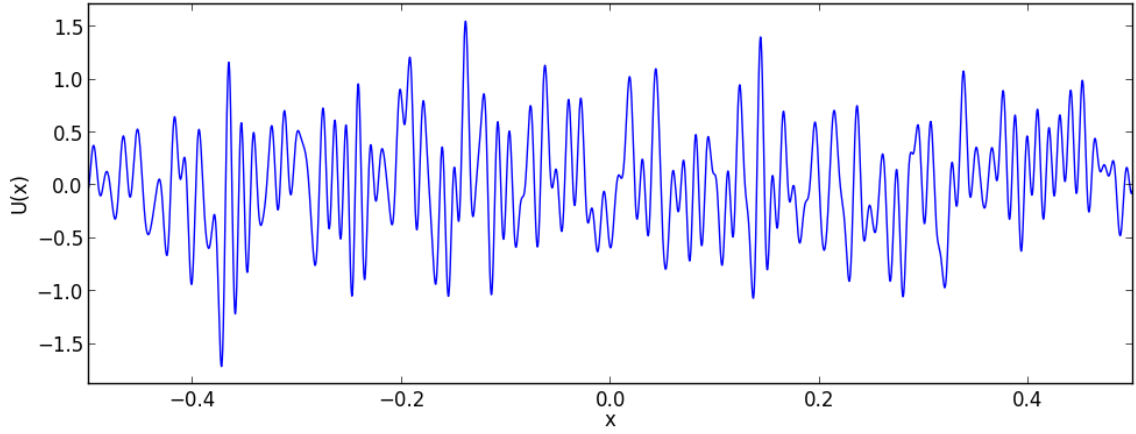


Figure 13: Example of a delta-correlated disorder realization with $N = 100$ and $r = 0.1$.

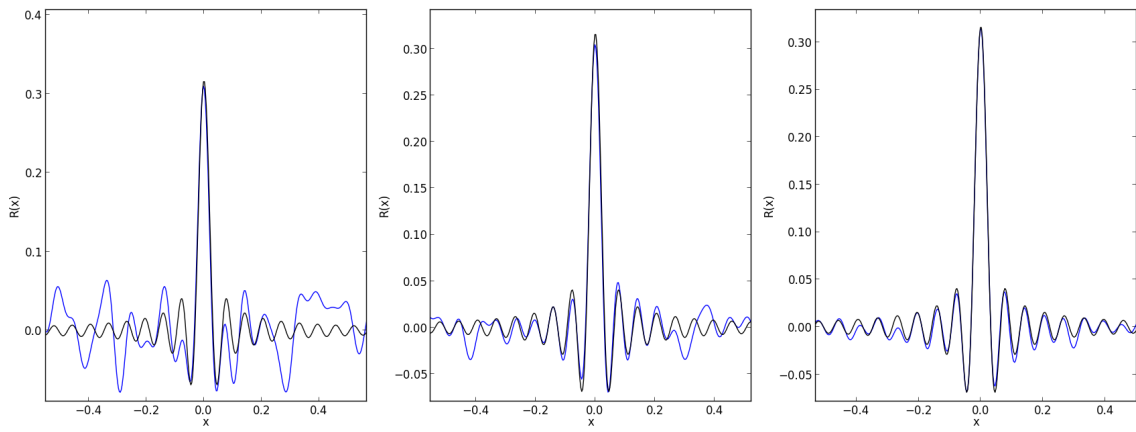


Figure 14: Comparison of the correlation $\langle U(x)U(0) \rangle$ (blue) with the expected correlation function $R(x)$ (black) with $N = 100$ and $r = 1$. Left to right: Averages of 100, 1000 and 10000 disorder realizations.

3.2 Imaginary-Time Propagation

We now discuss how the stationary ground state solution of the time-dependent Gross-Pitaevskii equation (11) is obtained numerically, reproducing the discussion in Ref. [29].

Following the procedure and notation of Ref. [29], we start with the partial differential equation (27), where we now take the time to be an imaginary parameter $t = -i\tau$ and move to dimensionless quantities as in section 2.5, where we additionally introduce

$$\tilde{\mu} = \frac{\mu}{\hbar\omega_0}, \quad \tilde{\tau} = \tau\omega_0, \quad \tilde{U} = \frac{U}{\hbar\omega_0}. \quad (108)$$

We then obtain the dimensionless equation

$$-\frac{\partial\psi}{\partial\tilde{\tau}} = \left[-\frac{\partial^2}{\partial\tilde{x}^2} - 2i\tilde{K}\frac{\partial}{\partial\tilde{x}} + \tilde{U}(\tilde{x}) - \tilde{\mu} + \tilde{g}|\psi(\tilde{x}, \tilde{\tau})|^2 \right] \psi(\tilde{x}, \tilde{\tau}). \quad (109)$$

In the following derivations, the tilde is omitted from the dimensionless quantities.

This equation has the form

$$-\frac{\partial\Psi(x, \tau)}{\partial\tau} = \hat{H}\Psi. \quad (110)$$

If some function ψ_i is an eigenstate of the Hamiltonian \hat{H} with eigenvalue E_i , we have

$$-\frac{\partial\psi_i}{\partial\tau} = \hat{H}\psi_i = E_i\psi_i \quad (111)$$

$$\Rightarrow \psi_i(\tau) = \psi_i(0)e^{-E_i\tau}. \quad (112)$$

We start with some arbitrary initial guess for the wavefunction $\Psi(0)$, which can be thought of as a linear combination of eigenstates. We see that propagation under this imaginary time parameter leads to an exponential decay of the amplitude of each eigenstate of the Hamiltonian. Since the rate of the decay increases with the eigenvalue according to (112), after a sufficient amount of imaginary time, only the state with the lowest energy eigenvalue contributes significantly to $\Psi(x, \tau)$.

Since all eigenstates decay according to (112), the normalization of the initial guess $\Psi(x, \tau)$ is not conserved. Therefore, the wavefunction must be normalized after each iteration of the imaginary time propagation.

3.3 Crank-Nicolson Method

The Crank-Nicolson method [32] is a finite-difference method, meaning that it uses a discretized grid of space points. Spatial derivatives are approximated by differences of the function values at adjacent grid points, yielding a set of algebraic difference equations in place of the initial differential equation.

We will split the time-propagation of the Hamiltonian in two steps, where the first step only involves the terms of the Hamiltonian containing no derivatives. In a second step, the spatial derivatives are taken into account. The Hamiltonian \hat{H} is separated into the

operators \hat{H}_1 and \hat{H}_2 , such that

$$\hat{H}_1 = U(x) - \mu + g|\psi(x, \tau)|^2, \quad (113)$$

$$\hat{H}_2 = -\frac{\partial^2}{\partial x^2} - 2iK \frac{\partial}{\partial x}. \quad (114)$$

We next discretize equation (109), introducing a grid of n_x space points x_i separated by a distance Δ_x and a grid of n_t time points τ_n separated by a distance Δ_τ , where we use the shorthand notation

$$\psi(i\Delta_x, n\Delta_\tau) = \psi_i^n. \quad (115)$$

With the usual definition for the time evolution operator for a time-independent Hamiltonian [33, p. 290]

$$U(t, t_0) = \exp\left(-\frac{i}{\hbar}\hat{H}(t - t_0)\right) \quad (116)$$

we find that one step in the time evolution with respect to \hat{H}_1 can be performed in a straightforward manner with the evolution operation

$$U_1 = \exp\left(-i\hat{H}_1\Delta_\tau\right), \quad (117)$$

where the constant \hbar is absorbed by our dimensionless units.

In the Crank-Nicolson method, space derivatives are discretized using central differences averaged over the wavefunction ψ_i^{n+1} and the wavefunction $\psi_i^{n+1/2} = U_1\psi_i^n$ after application of the non-derivative Hamiltonian \hat{H}_1 . We therefore replace

$$\left.\frac{\partial\psi}{\partial x}\right|_{x=x_i} \rightarrow \frac{1}{4\Delta_x}(\psi_{i+1}^{n+1} - \psi_{i-1}^{n+1} + \psi_{i+1}^{n+1/2} - \psi_{i-1}^{n+1/2}), \quad (118)$$

$$\left.\frac{\partial^2\psi}{\partial x^2}\right|_{x=x_i} \rightarrow \frac{1}{2\Delta_x^2} \left[(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) + (\psi_{i+1}^{n+1/2} - 2\psi_i^{n+1/2} + \psi_{i-1}^{n+1/2}) \right]. \quad (119)$$

The time derivative in (109) is approximated by a two-point forward difference involving the partially propagated wavefunction $\psi_i^{n+1/2}$ and the future wavefunction ψ_i^{n+1} . The discretized version of (109) then reads

$$-\frac{(\psi_i^{n+1} - \psi_i^{n+1/2})}{\Delta_\tau} = -\frac{1}{2\Delta_x^2} \left[(\psi_{i+1}^{n+1} - 2\psi_i^{n+1} + \psi_{i-1}^{n+1}) + (\psi_{i+1}^{n+1/2} - 2\psi_i^{n+1/2} + \psi_{i-1}^{n+1/2}) + \frac{i\Delta_x K}{2} (\psi_{i+1}^{n+1} - \psi_{i-1}^{n+1} + \psi_{i+1}^{n+1/2} - \psi_{i-1}^{n+1/2}) \right]. \quad (120)$$

This corresponds to the set of algebraic equation

$$A_i^- \psi_{i-1}^{n+1} + A_i^0 \psi_i^{n+1} + A_i^+ \psi_{i+1}^{n+1} = b_i, \quad (121)$$

with the coefficients

$$A_i^- = \frac{\Delta_t}{2\Delta_x^2} - \frac{iK\Delta_t}{4\Delta_x}, \quad (122)$$

$$A_i^0 = 1 - \frac{\Delta_t}{\Delta_x^2}, \quad (123)$$

$$A_i^+ = \frac{\Delta_t}{2\Delta_x^2} + \frac{iK\Delta_t}{4\Delta_x}, \quad (124)$$

and a quantity b_i involving only these coefficients and the known $\psi_i^{n+1/2}$:

$$b_i = A^+ \psi_{i+1}^{n+1/2} - A^0 \psi_i^{n+1/2} + A^- \psi_{i-1}^{n+1/2} \quad (125)$$

To solve (121), we assume the following relation between values of the wavefunction at neighbouring sites:

$$\psi_{i-1}^{n+1} = \alpha_i \psi_i^{n+1} + \beta_i. \quad (126)$$

After inserting this ansatz in (121), we can solve the resulting equation for α_i and β_i with the result

$$\alpha_{i+1} = \gamma_i A_i^+, \quad \beta_{i+1} = \gamma_i (A_i^- \beta_i - b_i), \quad \text{with} \quad \gamma_i = \frac{-1}{A_i^0 + A_i^- \alpha_i}. \quad (127)$$

We now have to make an initial choice for one of the α_i , ensuring that the periodic boundary conditions are met. To this end, we (implicitly) take the spatial indices of all quantities in (121) and (127) modulo n_x . We then fix the α_i , β_i and γ_i by choosing $\alpha_0 = 1$ and applying the recursion relations (127).

3.4 Numerical Results

Figure 15 shows the result of the numerical simulation. We find a linear increase of the momentum, after which the increase of the momentum gradually comes to the halt. The momentum then decays slowly. The onset of this decay seems to coincide with the critical velocity derived in (78).

While the numerical results do not fit the curve obtained from perturbation theory, the critical velocity seems to coincide with the onset of the momentum decay in the numerical simulation. Due to time constraints, this problem could not yet be resolved.

4 Conclusions

In this thesis, we set out to investigate the motion of a Bose-Einstein condensate trapped on a ring in the presence of a random disorder potential. We were able to derive the depletion and the momentum of the condensate as a function of the condensate's velocity around the ring. We were able to reproduce results first derived by K. Huang and H.-F. Meng for stationary disorder potentials and to extend them to the case of a relative motion between the condensate and the potential. We then attempted a numerical verification

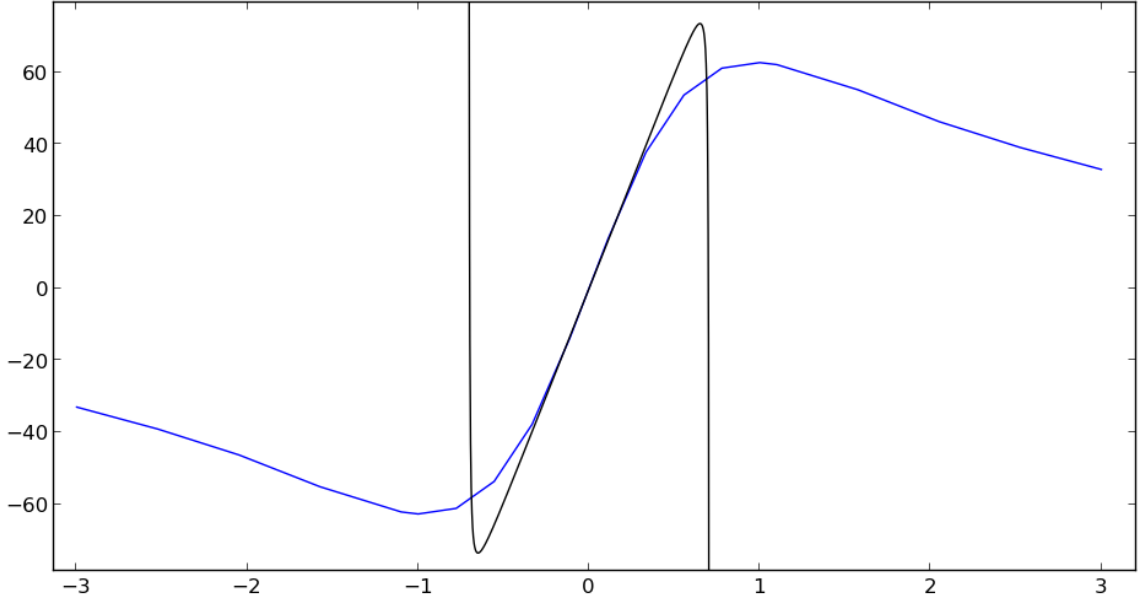


Figure 15: A plot of the numerical result for the momentum \tilde{p} as a function of \tilde{K} with the parameters $N = 100$, $r = 0.01$, $g = 1$. Blue: Numerical Results. Black: Theoretical curve from (91), stretched by a factor of 120.

of these results by solving the Gross-Pitaevskii equation numerically using the split-step Crank-Nicolson method. Unfortunately, this numerical approach could not be followed through to the extent desired due to time constraints.

We now present some possible future generalizations of the results derived in this thesis. In this work, we have only considered the limit of a very large ring trap and neglected a possible dependence of the physical quantities of interest on the ring size. Since the size of the ring is obviously an important and accessible experimental parameter, it would be insightful to generalize these results to ring traps of arbitrary size. This could be done with a more delicate treatment of the sums by using the Poisson summation formula to obtain a more accurate result.

Naturally, a derivation of higher order corrections in the perturbative framework could yield some interesting results, specifically regarding the derivation of the chemical potential, which was only incorporated in zeroth order in this work. Such a calculation could also directly yield an expression for the particle and condensate density in second order.

A more sophisticated approach to this problem would consider a separate perturbative expansion of the phase of the ground state wave function, allowing for the phase of the perturbative solution to vary in space. This possibility is neglected in our ansatz presented above. Another possible focus could be a closer examination of the impact of the disorder on the superfluid velocity.

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Declaration

I hereby confirm that I have completed this work on my own and that the references listed above include all sources of information that I used.

Berlin, August 13, 2013

Elias Döhne